Ensemble Averages of Narain CFTs & Holography

arXiv: 2104.14710+22XX.XXXXX
M. Ashwinkumar, M. Dodelson, A. Kidambi, M. Yamazaki

Jacob M. Leedom
Corfu Holography & Swampland, 09.09.2022

CLUSTER OF EXCELLENCE
QUANTUM UNIVERSE

Overview

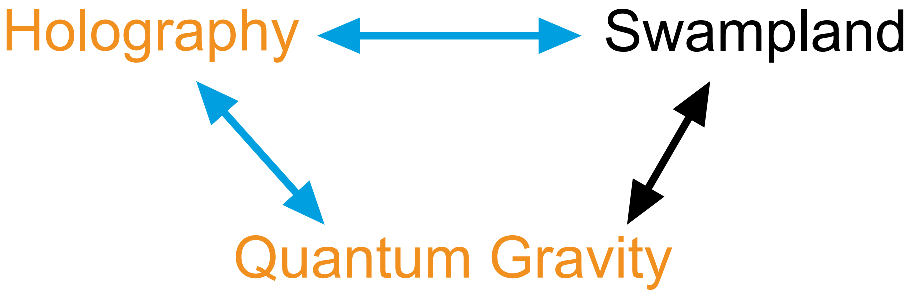
Holography

Swampland

Overview

Overview

Overview

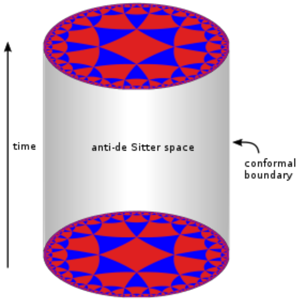


Overview

- Peculiar Duality between AdS_3 & CFT_2

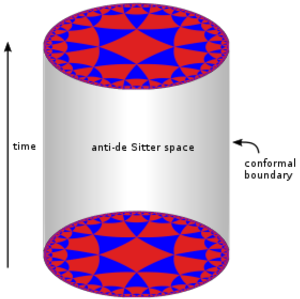
Overview

> Peculiar Duality between AdS_3 & CFT_2



Overview

> Peculiar Duality between AdS_3 & CFT_2

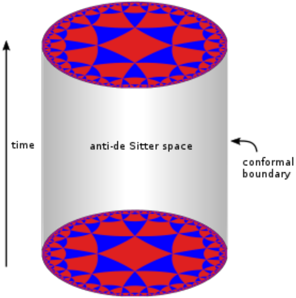


> Conventional Duality:

$$Z_{\text{Bulk}}[\phi(\text{boundary}) = J] = Z_{\text{CFT}}[J]$$

Overview

> Peculiar Duality between AdS_3 & CFT_2

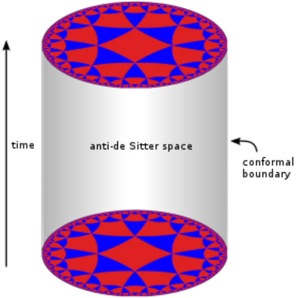


> Ensemble Duality:

$$\sum_{\text{AdS}_3 \text{ geometries}} Z_{\text{Bulk}}[\tau] = \int_{\text{moduli space}} [dm] Z_{\text{CFT}}[m;\tau]$$

Overview

- Peculiar Duality between AdS_3 & CFT_2



"You boil it in sawdust: you salt it in glue:

You condense it with locusts and tape:

Still keeping one principal object in view—

To preserve its symmetrical shape."

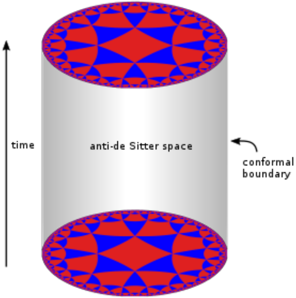
–*Hunting of the Snark*, Lewis Carroll

- Ensemble Duality:

$$\sum_{\substack{\mathrm{AdS}_3 \\ \text{geometries}}} Z_{\text{Bulk}}[\tau] = \int_{\substack{\text{moduli} \\ \text{space}}} [dm] Z_{\text{CFT}}[m;\tau]$$

Overview

- Peculiar Duality between AdS_3 & CFT_2



"You boil it in sawdust: you salt it in glue:

You condense it with locusts and tape:

Still keeping one principal object in view—

To preserve its symmetrical shape."

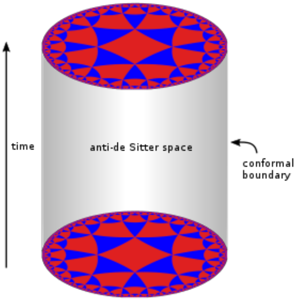
–*Hunting of the Snark*, Lewis Carroll

- Ensemble Duality:

$$\sum_{\text{AdS}_3 \text{ geometries}} Z_{\text{Bulk}}[\tau] = \int_{\text{moduli space}} [dm] Z_{\text{CFT}}[m;\tau]$$

Overview

> Peculiar Duality between AdS_3 & CFT_2



"You boil it in sawdust: you salt it in glue:

You condense it with locusts and tape:

Still keeping one principal object in view—

To preserve its symmetrical shape."

–*Hunting of the Snark*, Lewis Carroll

> Ensemble Duality:

$$\sum_{\text{AdS}_3 \text{ geometries}} Z_{\text{Bulk}}[\tau] = \int_{\text{moduli space}} [dm] Z_{\text{CFT}}[m; \tau]$$

$$\tau \to \tau_M = \frac{a\tau + b}{c\tau + d}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

AdS_3/CFT_2:

AdS$_3$/CFT$_2$: Ensemble of Even, Self-Dual CFTs

$\text{AdS}_3/\text{CFT}_2$: Ensemble of Even, Self-Dual CFTs

- Toy Illustration: 1 Compact Boson $\Rightarrow (c, \tilde{c}) = (1, 1)$ [2006.04839,2006.04855]

$$S_{CFT} = \frac{R^2}{2\pi} \int d^2\sigma \partial_a X \partial^a X$$

$$X \sim X + 2\pi$$

$\text{AdS}_3/\text{CFT}_2$: Ensemble of Even, Self-Dual CFTs

- Toy Illustration: 1 Compact Boson $\Rightarrow (c, \tilde{c}) = (1, 1)$ [2006.04839,2006.04855]

$$S_{CFT} = \frac{R^2}{2\pi} \int d^2\sigma \partial_a X \partial^a X$$

$$X \sim X + 2\pi$$

- Partition Function:

$\text{AdS}_3/\text{CFT}_2$: Ensemble of Even, Self-Dual CFTs

> Toy Illustration: 1 Compact Boson $\Rightarrow (c, \tilde{c}) = (1, 1)$ [2006.04839,2006.04855]

$$S_{CFT} = \frac{R^2}{2\pi} \int d^2\sigma \partial_a X \partial^a X$$

$$X \sim X + 2\pi$$

> Partition Function:

$$Z(R,\tau) = \frac{\Theta(R,\tau)}{\eta(\tau)\bar{\eta}(\bar{\tau})} = \frac{1}{\eta(\tau)\bar{\eta}(\bar{\tau})} \sum_{(p_L,p_R)\in\Gamma_{(1,1)}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}}$$

$$p_{L,R} = \frac{n}{2R} \pm wR$$

AdS_3/CFT_2: Ensemble of Even, Self-Dual CFTs

- Toy Illustration: 1 Compact Boson $\Rightarrow (c, \tilde{c}) = (1, 1)$ [2006.04839,2006.04855]

$$S_{CFT} = \frac{R^2}{2\pi} \int d^2\sigma \partial_a X \partial^a X$$

$$X \sim X + 2\pi$$

- Partition Function:

$$Z(R,\tau) = \frac{\Theta(R,\tau)}{\eta(\tau)\bar{\eta}(\bar{\tau})} = \frac{1}{\eta(\tau)\bar{\eta}(\bar{\tau})} \sum_{(p_L,p_R)\in\Gamma_{(1,1)}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}}$$

$$p_{L,R} = \frac{n}{2R} \pm wR$$

$$\Gamma_{(1,1)} = \begin{cases} \text{Even:} & p_L^2 - p_R^2 = 2nw \in 2\mathbb{Z} \\ \text{Self-Dual:} & \Gamma^*_{(1,1)} = \Gamma_{(1,1)} \end{cases}$$

