Workshop on noncommutative and generalized geometry in string theory, gauge theory and related physical models

N=2 Minkowski Superspace and its Quantization

Joint work with Rita Fioresi and María A. Lledó:

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N=2 Minkowski Superspace and its quantization

Motivation/Summary

Motivation

- Recall that, we can consider grassmannians as algebraic (projective) varieties using *Plücker embedding*.

- The image of a grassmannian under this embedding is characterized by well-known *Plücker relations*.

- From Physics' point of view, the grassmannian $Gr(2,4)$ is important. In this case, the *plücker map* is given by:

$$Gr(2,4) \longrightarrow \mathbb{P}(\bigwedge^2(\mathbb{C}^4)) \cong \mathbb{P}(\mathbb{C}^6)$$

$$(a,b) \mapsto [a \wedge b] \equiv [y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}]$$

where $y_{ij} : a_i b_j - b_i a_j$ and the *plücker relation* reads:

$$y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23} = 0.$$

Motivation

- The big cell is defined as $U_{12} := \{(a, b) : y_{12} \neq 0\}$. One can easily see that $U_{12} \cong \mathbb{C}^4$, which can be identified with the complex Minkowski space. (For more details, see [3] Chap. 2)

- In our work [1], we tried to extend this idea in super setting for grassmannian $Gr(2|0, 4|2)$.

  (A similar work for $Gr(2|0, 4|1)$ has already been done by D. Cervantes, R. Fioresi and M. A. Lledo in [5].)

Summary

- We prove that $Gr(2|0,4|2)$ is a supervariety by constructing *super Plücker map* $Gr(2|0,4|2) \longrightarrow \mathbb{P}(\bigwedge^2(\mathbb{C}^{4|2}))$ and finding super plucker relations.

- We notice that we can consider $\mathbb{C}(Gr)$ as a subalgebra of $\mathbb{C}(SL(4|2))$.

- We construct the quantum grassmannian $Gr_q(2|0,4|2)$ by considering the Manin relations among the entries of $\mathbb{C}(SL(4|2))$.

- Then, we consider *big cell* S *in this setting similar to the classical case and construct* $\mathbb{M}$ *what we call the* $N = 2$ *Minkowski superspace.*

- *At the end, we develop a quantization* $\mathbb{C}_q[\mathbb{M}]$ *of* $\mathbb{M}$ *and shown that* $\mathbb{C}_q[S] \longrightarrow \mathbb{C}_q[\mathbb{M}]$ *is a quantum principal bundle.*

Super Plücker Embedding for $Gr(2|0,4|2)$

Super Plücker Embedding

- Let $\{e_1, e_2, e_3, e_4, \epsilon_5, \epsilon_6\}$ be a basis for $\mathbb{C}^{4|2}$ and $E := \bigwedge^2(\mathbb{C}^{4|2})$.

  We then have following set as a basis for E:

$$(Even): \quad e_i \wedge e_j \quad 1 \le i < j \le 4, \;\; \epsilon_5 \wedge \epsilon_5, \quad \epsilon_6 \wedge \epsilon_6, \quad \epsilon_5 \wedge \epsilon_6$$

$$(Odd): \quad e_k \wedge \epsilon_l \quad 1 \le k \le 4 \;\; and \;\; 5 \le l \le 6$$

- A general element Q of E can be written as:

$$Q = q + \lambda_5 \wedge \epsilon_5 + \lambda_6 \wedge \epsilon_6 + a_{55}\epsilon_5 \wedge \epsilon_5 + a_{66}\epsilon_6 \wedge \epsilon_6 + a_{56}\epsilon_5 \wedge \epsilon_6$$

  . with

$$q = q_{ij} e_i \wedge e_j, \quad \lambda_m = \lambda_{mi} e_i, \qquad i, j = 1, \ldots, 4, \quad m = 5, 6\,.$$

- The element Q is decomposable if $Q = a \wedge b$, where

$$a = r + \xi_5\epsilon_5 + \xi_6\epsilon_6, \qquad b = s + \eta_5\epsilon_5 + \eta_6\epsilon_6,$$

  with $r = r_i e_i$, $s = s_i e_i$.

