CORFU **JERZY LUKIERSKI**
22.09.2022

QUANTUM-DEFORMED PHASE SPACES WITH NON-COMMUTATIVE COORDINATES AND MOMENTA

1. **Introduction**
2. **Quantum phase spaces with noncommutative coordinates and commutative momenta**
   a. **κ-deformed quantum phase space**
   b. **Snyder quantum phase space**
   c. **Extended Snyder models, spontanous Lorentz symmetry breaking and $\hbar$-expansion.**
3. **Yang quantum phase spaces with Snyder type noncommutativity of coordinates and momenta**
4. **Triple Special Relativity model and quantum phase spaces**
5. **Outlook (SUSY extensions, quantum superspace, quantum spinors)**

(collaboration with S. Meljanac, S. Mignemi, A. Pachoł, M. Woronowicz)

1. Introduction

Three phase space algebra frameworks:

a) Classical theories with commuting phase space variables (x_μ, p_μ)

b) Standard quantum theories (QM, standard QFT) - with canonically quantized phase space coordinates

$$[x_\mu, p^\nu] = i\delta_\mu{}^\nu$$

but $x_\mu \in X$ and $p_\mu \in P$ as subalgebras (X, P) remain commutative

c) Noncanonical quantum-deformed phase spaces $(\hat{x} \in \hat{X}, \hat{p} \in \hat{P})$ with general noncommutativity relations, in presence of Quantum Gravity (QG) corrections

$$[\hat{x}_\mu, \hat{x}_\nu] = iF_{\mu\nu}(\hat{X}, \hat{P}) \qquad [\hat{p}^\mu, \hat{p}^\nu] = iH^{\mu\nu}(\hat{X}, \hat{P})$$
$$[\hat{x}_\mu, \hat{p}^\nu] = iG_\mu{}^\nu(\hat{X}, \hat{P})$$

where the algebra above satisfies the Jacobi identities.

2. Quantum phase spaces $(\hat{X}, P)$ with noncommutative coordinates and commutative momenta

We assume that $H^{\mu\nu} = 0$ i.e. $(\hat{p}_\mu = p_\mu)$, and firstly consider $G_\mu{}^\nu(\hat{X}, \hat{P}) = G_\mu{}^\nu(\hat{P})$

$$[\hat{x}_\mu, \hat{x}_\nu] = iF_{\mu\nu}(\hat{X}) \qquad [\hat{x}_\mu, p^\nu] = iG_\mu{}^\nu(P) \qquad \rightarrow \qquad \hat{x}_\mu = G_\mu{}^\nu(P)x_\nu$$

In such case $\hat{x} \in \hat{X}$ span the algebraic basis of the coordinate subalgebra $\hat{X}$ of $(\hat{X}, P)$ and second relation contains on rhs the constant term.

Such associative phase space algebras should be described by Hopf algebroid structure - one can choose the nonsymmetric bialgebroid canonical coproduct

$$\Delta(\hat{X}) = \hat{X} \otimes 1$$

and one can use $\hat{X}$ as the base algebra of bialgebroid $\mathcal{H}$ defined by the structure

$$\mathcal{H} = (H, A; \alpha, \beta; \Delta_A; S; \epsilon)$$

where $H(\hat{X}, P)$ defines total algebra, $A = \hat{X}$ is the base algebra; further
α - source map (homomorphism) $A \rightarrow H \in \alpha(a)$
β - target map (antihomomorphism) $A \rightarrow H \in \beta(a)$ with the property $[\alpha(a), \beta(a)] = 0$ for any $a, b \in A$

$\Delta_A :$ coproduct map from H into (A, A)-bimodule product $H \otimes_A H$ (Takeushi, 1977) or into standard tensor product $H \otimes H$ with coproduct gauge freedom.

a) κ–deformed quantum phase space (with Hopf algebroid structure)

An example of such quantum phase space with closed quantum Minkowski space-time algebra is provided by the generalized κ deformation of Poincare algebra, where

$$F_{\mu\nu}(\hat{X}) = \frac{1}{\kappa}(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) \qquad a_\mu - \text{constant fourvector}$$

For standard κ-deformation we choose $a_\mu = (1, 0, 0, 0) \leftrightarrow a_\mu a^\mu = 1$
and for light-cone κ- deformation $a_\mu = (1, 1, 0, 0) \leftrightarrow a_\mu a^\mu = 0$

The function $G_\mu{}^\nu = \delta_\mu{}^\nu - \frac{1}{\kappa} p_\mu a^\nu$ defines κ-deformed noncanonical commutation relations.

