# Quasinormal modes of black holes in nonlinear electrodynamics

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**KN, Daisuke Yoshida, Jiro Soda, Phys.Rev.D 101 (2020) 124026, [2004.07560]**
**KN, Daisuke Yoshida, Phys.Rev.D 105 (2022) 044006, [2111.06273]**

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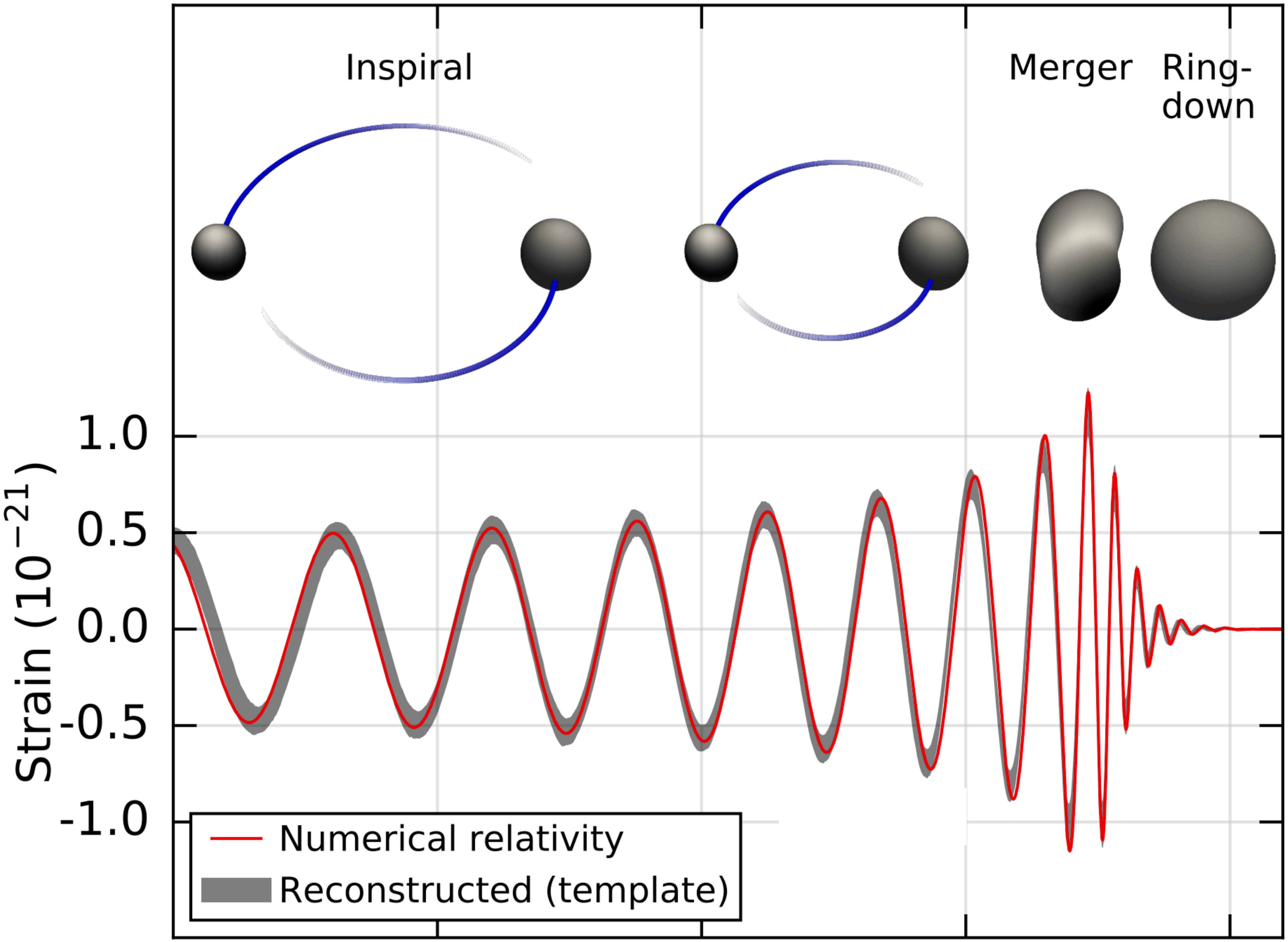
# Gravitational Waves from Black Holes



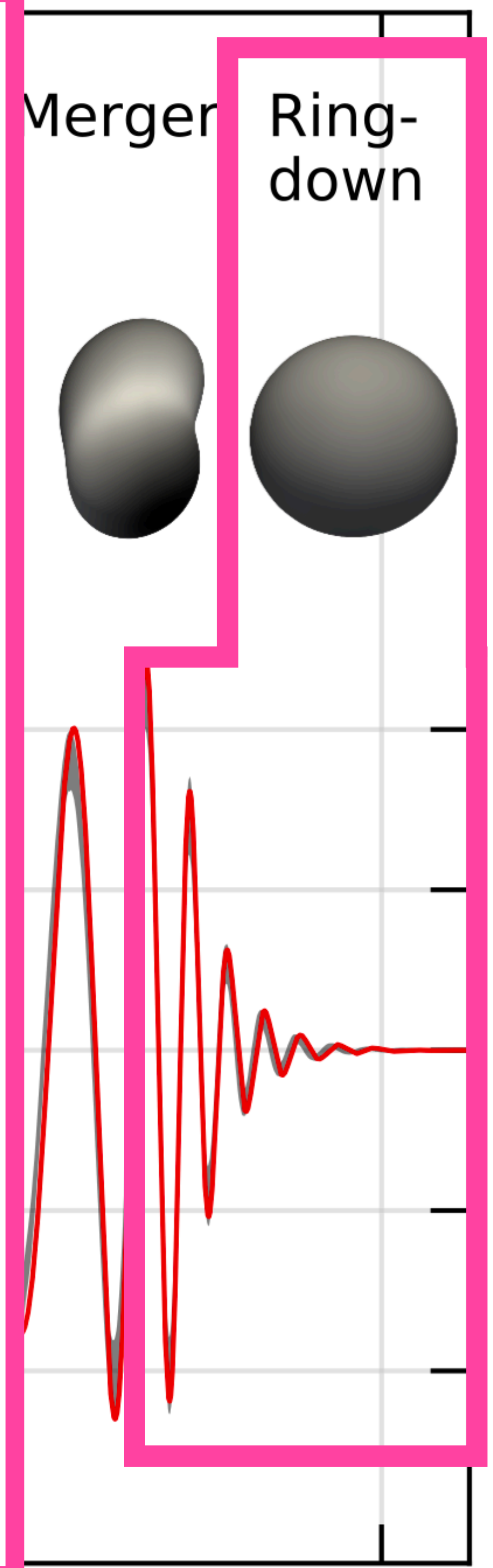
# Gravitational Waves from Black Holes

Ringdown phase is described by the perturbation of a single BH

- Superposition of **Quasi-Normal Modes (QNM)**

$$\sum_{l,m,n} (\mathrm{Amplitude})_{lmn} e^{i\omega_{lmn}(t-r)}$$

  - Based on GR, $\omega$'s are determined by a few parameters of the BH (mass, charge, spin): "Normal Modes" of BH spacetime
    - Important observables for estimating parameters of the BH
    - If once a deviation from the predictions is detected, there may be "new effects."



# Motivation

If there are "new effects," how much are QNMs altered by them?
- Effective field theoretical approach is useful.

**Earlier works** [Cardoso, Kimura, Maselli, Senatore (2018)][de Rham, Francfort, Zhang (2020)]
- "Einstein gravity" + "new effects in gravity"
  higher-order of spacetime curvature

$$\mathcal{L} = \frac{R}{16\pi G} + c_1 R_{\mu\nu}{}^{\alpha\beta} R_{\alpha\beta}{}^{\gamma\sigma} R_{\gamma\sigma}{}^{\mu\nu} + (R_{\mu\nu\rho\sigma}\text{-quartic}) + \cdots$$

  - QNMs of Schwarzschild BH are investigated.

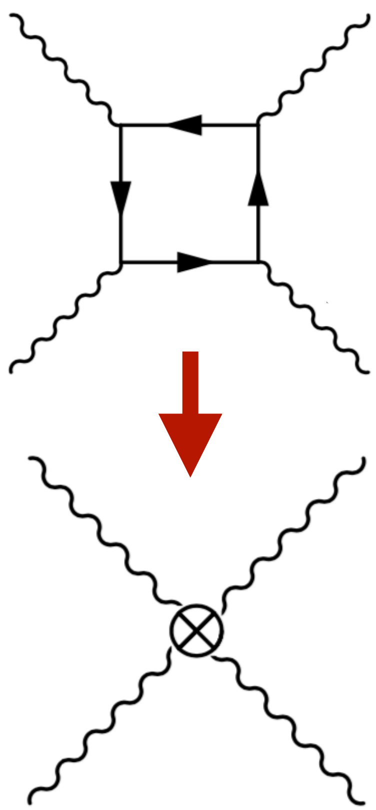
**Our work**
- BHs can have a charge. What about QNMs of charged BHs?

# Charged BHs and Nonlinear Electrodynamics

- Conventionally, BH solutions are constructed based on **"GR + Maxwell electrodynamics,"** and then, QNMs are calculated.

$$\mathcal{L} = \frac{R}{16\pi G} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

- However, **some effects beyond the above "standard framework" can actually appear,** for example, from quantum electrodynamics.