AdS$_3$/CFT$_2$: Ensemble of Even, Self-Dual CFTs

> Toy Illustration: 1 Compact Boson $\Rightarrow (c, \tilde{c}) = (1, 1)$

[2006.04839,2006.04855]

$$S_{CFT} = \frac{R^2}{2\pi} \int d^2\sigma \partial_a X \partial^a X$$

$$X \sim X + 2\pi$$

> Partition Function:

$$Z(R,\tau) = \frac{\Theta(R,\tau)}{\eta(\tau)\bar{\eta}(\bar{\tau})} = \frac{1}{\eta(\tau)\bar{\eta}(\bar{\tau})} \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell] - \pi\tau_2 H[\ell]}$$

$$Q[\ell] = 2nw$$

$$H[\ell] = \frac{n^2}{2R^2} + 2w^2R^2$$

$$\tau = \tau_1 + i\tau_2$$

$$\Gamma_{(1,1)} = \begin{cases} \text{Even:} & p_L^2 - p_R^2 = 2nw \in 2\mathbb{Z} \\ \text{Self-Dual:} & \Gamma^*_{(1,1)} = \Gamma_{(1,1)} \end{cases}$$

$\mathrm{AdS}_3/\mathrm{CFT}_2$: Ensemble of Even, Self-Dual CFTs

- Toy Illustration: 1 Compact Boson $\Rightarrow (c, \tilde{c}) = (1, 1)$ [2006.04839,2006.04855]

$$S_{CFT} = \frac{R^2}{2\pi} \int d^2\sigma \partial_a X \partial^a X$$

$$X \sim X + 2\pi$$

- Partition Function:

$$Z(R,\tau) = \frac{\Theta(R,\tau)}{\eta(\tau)\bar{\eta}(\bar{\tau})} = \frac{1}{\eta(\tau)\bar{\eta}(\bar{\tau})} \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell] - \pi\tau_2 H[\ell]}$$

$$Q[\ell] = 2nw$$

$$H[\ell] = \frac{n^2}{2R^2} + 2w^2 R^2$$

- Moduli Space & Metric:

AdS_3/CFT_2: Ensemble of Even, Self-Dual CFTs

- Toy Illustration: 1 Compact Boson $\Rightarrow (c, \tilde{c}) = (1, 1)$

[2006.04839,2006.04855]

$$S_{CFT} = \frac{R^2}{2\pi} \int d^2\sigma \partial_a X \partial^a X$$

$$X \sim X + 2\pi$$

- Partition Function:

$$Z(R,\tau) = \frac{\Theta(R,\tau)}{\eta(\tau)\bar{\eta}(\bar{\tau})} = \frac{1}{\eta(\tau)\bar{\eta}(\bar{\tau})} \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell] - \pi\tau_2 H[\ell]}$$

$$Q[\ell] = 2nw$$

$$H[\ell] = \frac{n^2}{2R^2} + 2w^2 R^2$$

- Moduli Space & Metric:

$$\mathcal{M}_{(1,1)} : 1 \leq R < \infty$$

$$ds^2 = 4\frac{dR^2}{R^2}$$

T-Duality: $R \leftrightarrow 1/R$

AdS_3/CFT_2: Ensemble of Even, Self-Dual CFTs

- Toy Illustration: 1 Compact Boson $\Rightarrow (c, \tilde{c}) = (1, 1)$

[2006.04839,2006.04855]

$$S_{CFT} = \frac{R^2}{2\pi} \int d^2\sigma \partial_a X \partial^a X$$

$$X \sim X + 2\pi$$

- Partition Function:

$$Z(R,\tau) = \frac{\Theta(R,\tau)}{\eta(\tau)\bar{\eta}(\bar{\tau})} = \frac{1}{\eta(\tau)\bar{\eta}(\bar{\tau})} \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell] - \pi\tau_2 H[\ell]}$$

$$Q[\ell] = 2nw$$

$$H[\ell] = \frac{n^2}{2R^2} + 2w^2R^2$$

- Moduli Space & Metric:

$$\mathcal{M}_{(1,1)} : 1 \leq R < \infty$$

$$ds^2 = 4\frac{dR^2}{R^2} \Rightarrow dm = \frac{dR}{2R}$$

T-Duality: $R \leftrightarrow 1/R$

AdS$_3$/CFT$_2$: Ensemble of Even, Self-Dual CFTs

- Toy Illustration: 1 Compact Boson $\Rightarrow (c,\tilde{c}) = (1,1)$ [2006.04839,2006.04855]

$$S_{CFT} = \frac{R^2}{2\pi}\int d^2\sigma \partial_a X \partial^a X$$

$$X \sim X + 2\pi$$

- Partition Function:

$$Z(R,\tau) = \frac{\Theta(R,\tau)}{\eta(\tau)\bar{\eta}(\bar{\tau})} = \frac{1}{\eta(\tau)\bar{\eta}(\bar{\tau})}\sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell] - \pi\tau_2 H[\ell]}$$

$$Q[\ell] = 2nw$$

$$H[\ell] = \frac{n^2}{2R^2} + 2w^2R^2$$

- Moduli Space:

$$\mathcal{M}_{(1,1)} : 1 \leq R < \infty$$

- Ensemble Average:

$$\int_{\mathcal{M}_{(1,1)}} dm Z[m,\tau] = \frac{1}{|\eta(\tau)|^2}\int_1^\infty \Theta(R,\tau)\frac{dR}{2R}$$

AdS$_3$/CFT$_2$: Ensemble of Even, Self-Dual CFTs

- Toy Illustration: 1 Compact Boson $\Rightarrow (c, \tilde{c}) = (1, 1)$ [2006.04839,2006.04855]

$$S_{CFT} = \frac{R^2}{2\pi} \int d^2\sigma \partial_a X \partial^a X$$

$$X \sim X + 2\pi$$

- Partition Function:

$$Z(R, \tau) = \frac{\Theta(R, \tau)}{\eta(\tau)\bar{\eta}(\bar{\tau})} = \frac{1}{\eta(\tau)\bar{\eta}(\bar{\tau})} \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell] - \pi\tau_2 H[\ell]}$$

$$Q[\ell] = 2nw$$

$$H[\ell] = \frac{n^2}{2R^2} + 2w^2 R^2$$

- Moduli Space:

$$\mathcal{M}_{(1,1)} : 1 \leq R < \infty$$

- Ensemble Average: Does not converge! Need $p > 1$ bosons

$$\int_{\mathcal{M}_{(1,1)}} dm Z[m, \tau] = \frac{1}{|\eta(\tau)|^2} \int_1^\infty \Theta(R, \tau) \frac{dR}{2R}$$

AdS$_3$/CFT$_2$: Ensemble of Even, Self-Dual CFTs

- p Compact Bosons $\Rightarrow (c, \tilde{c}) = (p, p)$ [2006.04839,2006.04855]

$$S_{CFT} = \frac{1}{2\pi} \int d^2\sigma \left(G_{\alpha\beta} \partial_a X^\alpha \partial^a X^\beta + i B_{\alpha\beta} \epsilon^{ab} \partial_a X^\alpha \partial_b X^\beta \right)$$

AdS$_3$/CFT$_2$: Ensemble of Even, Self-Dual CFTs

- p Compact Bosons $\Rightarrow (c, \tilde{c}) = (p, p)$ [2006.04839,2006.04855]

$$S_{CFT} = \frac{1}{2\pi} \int d^2\sigma \left(G_{\alpha\beta} \partial_a X^\alpha \partial^a X^\beta + i B_{\alpha\beta} \epsilon^{ab} \partial_a X^\alpha \partial_b X^\beta \right)$$

