Extended supersymmetry and Yano F-structures

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$$\{\bar{\mathbb{D}}_\pm, \bar{\bar{\mathbb{D}}}_\pm\} = 2i\partial_{\substack{+\\=}} \ ,$$

Left ℓ and Right $\mathfrak{r}$ Semichiral superfields

$$\bar{\mathbb{D}}_+\ell = 0 \ , \quad \bar{\mathbb{D}}_-\mathfrak{r} = 0$$

T. Buscher, U. L., M. Roček Phys.Lett.B 202 (1988) 94-98

Let $L = (\ell, \bar{\ell})$ and $R = (\mathfrak{r}, \bar{\mathfrak{r}})$. Equal number of left and right semichirals. Then a general action is

$$S = \int d^2x \mathbb{D}^2 \bar{\mathbb{D}}^2 K(L, R)$$

It reduces to a sigma model in $(1,1)$ superspace as follows:

The reduction

$$\bar{\mathbb{D}}_+\ell = (D_+ + iQ_+)\ell = 0\ , \quad \Rightarrow Q_+L = JD_+L$$
$$\bar{\mathbb{D}}_-\mathfrak{r} = (D_- + iQ_-)\mathfrak{r} = 0\ , \quad \Rightarrow Q_-R = JD_-R$$

where $L = (\ell, \bar{\ell})$ and $R = (\mathfrak{r}, \bar{\mathfrak{r}})$ as before. In addition we define

$$Q_-L = \Psi_-\ , \qquad Q_+R = \Psi_+$$

These are spinorial auxiliary fields. When they are integrated out of the action they become part of the complex structures.

Reduction

$$(L,R) \to (L,R)_|$$
$$(Q_-L, Q_+R)_| = (\Psi_-, \Psi_+) \Rightarrow (\Psi_-(L,R)_|, \Psi_+(L,R)_|)$$

The reduction of the model is

$$\int d^2x \mathbb{D}^2\bar{\mathbb{D}}^2 K(L,R)$$
$$\to \int d^2x D^2 Q^2 K(L,R)_|$$
$$= \int d^2x D^2 \big[(D_+LC_{LL} - D_+RK_{RL}J)\Psi_-^L - \Psi_+^R(C_{RR}D_-R + JK_{RL}D_-L)$$
$$+D_+LJK_{LR}JD_-R + \Psi_+^R K_{RL}\Psi_-^L\big]$$
$$\Rightarrow \int d^2x D^2 \Big(D_+X^A(G_{AB} + B_{AB})X^B\Big) = \int d^2x D^2 (D_+X^A E_{AB} X^B)$$

where $\to$ indicates reduction, $\Rightarrow$ implies solving for the auxiliary spinors and where $X^A = (L,R)$.

The potential

The metric plus B field are

$$
\begin{aligned}
E_{LL} &= C_{LL}K^{LR}JK_{RL} \\
E_{LR} &= JK_{LR}J + C_{LL}K^{LR}C_{RR} \\
E_{RL} &= -K_{RL}JK^{LR}JK_{RL} \\
E_{RR} &= -K_{RL}JK^{LR}C_{RR}
\end{aligned}
$$

with $C_{LL} := [J, K_{LL}]$ etc., while the complex structures read

$$
\mathbb{J}_+ = \begin{pmatrix} J & 0 \\ K^{RL}C_{LL} & K^{RL}JK_{LR} \end{pmatrix}, \quad \mathbb{J}_- = \begin{pmatrix} K^{LR}JK_{RL} & K^{LR}C_{RR} \\ 0 & J \end{pmatrix}
$$

and will not commute in general.

GHR Bihermitean geometry= GKG

A manifold endowed with two complex structures J_+ and J_-, a metric g and an antisymmetric B-field B carries a *bihermitian geometry* if g is hermitian with respect to both complex structures

$$J^t_\pm g J_\pm = g \ ,$$

and the two complex structures are covariantly constant with respect to two connections with torsion

$$\nabla_+ J_+ = 0 \ , \quad \nabla_- J_- = 0 \ ,$$

where the torsionfull connections are

$$\nabla_+ = \nabla_0 + T \ , \quad \nabla_- = \nabla_0 - T \ , \quad T_{ij}{}^k = \tfrac{1}{2} H_{ijn} g^{nk} \ .$$

There are two distinct cases of this geometry depending on whether the two complex structures commute or not.

Additional susy starting from $(1, 1)$.

J.Gates, C.Hull and M. Roček 1984, C.Hull and E. Witten 1985.

A general sigma model in $(1, 1)$ is

$$S = \int d^2xD_+D_-\Big(D_+\Phi^i(x,\theta)(G_{ij}+B_{ij})(\Phi)D_-\Phi^j(x,\theta)\Big)_|$$

$$= \int d^2x\partial_{+\!\!+}\phi^i E_{ij}(\phi)\partial_{=}\phi^j + \dots .$$

It has $(1, 1)$ supersymmetry manifest by construction. Additional supersymmetries will have the form (GHR),

$$\delta\Phi^i = \epsilon^+ J^i_{(+)k}D_+\Phi^k + \epsilon^- J^i_{(-)k}D_-\Phi^k$$

The conditions on J follow from two requirements:

- Closure of the algebra $[\delta_1, \delta_2]\Phi = -i2\epsilon_1\epsilon_2\partial\Phi$
- Invariance of the action $\delta S = 0$

Additional susy starting from $(2,2)$.

