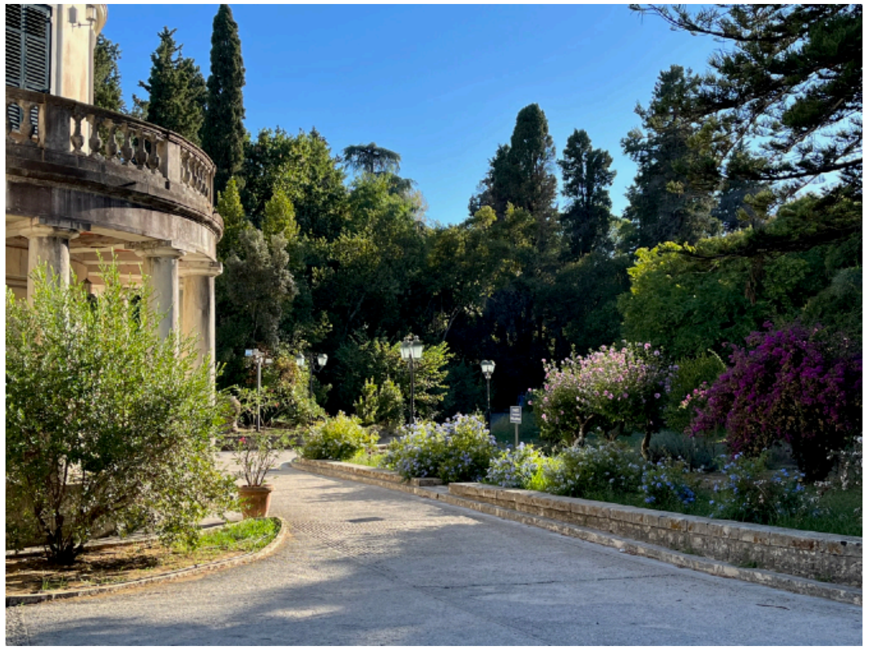
Noncommutative hamiltonian formalism for noncommutative gravity

L. Castellani

Università del Piemonte Orientale and INFN, Torino
Regge Center for Algebra, Geometry and Physics



Workshop on Noncommutative and generalized geometry in string theory, gauge theory and related physical models, Corfu, Sept. 2022

Both Hamiltonian methods and noncommutative geometry have a long history in theoretical physics (and mathematics)

brought together already many decades ago (quantum phase space, Dirac 1925)

In this talk: twisted hamiltonian formalism, application to twisted vierbein gravity

abelian twist $\longrightarrow$ star (exterior) product

The logic

Classical gravity:
forms, wedge product

$$\int R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd}$$

Legendre

Noncommutative gravity:
forms, $\star$ - wedge product

$$\int R^{ab} \wedge_\star V^c \wedge_\star V^d \varepsilon_{abcd}$$

twist

$\star$ - Legendre

Hamiltonian
Poisson bracket
Canonical transformations
Symmetry generators

$\star$ - Hamiltonian
$\star$ - Poisson bracket
$\star$ - Canonical transformations
$\star$ - Symmetry generators

NC field theories, $\star$ product

- Field theories on NC spaces become especially tractable when non commutativity is encoded in a twisted $\star$product (noncommutative, associative) between ordinary fields

- Example: Moyal-Groenewold $\star$ product:

$$f(x) \star g(x) \equiv f(x) \exp[\frac{i}{2}\theta^{\mu\nu} \overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}]\, g(x)$$

$$= fg + \frac{i}{2}\theta^{\mu\nu}\ \partial_\mu f\ \partial_\nu g + \frac{1}{2!}\left(\frac{i}{2}\right)^2 \theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\mu\partial_\rho f)\ (\partial_\nu\partial_\sigma g) + \cdots$$

- generalization: abelian twist

$$\partial_\mu \to X_A = X_A^\mu(x)\partial_\mu \quad \text{with} \quad [X_A, X_B] = 0$$

$$f(x) \star g(x) \equiv fg + \frac{i}{2} \boxed{\theta^{AB} X_A^\mu(x) X_B^\nu(x)} \, \partial_\mu f \; \partial_\nu g + \ldots$$

- extension to p-forms: $\wedge_\star$ - product

$$X_A \to \ell_{X_A} \quad \text{(Lie derivative)}$$

$$\tau \wedge_\star \tau' \equiv \tau \wedge \tau' + \frac{i}{2}\theta^{AB} \; \ell_{X_A}\tau \wedge \ell_{X_B}\tau' + \cdots$$

Properties

- Leibniz rule

$$d(\tau \wedge_\star \tau') = d\tau \wedge_\star \tau' + (-)^{deg(\tau)} \tau \wedge_\star d\tau'$$

- Integration: graded cyclicity

$$\int \tau \wedge_\star \tau' = (-)^{deg(\tau) deg(\tau')} \int \tau' \wedge_\star \tau$$

- Complex conjugation

$$(\tau \wedge_\star \tau')^* = (-)^{deg(\tau) deg(\tau')} \tau'^* \wedge_\star \tau^*$$

NB only need these properties in the sequel. Satisfied by abelian twist, but more general $\wedge_\star$ could still satisfy them ?