$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m_e)\psi + e\bar{\psi}\gamma^\mu A_\mu \psi$$

Low energy

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{2\alpha^2}{45 m_e^4}\left(4\mathcal{F}^2 + 7\mathcal{G}^2\right)$$

[Heisenberg, Euler (1936)]

$$\mathcal{F} := \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{G} := \frac{1}{8} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

# Charged BHs and Nonlinear Electrodynamics

- More generally, the effects beyond Maxwell electrodynamics can be summarized in a framework of **nonlinear electrodynamics.**

$$\mathcal{L} = \frac{R}{16\pi G} - \mathcal{F} + \underbrace{\alpha \mathcal{F}^2 + \beta \mathcal{G}^2}_{\text{parameters with dim.}-4}$$

$$\mathcal{F} := \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{G} := \frac{1}{8} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

- For charged BHs, the effects of nonlinear electrodynamics, $\alpha\mathcal{F}^2 + \beta\mathcal{G}^2$, can appear in QNMs.

**Purpose of our work**

- Calculate QNMs of charged BHs in **nonlinear electrodynamics.**

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   - Quasinormal modes of black holes
   - Nonlinear electrodynamics
2. **Black holes in nonlinear electrodynamics**
   - Black hole solutions
   - Perturbations of black holes
3. **Quasinormal modes of black holes in nonlinear electrodynamics**
   - Definition
   - Calculation results
4. **Summary**

# Black Hole Solutions

- GR + nonlinear electrodynamics

$$\mathcal{L} = \frac{R}{16\pi G} - \mathcal{F} + \alpha\mathcal{F}^2 + \beta\mathcal{G}^2$$

parameters with dim.–4

$$\mathcal{F} := \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\mathcal{G} := \frac{1}{8}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$$

- Spherically sym. BH with a magnetic charge
  - Spherically sym. magnetic field: $F = q\sin\theta d\theta \wedge d\phi$
    charge of the BH
  - Metric: $ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

mass of the BH

with $f(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2} - \frac{\bar{\alpha}Q^2(GM)^4}{10r^6}$, $\quad Q^2 := 4\pi G q^2$

- Dimensionless parameters: $\bar{\alpha} := \frac{\alpha q^2}{(GM)^4}, \quad \bar{\beta} := \frac{\beta q^2}{(GM)^4}$

# Black Hole Solutions

- GR + nonlinear electrodynamics

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$\frac{\alpha\mathcal{F}^2}{\mathcal{F}} \sim \bar{\alpha}$ → When $\bar{\alpha}, \bar{\beta} \ll 1$, our “effective” framework is valid.

- Dimensionless parameters: $\bar{\alpha} := \frac{\alpha q^2}{(GM)^4}, \quad \bar{\beta} := \frac{\beta q^2}{(GM)^4}$

# Black Hole Solutions

- GR + nonlinear electrodynamics

$\mathcal{L} = \frac{R}{\ldots} \ldots \mathcal{F}^2 + \beta\mathcal{G}^2$  $\mathcal{F} := \frac{1}{\ldots} F_{\mu\nu}F^{\mu\nu}$

BH with an electric charge *q* is given by the same metric up to linear order of *α*.

- Spherically sym. BH with a magnetic charge
  - Spherically sym. magnetic field: $F = q \sin\theta d\theta \wedge d\phi$

    charge of the BH
  - Metric: $ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

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# Perturbations of BHs

- **Perturbations of the metric/elemag field on BHs propagate as gravitational/elemag waves.**
  1. Add perturbations

     spherically sym. background

     metric: $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ elemag field: $F_{\mu\nu} = \bar{F}_{\mu\nu} + \delta F_{\mu\nu}$

     perturbation

  2. Expand on tensor spherical harmonics $[Y^a_{lm}]_{\mu\nu}(\theta, \phi)$

$$\delta g_{\mu\nu}(t, r, \theta, \phi) = \sum_a \sum_{l,m} \delta g^a(l, m; t, r)\, [Y^a_{lm}]_{\mu\nu}(\theta, \phi)$$

$$\delta F_{\mu\nu}(t, r, \theta, \phi) = \sum_a \sum_{l,m} \delta F^a(l, m; t, r)\, [Y^a_{lm}]_{\mu\nu}(\theta, \phi)$$

  3. EoMs of perturbations

metric perturbation (in terms of $\delta g^a$)

$$\left(-\frac{d^2}{dt^2} + \frac{d^2}{dr^{*2}} - f V_{\mathrm{I/II}}(r)\right) \begin{pmatrix} \mathcal{R}_{\mathrm{I/II}}(l, m; t, r) \\ \mathcal{E}_{\mathrm{I/II}}(l, m; t, r) \end{pmatrix} = 0$$

$r^*$: tortoise coordinate

2x2 matrix

elemag perturbation (in terms of $\delta F^a$)

# Perturbations of BHs

- **Perturbations of the metric/elemag field on BHs propagate as gravitational/elemag waves.**

1. Add perturbations

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perturbation

2. Expan

- 2-dof for each of metric and elemag perturbations.

$\mathcal{R}_\mathrm{I}$ & $\mathcal{R}_\mathrm{II}$ $\mathcal{E}_\mathrm{I}$ & $\mathcal{E}_\mathrm{II}$

- The system is separated according to parity.
  - type I: $\mathcal{R}_\mathrm{I}$ & $\mathcal{E}_\mathrm{I}$ couple through $V_\mathrm{I}$
  - type II: $\mathcal{R}_\mathrm{II}$ & $\mathcal{E}_\mathrm{II}$ couple through $V_\mathrm{II}$

3. EoMs

$$\left(-\frac{d^2}{dt^2} + \frac{d^2}{dr^{*2}} - fV_{\mathrm{I/II}}(r)\right)\begin{pmatrix}\mathcal{R}_{\mathrm{I/II}}(l,m;t,r)\\ \mathcal{E}_{\mathrm{I/II}}(l,m;t,r)\end{pmatrix} = 0$$

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# Quasinormal Modes—Definition

- EoMs of metric/elemag perturbations (Fourier transformed: $\partial_t \to -i\omega$)

metric perturbation

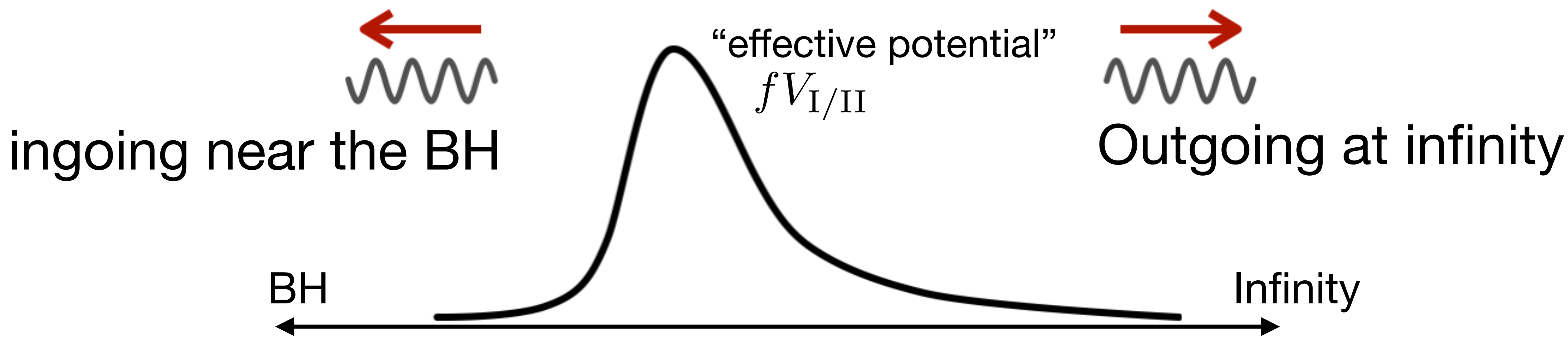
$$\left(-\frac{d^2}{dr^{*2}} + fV_{\mathrm{I/II}}(r)\right)\begin{pmatrix}\mathcal{R}_{\mathrm{I/II}} \\ \mathcal{E}_{\mathrm{I/II}}\end{pmatrix} = \omega^2 \begin{pmatrix}\mathcal{R}_{\mathrm{I/II}} \\ \mathcal{E}_{\mathrm{I/II}}\end{pmatrix}$$

2x2 matrix

elemag perturbation

QNMs are eigenmodes.

- Boundary conditions




➜ Eigenfrequencies, $\omega$'s, form a discrete spectrum.

- $\omega$'s are complex ➜ damping oscillations

# Quasinormal Modes — Results

$$\left(\frac{d^2}{dr^{*2}} + \omega^2 - f V_{\mathrm{I}/\mathrm{II}}(r)\right)\begin{pmatrix}\mathcal{R}_{\mathrm{I}/\mathrm{II}} \\ \mathcal{E}_{\mathrm{I}/\mathrm{II}}\end{pmatrix} = 0$$

The system is separated into I & II according to parity.

