Infrared Finite Scattering Theory in Quantum Field Theory and Quantum Gravity

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G.S., & R.M. Wald (to appear)

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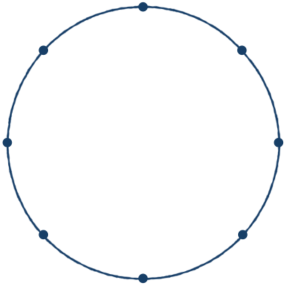
September 14, 2022

Gravitational and Electromagnetic Memory Effects



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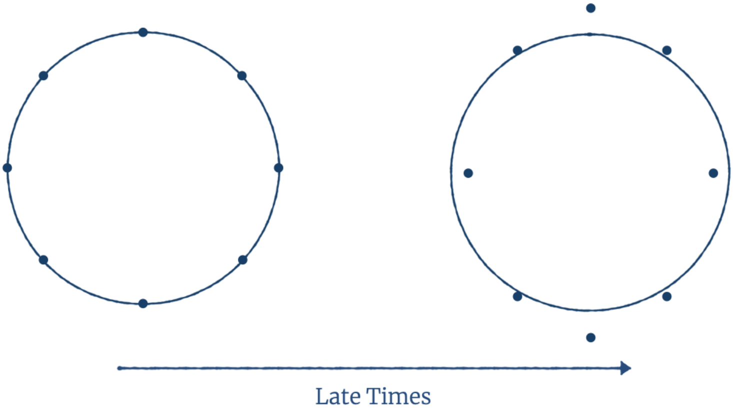

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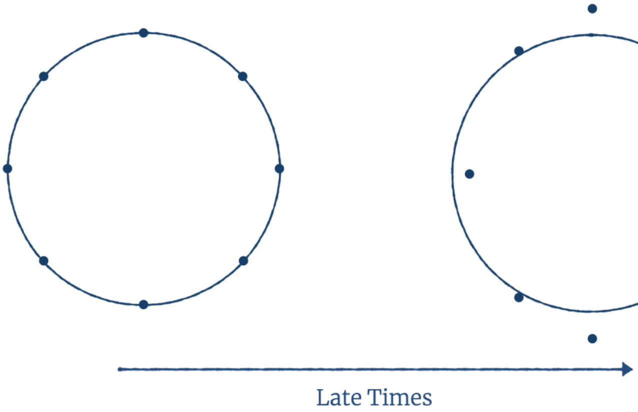
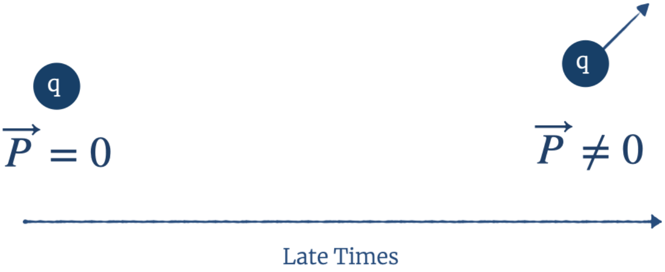


relative displacement → $$\Delta\xi_a = \frac{\Delta_{ab}\xi^b_{(0)}}{r}$$ ← Gravitational Memory

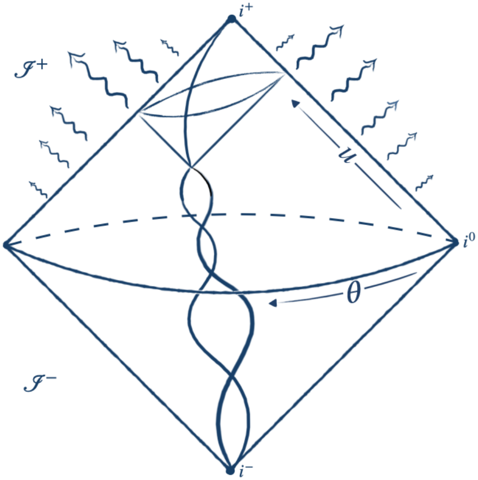
Gravitational and Electromagnetic Memory Effects



$$\Delta P_a = \frac{\Delta_a}{r}$$

Electromagnetic Memory

Classical Scattering, Radiation and Memory



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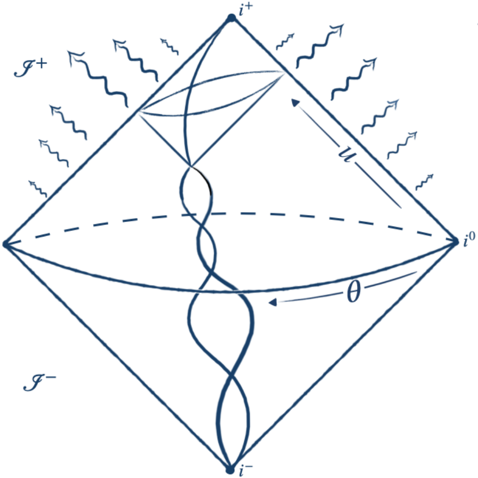
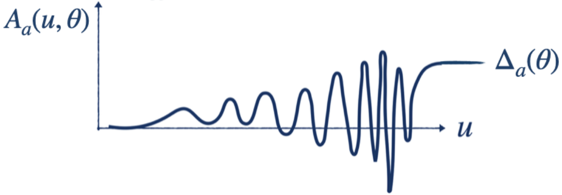