Sir William Rowan Hamilton, 1805 - 1865

★ —

Sir William Rowan Hamilton, 1805 - 1865

Index

1. covariant hamiltonian formalism for geometric theories (gravity). Review

2. NC covariant hamiltonian formalism.

3. NC hamiltonian for NC vierbein gravity.

4. NC Noether theorems
   NC canonical gauge generators

Covariant hamiltonian formalism

D'Adda, Nelson, Regge, Ann. Phys. 165, 384 (1985)

D'Adda and LC, *Phys.Rev.* D 101 (2020) 2, 025015, hep-th 1906.11852

LC, *JHEP* 04 (2020) 169, hep-th 2002.05523

Noncommutative vierbein gravity

P. Aschieri, LC, *JHEP* 0906 (2009) 086, hep-th 0902.3817
JHEP (2009) 06 087, hep-th 0902.3823

with SW map: P. Aschieri, LC, *JHEP* 07 (2012) 184, hep-th 1111.4822
P.Aschieri, M. Dimitrijevic, LC, *Phys. Rev. D* 87(2013)2, hep-th 1207.4346

Recent review: P. Aschieri, LC, *EPJ ST (to be subm.)*, hep-th 2209.03774

Noncommutative hamiltonian formalism

LC, hep-th 2209.02716

1. Covariant hamiltonian formalism

- The language of differential forms is natural for geometric theories, for ex. (super)gravity
- Forms suggest a covariant hamiltonian treatment, since derivatives of fields ϕ appear as $d\phi$
- Actions: integrals of *d*-form Lagrangians
- Forms are multiplied with the exterior (wedge) product $\wedge$

Geometric theories with p-form fields

$$S = \int_{\mathcal{M}^d} L(\phi_i, d\phi_i)$$

ϕ_i are p_i - forms

- Variational principle

$$\delta S = \int_{\mathcal{M}^d} \delta\phi_i \frac{\vec{\partial} L}{\partial \phi_i} + d(\delta\phi_i) \frac{\vec{\partial} L}{\partial (d\phi_i)}$$

all products are exterior products

form derivatives acting from the left

- Euler-Lagrange eq.s

$$d \frac{\vec{\partial} L}{\partial (d\phi_i)} - (-)^{p_i} \frac{\vec{\partial} L}{\partial \phi_i} = 0$$

Form Legendre transformation

- usual field momenta

$$\pi(x) = \frac{\partial L}{\partial(\partial_0 \phi(x))}$$

with *0* - form fields and Lagrangian

- form field momenta

$$\pi(x) = \frac{\partial L}{\partial(d\phi(x))}$$

(*d* - *p* -*1*) - form

with *p* - form fields and *d* - form Lagrangian

D'Adda, Nelson, Regge (1985)

Form Hamiltonian

- *(d - p_i -1)* - form momenta

$$\pi^i \equiv \frac{\vec{\partial} L}{\partial(d\phi_i)}$$

- *d* - form Hamiltonian

$$H = d\phi_i \; \pi^i - L = H(\phi_i, \pi^i)$$

- Hamilton eq.s

$$d\phi_i = (-)^{(d+1)(p_i+1)} \frac{\vec{\partial} H}{\partial \pi^i}$$

$$d\pi^i = (-)^{p_i+1} \frac{\vec{\partial} H}{\partial \phi_i}$$

equivalent to the Euler-Lagrange eq.s

Form Poisson bracket (FPB)

- from the on-shell differential:

$$dF = d\phi_i \frac{\vec{\partial} F}{\partial \phi_i} + d\pi^i \frac{\vec{\partial} F}{\partial \pi^i} =$$

$$\boxed{\frac{\overleftarrow{\partial} H}{\partial \pi^i} \frac{\vec{\partial} F}{\partial \phi_i} - (-)^{p_i d} \frac{\overleftarrow{\partial} H}{\partial \phi_i} \frac{\vec{\partial} F}{\partial \pi^i} \equiv \{F, H\}}$$

using $\frac{\overleftarrow{\partial} F}{\partial A} = (-)^{a(f+1)} \frac{\vec{\partial} F}{\partial A}$

- The FPB $\{A, B\}$ is a *(a + b - d + 1)* - form

- Canonical rel.s $\{\phi_i, \pi^j\} = \delta_i^j$

Properties of the form Poisson bracket

$$\{B, A\} = -(-)^{(a+d+1)(b+d+1)}\{A, B\}$$

(anti)symmetry

$$\{A, BC\} = B\{A, C\} + (-)^{c(a+d+1)}\{A, B\}C$$

derivation

$$\{AB, C\} = \{A, C\}B + (-)^{a(c+d+1)}A\{B, C\}$$

$$(-)^{(a+d+1)(c+d+1)}\{A, \{B, C\}\} + cyclic = 0$$

Jacobi

$$(-)^{(a+d+1)(b+d+1)}\{\{B, C\}, A\} + cyclic = 0$$

Gravity in d=4

- Lagrangian

$$L(\phi, d\phi) = R^{ab}V^cV^d\varepsilon_{abcd} = d\omega^{ab}V^cV^d\varepsilon_{abcd} - \omega^a{}_e\omega^{eb}V^cV^d\varepsilon_{abcd}$$