In each of I & II, two families of QNM

- I
  - QNM 1: $l = 2$, $l = 3$, …
    - Fundamental mode
    - Overtone modes
  - QNM 2: $l = 2$, $l = 3$, …
    - Fundamental mode
    - Overtone modes
- II
  - QNM 1: $l = 2$, $l = 3$, …
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    - Overtone modes
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# Quasinormal Modes — Results

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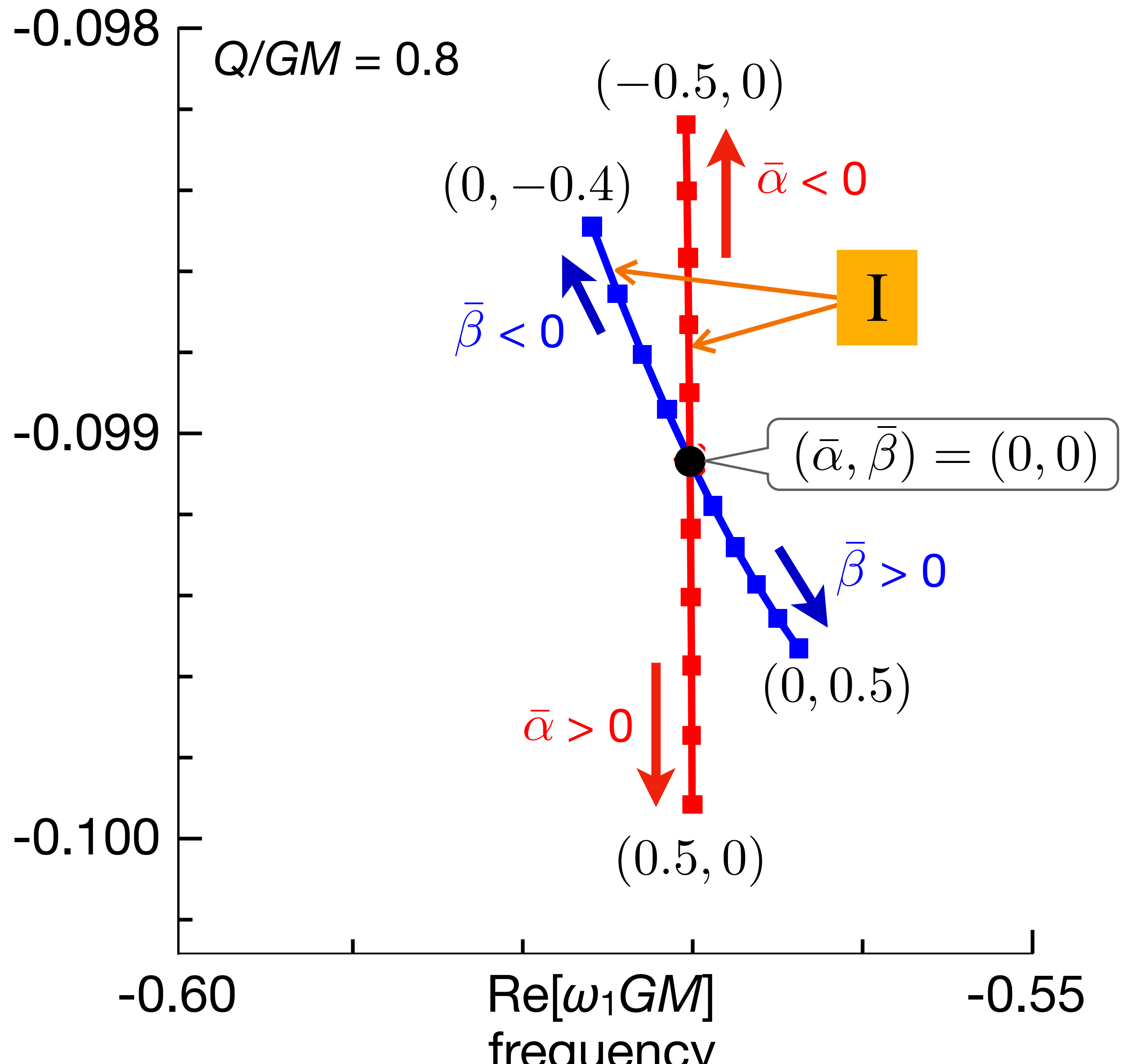


$$\mathcal{L} = \frac{R}{16\pi G} - \mathcal{F} + \alpha\mathcal{F}^2 + \beta\mathcal{G}^2$$

dimensionless parameters

$$\bar{\alpha} := \frac{\alpha q^2}{(GM)^4}, \quad \bar{\beta} := \frac{\beta q^2}{(GM)^4}$$

# Quasinormal Modes—Results

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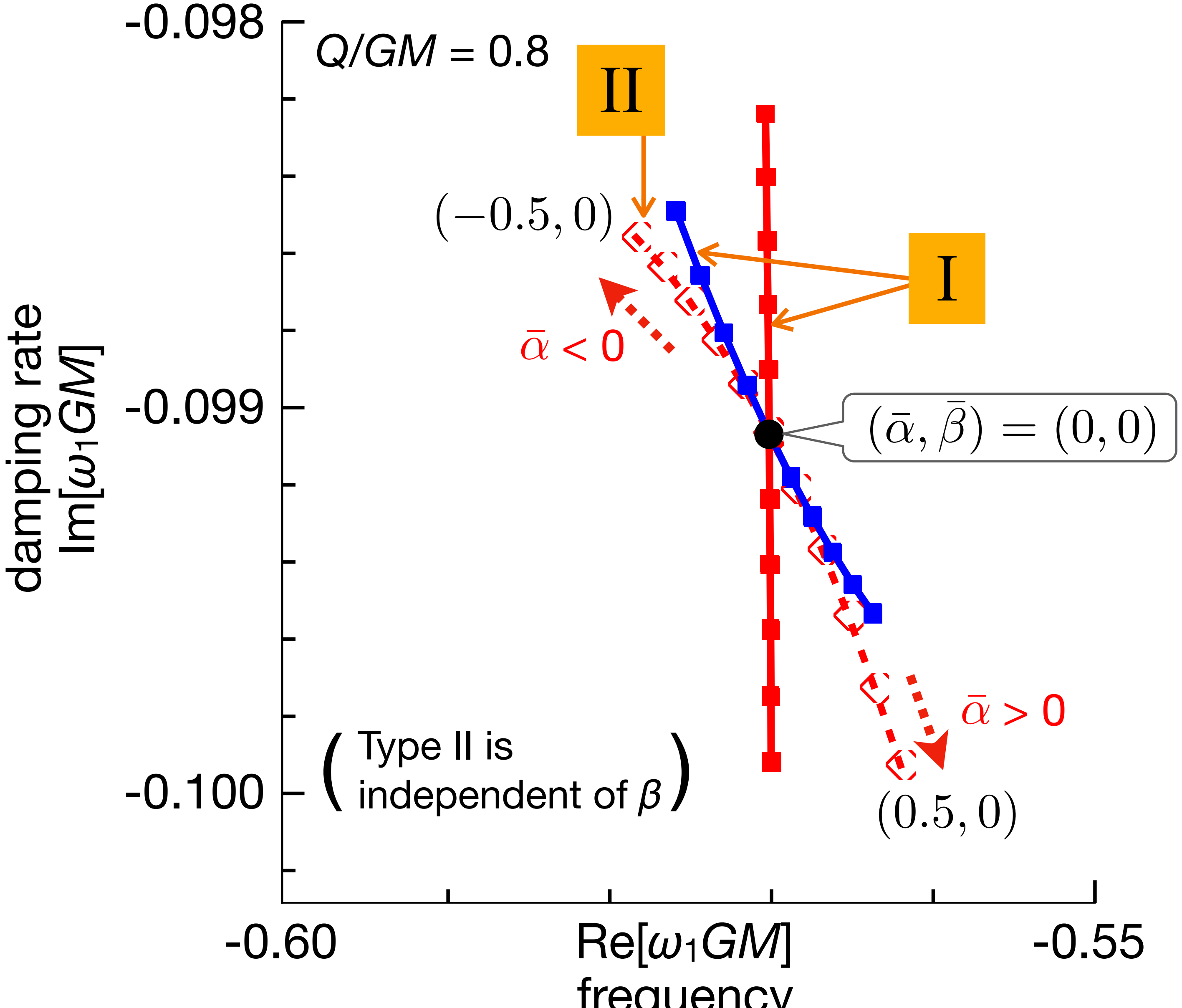