- One obtains the following equalities:

$$q = r \wedge s$$

$$\lambda_5 = \xi_5 s - \eta_5 r, \qquad \lambda_6 = \xi_6 s - \eta_6 r,$$

$$a_{55} = \xi_5 \eta_5, \qquad a_{66} = \xi_6 \eta_6, \qquad a_{56} = \xi_5 \eta_6 + \xi_6 \eta_5$$

which imply,

$$q \wedge q = 0,$$

$$q \wedge \lambda_5 = 0, \qquad q \wedge \lambda_6 = 0,$$

$$\lambda_5 \wedge \lambda_5 = -2a_{55} q, \qquad \lambda_6 \wedge \lambda_6 = -2a_{66} q, \qquad \lambda_5 \wedge \lambda_6 = -a_{56} q,$$

$$\lambda_5 a_{55} = 0, \qquad \lambda_6 a_{66} = 0,$$

$$\lambda_5 a_{66} = -\lambda_6 a_{56}, \qquad \lambda_6 a_{55} = -\lambda_5 a_{56},$$

$$a_{55}^2 = 0, \qquad a_{66}^2 = 0, \qquad a_{56} a_{56} = -2a_{55} a_{66}$$

$$a_{55} a_{56} = 0, \qquad a_{66} a_{56} = 0.$$

Super Plücker Embedding

- Above relations are the *super Plücker relations*. More explicitly, We can write them in coordinates in the following way (always $1 \leq i < j < k \leq 4$ and $5 \leq n \leq 6$):

$$
\begin{aligned}
&q_{12}q_{34} - q_{13}q_{24} + q_{14}q_{23} = 0, \\
&q_{ij}\lambda_{kn} - q_{ik}\lambda_{jn} + q_{jk}\lambda_{in} = 0, \\
&\lambda_{in}\lambda_{jn} = a_{nn}q_{ij}, \\
&\lambda_{i5}\lambda_{j6} + \lambda_{i6}\lambda_{j5} = a_{56}q_{ij}, \\
&\lambda_{in}a_{nn} = 0, \qquad \lambda_{i5}a_{66} = -\lambda_{i6}a_{56}, \qquad \lambda_{i6}a_{55} = -\lambda_{i5}a_{56} \\
&a_{56}a_{56} = -2a_{55}a_{66}, \qquad a_{55}a_{56} = 0, \qquad a_{66}a_{56} = 0, \\
&a_{nn}^2 = 0.
\end{aligned}
$$

Super Plücker Relations

We denote as $\mathbb{I}_P$ the ideal generated by them in the affine superspace $A^{9|8}$ (with generators $q_{ij}, a_{nm}, \lambda_{nk}$). They are homogeneous equations, so they are defined in the projective space $\mathbb{P}^{8|8}$.

Theorem
The superring associated to the image of Gr under the super Plücker embedding is

$$\mathbb{C}[Gr] \cong \mathbb{C}[q_{ij}, a_{nm}, \lambda_{nk}]/\mathbb{I}_P \,,$$

that is, the relations in $\mathbb{I}_P$ are all the relations satisfied by the generators $q_{ij}, a_{nm}, \lambda_{nk}$.

Theorem
The superring $\mathbb{C}[Gr]$ can be considered as a subalgebra of $\mathbb{C}(SL(4|2))$.

Proof:
Let us display the generators in matrix form:

$$\left(\begin{array}{cccc|cc} g_{11} & g_{12} & g_{13} & g_{14} & \gamma_{15} & \gamma_{16} \\ g_{21} & g_{22} & g_{23} & g_{24} & \gamma_{25} & \gamma_{26} \\ g_{31} & g_{32} & g_{33} & g_{34} & \gamma_{35} & \gamma_{36} \\ g_{41} & g_{42} & g_{43} & g_{44} & \gamma_{45} & \gamma_{46} \\ \hline \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & g_{55} & g_{56} \\ \gamma_{61} & \gamma_{62} & \gamma_{63} & \gamma_{64} & g_{65} & g_{66} \end{array}\right)$$

then $\mathbb{C}[SL(4|2)] = \mathbb{C}[g_{ij}, g_{mn}, \gamma_{im}, \gamma_{nj}]/(Ber - 1)$ where Ber is the Berezenian of the matrix and $1 \leq i, j \leq 4$ and $5 \leq m, n \leq 6$.