Important remark: the property that base algebra $\hat{X}$ is a closed subalgebra is quite limiting. In particular for so-called $\theta_{\mu\nu}$-deformation (Moyal deformation) due to the c-number nature of $\theta_{\mu\nu}$ such a property is not satisfied

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \notin \hat{X}$$

κ-deformation: JL+Z. Skoda+M. Woronowicz, arXiv:1507.02612
$\theta_{\mu\nu}$-deformation: JL+M. Woronowicz, arXiv:1902.12313

In the second paper it is shown that if for $\theta_{\mu\nu}$-deformation you wish to obtain quantum phase space as described by Hopf algebroid, we should consider the generalized 10+10 dimensional phase space, with added Lorentz sector $(\Lambda_{\mu\nu}, M_{\mu\nu})$

$$(\hat{x}_\mu; \hat{p}_\mu) \to (\hat{x}_\mu, \Lambda_{\mu\nu}; \hat{p}_\mu, M_{\mu\nu}), \qquad (\hat{x}_\mu, \Lambda_{\mu\nu}) \in A$$

and use so-called Heisenberg double construction.

b) Snyder quantum phase space (without Hopf algebroid structure)

Let us generalize the formula ($\hat{x}_\mu \in \hat{X}$)

$$[\hat{x}_\mu, \hat{x}_\nu] = iF_{\mu\nu}(\hat{X}) \qquad \Rightarrow \qquad [\hat{x}_\mu, \hat{x}_\nu] = iF_{\mu\nu}(\hat{X}, P)$$

We keep the relations $H^{\mu\nu} = 0$ ($[p^\mu, p^\nu] = 0$) as well as $[\hat{x}_\mu, p^\nu] = iG_\mu{}^\nu(P)$ valid.

In such a case quantum phase space can not be described by Hopf algebroid, because $\hat{X}$ is not a subalgebra of the phase space algebra $(\hat{X}, P)$.

Snyder quantum phase space provides a concrete example. For that purpose we introduce firstly Snyder quantum space-time algebra with $\hat{o}(4,1)$ generators

$$[\hat{x}_\mu, \hat{x}_\nu] = ig\hat{M}_{\mu\nu} \qquad \hat{M}_{\mu 4} = M\,\hat{x}_\mu \qquad \text{where} \qquad g = \frac{1}{M^2}$$

and M is the mass parameter with length dimensionality $[M] = -1$; $M_{\mu\nu}$ are Lorentz algebra generators. We keep D=4 relativistic covariance relation for $\hat{x}_\mu$

$$[\hat{M}_{\mu\nu}, \hat{x}_\rho] = i(\eta_{\mu\rho}\hat{x}_\nu - \eta_{\nu\rho}\hat{x}_\mu)\,.$$

To introduce historical Snyder quantum phase algebra (Snyder 1974) we introduce commutative p_μ, where $[x_\mu, p_\nu] = i\eta_{\mu\nu}$, and orbital canonical realization of $\hat{M}_{\mu\nu}$,

$$M_{\mu\nu} = i(x_\mu p_\nu - x_\nu p_\mu) = (G^{-1})_{[\mu|}{}^{\rho}\hat{x}_\rho p_{|\nu]}$$

NC Snyder quantum spacetime is realized by canonical (x_μ, p^ν) as follows

$$\hat{x}_\mu = G_\mu{}^\nu(p)x_\nu \quad \text{or} \quad x_\mu = (G^{-1})_\mu{}^\rho \hat{x}_\rho , \quad \text{where} \quad G_\mu{}^\nu(p) = \delta_\mu^\nu + g p_\mu p^\nu .$$