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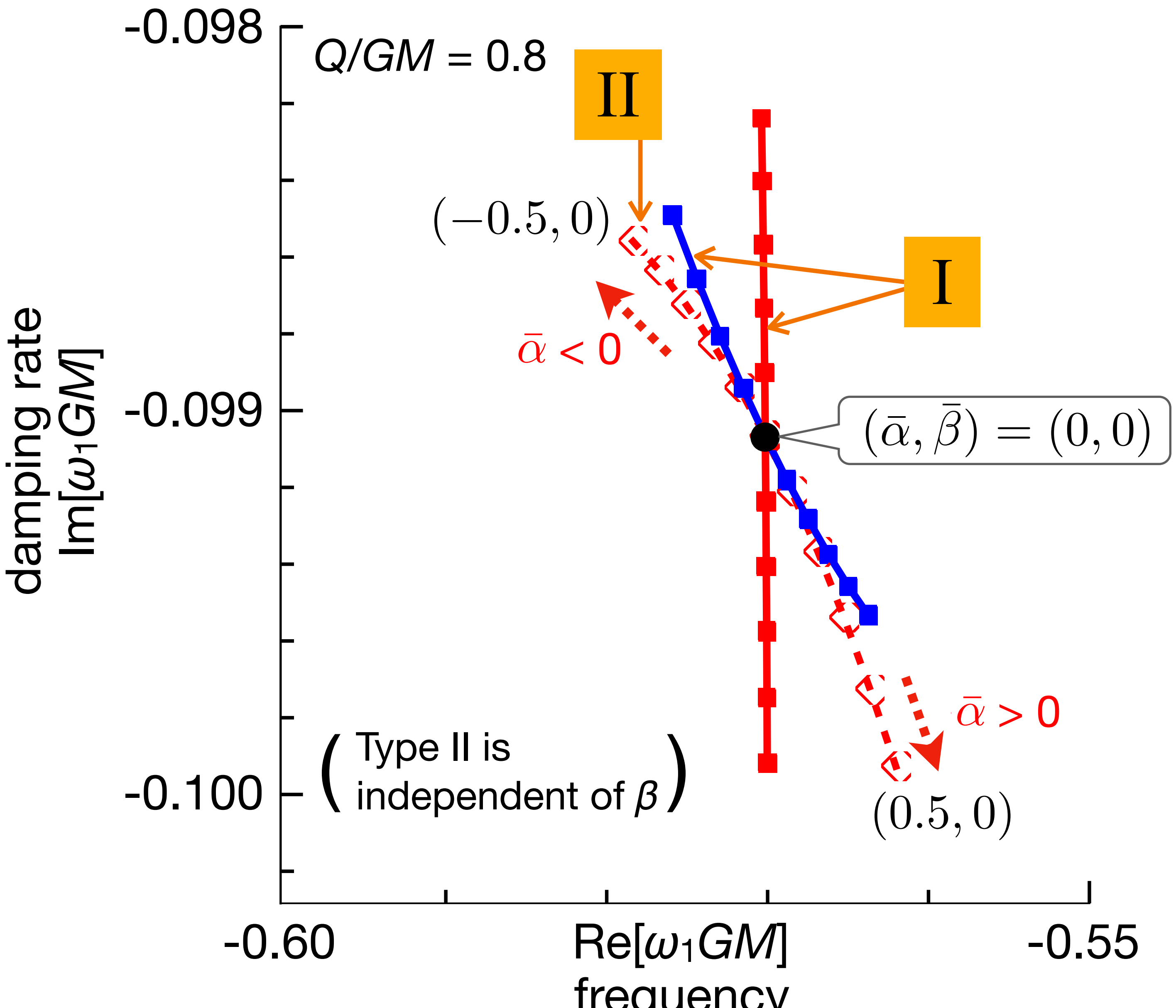
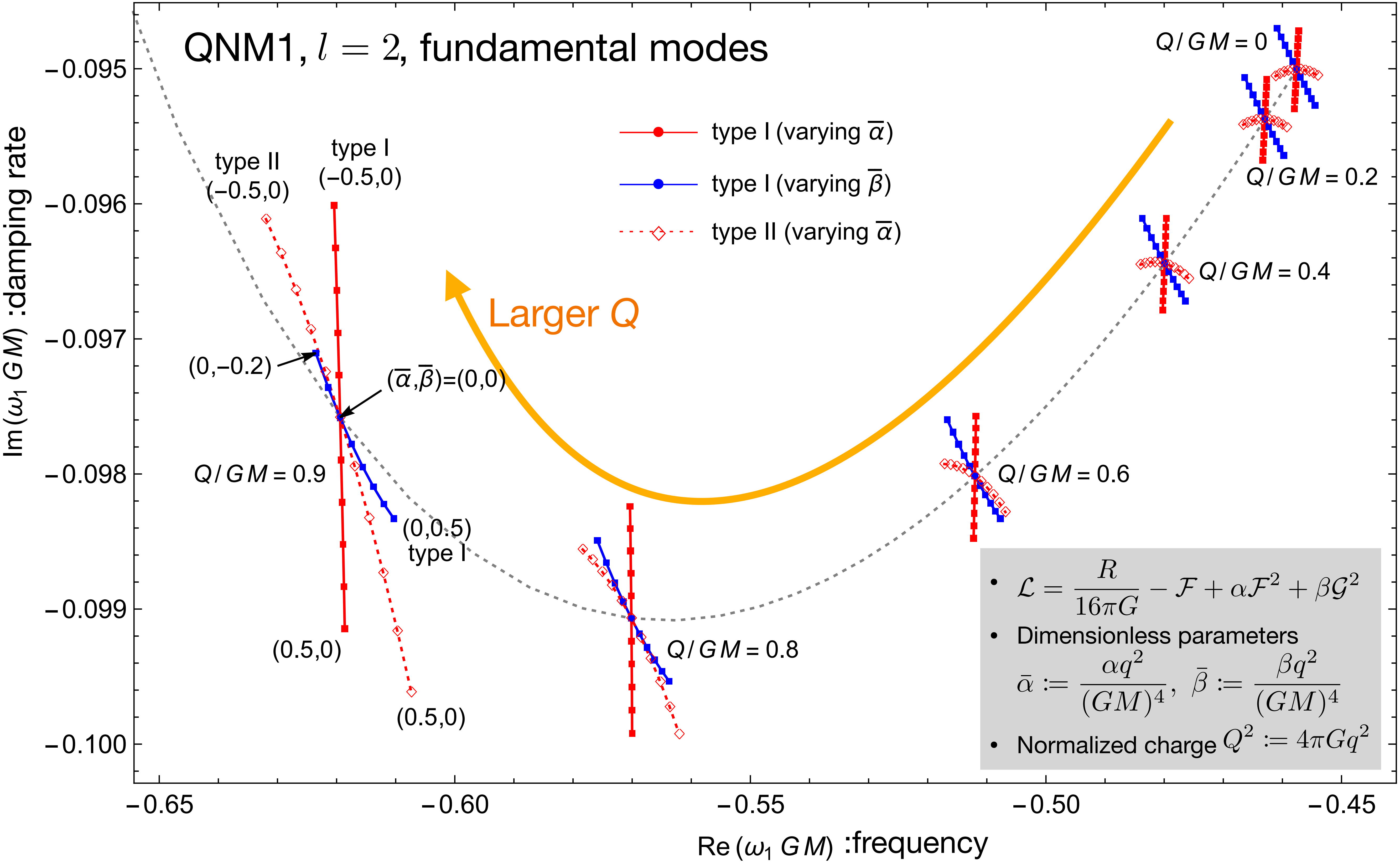


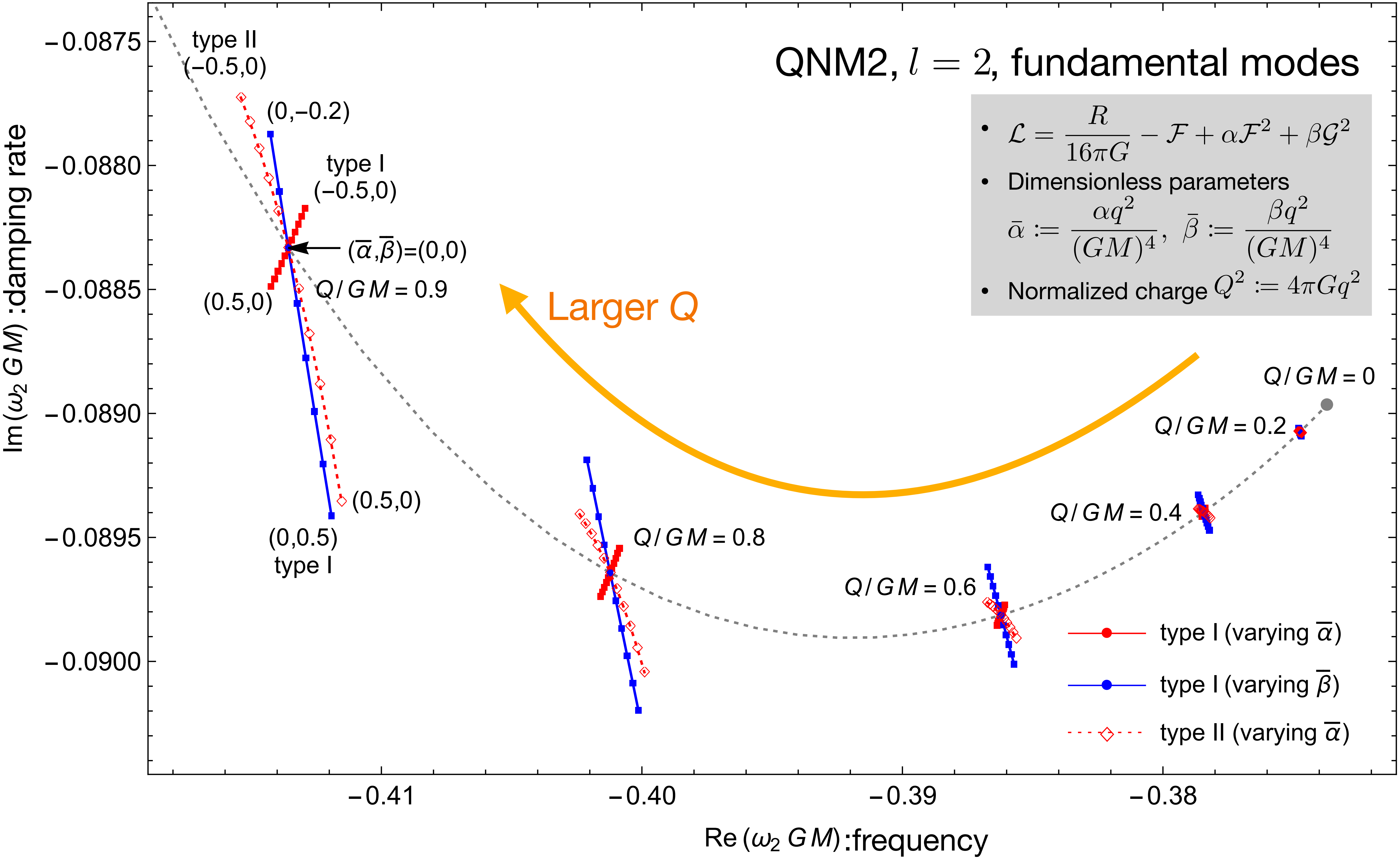

- In the case of $(\bar{\alpha}, \bar{\beta}) = (0, 0)$, QNMs of I and II agree: *isospectrality*
- Nonlinear electrodynamics (nonzero *α* or *β*) violate isospectrality.