Proof: (continued)
Let

$$y_{ij} = g_{i1}g_{j2} - g_{i2}g_{j1}, \quad \eta_{kn} = g_{i1}\gamma_{n2} - g_{i2}\gamma_{n1}$$

$$x_{55} = \gamma_{51}\gamma_{52}, \qquad x_{66} = \gamma_{61}\gamma_{62} \qquad x_{56} = \gamma_{51}\gamma_{62} + \gamma_{61}\gamma_{52}$$

Define a homomorphism:

$$\mathbb{C}[G] \longrightarrow \mathbb{C}[SL(4|2)]$$

$$q_{ij}, \lambda_{kn} \mapsto y_{ij}, \eta_{kn}$$

$$a_{55}, a_{66}, a_{56} \mapsto x_{55}, x_{66}, x_{56}$$

This completes the proof.

Quantum Grassmannian

$Gr_q(2|0,4|2)$

Quantum Matrix Superalgebra

Definition
The quantum matrix superalgebra $M_q(r|s)$ is defined as

$$M_q(r|s) := \mathbb{C}_q\langle z_{ij}, \xi_{kl}\rangle / \mathbb{I}_M$$

where $\mathbb{C}_q\langle z_{ij}, \xi_{kl}\rangle$ denotes the free superalgebra over $\mathbb{C}_q = \mathbb{C}[q, q^{-1}]$ generated by the even variables

$$z_{ij}, \qquad \text{for} \quad 1 \leq i, j \leq r \quad \text{or} \qquad r+1 \leq i, j \leq r+s.$$

and by the odd variables

$$\xi_{kl} \qquad \begin{aligned} &\text{for} \quad 1 \leq k \leq r, \quad r+1 \leq l \leq r+s \\ &\text{or} \;\; r+1 \leq k \leq r+s, \quad 1 \leq l \leq r, \end{aligned}$$

satisfying the relations $\xi_{kl}^2 = 0$ and $\mathbb{I}_M$ is an ideal that we describe below.

We can visualize the generators as a matrix $\begin{bmatrix} z_{m\times m} & \xi_{m\times n} \\ \xi_{n\times m} & z_{n\times n} \end{bmatrix}$.

It is convenient sometimes to have a common notation for even and odd variables.

$$a_{ij} = \begin{cases} z_{ij} & 1 \le i,j \le r, \text{ or } \quad r+1 \le i,j \le r+s, \\ \\ \xi_{ij} & 1 \le i \le r, \quad r+1 \le j \le r+s, \text{ or} \\ & r+1 \le i \le r+s, \quad 1 \le j \le r\,. \end{cases}$$

We assign a parity to the indices: $p(i) = 0$ if $1 \le i \le r$ and $p(i) = 1$ if $r+1 \le i \le r+s$. The parity of a_{ij} is $\pi(a_{ij}) = p(i) + p(j)$ mod 2.

Quantum Matrix Superalgebra

Then, the ideal $\mathbb{I}_M$ is generated by the following relations:

- For $j < l$:

$$a_{ij} a_{il} = (-1)^{\pi(a_{ij})\pi(a_{il})} q^{(-1)^{p(i)+\mathbf{1}}} a_{il} a_{ij}$$

- For $i < k$:

$$a_{ij} a_{kj} = (-1)^{\pi(a_{ij})\pi(a_{kj})} q^{(-1)^{p(j)+\mathbf{1}}} a_{kj} a_{ij}$$

- For $i < k,\ j > l$ *or* $i > k,\ j < l$

$$a_{ij} a_{kl} = (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl} a_{ij}$$

- For $i < k,\ j < l$:

$$a_{ij} a_{kl} - (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl} a_{ij} = (-1)^{\pi(a_{ij})\pi(a_{kl})} (q^{-1} - q) a_{kj} a_{il}$$

- There is also a comultiplication defined $\Delta(a_{ij}) := \sum_k a_{ik} \otimes a_{kj}$ and a counit $\varepsilon(a_{ij}) = \delta_{ij}$.
- One can restrict to $SL_q(r|s)$ by setting the quantum Berezinian to 1.