$$\Delta_a(\theta) = -\int_{-\infty}^{\infty} du\ E_a(u,\theta) \qquad E_a(u,\theta) = \partial_u A_a$$



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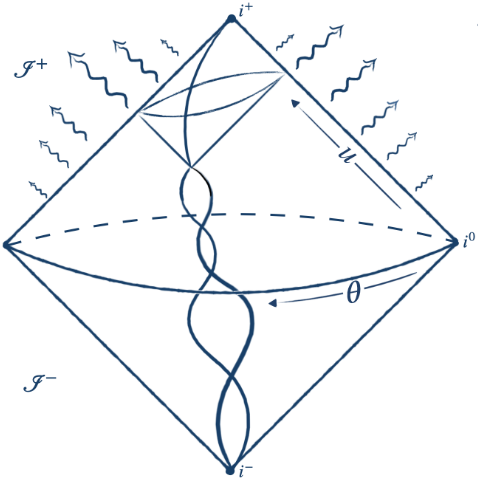

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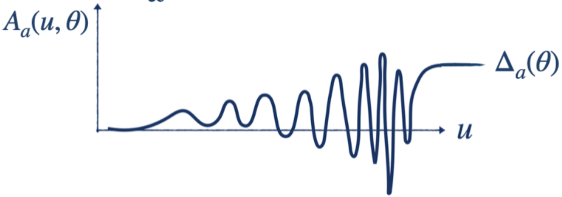
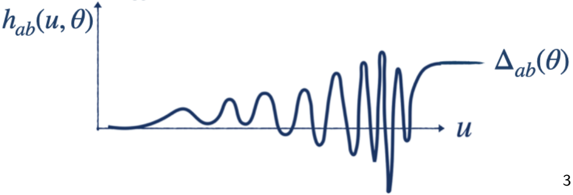
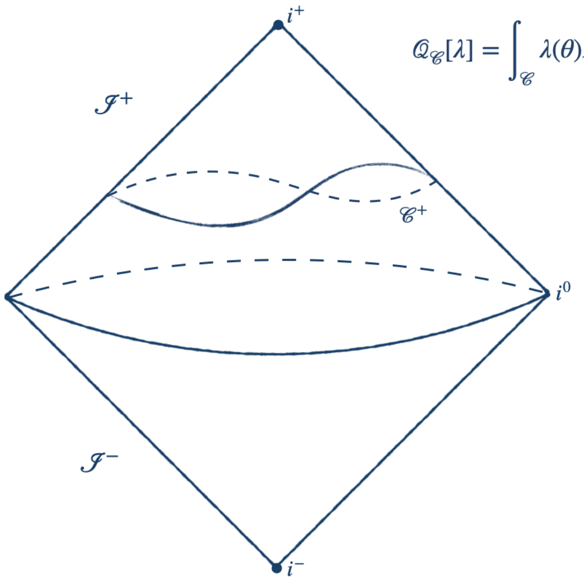


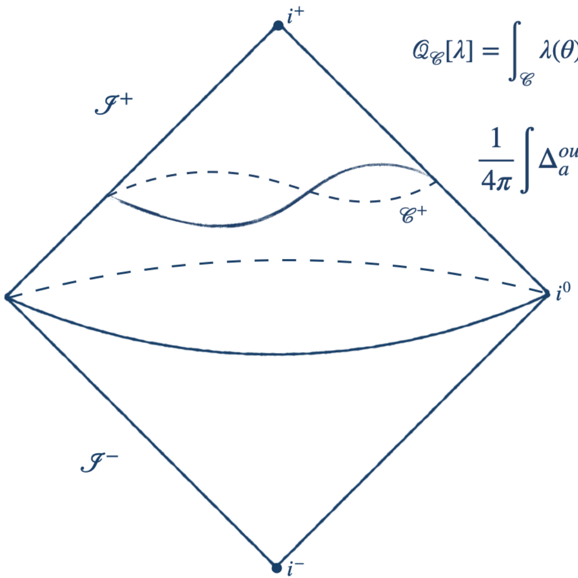
$$\Delta_{ab}(\theta) = \int_{-\infty}^{\infty} du\ N_{ab}(u,\theta) \qquad N_{ab}(u,\theta) = \partial_u h_{ab}$$