# Quasinormal Modes — Results




- $\mathcal{L} = \dfrac{R}{16\pi G} - \mathcal{F} + \alpha\mathcal{F}^2 + \beta\mathcal{G}^2$
- Dimensionless parameters $\bar{\alpha} := \dfrac{\alpha q^2}{(GM)^4},\ \bar{\beta} := \dfrac{\beta q^2}{(GM)^4}$
- Normalized charge $Q^2 := 4\pi G q^2$

# Quasinormal Modes — Results



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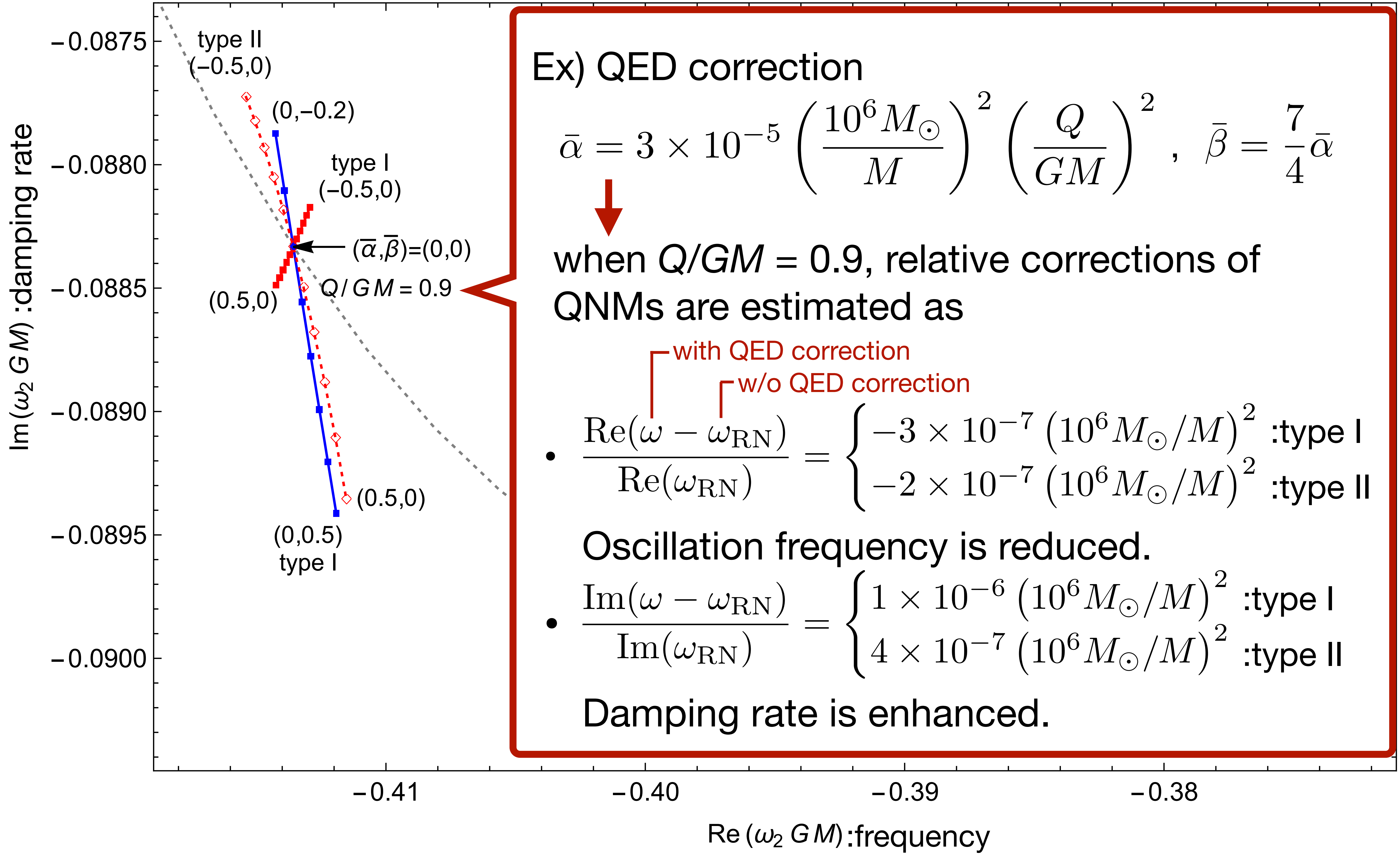



Ex) QED correction

$$\bar{\alpha} = 3 \times 10^{-5} \left( \frac{10^6 M_\odot}{M} \right)^2 \left( \frac{Q}{GM} \right)^2 , \quad \bar{\beta} = \frac{7}{4}\bar{\alpha}$$

when $Q/GM$ = 0.9, relative corrections of QNMs are estimated as

with QED correction

w/o QED correction

- $$\frac{\mathrm{Re}(\omega - \omega_{\mathrm{RN}})}{\mathrm{Re}(\omega_{\mathrm{RN}})} = \begin{cases} -3 \times 10^{-7} \left(10^6 M_\odot / M\right)^2 & \text{:type I} \\ -2 \times 10^{-7} \left(10^6 M_\odot / M\right)^2 & \text{:type II} \end{cases}$$

  Oscillation frequency is reduced.

- $$\frac{\mathrm{Im}(\omega - \omega_{\mathrm{RN}})}{\mathrm{Im}(\omega_{\mathrm{RN}})} = \begin{cases} 1 \times 10^{-6} \left(10^6 M_\odot / M\right)^2 & \text{:type I} \\ 4 \times 10^{-7} \left(10^6 M_\odot / M\right)^2 & \text{:type II} \end{cases}$$

  Damping rate is enhanced.

# Summary

- Calculate QNMs of charged BHs in **GR + nonlinear electrodynamics.**

$$\mathcal{L} = \frac{R}{16\pi G} - \mathcal{F} + \alpha\mathcal{F}^2 + \beta\mathcal{G}^2$$

$$\mathcal{F} := \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\mathcal{G} := \frac{1}{8}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$$

  - Metric & electromagnetic perturbations become coupled, unlike Maxwell (i.e., linear) electrodynamics.
  - Nonlinear electrodynamics (nonzero $\alpha$ or $\beta$) violates isospectrality.
  - $\alpha > 0$, $\beta > 0$ (such as QED correction) act to reduce the oscillation frequency and enhance the damping rate.
    - QED's prediction in strong gravity
    - GW is modified regardless of whether the charge is the electromagnetic charge in the Standard Model sector or *a possible charge in a dark sector.*

      → GW as a tool to probe a dark sector

# Backup

# Charged BHs

No observations so far.

However, the possibility of existence has been discussed.

- Electric BH: leading to a balance of gravitational+electric potential for electrons and protons [Zajaček, Tursunov (2019)]
- Magnetic BH: production in the early universe(?) [Maldacena (2020)]

$$
\begin{aligned}
\frac{\gamma_2 R_{\mu\nu} F^{\mu\rho} F^{\nu}{}_{\rho}}{\alpha \mathcal{F}^2} &\sim 32\pi \frac{G\gamma_2}{\alpha}, \\
\frac{\gamma_3 R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}}{\alpha \mathcal{F}^2} &\sim -64\pi \frac{G\gamma_3}{\alpha} + 128\pi \left(\frac{GM}{Q}\right)^2 \frac{G\gamma_3}{\alpha}
\end{aligned}
$$

# Perturbations of BHs

- $V_{\mathrm{I}}$ & $V_{\mathrm{II}}$ : “effective potentials”

$$V_{\mathrm{I}} = \begin{pmatrix} V_{\mathrm{I},11} & V_{\mathrm{I},12} \\ V_{\mathrm{I},21} & V_{\mathrm{I},22} \end{pmatrix}$$

$$\begin{aligned} V_{\mathrm{I},11} &= \frac{l(l+1)+3(f-1)}{r^2} + \frac{Q^2}{r^4} - \frac{\bar{\alpha}Q^2(GM)^4}{2r^8}\,, \\ V_{\mathrm{I},12} = V_{\mathrm{I},21} &= \frac{2\sqrt{(l+2)(l-1)Q^2[1-\bar{\alpha}(GM)^4/r^4]}}{r^3}\,, \\ V_{\mathrm{I},22} &= \frac{l(l+1)}{r^2}\left(1+\frac{2\bar{\beta}(GM)^4/r^4}{1-\bar{\alpha}(GM)^4/r^4}\right)^{-1} + \cdots \end{aligned}$$

$$V_{\mathrm{II}} = \begin{pmatrix} V_{\mathrm{II},11} & V_{\mathrm{II},12} \\ V_{\mathrm{II},21} & V_{\mathrm{II},22} \end{pmatrix}$$

$$\begin{aligned} V_{\mathrm{II},11} &= -\frac{1}{r^2}\left(f + l(l+1) - 3 + \frac{Q^2}{r^2} - \frac{\bar{\alpha}Q^2(GM)^4}{2r^6}\right) + \cdots\,, \\ V_{\mathrm{II},21} = V_{\mathrm{II},12} &= \cdots\,, \\ V_{\mathrm{II},22} &= \cdots \end{aligned}$$

- dependence on $\alpha$ & $\beta$ ← effects of nonlinear electrodynamics

$$\left(-\frac{d^2}{dt^2} + \frac{d^2}{dr^{*2}} - fV_{\mathrm{I/II}}(r)\right)\begin{pmatrix} \mathcal{R}_{\mathrm{I/II}}(l,m;t,r) \\ \mathcal{E}_{\mathrm{I/II}}(l,m;t,r) \end{pmatrix} = 0$$