One gets Snyder phase space $(\hat{X}, P)$ defined by the relations

$$[\hat{x}_\mu, \hat{x}_\nu] = igM_{\mu\nu} = ig((G^{-1})_\mu{}^\rho \hat{x}_\rho p_\nu - (G^{-1})_\nu{}^\rho \hat{x}_\rho p_\mu) \leftarrow \quad F_{\mu\nu}(\hat{X}, P)$$

$$[\hat{x}_\mu, p_\nu] = i(\eta_{\mu\nu} + g p_\mu p_\nu)$$

Jacobi identities permit the generalization of the formula for $G_\mu{}^\nu$ and more general quantum-deformed Heisenberg algebra relations is allowed. In general case one gets

$$[\hat{x}_\mu, p_\nu] = i(F(gp^2)\eta_{\mu\nu} + p_\mu p_\nu K(gp^2))$$

where functions K, F satisfy the local functional relation

$$K = \Phi(F, F') , \quad \text{where} \quad F' = \frac{dF}{dx} .$$

(Snyder choice: $F = K = 1$).

c) Extended Snyder models, spontaneous Lorentz symmetry breaking and $\hbar$-expansions

Snyder model is called extended if $\hat{M}_{\mu\nu}$ $(\mu,\nu = 0,1,2,3)$ and $\hat{x}_\mu$ are considered as independent generators, described by $D = 4$ de-Sitter $\hat{o}(4,1)$ Lie algebra $\hat{M}_{AB}$ $(A,B = 0,1,2,3,4)$, where

$$\hat{M}_{AB} = (\hat{M}_{\mu\nu}\,, \hat{M}_{\mu 4} = M\hat{x}_\mu) \qquad M - \quad \text{mass like parameter}$$

If we expand $\hat{M}_{AB}$ in $\hbar$-power series

$$\hat{M}_{AB} = \hat{M}^{(0)}_{AB} + \hbar\hat{M}^{(1)}_{AB} + \hbar^2\,\hat{M}^{(2)}_{AB} + \ldots$$

the classical part of the operator $\hat{M}^0_{AB}$ is given by

$$\hat{M}^{(0)}_{AB} = X_{AB} = \lim_{\hbar\mapsto 0}\hat{M}_{AB}$$

where in quantized theory $(\hbar \neq 0)$

$$X_{AB} = (x_\mu, x_{\mu\nu}) = \lim_{\hbar\mapsto 0}(\hat{x}_\mu\,, \hat{M}_{\mu\nu}) =< 0|\hat{M}_{AB}|0 >$$

describe the spontaneously symmetry breaking parameters (tensorial Goldston de-Sitter modes) with $x_{\mu\nu}$ describing spontaneously breaking of Lorentz symmetry (see, e.g. R. Bluhm, V.A. Kostelecky, PRD 2005).

The algebraic equations defining Snyder model in first order of $\hbar$ lead to the following equations for $\hat{x}_\mu\,, \hat{M}_{\mu\nu}$

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i\hbar}{M^2}\hat{M}_{\mu\nu} \longrightarrow \quad [x_\mu, \hat{x}_\nu^{(1)}] - [x_\nu, \hat{x}_\mu^{(1)}] = \frac{i}{M^2}x_{\mu\nu}$$

$$\text{Lorentz cov. relat.} \longrightarrow \quad [x_{\mu\nu}, \hat{x}_\rho^{(1)}] + [M_{\mu\nu}^{(1)}, \hat{x}_\rho^{(1)}] = i(\eta_{\mu\rho}x_\nu - \eta_{\nu\rho}x_\mu)$$

$$\text{Lorentz algebra} \longrightarrow \quad [M_{\mu\nu}^{(1)}, x_{\rho\tau}] + [X_{\mu\nu}, M_{\rho\tau}^{(1)}] = i(\eta_{\mu\rho}x_{\nu\tau} + \ldots)$$

From these equations we see that $\hat{x}_\nu^{(n)}, M_{\mu\nu}^{(n)}$ $(n \geq 1)$ should depend also on canonically dual momenta $(p_\mu, p_{\mu\nu})$ satisfying the generalized Heisenberg algebra

$$[x_\mu, p_\nu] = i\hbar\eta_{\mu\nu}\,, \qquad [x_{\mu\nu}, p_{\rho\tau}] = i\hbar(\eta_{\mu\rho}\eta_{\nu\tau} - \eta_{\nu\rho}\eta_{\mu\tau} + \ldots)$$

$$[x_\mu, x_{\nu\rho}] = [p_\mu, p_{\nu\rho}] = [x_\mu, x_\nu] = [p_\mu, p_\nu] = 0$$

One gets in first order of $\hbar$ the standard formula for Lorentz generators

$$\hat{M}_{\mu\nu}^{(1)} = x_\mu p_\nu + x_{\mu\rho}p_\nu^\rho$$

i.e. terms linear in $\hbar$ in $\hbar$-expansion of $\hat{M}_{\mu\nu}$ describe sum of standard and tensorial angular momentum.