Definition
The *quantum super Grassmannian* $Gr_q(2|0,4|2)$ is the subalgebra of $SL_q(4|2)$ generated by the elements

$$D_{ij} := a_{i1}a_{j2} - q^{-1}a_{i2}a_{j1} \qquad D_{in} := a_{i1}a_{n2} - q^{-1}a_{i2}a_{n1}$$

$$D_{55} := a_{51}a_{52} \qquad D_{66} := a_{61}a_{62}$$

$$D_{56} = a_{51}a_{62} - q^{-1}a_{52}a_{61}$$

with $1 \leq i < j \leq 4$ and $n = 5, 6$.

$Gr_q(2|0,4|2)$

After some (tedious) calculations we arrive at:

- Let $1 \leq i,j,k,l \leq 6$ be not all distinct, and D_{ij}, D_{kl} not both odd. $D_{ij}D_{kl} = q^{-1}D_{kl}D_{ij}, \qquad (i,j) < (k,l),\ i<j, k<l$ where the ordering '$<$' of pairs is the lexicographical ordering.

- Let $1 \leq i,j,k,l \leq 6$ be all distinct, and D_{ij}, D_{kl} not both odd and $D_{ij}, D_{kl} \neq D_{56}$. Then

$$\begin{aligned}
D_{ij}D_{kl} &= q^{-2}D_{kl}D_{ij}, && 1 \leq i<j<k<l \leq 6,\\
D_{ij}D_{kl} &= q^{-2}D_{kl}D_{ij} - (q^{-1}-q)D_{ik}D_{jl} && 1 \leq i<k<j<l \leq 6,\\
D_{ij}D_{kl} &= D_{kl}D_{ij} && 1 \leq i<k<l<j \leq 6,
\end{aligned}$$

- Let $1 \leq i < j \leq 4$, $5 \leq n \leq m \leq 6$. Then

$$
\begin{aligned}
&D_{in}D_{jn} = -q^{-1}D_{jn}D_{in} - (q^{-1}-q)D_{ij}D_{nn} = -qD_{jn}D_{in},\\
&D_{ij}D_{nm} = q^{-2}D_{nm}D_{ij},\\
&D_{i5}D_{j6} = -q^{-2}D_{j6}D_{i5} - (q^{-1}-q)D_{ij}D_{56},\\
&D_{i6}D_{j5} = -D_{j5}D_{i6},\\
&D_{i5}D_{i6} = -q^{-1}D_{i6}D_{i5},\\
&D_{i5}D_{i6} = -q^{-1}D_{i6}D_{i5},\\
&D_{55}D_{66} = -q^{-2}D_{66}D_{55}\,,\\
&D_{55}D_{56} = 0\,.
\end{aligned}
$$

$Gr_q(2|0,4|2)$

- The *super plucker relations* are quantized as follows: One has for $1 \leq i < j \leq 4$ and $n = 5, 6$:

$$
\begin{aligned}
&D_{12}D_{34} - q^{-1}D_{13}D_{24} + q^{-2}D_{14}D_{23} = 0,\\
&D_{ij}D_{kn} - q^{-1}D_{ik}D_{jn} + q^{-2}D_{in}D_{jk} = 0,\\
&D_{i5}D_{j6} + q^{-1}D_{i6}D_{j5} = qD_{ij}D_{56},\\
&D_{in}D_{jn} = qD_{ij}D_{nn},\\
&D_{in}D_{nn} = 0,\\
&D_{i5}D_{66} = -q^{-1}D_{i6}D_{56},\\
&D_{i6}D_{55} = -q^{2}D_{i5}D_{56},\\
&D_{nn}^2 = 0, \qquad D_{55}D_{56} = 0, \qquad D_{66}D_{56} = 0,\\
&D_{56}D_{56} = (q^{-1} - 3q)D_{55}D_{66}.
\end{aligned}
$$

