Quantum properties of $U(1)$-like gauge theory on κ-Minkowski

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September, 22nd 2022

Workshop on Noncommutative and generalized geometry in string theory, gauge theory and related physical models - Corfu (Grece)

K. Hersent, P. Mathieu, J.-C. Wallet, "*Quantum instability of gauge theories on κ-Minkowski space*", Phys. Rev. D **105** (2021) 106013, 10.1103/PhysRevD.105.106013, arXiv:2107.14462

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Toward quantum gravity ?

→ Good candidate for a quantum space underlying the description of quantum gravity, at least in some regime.

[1] A. Addazi *et al.*, "*Quantum gravity phenomenology at the dawn of the multi-messenger era - A review*", Prog. in Part. and Nuc. Phys. **125** (2022) 103948, 10.1016/j.ppnp.2022.103948, arXiv:2111.05659

[2] L. Freidel and E. R. Livine, "*3D Quantum Gravity and Effective Noncommutative Quantum Field Theory*", Phys. Rev. Lett. **96** (2006), 10.1103/PhysRevLett.96.221301, arXiv:hep-th/0512113

Toward quantum gravity ?

→ Good candidate for a quantum space underlying the description of quantum gravity, at least in some regime.

Motivations of studying κ-Minkowski:

- Its low energy limit ($\kappa \to +\infty$) is the Minkowski space.
- A κ-Poincaré-invariant NCFT on κ-Minkowski would easily satisfy Poincaré-invariance at low energies.
- κ-Poincaré realises a Doubly Special Relativity (DSR) giving a testable framework.[1]
- In 2+1 dimensions, an effective NCFT containing matter must be κ-Poincaré invariant.[2]

[1] A. Addazi *et al.*, "*Quantum gravity phenomenology at the dawn of the multi-messenger era - A review*", Prog. in Part. and Nuc. Phys. **125** (2022) 103948, 10.1016/j.ppnp.2022.103948, arXiv:2111.05659

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The Poincaré algebra

Generators:

P_0 (Time translation)	$(M_j)_{1\leqslant j\leqslant d}$ (Rotations)
$(P_j)_{1\leqslant j\leqslant d}$ (Space translations)	$(N_j)_{1\leqslant j\leqslant d}$ (Boosts)

The Poincaré algebra

<u>Generators:</u> P_0 (Time translation), $(P_j)_{1\leqslant j\leqslant d}$ (Space translations), $(M_j)_{1\leqslant j\leqslant d}$ (Rotations), $(N_j)_{1\leqslant j\leqslant d}$ (Boosts)

<u>Commutation relations:</u>

$$[M_j, M_k] = i\epsilon_{jk}{}^l M_l, \quad [M_j, N_k] = i\epsilon_{jk}{}^l N_l,$$
$$[M_j, P_k] = i\epsilon_{jk}{}^l P_l, \quad [M_j, P_0] = 0,$$
$$[N_j, N_k] = -i\epsilon_{jk}{}^l M_l, \quad [P_j, P_k] = 0,$$

$$[P_j, P_0] = 0, \quad [N_j, P_0] = -iP_j,$$
$$[N_j, P_k] = -i\delta_{jk} P_0.$$

The κ-Poicaré algebra[3] with Majid-Ruegg basis[4]:

Generators: $\mathcal{E} = e^{-P_0/\kappa}$ $\quad (M_j)_{1\leqslant j\leqslant d}$

$(P_j)_{1\leqslant j\leqslant d}$ $\quad (N_j)_{1\leqslant j\leqslant d}$

[3] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoy, "*q-deformation of Poincaré algebra*", Phys. Lett. B **264** (1991) 331-338, 10.1016/0370-2693(91)90358-W

[4] S. Majid and H. Ruegg, "*Bicrossproduct structure of κ-Poincaré group and non-commutative geometry*", Phys. Lett. B **334** (1994) 348-354, 10.1016/0370-2693(94)90699-8, arXiv:hep-th/9405107

The κ-Poincaré algebra

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$$[M_j, P_k] = i\epsilon_{jk}{}^l P_l, \quad [P_j, P_k] = 0,$$
$$[N_j, N_k] = -i\epsilon_{jk}{}^l M_l,$$

$$[P_j, \mathcal{E}] = 0, \quad [M_j, \mathcal{E}] = 0, \quad [N_j, \mathcal{E}] = -\frac{i}{\kappa} P_j \mathcal{E},$$
$$[N_j, P_k] = -\frac{i}{2}\delta_{jk}\left(\kappa(1-\mathcal{E}^2) + \frac{1}{\kappa}\vec{P}^2\right) + \frac{i}{\kappa} P_j P_k.$$

with $\vec{P}^2 = P^k P_k$ and κ a real parameter of mass dimension 1.