M. Goteman, U. L., M. Roček and I. Ryb, JHEP 09 (2010) 055:

U.L. 2207.11780 [hep-th] (2022)

We now ask instead under what conditions the $(2,2)$ model

$$\int d^2x\mathbb{D}^2\bar{\mathbb{D}}^2K(L,R)$$

has additional supersymmetries δL and δR, thinking of the $(2,2)$ semi chiral fields $(L,R)=\mathbb{X}^i$, $i=1\dots 4d$ as coordinates on the target space. The ansatz for such a symmetry is dictated by the chirality constraints on the superfields

$$\delta\mathbb{X}^i := \bar{\epsilon}^\alpha U^{(\alpha)i}{}_j\bar{\mathbb{D}}_\alpha\mathbb{X}^j + \epsilon^\alpha V^{(\alpha)i}{}_j\mathbb{D}_\alpha\mathbb{X}^j$$

where $\alpha=\pm$. The conditions on the functions U and V are again found from closure of the algebra and invariance of the action, but are rather different than in the $(1,1)$ formulation.

Yano F-structures

The matrices $U^{(\alpha)}$ (and $V^{(\alpha)}$) are degenerate and satisfy

$$U^{(+)}V^{(+)} = -\mathrm{diag}(1,0,1,1), \quad V^{(+)}U^{(+)} = -\mathrm{diag}(0,1,1,1),$$
$$U^{(-)}V^{(-)} = -\mathrm{diag}(1,1,1,0), \quad V^{(-)}U^{(-)} = -\mathrm{diag}(1,1,0,1)$$

where each entry is a $d \times d$ matrix. This prevents a direct interpretation in terms of complex structures on the tangent space TM. If the coordinates of $T_{(\alpha)}M$ are $(\mathbb{X}^i, \mathbb{D}_\alpha \mathbb{X}^i)$ and those of $\bar{T}_{(\alpha)}M$ are $(\mathbb{X}^i, \bar{\mathbb{D}}_\alpha \mathbb{X}^i)$ then $U^{(\alpha)}$ acts on $T_{(\alpha)}M$ and $V^{(\alpha)}$ on $\bar{T}_{(\alpha)}M$ so that the combined action is on $T_{(\alpha)}M \oplus \bar{T}_{(\alpha)}M$. This allows an interpretation of the transformation matrices as a Yano F-structures on the doubled tangent bundles.

$$\mathcal{F}_{(\alpha)} := \begin{pmatrix} 0 & U^{(\alpha)} \\ V^{(\alpha)} & 0 \end{pmatrix} \quad \Longrightarrow \mathcal{F}^3_{(\alpha)} + \mathcal{F}_{(\alpha)} = 0\,.$$

Projection operators

For each α then follow two complementary projection operators

$$m_{(\alpha)} := 1 + \mathcal{F}^2_{(\alpha)} \quad l_{(\alpha)} := -\mathcal{F}^2_{(\alpha)}$$

satisfying

$$l_{(\alpha)} + m_{(\alpha)} = 1, \quad l^2_{(\alpha)} = l_{(\alpha)}, \quad m^2_{(\alpha)} = m_{(\alpha)}\ , \quad l_{(\alpha)} m_{(\alpha)} = 0$$

and

$$\mathcal{F}_{(\alpha)} l_{(\alpha)} = l_{(\alpha)} \mathcal{F}_{(\alpha)} = \mathcal{F}_{(\alpha)}, \quad m_{(\alpha)} \mathcal{F}_{(\alpha)} = \mathcal{F}_{(\alpha)} m_{(\alpha)} = 0.$$

Invariance of the action

Invariance of

$$S = \int d^2x \mathbb{D}^2 \bar{\mathbb{D}}^2 K(L, R)$$

under $+$ transformations gives the following condition

$$\left(K_i U^{(+)i}{}_{[j} \right)_{k]} = 0, \quad j, k \neq \ell ,$$

and similarly for other $U^{(-)}$ and the Vs. This may be interpreted as the existence of a symplectic structure Ω

$$\Omega = \begin{pmatrix} 0 & \mathbb{K} \\ -\mathbb{K}^t & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{K} & 0 \\ 0 & -\mathbb{K}^t \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1}_{4d} \\ \mathbf{1}_{4d} & 0 \end{pmatrix} := \mathcal{K}\eta ,$$

with $\mathbb{K}$ formed from second derivatives of the potential $K(L, R)$, $\mathcal{K}$ a neutral metric and η a local product structure.