Iteratively, one finds that the terms $\hat{x}_\mu$ and $\hat{M}^{(n)}_{\mu\nu}$ as functions of $(x_\mu, p_\mu; x_{\mu\nu}, p_{\mu\nu})$ which are n-linear in the momenta $(p_\mu, p_{\mu\nu})$. In particular, one gets the following term linear in $\hbar$ for Snyder quantum space

$$\hat{x}^{(1)}_\mu = \frac{1}{2}(x^\nu p_{\mu\nu} - \beta x^\nu_\mu p_\nu)\,.$$

Such power expansion were used firstly by S. Meljanac and his collaborators (JMP (2020), PRD 102, 104 (2021)). Only very recently the application of such power series to $\hbar$-expansion with spontanously broken Lorentz symmetry has been considered (J.L., S. Meljanac, S. Mignemi, A. Pachoł, in preparation).

Remark: It appears that by modification of the basic Snyder relations (a_μ- constant fourvector)

$$[\hat{x}_\mu, \hat{x}_\nu] = i\hbar\frac{\hat{M}_{\mu\nu}}{M^2} + \frac{i\hbar}{\kappa}(a_\mu\hat{x}_\nu - a_\nu\hat{x}_\mu)$$

$$[\hat{M}_{\mu\nu}, \hat{x}_\rho] = i\hbar(\eta_{\mu\rho}\hat{x}_\nu - \eta_{\nu\rho}\hat{x}_\mu) + \frac{i\hbar}{\kappa}(a_\nu\hat{M}_{\mu\rho} - a_\mu\hat{M}_{\nu\rho})$$

one gets κ-deformed Snyder model (S. Meljanac, S. Mignemi, PLB814 (2021)) which in the extended version of this model allows as well the iterative solutions as $\hbar$-expansion.

3. Yang quantum phase spaces with both noncommutative coordinates $\hat{x}_\mu$ and momenta $\hat{p}_\mu$

C.N. Yang (half year after appearance in 1947 of Snyder paper) applied the Snyder description of noncommutativity of space-times in terms of $\hat{o}(4,1)$ Lie algebra generators as well to the fourmomenta

$$\hat{M}_{\mu 4} = M\hat{x}_\mu \xrightarrow{D=4\to D=5} (\hat{M}_{\mu 4} = M\hat{x}_\mu, \hat{M}_{\mu 5} = R\hat{p}_\mu) \qquad [M] = -1, \ [R] = 1$$

provided that we pass from $\hat{o}(4,1)$ to $\hat{o}(5,1)$ algebra $D = 5$ dS.

We get the following algebraic relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\frac{\hat{M}_{\mu\nu}}{M^2} \qquad [\hat{p}^\mu, \hat{p}^\nu] = i\frac{\hat{M}_{\mu\nu}}{R^2} \qquad (A)$$

$$[\hat{x}_\mu, \hat{p}^\nu] = i\frac{1}{MR}\eta_{\mu\nu}\hat{M}_{45} = i\eta_{\mu\nu}\hat{h} \qquad (B)$$

where scalar $\hat{h}$ ($[\hat{h}] = 0$) is the operator-valued extension of Planck constant $\hbar$

$$\hat{M}_{45} = MR\hat{h}\,.$$

and $\hat{h}$ enters in two additional commutators describing internal $\hat{o}(2)$ rotations

$$[\hat{h}, \hat{x}_\mu] = iR^2 \hat{p}_\mu \qquad [\hat{h}, \hat{p}_\mu] = -i\kappa^2 \hat{x}_\mu$$

The exchange $\hat{x}_\mu \overset{B}{\leftrightarrow} \hat{p}_\mu$, $\hat{h} \overset{B}{\leftrightarrow} -\hat{h}$ supplemented with the map $R \overset{B}{\leftrightarrow} \kappa$ describes so-called Born map, corresponding to the replacement of fourth axis in $\hat{o}(5,1)$ by fifth one ($\hat{M}_{4\mu} \leftrightarrow \hat{M}_{5\mu}$) by $o(2)$ rotation in (4,5) plane. Yang quantum phase space algebra (see (A), (B)) appears to be Born-selfdual.