$r^*$: tortoise coordinate

2x2 matrix

elemag perturbation (in terms of $\delta F^a$)

$$\left(\frac{d^2}{dr^{*2}}+\omega^2-fV_{\mathrm{I,II}}\right)\begin{pmatrix}\mathcal{R}_{\mathrm{I,II}}\\ \mathcal{E}_{\mathrm{I,II}}\end{pmatrix}=0$$

$$V_{\mathrm{I},11}=\frac{l(l+1)+3(f-1)}{r^2}+8\pi G\mathcal{L}\,,$$

$$V_{\mathrm{I},12}=V_{\mathrm{I},21}=\frac{\sqrt{16\pi Gq^2\mathcal{L}_{\mathcal{F}}(l+2)(l-1)}}{r^3}\,,$$

$$V_{\mathrm{I},22}=\frac{l(l+1)}{r^2}\frac{1}{1-q^2\mathcal{L}_{\mathcal{GG}}/(r^4\mathcal{L}_{\mathcal{F}})}+\frac{16\pi Gq^2\mathcal{L}_{\mathcal{F}}}{r^4}$$

$$V_{\mathrm{II},11}=\frac{\zeta(r)}{r^2}-\frac{2(2\lambda-f+1)}{r^2}+\frac{8\lambda(\lambda-f+1)}{r^2\zeta(r)}+\frac{8\lambda^2 f}{r^2\zeta^2(r)}+\frac{64\pi Gfq^2\lambda\mathcal{L}_{\mathcal{F}}}{r^4\zeta^2(r)}\,,$$

$$V_{\mathrm{II},12}=V_{\mathrm{II},21}=\sqrt{32\pi Gq^2\lambda\mathcal{L}_{\mathcal{F}}}\left[-\frac{1}{r^3}\left(1-\frac{2(2\lambda-f+2)}{\zeta(r)}-\frac{4\lambda f}{\zeta^2(r)}\right)+\frac{32\pi Gfq^2\mathcal{L}_{\mathcal{F}}}{r^5\zeta^2(r)}+\frac{2fq^2\mathcal{L}_{\mathcal{FF}}}{r^7\zeta(r)\mathcal{L}_{\mathcal{F}}}\right]\,,$$

$$V_{\mathrm{II},22}=\frac{2(\lambda+1)}{r^2}-\frac{16\pi Gq^2\mathcal{L}_{\mathcal{F}}}{r^4}\left(1-\frac{4(\lambda+1)}{\zeta(r)}-\frac{4\lambda f}{\zeta^2(r)}\right)+\frac{512\pi^2G^2fq^4\mathcal{L}_{\mathcal{F}}^2}{r^6\zeta^2(r)}$$
$$-\frac{q^2\mathcal{L}_{\mathcal{FF}}}{r^6\mathcal{L}_{\mathcal{F}}}\left[\zeta(r)-3f-4(\lambda+1)\right]+\frac{64\pi Gfq^4\mathcal{L}_{\mathcal{FF}}}{r^8\zeta(r)}+\frac{2fq^4\mathcal{L}_{\mathcal{FFF}}}{r^{10}\mathcal{L}_{\mathcal{F}}}-\frac{fq^4\mathcal{L}_{\mathcal{FF}}^2}{r^{10}\mathcal{L}_{\mathcal{F}}^2}\,,$$

$$\lambda:=\frac{1}{2}(l+2)(l-1)\,,$$

$$\zeta(r):=-3f+2\lambda+3-8\pi Gr^2\mathcal{L}\,.$$

$$\left(\frac{d^2}{dr^{*2}}+\omega^2-fV_{\mathrm{I,II}}\right)\begin{pmatrix}\mathcal{R}_{\mathrm{I,II}}\\ \mathcal{E}_{\mathrm{I,II}}\end{pmatrix}=0$$

$$\lambda := \frac{1}{2}(l+2)(l-1)\,,$$

$$\zeta(r) = -3f+2\lambda+3-\frac{Q^2}{r^2}+\frac{\bar{\alpha}Q^2(GM)^4}{2r^6}\,,$$

$$V_{\mathrm{I},11} = \frac{l(l+1)+3(f-1)}{r^2}+\frac{Q^2}{r^4}-\frac{\bar{\alpha}Q^2(GM)^4}{2r^8}\,,$$

$$V_{\mathrm{I},12} = V_{\mathrm{I},21} = \frac{2\sqrt{(l+2)(l-1)Q^2[1-\bar{\alpha}(GM)^4/r^4]}}{r^3}\,,$$

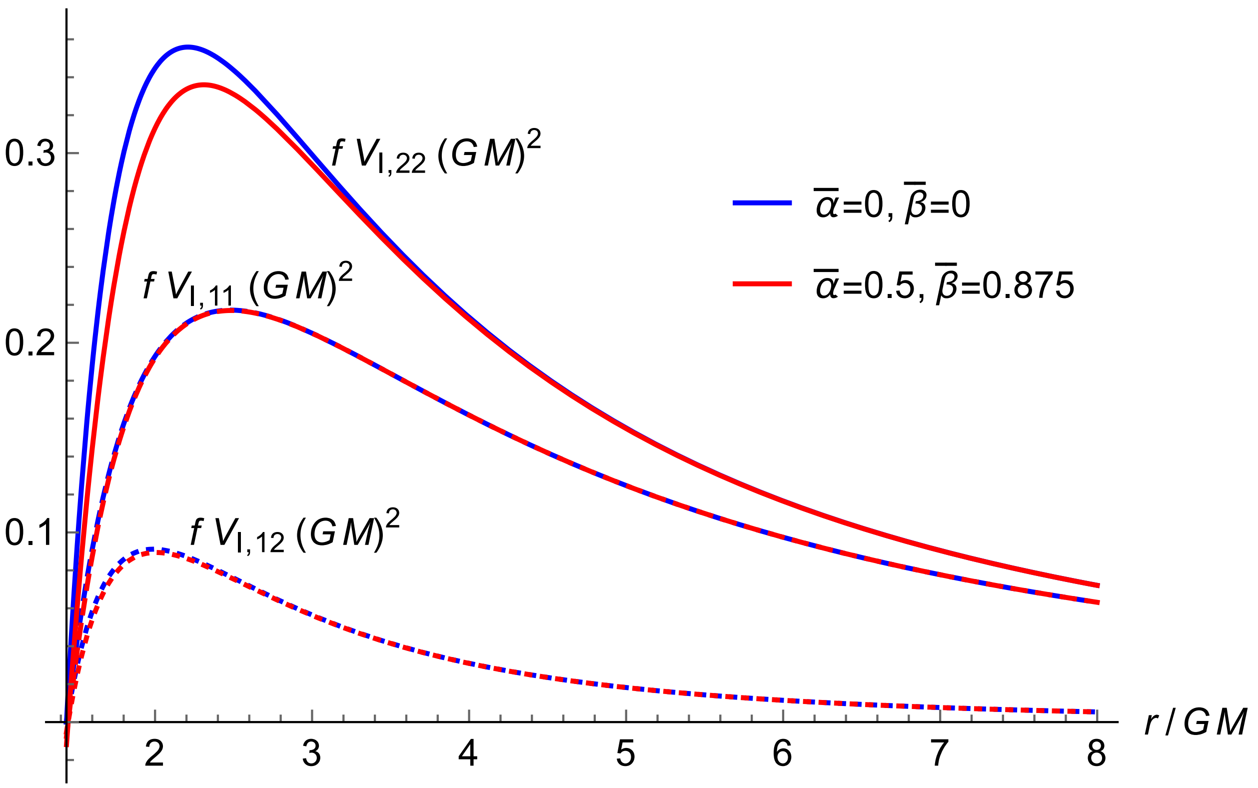
$$V_{\mathrm{I},22} = \frac{l(l+1)}{r^2}\left(1+\frac{2\bar{\beta}(GM)^4/r^4}{1-\bar{\alpha}(GM)^4/r^4}\right)^{-1}+\frac{4Q^2}{r^4}\left(1-\frac{\bar{\alpha}(GM)^4}{r^4}\right)$$
$$+\frac{2\bar{\alpha}(GM)^4}{r^6}\left(1-\frac{\bar{\alpha}(GM)^4}{r^4}\right)^{-1}\left(6f-1+\frac{Q^2}{r^2}-\frac{\bar{\alpha}Q^2(GM)^4}{2r^6}\right)+\frac{12\bar{\alpha}^2f(GM)^8}{r^{10}}\left(1-\frac{\bar{\alpha}(GM)^4}{r^4}\right)^{-2},$$