[3] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoy, "*q-deformation of Poincaré algebra*", Phys. Lett. B **264** (1991) 331-338, 10.1016/0370-2693(91)90358-W

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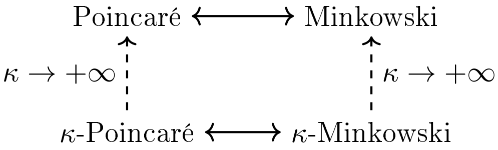
From the symmetries to the space-time

The κ-Minkowski $\mathcal{M}_\kappa$ space is built as the space having κ-Poincaré $\mathcal{P}_\kappa$ as symmetry group[5]:



[5] S. Majid and H. Ruegg, "*Bicrossproduct structure of κ-Poincaré group and non-commutative geometry*", Phys. Lett. B **334** (1994) 348-354, 10.1016/0370-2693(94)90699-8, arXiv:hep-th/9405107

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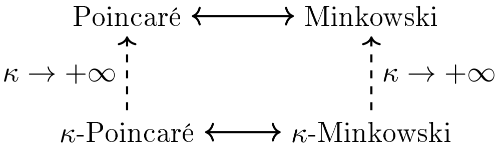

$\mathcal{M}_\kappa$ is generated by the x^μ's, $0 \leqslant \mu \leqslant d$, through

$$[x^j, x^k] = 0, \qquad [x^0, x^j] = \frac{i}{\kappa} x^j. \qquad 1 \leqslant j, k \leqslant d$$

[5] S. Majid and H. Ruegg, "*Bicrossproduct structure of κ-Poincaré group and non-commutative geometry*", Phys. Lett. B **334** (1994) 348-354, 10.1016/0370-2693(94)90699-8, arXiv:hep-th/9405107

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Star product on κ-Minkowski

To model a quantum version of a finite dimensional manifold $\mathcal{M}$, we work at the level of coordinates $f \in \mathcal{C}^\infty(\mathcal{M})$ rather than points $x \in \mathcal{M}$. The quantum space is then a general algebra of functions $\mathcal{A}$ with a noncommutative product $\star$.

[6] T. Poulain, J.-C.Wallet, *"κ-Poincaré invariant quantum field theories with KMS weight"*, Phys. Rev. D **98** (2018) 025002, 10.1103/PhysRevD.98.025002, arXiv:1801.02715

Star product on κ-Minkowski

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We take the star product on κ-Minkowski given by[6]

$$(f \star g)(x^0, \vec{x}) = \int \mathrm{d}p_0 \mathrm{d}y^0 \; e^{-iy^0 p_0} f(x^0 + y^0, \vec{x}) g(x^0, e^{-p_0/\kappa}\vec{x}),$$

$$f^\dagger(x^0, \vec{x}) = \int \mathrm{d}p_0 \mathrm{d}y^0 \; e^{-iy^0 p_0} \overline{f}(x^0 + y^0, e^{-p_0/\kappa}\vec{x}).$$

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Properties of the star product

Non-cyclic trace

$$\int \mathrm{d}^{d+1}x \; f \star g = \int \mathrm{d}^{d+1}x \; (\mathcal{E}^{d} \triangleright g) \star f$$

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Non-cyclic trace

$$\int \mathrm{d}^{d+1}x \; f \star g = \int \mathrm{d}^{d+1}x \; (\mathcal{E}^d \triangleright g) \star f$$

Gauge invariant action is not straightforward anymore:

$$S = \int \mathrm{d}^{d+1}x \; F^\dagger \star F$$

with F the curvature transforming as $F^g = g^\dagger \star F \star g$ and $g^\dagger \star g = 1$.
Then,

$$S^g = \int \mathrm{d}^{d+1}x \; (F^g)^\dagger \star F^g = \int \mathrm{d}^{d+1}x \; g^\dagger \star F^\dagger \star g \star g^\dagger \star F \star g$$
$$= \int \mathrm{d}^{d+1}x \; (\mathcal{E}^d \triangleright g) \star g^\dagger \star F^\dagger \star F \neq S$$