If we put $\frac{1}{\lambda_P} = m_P = M$ (Planck mass), and length dimensionalities as inverse to mass dimensionalities, after assuming that R is the radius of Universe, we obtain the link of physics at Planck distances and cosmological distances (see Y.B. Zeldovich (1968), P.A.M. Dirac (1974)).

Yang model can be described as embedding of two semidual $D = 4$ dS algebras ($\hat{o}^{(x)}(4,1)$ with NC space-time and $\hat{o}^{(p)}(4,1)$ with NC momenta) into D=5 dS algebra $\hat{o}(5,1)$, with the relations (A) satisfied and new generator $\hat{h}$ promoted from Planck constant $\hbar$ to the $\hat{o}(2)$ generator.

4. Triple Special Relativity (TRS) model and quantum phase space

In 2004 Kowalski-Glikman and Smolin did show that one can extend the relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\frac{\hat{M}_{\mu\nu}}{\kappa^2} \qquad [\hat{p}^\mu, \hat{p}^\nu] = i\frac{\hat{M}_{\mu\nu}}{R^2}$$

not in group-theoretic way (see Yang model) but by the calculation of the corresponding deformed Heisenberg algebra commutator $[\hat{x}_\mu, \hat{p}_\nu]$ using Jacobi identities, under the assumption that besides $\hat{x}_\mu, \hat{p}_\nu, \hat{M}_{\mu\nu}$ any new generators are not employed.

It appears that the commutator $[\hat{x}_\mu, \hat{p}_\nu]$ should be linear in the classical Lorentz generators $\hat{M}_{\mu\nu}$ and bilinear in the quantum phase coordinates $(\hat{x}_\mu, \hat{p}_\nu)$. One obtains the following quantum-deformed Heisenberg relation:

$$[\hat{x}_\mu, \hat{p}_\nu] = i g_{\mu\nu}(\hat{X}, \hat{P}, \hat{M})$$

where

$$g_{\mu\nu}(\hat{X}, \hat{P}, \hat{M}) = \eta_{\mu\nu} - \frac{1}{\kappa^2}\hat{p}_\mu\hat{p}_\nu - \frac{1}{R^2}\hat{x}_\mu\hat{x}_\nu + \frac{1}{\kappa R}(\hat{x}_\mu\hat{p}_\nu + \hat{p}_\mu\hat{x}_\nu + \hat{M}_{\mu\nu})$$

One can ask the question whether one can express consistently the generators $\hat{M}_{\mu\nu}$ in TSR model as bilinear in $(\hat{x}_\mu, \hat{p}_\nu)$. Indeed one can obtain the following result

$$\hat{M}_{\mu\nu} = \frac{i}{2\kappa R}([\hat{x}_\mu, \hat{p}_\nu] - [\hat{x}_\nu, \hat{p}_\mu]) = \frac{i}{2\kappa R}(\hat{x}_\mu \hat{p}_\nu + \hat{p}_\mu \hat{x}_\nu - \hat{x}_\nu \hat{p}_\mu - \hat{p}_\nu \hat{x}_\mu)\,.$$

By simple algebraic operations one can get also an alternative formula (Mignemi 2022)

$$\hat{M}_{\mu\nu} = \frac{1}{2}(\{\hat{x}_\mu, \hat{p}_\nu\} - \{\hat{x}_\nu, \hat{p}_\mu\}) = \frac{i}{2}(\hat{x}_\mu \hat{p}_\nu - \hat{p}_\mu \hat{x}_\nu - \hat{x}_\nu \hat{p}_\mu + \hat{p}_\nu \hat{x}_\mu)$$