- Partition Function:

$$Z(m, \tau) = \frac{\Theta(m, \tau)}{|\eta(\tau)|^{2p}} = \frac{1}{|\eta(\tau)|^{2p}} \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell] - \pi\tau_2 H[m,\ell]}$$

$$Q[\ell] = \ell^T Q \ell = 2 n_i w^i$$

$$H[m, \ell] = \ell^T H[m] \ell$$

$\text{AdS}_3/\text{CFT}_2$: Ensemble of Even, Self-Dual CFTs

- p Compact Bosons $\Rightarrow (c, \tilde{c}) = (p, p)$ [2006.04839,2006.04855]

$$S_{CFT} = \frac{1}{2\pi} \int d^2\sigma \left(G_{\alpha\beta} \partial_a X^\alpha \partial^a X^\beta + i B_{\alpha\beta} \epsilon^{ab} \partial_a X^\alpha \partial_b X^\beta \right)$$

- Partition Function:

$$Z(m,\tau) = \frac{\Theta(m,\tau)}{|\eta(\tau)|^{2p}} = \frac{1}{|\eta(\tau)|^{2p}} \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell] - \pi\tau_2 H[m,\ell]}$$

$$Q[\ell] = \ell^T Q \ell = 2 n_i w^i$$

$$H[m,\ell] = \ell^T H[m] \ell$$

- Moduli Space & Metric:

$$\mathcal{M}_{(p,p)} = O(p,p;\mathbb{Z}) \backslash O(p,p;\mathbb{R}) / O(p) \times O(p)$$

$$ds^2 = G^{\alpha\beta} G^{\rho\sigma} (dG_{\alpha\rho} dG_{\beta\sigma} + dB_{\alpha\rho} dB_{\beta\sigma})$$

AdS$_3$/CFT$_2$: Ensemble of Even, Self-Dual CFTs

- p Compact Bosons $\Rightarrow (c, \tilde{c}) = (p, p)$ [2006.04839,2006.04855]

$$S_{CFT} = \frac{1}{2\pi} \int d^2\sigma \left(G_{\alpha\beta} \partial_a X^\alpha \partial^a X^\beta + i B_{\alpha\beta} \epsilon^{ab} \partial_a X^\alpha \partial_b X^\beta \right)$$

- Partition Function:

$$Z(m, \tau) = \frac{\Theta(m, \tau)}{|\eta(\tau)|^{2p}} = \frac{1}{|\eta(\tau)|^{2p}} \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell] - \pi\tau_2 H[m,\ell]}$$

$$Q[\ell] = \ell^T Q \ell = 2 n_i w^i$$

$$H[m, \ell] = \ell^T H[m] \ell$$

- Moduli Space:

$$\mathcal{M}_{(p,p)} = O(p, p; \mathbb{Z}) \backslash O(p, p; \mathbb{R}) / O(p) \times O(p)$$

- Ensemble Average: Computable by Siegel-Weil Formula

$$\frac{1}{\text{Vol}(\mathcal{M}_{p,p})} \int_{\mathcal{M}_{(p,p)}} dm Z_{CFT}[m, \tau] = \frac{E_{p/2}(\tau)}{|\eta(\tau)|^{2p}}$$

$$E_s(\tau) = \sum_{(c,d)=1} \frac{1}{|c\tau + d|^{2s}}$$

AdS_3/CFT_2: $U(1)^{2p}$ Chern-Simons & a Sum over Geometries

$\text{AdS}_3/\text{CFT}_2$: U(1)^{2p} Chern-Simons & a Sum over Geometries

- Boundary has $U(1)^{2p}$ current algebra & $U(1)^{2p}$ global symmetry
  $\Rightarrow$ Bulk has $U(1)^{2p}$ gauge symmetry

$\text{AdS}_3/\text{CFT}_2$: U(1)^{2p} Chern-Simons & a Sum over Geometries

- Boundary has $U(1)^{2p}$ current algebra & $U(1)^{2p}$ global symmetry
  $\Rightarrow$ Bulk has $U(1)^{2p}$ gauge symmetry
- The level matrix of Chern-Simons is quantized, so a natural guess is:

$$S_{bulk} \supset \sum_{M,N} \frac{Q_{MN}}{2\pi} \int A^M \wedge dA^N$$

AdS$_3$/CFT$_2$: U(1)^{2p} Chern-Simons & a Sum over Geometries

- Boundary has $U(1)^{2p}$ current algebra & $U(1)^{2p}$ global symmetry
  $\Rightarrow$ Bulk has $U(1)^{2p}$ gauge symmetry
- The level matrix of Chern-Simons is quantized, so a natural guess is:

$$S_{bulk} \supset \sum_{M,N} \frac{Q_{MN}}{2\pi} \int A^M \wedge dA^N$$

- Not enough: Thermal AdS$_3$ - hyperbolic manifold filling in $\tau \sim \tau + 1$ cycle
  Boundary current algebra (Brown-Henneaux-like photon modes) give:

$$Z_{ThAdS} = \frac{1}{|\eta(\tau)|^{2p}}$$

AdS$_3$/CFT$_2$: U(1)^{2p} Chern-Simons & a Sum over Geometries

- Boundary has $U(1)^{2p}$ current algebra & $U(1)^{2p}$ global symmetry
  $\Rightarrow$ Bulk has $U(1)^{2p}$ gauge symmetry
- The level matrix of Chern-Simons is quantized, so a natural guess is:

$$S_{bulk} \supset \sum_{M,N} \frac{Q_{MN}}{2\pi} \int A^M \wedge dA^N$$

- Not enough: Thermal AdS$_3$ - hyperbolic manifold filling in $\tau \sim \tau + 1$ cycle
  Boundary current algebra (Brown-Henneaux-like photon modes) give:

$$Z_{ThAdS} = \frac{1}{|\eta(\tau)|^{2p}}$$

- Idea: what about other fillings?

$\text{AdS}_3/\text{CFT}_2$: U(1)^{2p} Chern-Simons & a Sum over Geometries

- Described by so-called $\text{PSL}_2(\mathbb{Z})$ black holes $\text{M}_{(c,d)}$
  - solid tori defined by filling different cycles of boundary torus (genus 1 handlebodies)

AdS_3/CFT_2: $\mathrm{U}(1)^{2p}$ Chern-Simons & a Sum over Geometries

- Described by so-called $\mathrm{PSL}_2(\mathbb{Z})$ black holes $\mathrm{M}_{(c,d)}$
  - solid tori defined by filling different cycles of boundary torus (genus 1 handlebodies)
  - any handlebody can be obtained from thermal AdS by modular transformation of boundary up to the action of

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

  - Thus geometries labeled by elements of $\Gamma_\infty \backslash \mathrm{PSL}_2(\mathbb{Z}) \Rightarrow$ coprime integers (c, d)
  - $\mathrm{M}_{(0,1)}$ = thermal AdS_3, $\mathrm{M}_{(1,0)}$ = BTZ black hole

AdS$_3$/CFT$_2$: U(1)^{2p} Chern-Simons & a Sum over Geometries

> Described by so-called PSL$_2(\mathbb{Z})$ black holes M$_{(c,d)}$

- solid tori defined by filling different cycles of boundary torus (genus 1 handlebodies)
- any handlebody can be obtained from thermal AdS by modular transformation of boundary up to the action of

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

- Thus geometries labeled by elements of $\Gamma_\infty \backslash$PSL$_2(\mathbb{Z}) \Rightarrow$ coprime integers (c,d)
- M$_{(0,1)}$ = thermal AdS$_3$, M$_{(1,0)}$ = BTZ black hole

> Sum over geometries$\Rightarrow$ sum over images under $\Gamma_\infty \backslash$PSL$_2(\mathbb{Z})$:

$$Z(\tau) = \sum_{g \in \Gamma_\infty \backslash \mathrm{PSL}_2(\mathbb{Z})} \frac{1}{|\eta(g\cdot\tau)|^{2p}}$$
$$= \frac{E_{p/2}(\tau)}{|\eta(\tau)|^{2p}}$$