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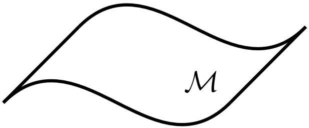
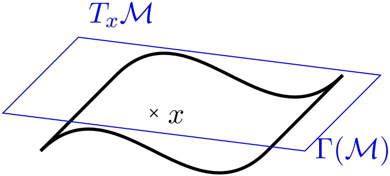
Noncommutative geometry based on derivations

Differential Geometry	Noncommutative geometry
	$\times$
$\mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$	Noncommutative algebra $\mathcal{A}$
	Derivations $\mathrm{Der}(\mathcal{A})$

Twisted derivations

Derivations

$X \in \mathrm{Der}(\mathcal{M}_\kappa)$ linear and satisfies the Leibniz rule :

$X(f \star g) = X(f) \star g + f \star X(g), \; f, g \in \mathcal{M}_\kappa.$

Non-cyclic trace

$$\int \mathrm{d}^{d+1}x \; f \star g = \int \mathrm{d}^{d+1}x \; (\mathcal{E}^d \triangleright g) \star f$$

Twisted derivations

Twisted derivations

$X \in \mathfrak{Der}(\mathcal{M}_\kappa)$ linear and satisfies the twisted Leibniz rule :

$$X(f \star g) = X(f) \star g + (\mathcal{E} \triangleright f) \star X(g), \ f, g \in \mathcal{M}_\kappa.$$

Non-cyclic trace

$$\int \mathrm{d}^{d+1}x \ f \star g = \int \mathrm{d}^{d+1}x \ (\mathcal{E}^d \triangleright g) \star f$$

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$X \in \mathfrak{Der}(\mathcal{M}_\kappa)$ linear and satisfies the twisted Leibniz rule :

$$X(f \star g) = X(f) \star g + (\mathcal{E} \triangleright f) \star X(g),\ f, g \in \mathcal{M}_\kappa.$$

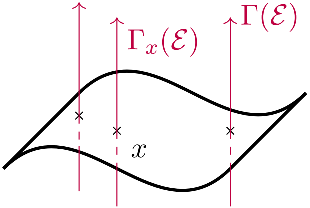
Twisted derivations of κ-Minkowski

$$X_0 = \kappa(1 - \mathcal{E}),\ X_j = P_j,\ 1 \leqslant j \leqslant d.$$

Note: $X_0 = P_0$ at the commutative limit $\kappa \to +\infty$.

The construction of this gauge theory on κ-Minkowski with twisted derivations can be found in:
P. Mathieu, J.-C. Wallet, "*Gauge theories on κ-Minkowski spaces: Twist and modular operators*", JHEP **05** (2020) 115, 10.1007/JHEP05(2020)112, arXiv:2002.02309

Noncommutative geometry based on derivations

Differential Geometry	Noncommutative geometry
	Module $\mathbb{E}$ over $\mathcal{A}$
$\Omega^\bullet(\mathcal{M})$	$\Omega^\bullet(\mathcal{A})$

Connection and curvature

Consider the module $\mathbb{E} = \mathcal{M}_\kappa$, with action $m \triangleleft f = m \star f$.

Expression of a connection and its curvature

$$\nabla_{X_\mu}(f) = X_\mu(f) + A_\mu \star f$$
$$F_{\mu\nu} = X_\mu(A_\nu) - X_\nu(A_\mu) + A_\mu \star A_\nu - A_\nu \star A_\mu$$

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Gauge transformation

$$A_\mu^g = g^\dagger \star A_\mu \star g + g^\dagger \star X_\mu(g) \qquad F_{\mu\nu}^g = g^\dagger \star F_{\mu\nu} \star g$$

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Unitary gauge group

$$\mathcal{U}(1) = \{g \in \mathbb{E}^\times, g^\dagger \star g = g \star g^\dagger = 1\}.$$

Twisted connection and curvature

Consider the module $\mathbb{E} = \mathcal{M}_\kappa$, with action $m \triangleleft f = m \star f$.