If we introduce two orbital formulas for $\hat{M}^{(x)}_{\mu\nu}$ and $\hat{M}^{(p)}_{\mu\nu}$ which enter $\hat{o}^{(x)}(4,1)$ and $\hat{o}^{(p)}(4,1)$ subalgebras of $\hat{o}(5,1)$

$$\hat{M}^{(x)}_{\mu\nu} = -i(\hat{x}_\mu \hat{p}_\nu - \hat{x}_\nu \hat{p}_\mu), \qquad \hat{M}^{(p)}_{\mu\nu} = -i(\hat{p}_\mu \hat{x}_\nu - \hat{p}_\nu \hat{x}_\mu)$$

one obtains that the Lorentz generators $\hat{M}_{\mu\nu}$ in TSR model are

$$\hat{M}_{\mu\nu} = \frac{R\kappa}{2}(\hat{M}^{(x)}_{\mu\nu} + \hat{M}^{(p)}_{\mu\nu})\,.$$

The Born duality map implies the relation $\hat{M}^{(x)}_{\mu\nu} \overset{B}{\longleftrightarrow} \hat{M}^{(p)}_{\mu\nu}$ between D=4 de-Sitter algebras in spacetime and momenta spaces and leads to the selfduality relation $\hat{M}_{\mu\nu} \overset{B}{\longleftrightarrow} \hat{M}_{\mu\nu}$.

5. Outlook

There were proposed recently some generalizations of considered phase space algebras, in particular:

i) SUSY extensions and quantum fermionic spinors

In Snyder type models one replaces in Lie-algebraic framework the coset generators by NC space-time and NC quantum phase space generators (Snyderization procedure). These relations can be extended supersymmetrically, with odd coset generators (supercharges) replaced by quantum-deformed fermionic spinorial coordinates. For example one can $D = 4$ AdS superalgebra $OSp(1;4)$ decompose into the sum of Lorentz sector $sl(2,\mathbb{C}) \simeq O(3,1)$ and the generators in the pair of cosets $\frac{Sp(4)}{sl(2;\mathbb{C})} \oplus \frac{OSp(1;4)}{Sp(4)}$, which via "Snyderization procedure" can be promoted respectively to quantum $D = 4$ AdS space-time coordinates $\hat{x}^\mu$ and quantum-deformed D=4 AdS $Sp(4)$ fermionic Majorana spinors $\hat{\xi}_\alpha$ (obtained from $OSp(1;4)$ supercharges). One can say that in such a way the coset $\frac{OSp(1,4)}{SL(2,C)}$ generators describes after Snyderization the quantum relativistic Snyder superspace coordinates $(\hat{x}^\mu, \hat{\xi}_\alpha)$.
In such a way we obtain various quantum-deformed superspaces containing quantum fermionic spinors which can be used in order to describe supersymmetric theory in the presence of QG effects (for various examples of supersymmetric Snyder and Yang models see J.L, M. Woronowicz, arXiv:2110.13679 and arXiv:2204.07787).

One can also introduce quantum bosonic spinors, for example quantum-deformed D=4 conformal spinors called (Penrose) twistors, as described by the coset $\frac{SU(2,3)}{SU(2,2)\otimes U(1)}$ (see J.L. arXiv:2104.14306). It is known for a long time that the supercoset $\frac{SU(2,2;1)}{SU(2,2)\otimes U(1)}$ provides quantum fermionic twistors (see J.L. J. Math. Phys. 21, 561 (1979)).

ii) Quantum phase space and generalizations of Hopf algebras

If we assume that generalized coordinates describe the algebraic sector of a Hopf algebra H, the algebraic sector of dual Hopf algebra $\tilde{H}$ describes the generalized momenta. In such a case one can obtain the quantum phase space as Hopf algebroid described by Heisenberg double $H \rtimes \tilde{\mathrm{H}}$, where we denote by $\rtimes$ a particular semidirect product of Hopf algebras which is called smash product. Heisenberg doubles belong to the algebraic category of Hopf algebroids, i.e. one can construct the following sequence of inclusions

$$\text{quantum phase spaces} \quad \supset \quad \text{Hopf algebroids} \quad \supset \quad \text{Heisenberg doubles}$$

In particular TSR model algebra from Sect. 4 describes algebraic structure which is neither Heisenberg double nor Hopf algebroid.

THANK YOU !