$Gr_q(2|0,4|2)$ as quantum homogeneous space

Theorem
The restriction of the comultiplication in $SL_q(4|2)$ *to* $Gr_q(2|0,4|2)$ *is of the form:*

$$Gr_q(2|0,4|2) \longrightarrow SL_q(4|2) \otimes Gr_q(2|0,4|2).$$

Proof.
Some tedious calculations. □

N=2 Minkowski Superspace and its quantization

Quantum Super Bundles

- Let (H, Δ, ϵ, S) be a Hopf superalgebra and A be an H-comodule superalgebra with coaction $\delta : A \longrightarrow A \otimes H$. Let

$$B := A^{coinv(H)} := \{a \in A \,|\, \delta(a) = a \otimes 1\}$$

The extension A of the superalgebra B is called *H-Hopf-Galois* (or simply *Hopf-Galois*) if the map

$$\chi : A \otimes_B A \longrightarrow A \otimes H, \qquad \chi = (m_A \otimes \mathrm{id})(\mathrm{id} \otimes_B \delta)$$

called the *canonical map*, is bijective (m_A denotes the multiplication in A).

- We define quantum principal superbundle as a pair (A, B), where A is an H-Hopf Galois extension and A is H-equivariantly projective as a left B-supermodule.

- Let H be a Hopf superalgebra and A an H-comodule superalgebra. The algebra extension $A^{coinv\ H} \subset A$ is called a *cleft extension* if there is a right H-comodule map $j : H \longrightarrow A$, called *cleaving map*, that is convolution invertible, , i.e. there exists a map $h : H \longrightarrow A$ such that the convolution product $j \star h$ satisfies:

$$(j \star h)(f) := (m_A \circ (j \otimes h) \circ \Delta)(f) = \epsilon(f).1$$

  for all $f \in H$.

- An extension $A^{coinv\ H} \subset A$ is called a *trivial extension* if there is an H-comodule algebra map $j : H \to A$. In this case, the convolution inverse is just $h = j \circ S$.

$N = 2$ Minkowski Superspace

- Consider the set S of $(4|2) \times 2$ matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ \alpha_{51} & \alpha_{52} \\ \alpha_{61} & \alpha_{62} \end{bmatrix}$$

with $a_{11}a_{22} - a_{12}a_{21}$ invertible.

- There is a natural right action of $GL_2(\mathbb{C})$ on S.

$N = 2$ Minkowski Superspace

- Every element of S can be writeen uniquely as:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ u_{31} & u_{32} \\ u_{41} & u_{42} \\ \nu_{51} & \nu_{52} \\ \nu_{61} & \nu_{62} \end{bmatrix}$$

- In other words, the quotient of S under the action of $GL_2(\mathbb{C})$ is an affine superspace $\mathbb{M} \cong \mathbb{C}^{4|4}$. We call $\mathbb{M}$ the $N = 2$ Minkowski superspace.

- One can easily compute u_{ij} and ν_{kl} for an arbitrary element of S:

$$u_{i1} = -d_{2i} d_{12}^{-1} \quad \textit{and} \quad u_{i2} = d_{1i} d_{12}^{-1}$$
$$\nu_{k1} = -d_{2k} d_{12}^{-1} \quad \textit{and} \quad u_{k2} = d_{1k} d_{12}^{-1}.$$

Lemma
The coordinate superalgebra $\mathbb{C}[\mathbb{M}] = \mathbb{C}[u_{ij}, \nu_{kl}]$ *is isomorphic to the superalgebra* $\mathbb{C}[S]^{coinv(\mathbb{C}[GL_2(\mathbb{C})])}$ *with respect to the* $\mathbb{C}[GL_2(\mathbb{C})]$ *right coaction defined by:*

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ \alpha_{51} & \alpha_{52} \\ \alpha_{61} & \alpha_{62} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ \alpha_{51} & \alpha_{52} \\ \alpha_{61} & \alpha_{62} \end{bmatrix} \otimes \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}.$$

Theorem
The natural projection $p : S \longrightarrow S/GL_2(\mathbb{C}) \cong \mathbb{M}$ *is a trivial principal bundle.*

Let $\mathbb{C}_q[S]$ be the quantization of S obtained by taking the Manin relations among the entries with D_{12} invertible.