The projections of Ω are

$$I_{(\alpha)}\,\Omega\,I_{(\alpha)} =: \Omega^{(\alpha)}~,$$

they satisfy

$$\mathcal{F}^t_{(\alpha)}\Omega^{(\alpha)}\mathcal{F}_{(\alpha)} = \Omega^{(\alpha)}~,$$

i.e. the F-structures preserve Ω on the subspaces $\Lambda^{(\alpha)}$ defined by the projectors $I_{(\alpha)}$.

Integrability of $\mathcal{F}$

To discuss integrability we introduce a doubled manifold $\mathbb{M} = M^2$ and integrability of endomorphisms of $T\mathbb{M}$ via the Lie bracket. The doubled tangent space then follows from a double foliation of $\mathbb{M}$, the identification of coordinates on $\mathbb{M}$ as coordinates on two leafs S and $\tilde{S}$ of the foliation and finally the restriction to the physical $\mathcal{M}$ as a base via a section condition. This entails splitting the tangent space using the projection operators

$$P = \tfrac{1}{2}\Big(\mathbf{1} + \eta\Big)~, \qquad \tilde{P} = \tfrac{1}{2}\Big(\mathbf{1} - \eta\Big)~.$$

Using these we have

$$T\mathbb{M} = L \oplus \tilde{L}~,$$

where the ± 1 eigenspaces L and $\tilde{L}$ are integrable distributions.

By Frobenius theorem L and $\tilde{L}$ define a double foliation structure in $\mathbb{M}$ so that $L = TS$, $\tilde{L} = T\tilde{S}$. The spacetime M is identified with a leaf of the foliation. This introduces coordinates $(\mathbb{X}^i, \mathbb{Y}^i)$ on $\mathbb{M}$. The physical spacetime is then choosen by a section constraint.
So now we define integrability of $\mathcal{F}$ by requiring that its Nijenhuis tensor $\mathcal{N}_{IJ}{}^P(\mathcal{F})$ on $T\mathbb{M}$ vanishes when we impose the section constraint which we take to be $\frac{\partial}{\partial \mathbb{Y}^i} \sim 0$.

The conditions on $U^{(\alpha)}$ and $V^{(\alpha)}$ from closure of the algebra ensures the integrability of $\mathcal{F}$

For each α they define two complementary distributions of dimensions $6d$ and $2d$, respectively. They trivially fulfil the integrability conditions

$$m^i_{(\pm)l}\mathcal{N}^l_{\mathcal{F}_{(\pm)}jk} = 0, \qquad \mathcal{N}^i_{\mathcal{F}_{(\pm)}jk}m^j_{(\pm)l}m^k_{(\pm)m} = 0.$$

where $\mathcal{N}_{\mathcal{F}}$ is the Nijenhuis tensor for $\mathcal{F}$.

Closure of the algebra has led to two integrable Yano *F*-structures on a doubled tangent space with associated splitting of this tangent space into involutive subspaces

The bi-quaternionic structure at (1, 1)

$$\delta^{\mathcal{A}}\mathbb{X}^i_| = \epsilon^+_{\mathcal{A}} J^{i(\mathcal{A})}_{(+)j} D_+ \mathbb{X}^j_| + \epsilon^-_{\mathcal{A}} J^{i(\mathcal{A})}_{(-)j} D_- \mathbb{X}^j_| \,,$$

$$J^{(\mathcal{A})}_{(\pm)} J^{(\mathcal{B})}_{(\pm)} = -\delta^{\mathcal{AB}} + \epsilon^{\mathcal{ABC}} J^{(\mathcal{C})}_{(\pm)} \,,$$

$$J^{(1)}_{(\alpha)} = U^{(\alpha)}\pi_{(\alpha)} + V^{(\alpha)}\bar{\pi}_{(\alpha)}$$

$$J^{(2)}_{(\alpha)} = i\left(U^{(\alpha)}\pi_{(\alpha)} - V^{(\alpha)}\bar{\pi}_{(\alpha)}\right) .$$

where

$$\pi_{(\pm)} := \tfrac{1}{2}\left(\mathbf{1} + iJ^{(3)}_{(\alpha)}\right) \ .$$

Summary

Additional off-shell supersymmetry for the symplectic sigma model requires

1. That the manifold has dimension $4d$ with $d > 1$.
2. That the doubled tangent bundles $\mathbb{T}_{(\alpha)} = T_{(\alpha)}M \oplus \bar{T}_{(\alpha)}M$ carry integrable Yano F-stuctures $\mathcal{F}_{(\alpha)}$ and a symplectic form Ω.
3. That $\mathcal{F}_{(\alpha)}$ preserves Ω.
4. That each $\mathcal{F}$ splits $\mathbb{T}$ into invariant complementary distributions defined by the projection operators $l_{(\alpha)}$ and $m_{(\alpha)}$.
5. Reduction to $(1,1)$ gives an interpretation of this structure in terms of the pre-existing bi-hermitian geometry.

Σας ευχαριστώ για την προσοχή σας