Memory and Charges



$$\mathcal{Q}_{\mathscr{C}}[\lambda] = \int_{\mathscr{C}} \lambda(\theta) F^{(2)}_{ur}(u,\theta)\, d\Omega$$



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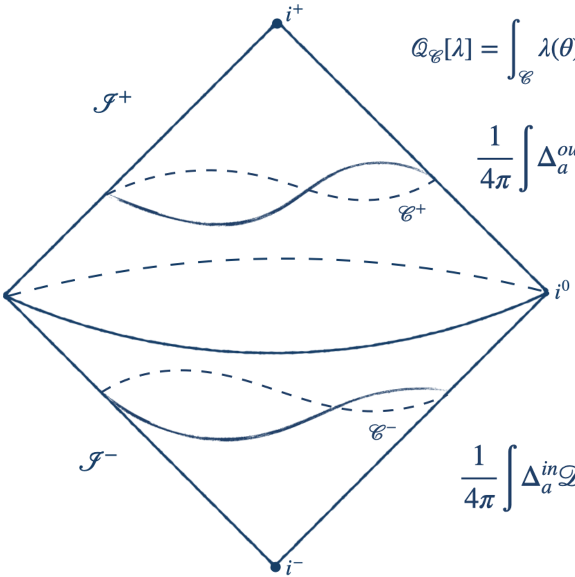
massless charge-current flux

$$\frac{1}{4\pi}\int \Delta_a^{out}\mathscr{D}^a\lambda\, d\Omega = \mathcal{Q}_{i^+}[\lambda] - \mathcal{Q}_{i^0}[\lambda] + \int_{\mathscr{I}^+} J_{out}\lambda$$



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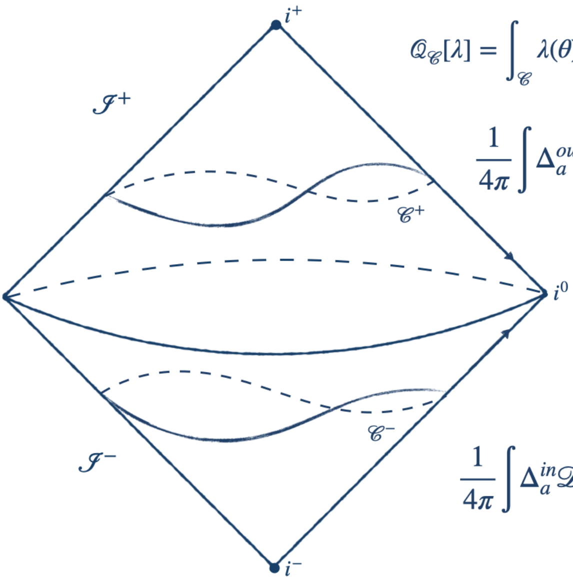

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$$\lim_{\mathscr{C}^+ \to i^0} \mathcal{Q}_{\mathscr{C}^+}[\lambda] = \lim_{\mathscr{C}^- \to i^0} \mathcal{Q}_{\mathscr{C}^-}[\tilde{\lambda}]$$

where $\tilde{\lambda}(\theta) = \lambda(-\theta)$

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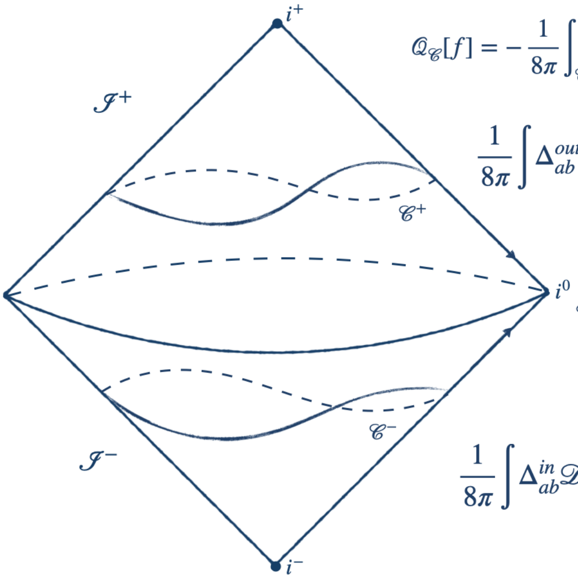

$$\mathcal{Q}_{\mathscr{C}}[f] = -\frac{1}{8\pi}\int_{\mathscr{C}} f(\theta) C^{(3)}_{urur}(u,\theta)\, d\Omega \quad \text{when} \quad N_{AB} = 0$$

$$\frac{1}{8\pi}\int \Delta^{out}_{ab}\mathscr{D}^a\mathscr{D}^b f\, d\Omega = \mathcal{Q}_{i^+}[f] - \mathcal{Q}_{i^0}[f] + \int_{\mathscr{I}^+} N^2 f$$