- 2 - form momenta

$$\pi_a = \frac{\partial L}{\partial(dV^a)} = 0$$

$$\pi_{ab} = \frac{\partial L}{\partial(d\omega^{ab})} = V^cV^d\varepsilon_{abcd}$$

both momenta definitions are primary constraints

$$\Phi_a \equiv \pi_a = 0, \quad \Phi_{ab} \equiv \pi_{ab} - V^cV^d\varepsilon_{abcd} = 0$$

- Hamiltonian

$$H = dV^a \ \pi_a + d\omega^{ab} \ \pi_{ab} - d\omega^{ab} \ V^c V^d \varepsilon_{abcd} + \omega^a{}_e \ \omega^{eb} \ V^c V^d \varepsilon_{abcd} =$$

$$= dV^a \ \Phi_a + d\omega^{ab} \ \Phi_{ab} + \omega^a{}_e \ \omega^{eb} \ V^c V^d \varepsilon_{abcd}$$

- Hamilton eq.s

$$d\pi_a = \frac{\partial H}{\partial V^a} = -2R^{bc} V^d \epsilon_{abcd}$$

$$d\pi_{ab} = \frac{\partial H}{\partial \omega^{ab}} = 2\omega^c{}_{[a} V^d V^e \epsilon_{b]cde}$$

velocities dV^a and $d\omega^{ab}$ undetermined at this stage

- using the constraints in the Hamilton eq.s yields the field equations :

$$d\Phi_a = \{\Phi_a, H\} = 0 \quad \Rightarrow \quad R^{bc}\, V^d \varepsilon_{abcd} = 0$$ Einstein eq.s

$$d\Phi_{ab} = \{\Phi_{ab}, H\} = 0 \quad \Rightarrow \quad R^c\, V^d \varepsilon_{abcd} = 0$$ zero torsion

- these eq.s partially determine the “velocities”

$$dV^a = \omega^a{}_b\, V^b$$

$d\omega^{ab}$ constrained by Einstein eq.s

- constraint algebra

$$\{\Phi_a, \Phi_b\} = \{\Phi_{ab}, \Phi_{cd}\} = 0; \quad \{\Phi_a, \Phi_{bc}\} = -2\varepsilon_{abcd} V^d$$

Thus constraints are not all first-class, and this is consistent with some ``velocities” getting fixed by requiring “conservation” of the primary constraints.

Infinitesimal canonical transformations

- on any *a* - form *A* :

$$\delta A = \varepsilon \{A, G\}$$

where the generator *G* is a *(d - 1)* - form

Then $\{A, G\}$ is a *a* - form like *A*

- preserve commutation relations with FPB

- the commutator of two infinitesimal canonical transformations generated by G_1 and G_2 is again a canonical transformation, generated by

$$\{G_1, G_2\}$$

Action invariance and Noether theorems

- canonical global transformations

$$\delta\phi_i = \{\phi_i, G\} = \frac{\overleftarrow{\partial}\, G}{\partial\pi^i}, \qquad \delta\pi^i = \{\pi^i, G\} = -(-)^{p_i d}\frac{\overleftarrow{\partial}\, G}{\partial\phi_i},$$

$$S = \int_{\mathcal{M}^d} d\phi_i\ \pi^i - H$$

$$\delta S = \int_{\mathcal{M}^d} d(\delta\phi_i)\pi^i + d\phi_i\delta\pi^i - \delta H =$$

$$= \int_{\mathcal{M}^d} d(\delta\phi_i\pi^i - G) - \{H, G\}$$

→ The action is invariant, up to bdy term, under the canonical form-transformation generated by G iff

$$\{H, G\} = 0 \qquad \text{up to total der}$$

- Noether charges

on shell (i.e. using Hamilton eq.s):

$$dG = \{G, H\}$$

thus if G generates a symmetry, on shell *dG=0*.
Then by Stokes theorem, on shell the integral

$$\boxed{\mathcal{G}(t) = \int_{\mathcal{S}} G}$$

with $\mathcal{S}$ (d-1)-dim slice of the manifold $\mathcal{M}^d$, and fields with suitable bdy conditions “at the end of the slice” ,

is conserved in *t,* the coordinate parametrizing the slice foliation

- canonical local transf. generated by $\varepsilon(x)G + (d\varepsilon)\ F$

$$\delta S = \int_{\partial\mathcal{M}^d} \varepsilon(\{\phi_i, G\}\pi^i - G) + d\varepsilon(\{\phi_i, F\}\pi^i - F)$$

$$+ \int_{\mathcal{M}^d} [d\varepsilon\ (G - \{H, F\}) - \varepsilon\{H, G\}]$$

are symmetries (for arbitrary $\varepsilon(x)$) iff

$$\boxed{G - \{H, F\} = 0, \quad \{H, G\} = 0}$$

LC, Ann. of Phys. (1982)