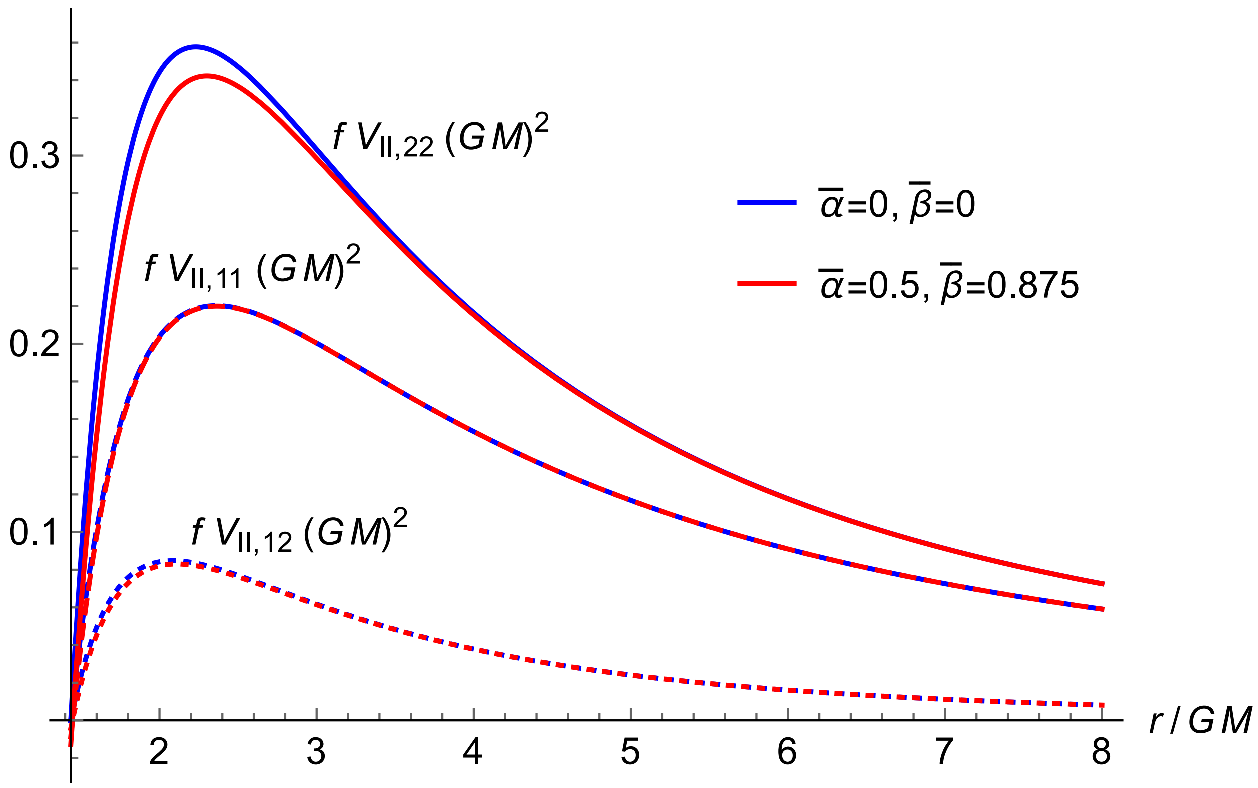
$$V_{\mathrm{II},11} = \frac{\zeta(r)}{r^2}-\frac{2(2\lambda-f+1)}{r^2}+\frac{8\lambda(\lambda-f+1)}{r^2\zeta(r)}+\frac{8\lambda^2 f}{r^2\zeta^2(r)}+\frac{16\lambda fQ^2}{r^4\zeta^2(r)}\left(1-\frac{\bar{\alpha}(GM)^4}{r^4}\right),$$

$$V_{\mathrm{II},12} = V_{\mathrm{II},21} = \sqrt{8\lambda Q^2\left[1-\frac{\bar{\alpha}(GM)^4}{r^4}\right]}\left\{-\frac{1}{r^3}\left(1-\frac{2(2\lambda-f+2)}{\zeta(r)}-\frac{4\lambda f}{\zeta^2(r)}\right)+\frac{8fQ^2}{r^5\zeta^2(r)}\left(1-\frac{\bar{\alpha}(GM)^4}{r^4}\right)\right.$$
$$\left.-\frac{4\bar{\alpha}f(GM)^4}{r^7\zeta(r)}\left(1-\frac{\bar{\alpha}(GM)^4}{r^4}\right)^{-1}\right\},$$

$$V_{\mathrm{II},22} = \frac{2(\lambda+1)}{r^2}-\frac{4Q^2}{r^4}\left(1-\frac{\bar{\alpha}(GM)^4}{r^4}\right)\left(1-\frac{4(\lambda+1)}{\zeta(r)}-\frac{4\lambda f}{\zeta^2(r)}\right)+\frac{32fQ^4}{r^6\zeta^2(r)}\left(1-\frac{\bar{\alpha}(GM)^4}{r^4}\right)^2$$
$$+\frac{2\bar{\alpha}(GM)^4(\zeta(r)-3f-4(\lambda+1))}{r^6}\left(1-\frac{\bar{\alpha}(GM)^4}{r^4}\right)^{-1}-\frac{32\bar{\alpha}fQ^2(GM)^4}{r^8\zeta(r)}-\frac{4\bar{\alpha}^2f(GM)^8}{r^{10}}\left(1-\frac{\bar{\alpha}(GM)^4}{r^4}\right)^{-2},$$

# Effective potentials



# QNM Calculation Method 1

$$\left(\frac{d^2}{dr^{*2}} + \omega^2 - fV\right)\psi = 0$$

2x2 matrix

**Method 1. Numerical integration:** based on [Chaverra, Degollado, Moreno, Sarbach (2016)]

- **Idea:** we can construct 2x2 matrix-valued solutions satisfying boundary conditions as

Outgoing at infinity: $\psi_{+\infty}(\omega, r) = e^{+i\omega r^*} \lim_{N\to\infty} ((T_{+\infty,\omega})^N \mathbf{1})(r)$

operator on 2x2 matrices

$$(T_{+\infty,\omega}\xi)(r) = \mathbf{1} - \frac{1}{2i\omega}\int dr' \left[1 - \exp\left(2i\omega \int_r^{r'} \frac{dr''}{f(r'')}\right)\right] V(r')\xi(r')$$

Ingoing at the horizon: $\psi_{-\infty}(\omega, r) = e^{-i\omega r^*} \lim_{N\to\infty} ((T_{-\infty,\omega})^N \mathbf{1})(r)$

operator on 2x2 matrices

$$(T_{-\infty,\omega}\xi)(r) = \mathbf{1} + \frac{1}{2i\omega}\int dr' \left[1 - \exp\left(-2i\omega \int_r^{r'} \frac{dr''}{f(r'')}\right)\right] V(r')\xi(r')$$

To find eigenfrequencies (QNMs), we seek $\omega$ such that $\psi_{+\infty}$ & $\psi_{-\infty}$ are matched.

(= zeros of the Wronskian of $\psi_{+\infty}$ & $\psi_{-\infty}$)

# QNM Calculation Method 2

$$\left(\frac{d^2}{dr^{*2}} + \omega^2 - fV\right)\vec{\psi} = 0$$

2x2 matrix

**<u>Method 2.</u> Continued fractions method:** originally [Leaver (1985)]; matrix-valued version [Pani (2013)] etc.

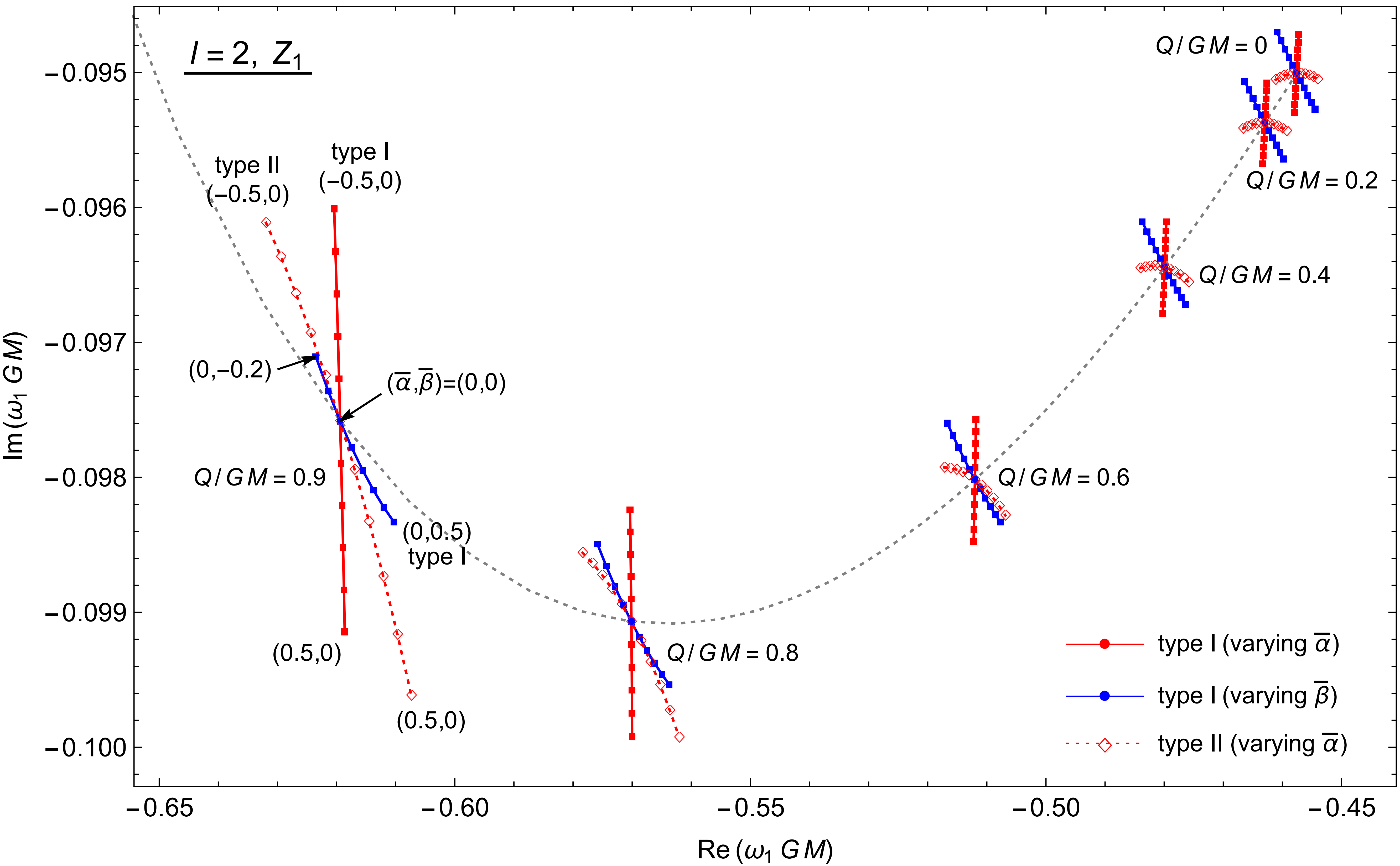
- **Idea:** series ansatz suited for boundary conditions

$$\psi = \left(\frac{r}{2GM}\right)^{2GMi\omega} e^{i\omega r}[(r-r_{\mathrm{h}})/r]^{-i\omega/f'(r_{\mathrm{h}})} \sum_{n=0}^{\infty} \begin{pmatrix} a_n^{(1)}[(r-r_{\mathrm{h}})/r]^n \\ a_n^{(2)}[(r-r_{\mathrm{h}})/r]^n \end{pmatrix}$$