AdS$_3$/CFT$_2$: Duality Established

> U(1)^{2p} Chern-Simons Theory

$$S_{Bulk} = \sum_{MN} \frac{Q_{MN}}{2\pi} \int A^M \wedge dA^N$$

> Even, Self-Dual, (p,p) CFTs

$$S_{CFT} = \frac{1}{2\pi} \int d^2\sigma \Big(G_{\alpha\beta} \partial_a X^\alpha \partial^a X^\beta + i B_{\alpha\beta} \epsilon^{ab} \partial_a X^\alpha \partial_b X^\beta \Big)$$

AdS$_3$/CFT$_2$: Duality Established

> U(1)^{2p} Chern-Simons Theory

$$S_{Bulk} = \sum_{MN} \frac{Q_{MN}}{2\pi} \int A^M \wedge dA^N$$

> Geometries

M$_{(c,d)}$:PSL$_2(\mathbb{Z})$ black holes

$(c,d) \leftrightarrow \Gamma_\infty \backslash$PSL$_2(\mathbb{Z})$

> Even, Self-Dual, (p,p) CFTs

$$S_{CFT} = \frac{1}{2\pi} \int d^2\sigma \Big(G_{\alpha\beta} \partial_a X^\alpha \partial^a X^\beta + i B_{\alpha\beta} \epsilon^{ab} \partial_a X^\alpha \partial_b X^\beta \Big)$$

> Moduli Space $\mathcal{M}_{(p,p)}$

O$(p,p;\mathbb{Z})\backslash$O$(p,p;\mathbb{R})/O(p) \times$ O(p)

AdS$_3$/CFT$_2$: Duality Established

> U(1)^{2p} Chern-Simons Theory

$$S_{Bulk} = \sum_{MN} \frac{Q_{MN}}{2\pi} \int A^M \wedge dA^N$$

> Even, Self-Dual, (p,p) CFTs

$$S_{CFT} = \frac{1}{2\pi} \int d^2\sigma \Big(G_{\alpha\beta} \partial_a X^\alpha \partial^a X^\beta + i B_{\alpha\beta} \epsilon^{ab} \partial_a X^\alpha \partial_b X^\beta \Big)$$

> Geometries

$M_{(c,d)}$:PSL$_2(\mathbb{Z})$ black holes

$(c,d) \leftrightarrow \Gamma_\infty \backslash \mathrm{PSL}_2(\mathbb{Z})$

> Moduli Space $\mathcal{M}_{(p,p)}$

$\mathrm{O}(p,p;\mathbb{Z}) \backslash \mathrm{O}(p,p;\mathbb{R}) / \mathrm{O}(p) \times \mathrm{O}(p)$

> Sum Over Geometries:

$$\sum_{g \in \Gamma_\infty \backslash \mathrm{PSL}_2(\mathbb{Z})} \frac{1}{|\eta(g\cdot\tau)|^{2p}} = \frac{E_{p/2}(\tau)}{|\eta(\tau)|^{2p}}$$

$\mathrm{AdS}_3/\mathrm{CFT}_2$: Duality Established

> $U(1)^{2p}$ Chern-Simons Theory

$$S_{Bulk} = \sum_{MN} \frac{Q_{MN}}{2\pi} \int A^M \wedge dA^N$$

> Even, Self-Dual, (p,p) CFTs

$$S_{CFT} = \frac{1}{2\pi} \int d^2\sigma \Big(G_{\alpha\beta}\partial_a X^\alpha \partial^a X^\beta + iB_{\alpha\beta}\epsilon^{ab}\partial_a X^\alpha \partial_b X^\beta \Big)$$

> Geometries

$M_{(c,d)}$:$PSL_2(\mathbb{Z})$ black holes

$(c,d) \leftrightarrow \Gamma_\infty \backslash PSL_2(\mathbb{Z})$

> Moduli Space $\mathcal{M}_{(p,p)}$

$$O(p,p;\mathbb{Z})\backslash O(p,p;\mathbb{R})/O(p)\times O(p)$$

> Sum Over Geometries:

> Average over moduli space:

$$\frac{E_{p/2}(\tau)}{|\eta(\tau)|^{2p}} = \frac{1}{\mathrm{Vol}(\mathcal{M}_{p,p})} \int_{\mathcal{M}_{(p,p)}} dm Z_{CFT}[m,\tau]$$

AdS$_3$/CFT$_2$: Duality Established

> $U(1)^{2p}$ Chern-Simons Theory

$$S_{Bulk} = \sum_{MN} \frac{Q_{MN}}{2\pi} \int A^M \wedge dA^N$$

> Even, Self-Dual, (p,p) CFTs

$$S_{CFT} = \frac{1}{2\pi} \int d^2\sigma \Big(G_{\alpha\beta} \partial_a X^\alpha \partial^a X^\beta + i B_{\alpha\beta} \epsilon^{ab} \partial_a X^\alpha \partial_b X^\beta \Big)$$

> Geometries

$M_{(c,d)}$:$\mathrm{PSL}_2(\mathbb{Z})$ black holes

$(c,d) \leftrightarrow \Gamma_\infty \backslash \mathrm{PSL}_2(\mathbb{Z})$

> Moduli Space $\mathcal{M}_{(p,p)}$

$\mathrm{O}(p,p;\mathbb{Z}) \backslash \mathrm{O}(p,p;\mathbb{R}) / \mathrm{O}(p) \times \mathrm{O}(p)$

> Sum Over Geometries:

> Average over moduli space:

$$\sum_{g \in \Gamma_\infty \backslash \mathrm{PSL}_2(\mathbb{Z})} \frac{1}{|\eta(g \cdot \tau)|^{2p}} = \frac{1}{\mathrm{Vol}(\mathcal{M}_{p,p})} \int_{\mathcal{M}_{(p,p)}} dm Z_{CFT}[m,\tau]$$

AdS$_3$/CFT$_2$: Duality Established

> U(1)^{2p} Chern-Simons Theory

$$S_{Bulk} = \sum_{MN} \frac{Q_{MN}}{2\pi} \int A^M \wedge dA^N$$

> Geometries

M$_{(c,d)}$:PSL$_2(\mathbb{Z})$ black holes

$(c, d) \leftrightarrow \Gamma_\infty \backslash$PSL$_2(\mathbb{Z})$

> Sum Over Geometries:

> Even, Self-Dual, (p, p) CFTs

$$S_{CFT} = \frac{1}{2\pi} \int d^2\sigma \Big(G_{\alpha\beta}\partial_a X^\alpha \partial^a X^\beta + iB_{\alpha\beta}\epsilon^{ab}\partial_a X^\alpha \partial_b X^\beta \Big)$$

> Moduli Space $\mathcal{M}_{(p,p)}$

$$\mathrm{O}(p, p; \mathbb{Z})\backslash \mathrm{O}(p, p; \mathbb{R})/\mathrm{O}(p) \times \mathrm{O}(p)$$

> Average over moduli space:

$$\sum_{\substack{\text{AdS}_3 \\ \text{geometries}}} Z_{\text{Bulk}}[\tau] = \int_{\substack{\text{moduli} \\ \text{space}}} [dm] Z_{\text{CFT}}[m; \tau]$$

Number Theory Detour: The Siegel-Weil Formula

Number Theory Detour: The Siegel-Weil Formula

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h]-\pi\tau_2 H[m;\ell+h]}$$

$Q[\ell] \in 2\mathbb{Z}$

$H : H^T Q H = Q$

$h \in \mathcal{D} = \Lambda^*/\Lambda$

Number Theory Detour: The Siegel-Weil Formula

> General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h] - \pi\tau_2 H[m;\ell+h]}$$