- moreover $\boxed{\{constraints, G\} \approx 0, \quad \{constraints, F\} \approx 0}$

i.e. *G* and *F* must be first class, but not necessarily constraints

Gauge generators for d=4 vierbein gravity

$$\mathbb{G} = (d\varepsilon)F + \varepsilon(x)G$$

3-form 2-form 3-form

- Lorentz rotations

$$\mathbb{G} = \varepsilon^{ab}G_{ab} + d\varepsilon^{ab}F_{ab} = \varepsilon^{ab}(2\omega^{c}{}_{a}\pi_{bc} - V_a\pi_b) + (d\varepsilon^{ab})\pi_{ab} = \mathcal{D}\varepsilon^{ab}\pi_{ab} - \varepsilon^{ab}V_a\pi_b$$

$$\delta V^a = \{V^a, \mathbb{G}\} = \varepsilon^{a}{}_{b}V^b, \quad \delta\omega^{ab} = \{\omega^{ab}, \mathbb{G}\} = \mathcal{D}\varepsilon^{ab}$$

$$\delta\pi_a = \{\pi_a, \mathbb{G}\} = \varepsilon_{a}{}^{b}\pi_b, \qquad \delta\pi_{ab} = \varepsilon^{c}{}_{[a}\pi_{b]c}$$

2. NC covariant hamiltonian formalism

Geometric theories with p-form fields, with twisted $\wedge_\star$

$$S = \int_{\mathcal{M}^d} L(\phi_i, d\phi_i, \wedge_\star)$$

- Variational principle

$$\delta S = \int_{\mathcal{M}^d} \delta\phi_i \wedge_\star \boxed{\frac{\vec{\partial}\ L}{\partial\phi_i}} + d(\delta\phi_i) \wedge_\star \boxed{\frac{\vec{\partial}\ L}{\partial(d\phi_i)}}$$

all products are exterior products

form derivatives acting from the left
with CYCLIC REORDERING

- Euler-Lagrange eq.s

$$d\frac{\vec{\partial} L}{\partial(d\phi_i)} - (-)^{p_i}\frac{\vec{\partial} L}{\partial\phi_i} = 0$$

$\star$ - Hamiltonian

- *(d - p_i -1)* - form momenta

$$\pi^i \equiv \frac{\vec{\partial} L}{\partial (d\phi_i)}$$

- *d* - form Hamiltonian

$$\boxed{H = d\phi_i \wedge_\star \pi^i - L}$$

- Hamilton eq.s

$$d\phi_i = (-)^{(d+1)(p_i+1)} \frac{\vec{\partial} H}{\partial \pi^i}$$

$$d\pi^i = (-)^{p_i+1} \frac{\vec{\partial} H}{\partial \phi_i}$$

equivalent to the Euler-Lagrange eq.s

$\star$ - Poisson bracket

$$\{A, B\}_\star \equiv \frac{\overleftarrow{\partial} B}{\partial \pi^i} \wedge_\star \frac{\overrightarrow{\partial} A}{\partial \phi_i} - (-)^{p_i d} \frac{\overleftarrow{\partial} B}{\partial \phi_i} \wedge_\star \frac{\overrightarrow{\partial} A}{\partial \pi^i}$$

$\star$ - canonical transformations

$$\delta \phi_i = \{\phi_i, G\}_\star = \frac{\overleftarrow{\partial} G}{\partial \pi^i}$$

the generator G is a d-1 form

$$\delta \pi^i = \{\pi^i, G\}_\star = -(-)^{p_i d} \frac{\overleftarrow{\partial} G}{\partial \phi_i}$$

$$\delta A = \delta \phi_i \wedge_\star \frac{\overrightarrow{\partial} A}{\partial \phi_i} + \delta \pi^i \wedge_\star \frac{\overrightarrow{\partial} A}{\partial \pi^i} = \{A, G\}_\star$$

holds only under integration

preserve $\{\phi_i, \pi^j\}_\star = \delta_i^j$

$\star$ - symmetries

The conditions for gauge generators are formally as in the classical case, with Poisson brackets replaced by their twisted version.

In particular $\varepsilon(x)G + (d\varepsilon)\ F$ generates a symmetry iff

$$G - \{H, F\}_\star = 0, \quad \{H, G\}_\star = 0$$

$$\{constraints, G\}_\star \approx 0, \quad \{constraints, F\}_\star \approx 0$$

3. NC hamiltonian for NC vierbein gravity

NC vierbein gravity

- Classical action

$$S = \int R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} = -4 \int R \sqrt{-g} \; d^4x$$

with $V^a = V^a_\mu \; dx^\mu, \quad \omega^{ab} = \omega^{ab}_\mu \; dx^\mu$

$$R^{ab} = R^{ab}{}_{\mu\nu} \; dx^\mu \wedge dx^\nu = d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b$$

- Index-free

$$S = \int Tr \; (i\gamma_5 \; R \wedge V \wedge V)$$

with $V = V^a \gamma_a, \quad \Omega = \frac{1}{4}\omega^{ab}\gamma_{ab}, \quad R = d\Omega - \Omega \wedge \Omega$

$$S = \int Tr\ (i\gamma_5\ R \wedge V \wedge V)$$

- Invariances:

  - Diff.s

$$\delta_\epsilon V^a = \epsilon^a{}_b V^b, \quad \delta_\epsilon \omega^{ab} = d\epsilon^{ab} - \omega^a{}_c \epsilon^{cb} + \omega^b{}_c \epsilon^{ca} \equiv \mathcal{D}\epsilon^{ab}$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = -\delta_{[\epsilon_1, \epsilon_2]}$$

  - Local Lorentz rotations

$$\delta_\varepsilon V = -V\varepsilon + \varepsilon V, \quad \delta_\varepsilon \Omega = d\varepsilon - \Omega\varepsilon + \varepsilon\Omega$$

with $\varepsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab}$

$\Rightarrow$ $$\delta_\varepsilon R = -R\varepsilon + \varepsilon R$$

$\Rightarrow$ $$\delta_\varepsilon \int Tr\ (i\gamma_5\ R \wedge V \wedge V) = 0$$

by cyclicity of Tr and $[\gamma_5, \varepsilon] = 0$

- $\star$ vierbein gravity:

$$S = \int Tr\ (i\gamma_5\ R \wedge_\star V \wedge_\star V)$$

with $R = d\Omega - \Omega \wedge_\star \Omega$

- Invariances:

1) Diff.s: the action is an integral of a 4-form on a 4-manifold

2) $\star$ - gauge invariance under:

$$\delta_\varepsilon V = -V \star \varepsilon + \varepsilon \star V, \quad \delta_\varepsilon \Omega = d\varepsilon - \Omega \star \varepsilon + \varepsilon \star \Omega$$

$\Rightarrow$ $$\delta_\varepsilon R = -R \star \varepsilon + \varepsilon \star R$$

$\Rightarrow$ $$\delta_\varepsilon \int Tr\ (i\gamma_5\ R \wedge_\star V \wedge_\star V) = 0$$

by cyclicity of Tr and $\int$, and if $[\gamma_5, \varepsilon] = 0$

Note:

$\Omega \wedge_\star \Omega$ contains $[\gamma^{ab}, \gamma^{cd}] \to \gamma^{ef}$ and

$\{\gamma^{ab}, \gamma^{cd}\} \to 1, \gamma_5$

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} + i\;\omega 1 + \tilde{\omega}\gamma_5$$

$$\varepsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} + i\;\varepsilon 1 + \tilde{\varepsilon}\gamma_5$$

$$V = V^a\gamma_a + \tilde{V}^a\gamma_a\gamma_5$$

$$R = \frac{1}{4}R^{ab}\gamma_{ab} + i\;r1 + \tilde{r}\gamma_5$$

New fields: $\omega, \quad \tilde{\omega}, \quad \tilde{V}^a$

gauge invariance: $SL(2,C) \to \star GL(2,C)$

NC vierbein gravity action in components

$$S = \int R^{ab} \wedge_\star (V^c \wedge_\star V^d - \widetilde{V}^c \wedge_\star \widetilde{V}^d)\epsilon_{abcd} - 2i\ R^{ab} \wedge_\star (V_a \wedge_\star \widetilde{V}_b - \widetilde{V}_a \wedge_\star V_b)$$
$$-4\ r \wedge_\star (V^a \wedge_\star \widetilde{V}_a - \widetilde{V}^a \wedge_\star V_a) + 4i\ \tilde{r} \wedge_\star (V^a \wedge_\star V_a - \widetilde{V}^a \wedge_\star \widetilde{V}_a)$$

where the curvatures are

$$R^{ab} = d\omega^{ab} - \frac{1}{2}\omega^a{}_c \wedge_\star \omega^{cb} + \frac{1}{2}\omega^b{}_c \wedge_\star \omega^{ca} - i(\omega^{ab} \wedge_\star \omega + \omega \wedge_\star \omega^{ab}) -$$
$$- \frac{i}{2}\epsilon^{ab}{}_{cd}(\omega^{cd} \wedge_\star \widetilde{\omega} + \widetilde{\omega} \wedge_\star \omega^{cd})$$
$$r = d\omega - \frac{i}{8}\omega^{ab} \wedge_\star \omega_{ab} - i(\omega \wedge_\star \omega - \widetilde{\omega} \wedge_\star \widetilde{\omega})$$
$$\tilde{r} = d\widetilde{\omega} + \frac{i}{16}\varepsilon_{abcd}\omega^{ab} \wedge_\star \omega^{cd} - i(\omega \wedge_\star \widetilde{\omega} + \widetilde{\omega} \wedge_\star \omega)$$