Expression of a twisted connection and its curvature

$$\nabla_{X_\mu}(f) = X_\mu(f) + A_\mu \star f$$
$$F_{\mu\nu} = X_\mu(A_\nu) - X_\nu(A_\mu) + (\mathcal{E} \triangleright A_\mu) \star A_\nu - (\mathcal{E} \triangleright A_\nu) \star A_\mu$$

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$$A_\mu^g = (\mathcal{E} \triangleright g^\dagger) \star A_\mu \star g + (\mathcal{E} \triangleright g^\dagger) \star X_\mu(g) \quad F_{\mu\nu}^g = (\mathcal{E}^2 \triangleright g^\dagger) \star F_{\mu\nu} \star g$$

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$$A_\mu^g = (\mathcal{E} \triangleright g^\dagger) \star A_\mu \star g + (\mathcal{E} \triangleright g^\dagger) \star X_\mu(g) \quad F_{\mu\nu}^g = (\mathcal{E}^2 \triangleright g^\dagger) \star F_{\mu\nu} \star g$$

Unitary gauge group

$$\mathcal{U}(1) = \{g \in \mathbb{E}^\times, g^\dagger \star g = g \star g^\dagger = 1\}.$$

The gauge invariant action

Noncommutative $\mathcal{U}(1)$-Yang-Mills-like action

$$S = \int \mathrm{d}^5x \; F \star F^\dagger$$

which requires that the **space-time dimension is 5**.

This action satisfies the following properties:

1. κ-Poincaré invariance,
2. Gauge invariance under the unitary gauge group $\mathcal{U}(1)$,
3. The commutative limit $\kappa \to +\infty$ coincide with standard Abelian gauge theory (in $(d+1)$-dimensional Minkowski space).

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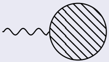
Non-vanishing tadpole

At first, we assume that A is real-valued, *i.e.* $\overline{A} = A$.

$$S = \int K^{\mu\nu} A_\mu A_\nu + V_{(3)}^{\mu\nu\rho} A_\mu A_\nu A_\rho + V_{(4)}^{\mu\nu\rho\sigma} A_\mu A_\nu A_\rho A_\sigma$$

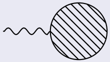
We use the Fadeev-Popov procedure with BRST gauge fixing.

Deformed Lorenz gauge
$X^\mu A_\mu = 0$,



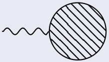
$$= \int \mathrm{d}^5 x \, \mathcal{I}(\kappa) A_0(x).$$

Parametrized temporal gauge
$A_0 = \lambda$,



$$= \Gamma_1^{\mathrm{gh}}(A_0) + \lambda \int \mathrm{d}^5 x \, \mathcal{J}(\kappa) A_0(x).$$

Comments on the tadpole



$$= \begin{cases} \int \mathrm{d}^5x \, \mathcal{I}(\kappa) A_0(x) & (X^\mu A_\mu = 0) \\ \Gamma_1^{\mathrm{gh}}(A_0) + \lambda \int \mathrm{d}^5x \, \mathcal{J}(\kappa) A_0(x) & (A_0 = \lambda) \end{cases}$$

Comments:

- Good commutative limit: the tadpole vanish when $\kappa \to +\infty$.
- The tapole is still non-vanishing when relaxing $\overline{A} = A$.
- If matter is added, it does not contribute.
- Non-vanishing tadpole has already been encountered in other quantum spaces like for massless gauge theory[7] on $\mathbb{R}^3_\lambda$ or for matrix gauge theory on the Moyal plane[8] $\mathbb{R}^2_\theta$.
- The gauge (BRST) symmetry is broken.
- The temporal gauge $A_0 = 0$ is recovered taking $\lambda \to 0$. Doing so the ghosts decouples $\Gamma_1^{\mathrm{gh}}(A_0) = 0$ and the tadpole vanish.

[7] A. Géré, P. Vitale, J.-C. Wallet, "*Quantum gauge theories on noncommutative 3-d space*", Phys. Rev. D **90** (2014) 045019, 10.1103/PhysRevD.90.045019, arXiv:1312.6145

[8] P. Martinetti, P. Vitale, J.-C. Wallet, "*Noncommutative gauge theories on $\mathbb{R}^2_\theta$ as matrix models*", JHEP **09** (2013) 051, 10.1007/JHEP09(2013)051, arXiv:1303.7185

The new vacuum

The vacuum expectation value of the gauge potential is non-vanishing: $\langle A_\mu \rangle \neq 0$.

$$\Downarrow$$

The classical vacuum is unstable against quantum fluctuations (through linear term in A_μ).

$$\Downarrow$$

The theory is expended around a new vacuum $A_\mu = \langle A_\mu \rangle + \alpha_\mu$, with a new field variable α_μ, through 1-loop renormalization.

Thanks for your attention !