$Q[\ell] \in 2\mathbb{Z}$
$H : H^T Q H = Q$
$h \in \mathcal{D} = \Lambda^*/\Lambda$
$\sigma = p - q$

$$\Theta_{Q,h}(m;\tau_M) = \frac{e^{-i\pi\sigma/4} c^{-\frac{p+q}{2}}}{|Q|^{1/2}} (c\tau + d)^{p/2} (c\bar{\tau} + d)^{q/2} \sum_{h'\in\mathcal{D}} \lambda_{hh'} \Theta_{Q,h'}(\tau)$$

$$\lambda_{hh'} = \sum_{g\in\Lambda/c\Lambda} \text{exp}\left(\frac{i\pi}{\gamma}(aQ[g+h] - 2Q[h', g+h] + dQ[h'])\right)$$

$$\tau_M = \frac{a\tau + b}{c\tau + d} \quad \& \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

Number Theory Detour: The Siegel-Weil Formula

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h] - \pi\tau_2 H[m;\ell+h]}$$

- Siegel-Weil Formula:

$$\langle\Theta_{Q,h}(m;\tau)\rangle = \frac{1}{\text{Vol}(\mathcal{M}_Q)}\int_{\mathcal{M}_Q}[dm]\Theta_{Q,h}(m;\tau)$$

$$= \delta_{h\in\Lambda} + \sum_{\substack{(c,d)=1\\ c>0}} \frac{\gamma_{Q,h}(c,d)}{(c\tau+d)^{\frac{p}{2}}(c\bar{\tau}+d)^{\frac{q}{2}}} := E_{Q,h}(\tau)$$

$$\gamma_{Q,h} := e^{i\pi\sigma/4}|Q|^{-\frac{1}{2}}c^{-\frac{p+q}{2}} \sum_{g\in\Lambda/c\Lambda} \exp\Big(-i\pi\frac{d}{c}Q[g+h]\Big)$$

$Q[\ell]\in 2\mathbb{Z}$

$H : H^T Q H = Q$

$h\in\mathcal{D} = \Lambda^*/\Lambda$

$\sigma = p - q$

Number Theory Detour: The Siegel-Weil Formula

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h]-\pi\tau_2 H[m;\ell+h]}$$

$Q[\ell] \in 2\mathbb{Z}$

$H : H^T Q H = Q$

$h \in \mathcal{D} = \Lambda^*/\Lambda$

- Siegel-Weil Formula:

$$\langle\Theta_{Q,h}(m;\tau)\rangle = \frac{1}{\text{Vol}(\mathcal{M}_Q)}\int_{\mathcal{M}_Q}[dm]\Theta_{Q,h}(m;\tau)$$

$$= \delta_{h\in\Lambda} + \sum_{\substack{(c,d)=1\\ c>0}} \frac{\gamma_{Q,h}(c,d)}{(c\tau+d)^{\frac{p}{2}}(c\bar{\tau}+d)^{\frac{q}{2}}} := E_{Q,h}(\tau)$$

$$\gamma_{Q,h} = c^{-p}\sum_{n_i,w^i=0}^{c-1} \exp\left(-i\pi\frac{d}{c}n_i w^i\right) = 1$$

if $q=p$ and $|Q|=1$

Number Theory Detour: The Siegel-Weil Formula

> General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h] - \pi\tau_2 H[m;\ell+h]}$$

$Q[\ell] \in 2\mathbb{Z}$

$H : H^T Q H = Q$

$h \in \mathcal{D} = \Lambda^*/\Lambda$

> Siegel-Weil Formula:

$$\langle \Theta_{Q,h}(m;\tau) \rangle = E_{Q,h}(\tau)$$

> Sketch of Proof (without explicit integration):

Number Theory Detour: The Siegel-Weil Formula

> General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h] - \pi\tau_2 H[m;\ell+h]}$$

> Siegel-Weil Formula:

$$\langle\Theta_{Q,h}(m;\tau)\rangle = E_{Q,h}(\tau)$$

> Sketch of Proof (without explicit integration):

$$\lim_{\tau\to i\infty} \Theta_{Q,h}(m;\tau) = \delta_{h\in\Lambda}$$

$Q[\ell] \in 2\mathbb{Z}$

$H : H^T Q H = Q$

$h \in \mathcal{D} = \Lambda^*/\Lambda$

Number Theory Detour: The Siegel-Weil Formula

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h]-\pi\tau_2 H[m;\ell+h]}$$

$Q[\ell]\in 2\mathbb{Z}$

$H : H^T Q H = Q$

$h\in\mathcal{D}=\Lambda^*/\Lambda$

- Siegel-Weil Formula:

$$\langle\Theta_{Q,h}(m;\tau)\rangle = E_{Q,h}(\tau)$$

- Sketch of Proof (without explicit integration):

$$\lim_{\tau\to i\infty}\Theta_{Q,h}(m;\tau) = \delta_{h\in\Lambda}$$

$$\Theta_{Q,h}(m;\tau) \sim \frac{e^{i\pi\sigma/4}|Q|^{-1/2}c^{-\frac{p+q}{2}}}{(c\tau+d)^{p/2}(c\bar{\tau}+d)^{q/2}}\sum_{h'\in\mathcal{D}}\lambda_{hh'}\delta_{h'\in\Lambda}$$

Number Theory Detour: The Siegel-Weil Formula

> General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell+h] - \pi\tau_2 H[m;\ell+h]}$$

$$Q[\ell] \in 2\mathbb{Z}$$
$$H : H^T Q H = Q$$
$$h \in \mathcal{D} = \Lambda^*/\Lambda$$

> Siegel-Weil Formula:

$$\langle \Theta_{Q,h}(m;\tau) \rangle = E_{Q,h}(\tau)$$

> Sketch of Proof (without explicit integration):

$$\lim_{\tau \to i\infty} \Theta_{Q,h}(m;\tau) = \delta_{h\in\Lambda}$$

$$\Theta_{Q,h}(m;\tau) \sim \frac{e^{i\pi\sigma/4}|Q|^{-1/2} c^{-\frac{p+q}{2}}}{(c\tau+d)^{p/2}(c\bar{\tau}+d)^{q/2}} \lambda_{h0} = \frac{\gamma_{Q,h}(c,d)}{(c\tau+d)^{p/2}(c\bar{\tau}+d)^{q/2}}$$

Number Theory Detour: The Siegel-Weil Formula

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h]-\pi\tau_2 H[m;\ell+h]}$$

$Q[\ell]\in 2\mathbb{Z}$

$H: H^T Q H = Q$

$h\in\mathcal{D}=\Lambda^*/\Lambda$

- Siegel-Weil Formula:

$$\langle\Theta_{Q,h}(m;\tau)\rangle = E_{Q,h}(\tau)$$

- Sketch of Proof (without explicit integration):

$$\lim_{\tau\to i\infty}\Theta_{Q,h}(m;\tau) = \delta_{h\in\Lambda}$$

$$\Theta_{Q,h}(m;\tau) \sim \frac{e^{i\pi\sigma/4}|Q|^{-1/2}c^{-\frac{p+q}{2}}}{(c\tau+d)^{p/2}(c\bar{\tau}+d)^{q/2}}\lambda_{h0} = \frac{\gamma_{Q,h}(c,d)}{(c\tau+d)^{p/2}(c\bar{\tau}+d)^{q/2}}$$

$$E_{Q,h}(\tau) - \langle\Theta_{Q,h}(m;\tau)\rangle = 0 \text{ at cusps}$$

Number Theory Detour: The Siegel-Weil Formula

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h] - \pi\tau_2 H[m;\ell+h]}$$

$Q[\ell] \in 2\mathbb{Z}$

$H : H^T Q H = Q$

$h \in \mathcal{D} = \Lambda^*/\Lambda$

- Siegel-Weil Formula:

$$\langle \Theta_{Q,h}(m;\tau) \rangle = E_{Q,h}(\tau)$$

- Sketch of Proof (without explicit integration):

$$\left(-\tau_2^2(\partial_1^2 + \partial_2^2) - \frac{(p+q)\tau_2}{2}\partial_2 - i\frac{(q-p)\tau_2}{2}\partial_1 + \Delta_{\mathcal{M}_Q} \right)\Theta(m;\tau) = 0$$