Definition
We define the $N = 2$ quantum chiral Minkowski superspace $\mathbb{C}_q[\mathbb{M}]$ as the superalgebra generated by the elements:

$$\tilde{u}_{i1} := -q^{-1} D_{2i} D_{12}^{-1} \qquad \tilde{u}_{i2} := D_{1i} D_{12}^{-1} \qquad i = 3, 4$$

$$\tilde{\nu}_{k1} := -q^{-1} D_{2k} D_{12}^{-1} \qquad \tilde{u}_{k2} := D_{1k} D_{12}^{-1} \qquad k = 5, 6.$$

Using our previous computations for commutation relations among $D'_{rs}s$, we get the following commutation relations among $\tilde{u}'_{ij}s$ and $\tilde{\nu}'_{kl}s$:

$N = 2$ Quantum Chiral Minkowski Superspace

$$\tilde{u}_{i2}\tilde{u}_{i1} = q^{-1}\tilde{u}_{i1}\tilde{u}_{i2} \qquad \textit{for } i = 3,4$$

$$\tilde{\nu}_{k1}\tilde{\nu}_{k2} = -q^{-1}\tilde{\nu}_{k2}\tilde{\nu}_{k1} \qquad \textit{for } k = 5,6$$

$$\tilde{\nu}_{5l}\tilde{\nu}_{6l} = -q^{-1}\tilde{\nu}_{6l}\tilde{\nu}_{5l} \qquad \textit{for } l = 1,2$$

$$\tilde{u}_{3j}\tilde{u}_{4j} = q^{-1}\tilde{u}_{4j}\tilde{u}_{3j} \qquad \textit{for } j = 1,2$$

$$\tilde{u}_{ij}\tilde{\nu}_{kj} = q^{-1}\tilde{\nu}_{kj}\tilde{u}_{ij} \qquad \textit{for } j = 1,2 \quad i = 3,4 \quad k = 5,6$$

$$\tilde{u}_{i1}\tilde{\nu}_{k2} = \tilde{\nu}_{k2}\tilde{u}_{i1} \qquad \textit{for } i = 3,4 \quad k = 5,6$$

$$\tilde{u}_{31}\tilde{u}_{42} = \tilde{u}_{42}\tilde{u}_{31}$$

$$\tilde{\nu}_{51}\tilde{\nu}_{62} = -\tilde{\nu}_{62}\tilde{\nu}_{51}$$

$$\tilde{u}_{32}\tilde{u}_{41} - \tilde{u}_{41}\tilde{u}_{32} = (q^{-1} - q)\tilde{u}_{42}\tilde{u}_{31}$$

$$\tilde{u}_{i2}\tilde{\nu}_{k1} - \tilde{\nu}_{k1}\tilde{u}_{i2} = (q^{-1} - q)\tilde{\nu}_{k2}\tilde{u}_{i1} \qquad \textit{for } i = 3,4 \quad k = 5,6$$

$$\tilde{\nu}_{52}\tilde{\nu}_{61} + \tilde{\nu}_{61}\tilde{\nu}_{52} = -(q^{-1} - q)\tilde{\nu}_{62}\tilde{\nu}_{51}$$

N=2 Quantum Chiral Minkowski Superspace

Theorem
$\mathbb{C}_q[\mathbb{M}]$ *is isomorphic as a superalgebra to the superalgebra of matrices* $M_q(2|2)$*.*

Theorem
The quantum superalgebra $\mathbb{C}_q[S]$ *is a trivial quantum principal bundle over* $N = 2$ *quantum chiral Minkowski superspace* $\mathbb{C}_q[\mathbb{M}]$*.*

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Thank You