NC gauge variations

$$\delta_\varepsilon V^a = \frac{1}{2}(\varepsilon^a{}_b \star V^b + V^b \star \varepsilon^a{}_b) + \frac{i}{4}\epsilon^a{}_{bcd}(\widetilde{V}^b \star \varepsilon^{cd} - \varepsilon^{cd} \star \widetilde{V}^b)$$
$$+ i(\varepsilon \star V^a - V^a \star \varepsilon) - \widetilde{\varepsilon} \star \widetilde{V}^a - \widetilde{V}^a \star \widetilde{\varepsilon}$$
$$\delta_\varepsilon \widetilde{V}^a = \frac{1}{2}(\varepsilon^a{}_b \star \widetilde{V}^b + \widetilde{V}^b \star \varepsilon^a{}_b) + \frac{i}{4}\epsilon^a{}_{bcd}(V^b \star \varepsilon^{cd} - \varepsilon^{cd} \star V^b)$$
$$+ i(\varepsilon \star \widetilde{V}^a - \widetilde{V}^a \star \varepsilon) - \widetilde{\varepsilon} \star V^a - V^a \star \widetilde{\varepsilon}$$
$$\delta_\varepsilon \omega^{ab} = d\varepsilon^{ab} + \frac{1}{2}(\varepsilon^a{}_c \star \omega^{cb} - \varepsilon^b{}_c \star \omega^{ca} + \omega^{cb} \star \varepsilon^a{}_c - \omega^{ca} \star \varepsilon^b{}_c)$$
$$+ i(\varepsilon^{ab} \star \omega - \omega \star \varepsilon^{ab}) + \frac{i}{2}\epsilon^{ab}{}_{cd}(\varepsilon^{cd} \star \widetilde{\omega} - \widetilde{\omega} \star \varepsilon^{cd})$$
$$+ i(\varepsilon \star \omega^{ab} - \omega^{ab} \star \varepsilon) + \frac{i}{2}\epsilon^{ab}{}_{cd}(\widetilde{\varepsilon} \star \omega^{cd} - \omega^{cd} \star \widetilde{\varepsilon})$$
$$\delta_\varepsilon \omega = d\epsilon - \frac{i}{8}(\omega^{ab} \star \varepsilon_{ab} - \varepsilon_{ab} \star \omega^{ab}) + i(\varepsilon \star \omega - \omega \star \varepsilon - \widetilde{\varepsilon} \star \widetilde{\omega} + \widetilde{\omega} \star$$
$$\delta_\varepsilon \widetilde{\omega} = d\widetilde{\varepsilon} + \frac{i}{16}\epsilon_{abcd}(\omega^{ab} \star \varepsilon^{cd} - \varepsilon^{cd} \star \omega^{ab}) + i(\varepsilon \star \widetilde{\omega} - \widetilde{\omega} \star \varepsilon + \widetilde{\varepsilon} \star \omega -$$

from

$$\delta_\epsilon V = -V \star \epsilon + \epsilon \star V, \quad \delta_\epsilon \Omega = d\epsilon - \Omega \star \epsilon + \epsilon \star \Omega$$

gauge invariance: $SL(2,C) \to \star GL(2,C)$

↑ Lorentz

generators: $\gamma^{ab}, 1, \gamma_5$

$\star$ Lie algebra: $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_2 \star \epsilon_1 - \epsilon_1 \star \epsilon_2}$

NC Legendre transformation and NC Hamiltonian

$$\pi_a = \frac{\vec{\partial}\, L}{\partial(dV^a)} = 0$$

$$\tilde{\pi}_a = \frac{\vec{\partial}\, L}{\partial(d\tilde{V}^a)} = 0$$

$$\pi_{ab} = \frac{\vec{\partial}\, L}{\partial(d\omega^{ab})} = \epsilon_{abcd}(V^c V^d - \tilde{V}^c \tilde{V}^d) + 2i(\tilde{V}^{[a} V^{b]} - V^{[a} \tilde{V}^{b]})$$

$$\pi = \frac{\vec{\partial}\, L}{\partial(d\omega)} = 4(\tilde{V}^c V^c - V^c \tilde{V}^c)$$

$$\tilde{\pi} = \frac{\vec{\partial}\, L}{\partial(d\tilde{\omega})} = 4i(V^c V^c - \tilde{V}^c \tilde{V}^c)$$

$$\begin{aligned}
H = {} & dV^a \pi_a + d\tilde{V}^a \tilde{\pi}_a + d\omega^{ab}\pi_{ab} + d\omega\ \pi + d\tilde{\omega}\ \tilde{\pi} - L = \\
& dV^a \Phi_a + d\tilde{V}^a \tilde{\Phi}_a + d\omega^{ab}\Phi_{ab} + d\omega\ \Phi + d\tilde{\omega}\ \tilde{\Phi} + \\
& + [\omega^{c[a}\omega^{b]}{}_c + i(\omega^{ab}\omega + \omega\omega^{ab}) + \frac{i}{2}(\omega_{ef}\tilde{\omega} + \tilde{\omega}\omega_{ef})\epsilon^{abef}] \cdot \\
& \cdot (V^c V^d - \tilde{V}^c \tilde{V}^d)\epsilon_{abcd} + 2i(\tilde{V}_a V_b - V_a \tilde{V}_b)] + \\
& + 4(\frac{i}{8}\omega^{ab}\omega_{ab} - i\tilde{\omega}\tilde{\omega} + i\omega\omega)(\tilde{V}^c V_c - V^c \tilde{V}_c) - \\
& - 4i(\frac{i}{16}\omega^{ab}\omega^{cd}\epsilon_{abcd} - i\omega\tilde{\omega} - i\tilde{\omega}\omega)(V^c V_c - \tilde{V}^c \tilde{V}_c)
\end{aligned}$$