Number Theory Detour: The Siegel-Weil Formula

$Q[\ell] \in 2\mathbb{Z}$
$H : H^T Q H = Q$
$h \in \mathcal{D} = \Lambda^*/\Lambda$

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell+h] - \pi\tau_2 H[m;\ell+h]}$$

- Siegel-Weil Formula:

$$\langle \Theta_{Q,h}(m;\tau) \rangle = E_{Q,h}(\tau)$$

- Sketch of Proof (without explicit integration):

$$\left(-\tau_2^2(\partial_1^2 + \partial_2^2) - \frac{(p+q)\tau_2}{2}\partial_2 - i\frac{(q-p)\tau_2}{2}\partial_1 \right) \langle \Theta(m;\tau) \rangle = 0$$

Number Theory Detour: The Siegel-Weil Formula

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h] - \pi\tau_2 H[m;\ell+h]}$$

$Q[\ell] \in 2\mathbb{Z}$

$H : H^T Q H = Q$

$h \in \mathcal{D} = \Lambda^*/\Lambda$

- Siegel-Weil Formula:

$$\langle \Theta_{Q,h}(m;\tau) \rangle = E_{Q,h}(\tau)$$

- Sketch of Proof (without explicit integration):

$$\left(-\tau_2^2(\partial_1^2 + \partial_2^2) - \frac{(p+q)\tau_2}{2}\partial_2 - i\frac{(q-p)\tau_2}{2}\partial_1 \right) E_{Q,h}(\tau) = 0$$

Number Theory Detour: The Siegel-Weil Formula

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h]-\pi\tau_2 H[m;\ell+h]}$$

- Siegel-Weil Formula:

$$\langle\Theta_{Q,h}(m;\tau)\rangle = E_{Q,h}(\tau)$$

- Sketch of Proof (without explicit integration):

$$\left(\Box_{(p-q)/2} + \frac{((p+q)/4-1)(p+q)}{4}\right)\left(\tau_2^{(p+q)/4} f(\tau)\right) = 0$$

$$\Box_k = -\tau_2^2(\partial_1^2+\partial_2^2) + ik\tau_2\partial_1$$

$Q[\ell] \in 2\mathbb{Z}$

$H : H^T Q H = Q$

$h \in \mathcal{D} = \Lambda^*/\Lambda$

Number Theory Detour: The Siegel-Weil Formula

> General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell+h] - \pi\tau_2 H[m;\ell+h]}$$

$$Q[\ell] \in 2\mathbb{Z}$$
$$H : H^T Q H = Q$$
$$h \in \mathcal{D} = \Lambda^*/\Lambda$$

> Siegel-Weil Formula:

$$\langle \Theta_{Q,h}(m;\tau) \rangle = E_{Q,h}(\tau)$$

> Sketch of Proof (without explicit integration):

$$\left(\Box_{(p-q)/2} + \frac{((p+q)/4 - 1)(p+q)}{4} \right) (\tau_2^{(p+q)/4} f(\tau)) = 0$$

$$\Box_k = -\tau_2^2(\partial_1^2 + \partial_2^2) + ik\tau_2\partial_1$$

Minimum eigenvalue (square normalizable) : $\lambda_{min,k} = \frac{|k|}{2}\left(1 - \frac{|k|}{2}\right)$

Number Theory Detour: The Siegel-Weil Formula

$Q[\ell] \in 2\mathbb{Z}$
$H : H^T Q H = Q$
$h \in \mathcal{D} = \Lambda^*/\Lambda$

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h] - \pi\tau_2 H[m;\ell+h]}$$

- Siegel-Weil Formula:

$$\langle\Theta_{Q,h}(m;\tau)\rangle = E_{Q,h}(\tau)$$

- Sketch of Proof (without explicit integration):

$$\left(\Box_{(p-q)/2} + \frac{((p+q)/4-1)(p+q)}{4}\right)(\tau_2^{(p+q)/4} f(\tau)) = 0$$

$$\Box_k = -\tau_2^2(\partial_1^2 + \partial_2^2) + ik\tau_2\partial_1$$

$$\frac{(1-(p+q)/4)(p+q)}{4} - \lambda_{min,(p-q)/2} = \frac{1}{4}(2-p)q \leq 0$$

Number Theory Detour: The Siegel-Weil Formula

$Q[\ell] \in 2\mathbb{Z}$

$H : H^T Q H = Q$

$h \in \mathcal{D} = \Lambda^*/\Lambda$

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell \in \Lambda} e^{i\pi\tau_1 Q[\ell+h] - \pi\tau_2 H[m;\ell+h]}$$

- Siegel-Weil Formula:

$$\langle \Theta_{Q,h}(m;\tau) \rangle = E_{Q,h}(\tau)$$

- Sketch of Proof (without explicit integration):

$$\left(\Box_{(p-q)/2} + \frac{((p+q)/4 - 1)(p+q)}{4} \right) \left(\tau_2^{(p+q)/4} f(\tau) \right) = 0$$

$$\Box_k = -\tau_2^2(\partial_1^2 + \partial_2^2) + ik\tau_2\partial_1$$

$$\frac{(1-(p+q)/4)(p+q)}{4} - \lambda_{min,(p-q)/2} = \frac{1}{4}(2-p)q \leq 0$$

no normalizable eigenfunction satisfies equation

Number Theory Detour: The Siegel-Weil Formula

> General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h]-\pi\tau_2 H[m;\ell+h]}$$

$Q[\ell] \in 2\mathbb{Z}$

$H : H^T Q H = Q$

$h \in \mathcal{D} = \Lambda^*/\Lambda$

> Siegel-Weil Formula:

$$\langle\Theta_{Q,h}(m;\tau)\rangle = E_{Q,h}(\tau)$$

> Sketch of Proof (without explicit integration):

$$\left(\Box_{(p-q)/2} + \frac{((p+q)/4-1)(p+q)}{4}\right)(\tau_2^{(p+q)/4} f(\tau)) = 0$$

$$f_{Q,h}(\tau) = \tau_2^{(p+q)/4}(E_{Q,h}(\tau) - \langle\Theta_{Q,h}(m;\tau)\rangle) \Rightarrow \lambda < \lambda_{min}$$

Number Theory Detour: The Siegel-Weil Formula

- General Siegel-Narain Theta Function:

$$\Theta_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda} e^{i\pi\tau_1 Q[\ell+h]-\pi\tau_2 H[m;\ell+h]}$$

$Q[\ell] \in 2\mathbb{Z}$
$H : H^T Q H = Q$
$h \in \mathcal{D} = \Lambda^*/\Lambda$

- Siegel-Weil Formula:

$$\langle\Theta_{Q,h}(m;\tau)\rangle = E_{Q,h}(\tau)$$

- Sketch of Proof (without explicit integration):

$$\left(\Box_{(p-q)/2} + \frac{((p+q)/4-1)(p+q)}{4}\right)(\tau_2^{(p+q)/4} f(\tau)) = 0$$

$$f_{Q,h}(\tau) = \tau_2^{(p+q)/4}(E_{Q,h}(\tau) - \langle\Theta_{Q,h}(m;\tau)\rangle) \Rightarrow \lambda < \lambda_{min}$$

$f_{Q,h}(\tau)$ is square nomalizable $\Rightarrow$ zero at cusps $\Rightarrow$ $f_{Q,h}(\tau) = 0$

AdS$_3$/CFT$_2$: Even Q CFTs

- Consider a CFT with $(c, \tilde{c}) = (p, q)$
- Vertex Operators:

$$V_{k_L, k_R} = \exp(ik_L \cdot X_L(z) + ik_R \cdot X_R(\bar{z}))$$

- To close OPE, need to define theory by picking out vertex operators that fill out a lattice Λ
- For vertex operators to have integer spin, want an even quadratic form $Q[k] \in 2\mathbb{Z}$ of signature (p, q)
- Also have Hamiltonian H that need not have integer values on Λ and depends on moduli
- Moduli Space:

$$\mathcal{M}_Q = O(p, q, \mathbb{Z}) \backslash O(p, q, \mathbb{R}) / O(p, \mathbb{R}) \times O(q, \mathbb{R})$$