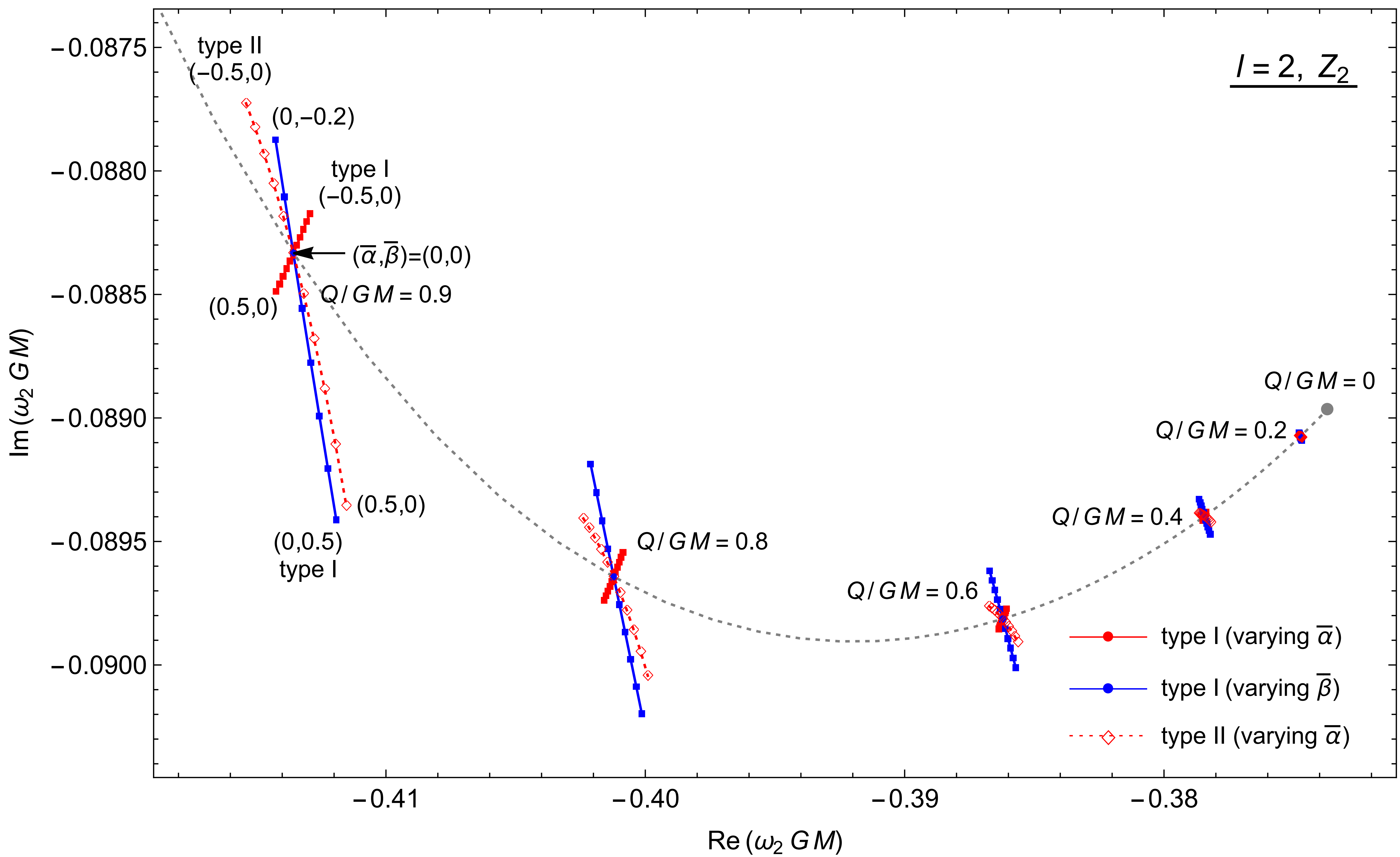
Inserting this ansatz into EoMs, we get recurrence relations for series coefficients $\begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \end{pmatrix}$.

By requiring that nontrivial $\vec{\psi}$s exist, we can find eigenfrequencies (QNMs) *semi-analytically*.

- For simplicity, we take linear approximations in EFT parameters, $\alpha$ & $\beta$.



l = 2, Z_1
type II
(−0.5,0)
type I
(−0.5,0)
(0,−0.2)
(ᾱ,β̄)=(0,0)
Q / G M = 0.9
(0,0.5)
type I
(0.5,0)
(0.5,0)
Q / G M = 0.8
Q / G M = 0.6
Q / G M = 0.4
Q / G M = 0.2
Q / G M = 0
type I (varying ᾱ)
type I (varying β̄)
type II (varying ᾱ)
Im(ω_1 G M)
Re (ω_1 G M)
−0.095
−0.096
−0.097
−0.098
−0.099
−0.100
−0.65
−0.60
−0.55
−0.50
−0.45



$l = 2,\ Z_2$
type II
(−0.5,0)
(0,−0.2)
type I
(−0.5,0)
$(\bar{\alpha},\bar{\beta})$=(0,0)
(0.5,0)
$Q/GM = 0.9$
(0.5,0)
(0,0.5)
type I
$Q/GM = 0.8$
$Q/GM = 0.6$
$Q/GM = 0.4$
$Q/GM = 0.2$
$Q/GM = 0$
type I (varying $\bar{\alpha}$)
type I (varying $\bar{\beta}$)
type II (varying $\bar{\alpha}$)
Re$(\omega_2 GM)$
Im$(\omega_2 GM)$

# Quasinormal modes — Results

- Parametrize as $\omega = \underbrace{\omega_{\mathrm{RN}}} + \bar{\alpha}\underbrace{\kappa_\alpha} + \bar{\beta}\underbrace{\kappa_\beta}$.

correction coefficients

Find $\kappa_\alpha, \kappa_\beta$ by fitting the calculation results.

| $l=2$ | QNM 1 | | | | | QNM 2 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | type I | | type II | | | type I | | type II | |
| $Q/GM$ | $\omega_{\mathrm{RN}}GM$ | $\kappa_\alpha GM$ | $\kappa_\beta GM$ | $\kappa_\alpha GM$ | $\kappa_\beta GM$ | $\omega_{\mathrm{RN}}GM$ | $\kappa_\alpha GM$ | $\kappa_\beta GM$ | $\kappa_\alpha GM$ | $\kappa_\beta GM$ |
| 0 | -0.45760 -0.09500i | – – | – – | – – | – – | -0.37367 -0.08896i | – – | – – | – – | – – |
| 0.2 | -0.46297 -0.09537i | -0.00068 -0.00060i | 0.00660 -0.00058i | 0.00735 -0.00001i | 0 +0i | -0.37474 -0.08907i | -0.00003 -0.00001i | 0.00011 -0.00003i | 0.00012 -0.00001i | 0 +0i |
| 0.4 | -0.47993 -0.09644i | -0.00061 -0.00068i | 0.00727 -0.00061i | 0.00819 -0.00010i | 0 +0i | -0.37844 -0.08940i | -0.00012 -0.00003i | 0.00043 -0.00014i | 0.00046 -0.00003i | 0 +0i |
| 0.6 | -0.51201 -0.09802i | -0.00041 -0.00091i | 0.00896 -0.00073i | 0.01032 -0.00035i | 0 +0i | -0.38622 -0.08981i | -0.00030 -0.00008i | 0.00101 -0.00039i | 0.00111 -0.00014i | 0 +0i |
| 0.8 | -0.57013 -0.09907i | 0.00036 -0.00168i | 0.01363 -0.00119i | 0.01637 -0.00135i | 0 +0i | -0.40122 -0.08964i | -0.00072 -0.00019i | 0.00220 -0.00113i | 0.00249 -0.00061i | 0 +0i |
| 0.9 | -0.61940 -0.09758i | 0.00180 -0.00314i | 0.01980 -0.00208i | 0.02469 -0.00349i | 0 +0i | -0.41357 -0.08833i | -0.00130 -0.00032i | 0.00340 -0.00226i | 0.00391 -0.00156i | 0 +0i |

# Black holes without singularity

$$\mathcal{L}(\mathcal{F}) = \frac{3}{2sg^2}\left(\frac{\sqrt{2g^2\mathcal{F}}}{1+\sqrt{2g^2\mathcal{F}}}\right)^{5/2} \qquad \mathcal{F} := \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2),$$

$$\rightarrow \quad ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \qquad f(r) = 1 - \frac{2mr^2}{(r^2+g^2)^{3/2}}$$

does not have the expansion in $\mathcal{F}$

$f(r) \xrightarrow{r\to\infty} 1 - \frac{2m}{r}$ : Schwarzchild-like

$f(r) \xrightarrow{r\to 0} 1 - \frac{2m}{g^3}r^2$ : de Sitter-like