4. NC canonical gauge generators

$$\mathbb{G} = (d\epsilon) \wedge_\star F + \epsilon \star G$$

Gauge generator for $\star$GL(2,C) transformations

$$
\begin{aligned}
\mathbb{G} = & \, d\varepsilon_{ab}\pi^{ab} + d\varepsilon\ \pi + d\widetilde{\varepsilon}\ \widetilde{\pi} + \\
& + \frac{1}{2}\varepsilon_{ab}(V^a\pi^b + \pi^b V^a) + \frac{i}{4}\varepsilon^{cd}(\widetilde{V}^b\pi^a - \pi^a\widetilde{V}^b)\epsilon_{abcd} + \\
& + \frac{1}{2}\varepsilon_{ab}(\widetilde{V}^a\widetilde{\pi}^b + \widetilde{\pi}^b\widetilde{V}^a) + \frac{i}{4}\varepsilon^{cd}(V^b\widetilde{\pi}^a - \widetilde{\pi}^a V^b)\epsilon_{abcd} - \\
& - \varepsilon^{ca}(\omega^b{}_c\pi_{ab} + \pi_{ab}\ \omega^b{}_c) - i\varepsilon^{ab}(\omega\pi_{ab} - \pi_{ab}\omega) - \frac{i}{2}\varepsilon_{cd}(\widetilde{\omega}\pi_{ab} - \pi_{ab}\widetilde{\omega})\epsilon^{abcd} - \\
& - \frac{i}{8}\varepsilon_{ab}(\omega^{ab}\pi - \pi\omega^{ab}) + \frac{i}{16}\varepsilon^{cd}(\omega^{ab}\widetilde{\pi} - \widetilde{\pi}\omega^{ab})\epsilon_{abcd} - \\
& - i\varepsilon(V^a\pi_a - \pi_a V^a) + \widetilde{\varepsilon}(\widetilde{V}^a\pi_a + \pi_a\widetilde{V}^a) - i\varepsilon(\widetilde{V}^a\widetilde{\pi}_a - \widetilde{\pi}_a\widetilde{V}^a) + \widetilde{\varepsilon}(V^a\widetilde{\pi}_a + \widetilde{\pi}_a V^a) - \\
& - i\varepsilon(\omega^{ab}\pi_{ab} - \pi_{ab}\omega^{ab}) - \frac{i}{2}\widetilde{\varepsilon}(\omega_{ab}\pi_{cd} - \pi_{cd}\omega_{ab})\epsilon^{abcd} \\
& - i\varepsilon(\omega\pi - \pi\omega + \widetilde{\omega}\widetilde{\pi} - \widetilde{\pi}\widetilde{\omega}) + i\widetilde{\varepsilon}(\widetilde{\omega}\pi - \pi\widetilde{\omega} - \omega\widetilde{\pi} + \widetilde{\pi}\omega)
\end{aligned}
$$

NC gauge variations

$$\delta_\varepsilon \phi_i = \{\phi_i, \mathbb{G}\} = -\frac{\overrightarrow{\partial}\,\mathbb{G}}{\partial \pi^i}$$

$$\delta_\varepsilon V^a = \frac{1}{2}(\varepsilon^a{}_b \star V^b + V^b \star \varepsilon^a{}_b) + \frac{i}{4}\epsilon^a{}_{bcd}(\widetilde{V}^b \star \varepsilon^{cd} - \varepsilon^{cd} \star \widetilde{V}^b)$$
$$+ i(\varepsilon \star V^a - V^a \star \varepsilon) - \widetilde{\varepsilon} \star \widetilde{V}^a - \widetilde{V}^a \star \widetilde{\varepsilon}$$
$$\delta_\varepsilon \widetilde{V}^a = \frac{1}{2}(\varepsilon^a{}_b \star \widetilde{V}^b + \widetilde{V}^b \star \varepsilon^a{}_b) + \frac{i}{4}\epsilon^a{}_{bcd}(V^b \star \varepsilon^{cd} - \varepsilon^{cd} \star V^b)$$
$$+ i(\varepsilon \star \widetilde{V}^a - \widetilde{V}^a \star \varepsilon) - \widetilde{\varepsilon} \star V^a - V^a \star \widetilde{\varepsilon}$$
$$\delta_\varepsilon \omega^{ab} = d\varepsilon^{ab} + \frac{1}{2}(\varepsilon^a{}_c \star \omega^{cb} - \varepsilon^b{}_c \star \omega^{ca} + \omega^{cb} \star \varepsilon^a{}_c - \omega^{ca} \star \varepsilon^b{}_c)$$
$$+ i(\varepsilon^{ab} \star \omega - \omega \star \varepsilon^{ab}) + \frac{i}{2}\epsilon^{ab}{}_{cd}(\varepsilon^{cd} \star \widetilde{\omega} - \widetilde{\omega} \star \varepsilon^{cd})$$
$$+ i(\varepsilon \star \omega^{ab} - \omega^{ab} \star \varepsilon) + \frac{i}{2}\epsilon^{ab}{}_{cd}(\widetilde{\varepsilon} \star \omega^{cd} - \omega^{cd} \star \widetilde{\varepsilon})$$
$$\delta_\varepsilon \omega = d\epsilon - \frac{i}{8}(\omega^{ab} \star \varepsilon_{ab} - \varepsilon_{ab} \star \omega^{ab}) + i(\varepsilon \star \omega - \omega \star \varepsilon - \widetilde{\varepsilon} \star \widetilde{\omega} + \widetilde{\omega} \star \widetilde{\varepsilon})$$
$$\delta_\varepsilon \widetilde{\omega} = d\widetilde{\varepsilon} + \frac{i}{16}\epsilon_{abcd}(\omega^{ab} \star \varepsilon^{cd} - \varepsilon^{cd} \star \omega^{ab}) + i(\varepsilon \star \widetilde{\omega} - \widetilde{\omega} \star \varepsilon + \widetilde{\varepsilon} \star \omega - \omega \star \widetilde{\varepsilon})$$