AdS$_3$/CFT$_2$: Even Q CFTs

- The partition function is built from the Siegel-Narain theta functions:

$$Z^{CFT}_{Q,0}(m;\tau) = \frac{\Theta_Q(m;\tau)}{\eta^p(\tau)\bar{\eta}^q(\bar{\tau})}$$

- this is not generally modular invariant and one needs to consider $Z^{CFT}_{Q,h}(m;\tau)$
- Ensemble averages are direct application of Siegel-Weil Formula:

$$\langle Z^{CFT}_{Q,h}(m;\tau)\rangle = \frac{E_{Q,h}(\tau)}{\eta^p(\tau)\bar{\eta}^q(\bar{\tau})}$$

AdS$_3$/CFT$_2$: U(1)$^{p+q}$ Chern-Simons Theory

- Averaged Partition functions can be written as

$$\langle Z^{CFT}_{Q,h}(m;\tau)\rangle = \sum_{g\in\Gamma_\infty\backslash \mathrm{PSL}_2(\mathbb{Z})} e^{\frac{i\pi\sigma}{12}\Phi(g)-\frac{i\pi\sigma}{4}} \frac{\gamma_{Q,h}(c,d)}{\eta^p(g\cdot\tau)\bar{\eta}^q(g\cdot\bar{\tau})}$$

$\Phi(g)$ = Rademacher Phi

AdS$_3$/CFT$_2$: U(1)$^{p+q}$ Chern-Simons Theory

> Averaged Partition functions can be written as

$$\langle Z_{Q,h}^{CFT}(m;\tau)\rangle = \sum_{g\in\Gamma_\infty\backslash \mathrm{PSL}_2(\mathbb{Z})} e^{\frac{i\pi\sigma}{12}\Phi(g)-\frac{i\pi\sigma}{4}} \frac{\gamma_{Q,h}(c,d)}{\eta^p(g\cdot\tau)\bar{\eta}^q(g\cdot\bar{\tau})}$$

$\Phi(g)$ = Rademacher Phi

> Still expect the bulk theory to be something like U(1)$^{p+q}$ Chern-Simons

- Above is clearly a sum over $M_{c,d}$ geometries of current algebra partition function
- But what does the phase correspond to in the bulk?

AdS$_3$/CFT$_2$: U(1)$^{p+q}$ Chern-Simons Theory

- Averaged Partition functions can be written as

$$\langle Z^{CFT}_{Q,h}(m;\tau)\rangle = \sum_{g\in\Gamma_\infty\backslash \mathrm{PSL}_2(\mathbb{Z})} e^{\frac{i\pi\sigma}{12}\Phi(g)-\frac{i\pi\sigma}{4}} \frac{\gamma_{Q,h}(c,d)}{\eta^p(g\cdot\tau)\bar{\eta}^q(g\cdot\bar{\tau})}$$

$\Phi(g)$ = Rademacher Phi

- Still expect the bulk theory to be something like U(1)$^{p+q}$ Chern-Simons
  - Above is clearly a sum over $M_{c,d}$ geometries of current algebra partition function
  - But what does the phase correspond to in the bulk?
- Educated guess - Eisenstein series constructed by matching cusps of theta function controlled by $\lambda_{h,0}$:

AdS$_3$/CFT$_2$: U(1)$^{p+q}$ Chern-Simons Theory

- Averaged Partition functions can be written as

$$\langle Z^{CFT}_{Q,h}(m;\tau)\rangle = \sum_{g\in\Gamma_\infty\backslash \mathrm{PSL}_2(\mathbb{Z})} e^{\frac{i\pi\sigma}{12}\Phi(g)-\frac{i\pi\sigma}{4}} \frac{\gamma_{Q,h}(c,d)}{\eta^p(g\cdot\tau)\bar{\eta}^q(g\cdot\bar{\tau})}$$

$\Phi(g)$ = Rademacher Phi

- Still expect the bulk theory to be something like U(1)$^{p+q}$ Chern-Simons
  - Above is clearly a sum over $M_{c,d}$ geometries of current algebra partition function
  - But what does the phase correspond to in the bulk?
- Educated guess - Eisenstein series constructed by matching cusps of theta function controlled by $\lambda_{h,0}$:

$$\gamma_{Q,h}(c,d) \sim \lambda_{h,0}$$

Want elements of a transformation matrix in the bulk

AdS$_3$/CFT$_2$: U(1)$^{p+q}$ Chern-Simons Theory

- Canonical Quantization: CS States for general even Q labeled by elements of $\mathcal{D}$:

$$T\,|h\rangle = e^{i\pi Q[h]} e^{-\frac{i\pi\sigma}{12}}\,|h\rangle$$
$$S\,|h\rangle = \frac{1}{\sqrt{|Q|}} \sum_{h' \in \mathcal{D}} e^{-2\pi i Q[h,h']}\,|h'\rangle$$

- The bulk analog of $\lambda_{h,0}$ is something like $\langle 0|\,U(g)\,|h\rangle$. In fact we have

$$\langle 0|\,U(g)\,|h\rangle^* = \langle h|\,U(g)^{-1}\,|0\rangle = e^{\frac{i\pi\Phi(g)}{12} - \frac{i\pi\sigma}{4}}\gamma_{Q,h}(c,d)$$

Partition functions of CS on Lens spaces! (see paper by Jeffery)

$\text{AdS}_3/\text{CFT}_2$: Duality for Even Q

$$\sum_{g \in \Gamma_\infty \backslash \mathsf{PSL}_2(\mathbb{Z})} \frac{\langle h | \, U(g)^{-1} \, |0\rangle}{\eta^p(g\cdot\tau)\bar{\eta}^q(g\cdot\bar{\tau})} = \frac{1}{\mathsf{Vol}(\mathcal{M}_Q)} \int_{\mathcal{M}_Q} \frac{\Theta_{Q,h}(m;\tau)}{\eta^p(\tau)\bar{\eta}^q(\bar{\tau})}$$

Global Symmetries, Orbifolds, & Ensembles

> The Bulk theories have global symmetries, i.e. $\mathbb{Z}_2$ symmetry $A \to -A$

Global Symmetries, Orbifolds, & Ensembles

- The Bulk theories have global symmetries, i.e. $\mathbb{Z}_2$ symmetry $A \to -A$
- One can gauge these symmetries to produce orbifolded dualities. Even, self dual case considered in [2103.15826,2105.12594]

Global Symmetries, Orbifolds, & Ensembles

- The Bulk theories have global symmetries, i.e. $\mathbb{Z}_2$ symmetry $A \to -A$
- One can gauge these symmetries to produce orbifolded dualities. Even, self dual case considered in [2103.15826,2105.12594]
- General Q theories lead to novel Eisenstein series. The "gauge" sector partition function has generalized lattice theta functions :

$$\vartheta^I_{(\delta,\eta)}(\tau,\bar{\tau};m) := \sum_{\ell \in I+\delta} e^{i\pi\tau Q_L(\ell) - i\pi\bar{\tau}Q_R(\ell)} e^{2i\pi Q_I(\ell,\eta)}$$

that average to "twisted" non-holomorphic Eisenstein series:

$$E^I_{(\delta,\eta)}(\tau,\bar{\tau}) = \delta_{\delta\in I} + \frac{1}{\sqrt{\det Q_I}} e^{-\frac{\pi i}{4}(p_I - q_I)} \sum_{(c,d)=1,c>0} c^{-\frac{p_I+q_I}{2}} (c\tau+d)^{-\frac{p_I}{2}} (c\bar{\tau}+d)^{-\frac{q_I}{2}} \mu_{(\delta,\eta)\cdot M^{-1}} \lambda^I_{0,-d\delta+c\eta}(M^{-1})$$