reproduce the NC gauge variations obtained from

$$\delta_\epsilon V = -V \star \epsilon + \epsilon \star V, \quad \delta_\epsilon \Omega = d\epsilon - \Omega \star \epsilon + \epsilon \star \Omega$$

NC gauge variations of the momenta $\delta_\varepsilon \pi^i = \{\pi^i, \mathbb{G}\} = \frac{\vec{\partial}\ \mathbb{G}}{\partial \phi_i}$

$$\delta_\varepsilon \pi_a = \frac{\vec{\partial}\ \mathbb{G}}{\partial V^a} = \frac{1}{2}(\pi^b \varepsilon_{ab} + \varepsilon_{ab}\pi^b) - \frac{i}{4}(\widetilde{\pi}^b \varepsilon^{cd} - \varepsilon^{cd}\widetilde{\pi}^b)\epsilon_{abcd} - \\ - i(\pi_a \varepsilon - \varepsilon \pi_a) + \widetilde{\pi}_a \widetilde{\varepsilon} + \widetilde{\varepsilon}\widetilde{\pi}_a$$

$$\delta_\varepsilon \widetilde{\pi}_a = \frac{\vec{\partial}\ \mathbb{G}}{\partial \widetilde{V}^a} = \frac{1}{2}(\widetilde{\pi}^b \varepsilon_{ab} + \varepsilon_{ab}\widetilde{\pi}^b) - \frac{i}{4}(\pi^b \varepsilon^{cd} - \varepsilon^{cd}\pi^b)\epsilon_{abcd} + \\ + \pi_a \widetilde{\varepsilon} + \widetilde{\varepsilon}\pi_a - i(\widetilde{\pi}_a \varepsilon - \varepsilon \widetilde{\pi}_a)$$

$$\delta_\varepsilon \pi_{ab} = \frac{\vec{\partial}\ \mathbb{G}}{\partial \omega^{ab}} = -\pi_{c[a}\varepsilon_{b]}{}^c - \varepsilon_{c[b}\pi_{a]}{}^c - \frac{i}{8}(\pi \varepsilon_{ab} - \varepsilon_{ab}\pi) + \frac{i}{16}(\widetilde{\pi}\varepsilon^{cd} - \varepsilon^{cd}\widetilde{\pi})\epsilon_{abcd} \\ - i(\pi_{ab}\varepsilon - \varepsilon\pi_{ab}) - \frac{i}{2}(\pi^{cd}\widetilde{\varepsilon} - \widetilde{\varepsilon}\pi^{cd})\epsilon_{abcd}$$

$$\delta_\varepsilon \pi = \frac{\vec{\partial}\ \mathbb{G}}{\partial \omega} = -i(\pi_{ab}\varepsilon^{ab} - \varepsilon^{ab}\pi_{ab}) - i(\pi\varepsilon - \varepsilon\pi + \widetilde{\pi}\widetilde{\varepsilon} - \widetilde{\varepsilon}\widetilde{\pi})$$

$$\delta_\varepsilon \widetilde{\pi} = \frac{\vec{\partial}\ \mathbb{G}}{\partial \widetilde{\omega}} = -\frac{i}{2}(\pi^{ab}\varepsilon^{cd} - \varepsilon^{cd}\pi^{ab})\epsilon_{abcd} + i(\pi\widetilde{\varepsilon} - \widetilde{\varepsilon}\pi - \widetilde{\pi}\varepsilon + \varepsilon\widetilde{\pi})$$

Conclusions

The covariant hamiltonian formalism (CHF) is *not* equivalent to the usual ("time singled out") hamiltonian formalism.

The definitions of momenta are different.

However the form Hamilton equations are equivalent to the usual Hamilton equations of motion. Thus both formalisms describe the same classical theory.

Canonical quantization however may not lead to the same quantum theory, since the canonical commutation relations are different. Thus next homework: quantization within CHF

And also: $\star$- canonical quantization for $\star$- deformed gravity

Thank you !