Global Symmetries, Orbifolds, & Ensembles

- The Bulk theories have global symmetries, i.e. $\mathbb{Z}_2$ symmetry $A \to -A$
- One can gauge these symmetries to produce orbifolded dualities. Even, self dual case considered in [2103.15826,2105.12594]
- General Q theories lead to novel Eisenstein series. The "gauge" sector partition function has generalized lattice theta functions :

$$\vartheta^I_{(\delta,\eta)}(\tau,\bar\tau;m) := \sum_{\ell\in I+\delta} e^{i\pi\tau Q_L(\ell)-i\pi\bar\tau Q_R(\ell)} e^{2i\pi Q_I(\ell,\eta)}$$

that average to "twisted" non-holomorphic Eisenstein series:

$$E^I_{(\delta,\eta)}(\tau,\bar\tau) = \delta_{\delta\in I} + \frac{1}{\sqrt{\det Q_I}} e^{-\frac{\pi i}{4}(p_I-q_I)} \sum_{(c,d)=1,c>0} c^{-\frac{p_I+q_I}{2}}(c\tau+d)^{-\frac{p_I}{2}}(c\bar\tau+d)^{-\frac{q_I}{2}}\mu_{(\delta,\eta)\cdot M^{-1}}\lambda^I_{0,-d\delta+c\eta}(M^{-1})$$

- Global symmetries not in contradiction with [1810.05338+] - this is not Einstein gravity
- However, this *is* holographic - lesson for quantum gravity?

Understanding Ensembles: Duality before Averaging

> Ensemble averages are not encountered in typical holographic setups
  - 1 bulk theory & 1 boundary theory

Understanding Ensembles: Duality before Averaging

- Ensemble averages are not encountered in typical holographic setups
  - 1 bulk theory & 1 boundary theory
- Can one understand ensemble averaged duality as a coarse-graining of a usual holographic duality? [arXiv:2201.05093]

Understanding Ensembles: Duality before Averaging

- Ensemble averages are not encountered in typical holographic setups
  - 1 bulk theory & 1 boundary theory
- Can one understand ensemble averaged duality as a coarse-graining of a usual holographic duality? [arXiv:2201.05093]

Need to describe duality *before* averaging

Understanding Ensembles: Duality before Averaging

- Ensemble averages are not encountered in typical holographic setups
  - 1 bulk theory & 1 boundary theory
- Can one understand ensemble averaged duality as a coarse-graining of a usual holographic duality? [arXiv:2201.05093]

Need to describe duality *before* averaging

- Regular CS-CFT duality does not work - dual of CS is rational CFT. At generic point in moduli space, CFT is not rational
- Must use *Maxwell-Chern-Simons* theory [arXiv:0403225]

$$S_{MCS} = \frac{1}{16\pi^2} \sum_{i,j} \int_M \left(-\frac{1}{2e^2} \lambda_{ij}^{-1} dA^i \wedge \star dA^j + 2\pi i Q_{ij} A^i \wedge dA^j \right)$$

Understanding Ensembles: Duality before Averaging

> Ensemble averages are not encountered in typical holographic setups
  - 1 bulk theory & 1 boundary theory

> Can one understand ensemble averaged duality as a coarse-graining of a usual holographic duality? [arXiv:2201.05093]

Need to describe duality *before* averaging

> Regular CS-CFT duality does not work - dual of CS is rational CFT. At generic point in moduli space, CFT is not rational

> Must use *Maxwell-Chern-Simons* theory [arXiv:0403225]

$$S_{MCS} = \frac{1}{16\pi^2}\sum_{i,j}\int_M \left(-\frac{1}{2e^2}\lambda_{ij}^{-1} dA^i \wedge \star dA^j + 2\pi i Q_{ij} A^i \wedge dA^j \right)$$

> Connection to string theory: MCS arises in $AdS_3 \times K_7$ compactifications, where $K_7 = S^3 \times (S^1)^4$ or $S^3 \times S^3 \times S^1$

Averaged Observables & The Positive Definite Case

- To really define these averaged theories, one should have some handle on the theory beyond the partition function, i.e. correlation functions

Averaged Observables & The Positive Definite Case

- To really define these averaged theories, one should have some handle on the theory beyond the partition function, i.e. correlation functions
- Can this be done? Yes, at least we know how in some cases

Averaged Observables & The Positive Definite Case

- To really define these averaged theories, one should have some handle on the theory beyond the partition function, i.e. correlation functions
- Can this be done? Yes, at least we know how in some cases
- Positive definite case: Chiral CFTs
  - These CFTs have a 0-dimensional moduli space, but one can still define an average

$$\langle\langle \vartheta_Q(Z) \rangle\rangle = \frac{1}{M(Q)} \sum_{j=1}^{h(Q)} \frac{\vartheta_{Q_j}(Z)}{|O_{Q_j}(\mathbb{Z})|}\,, \qquad M(Q) := \sum_{j=1}^{h(Q)} \frac{1}{|O_{Q_j}(\mathbb{Z})|}$$

Averaged Observables & The Positive Definite Case

- To really define these averaged theories, one should have some handle on the theory beyond the partition function, i.e. correlation functions
- Can this be done? Yes, at least we know how in some cases
- Positive definite case: Chiral CFTs
  - These CFTs have a 0-dimensional moduli space, but one can still define an average

$$\langle\!\langle \vartheta_Q(Z) \rangle\!\rangle = \frac{1}{M(Q)} \sum_{j=1}^{h(Q)} \frac{\vartheta_{Q_j}(Z)}{|O_{Q_j}(\mathbb{Z})|} \,, \qquad M(Q) := \sum_{j=1}^{h(Q)} \frac{1}{|O_{Q_j}(\mathbb{Z})|}$$

  - If $P(x)$ is a spherical polynomial with respect to Q, then $P(\partial X)$ is a primary operator and its one-point function & ensemble average are

$$\vartheta_{Q,P} = \sum_{\ell \in \Lambda} P(\ell) e^{iQ(\ell)\tau} \,, \qquad \langle\!\langle \vartheta_{Q,P^{\nu}_m}(\tau) \rangle\!\rangle = C^{(\nu)}_k |T_m$$

- Can also define average of orbifold twist operator correlation functions [2103.15826]

Things Left Unsaid

- Odd lattice CFTs & Spin Chern Simons

$$\Theta^{\epsilon_1,\epsilon_2}_{Q,h}(m;\tau) = \sum_{\ell\in\Lambda+h+\epsilon_1 W/2} e^{i\pi\tau Q_L[\ell]-i\pi\bar{\tau}Q_R[\ell]}(-1)^{\epsilon_2(W,\ell)}$$

- Higher genus

$$E^g_{Q,\vec{h}}(\Omega) = \sum_{\gamma\in\Gamma_\infty\backslash\mathsf{Sp}(2g,\mathbb{Z})} \frac{\gamma_{\vec{h}}(C,D)}{\mathsf{det}(C\Omega+D)^{p/2}\mathsf{det}(C\bar{\Omega}+D)^{q/2}}$$

Conclusion

CS Topological Invariants

Holography

CFTs with general Q

$$\sum_{\substack{\text{AdS}_3 \\ \text{geometries}}} Z_{\text{Bulk}}[\tau] = \int_{\substack{\text{moduli} \\ \text{space}}} [dm] Z_{\text{CFT}}[m;\tau]$$

Number Theory

Quantum Gravity

Conclusion

Stringy Fine-Grained Holography,
Global Symmetries,
Swampland?

CS Topological Invariants

Holography

CFTs with general Q

$$\sum_{\text{AdS}_3 \text{ geometries}} Z_{\text{Bulk}}[\tau] = \int_{\text{moduli space}} [dm] Z_{\text{CFT}}[m;\tau]$$

Quantum Gravity

Number Theory

Thank you!

Contact

DESY. Deutsches Elektronen-Synchrotron

www.desy.de

Jacob M. Leedom
0000–0003–4911–2188
Theory - Cosmology
jacob.michael.leedom@desy.de
Office O1.142