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Fock Quantization

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- The standard Fock space $\mathscr{F}_0^{\mathscr{I}^+}$ is constructed constructed from the "one"-particle Hilbert space $\mathscr{H}_0^{\mathscr{I}^+}$ of gravitons:

$$||h||^2 = 16\pi \int_0^\infty \int_{\mathbb{S}^2} d\omega d\Omega \; \omega |\tilde{h}_{ab}(\omega, \theta)|^2$$

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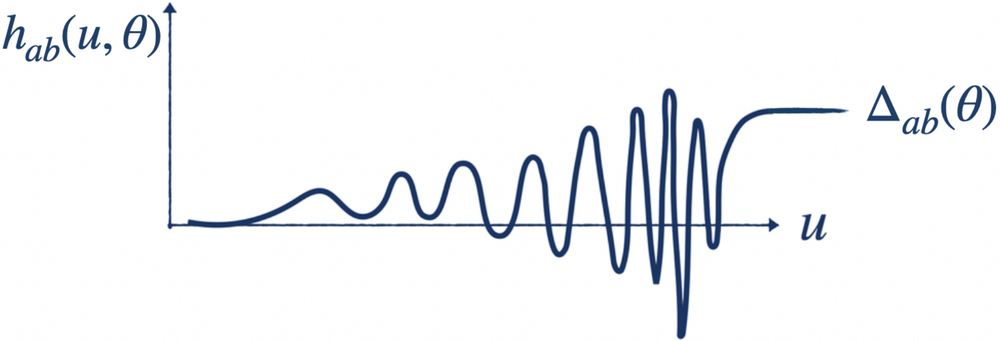
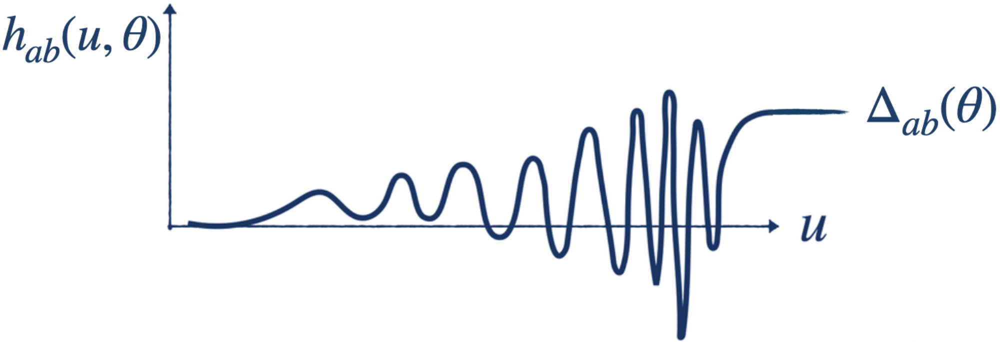


$$\Delta_{ab} \neq 0 \implies \Delta h_{ab} \neq 0 \implies \tilde{h}_{ab}(\omega,\theta) \sim \frac{1}{\omega}$$



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- There are an *uncountably infinite* number of of "in" and "out" Fock spaces $\mathscr{F}_\Delta^{\mathscr{I}^\pm}$ labeled by the "in/out" memory $\Delta^{in/out}$. The memory is *not* conserved and so the "standard" S-matrix does not exist! To go beyond "inclusive cross sections" and have a well-defined S-matrix we need to include states with memory.

Memory Representations

- States with memory are perfectly legitimate states and a Hilbert space of states with memory Δ_{ab} can be constructed by starting with $\mathscr{F}_0^{\mathscr{I}}$ and performing the field redefinition:

$$\boldsymbol{N}_{ab}(u,\theta) \rightarrow \boldsymbol{N}_{ab}(u,\theta) + N_{ab}(u,\theta)\mathbf{1} \text{ where } \int_{-\infty}^{\infty} du \; N_{ab}(u,\theta) = \Delta_{ab}(\theta)$$

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$$\boldsymbol{\Delta}_{ab}(\theta)\,|\Psi_\Delta^{\mathscr{I}}\rangle = \Delta_{ab}(\theta)\,|\Psi_\Delta^{\mathscr{I}}\rangle \quad \forall\,|\Psi_\Delta^{\mathscr{I}}\rangle \in \mathscr{F}_\Delta^{\mathscr{I}}$$

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A Hilbert Space for Scattering

- What sort of Hilbert space should we choose? Need to include a sufficiently many number of $\mathscr{F}_{\Delta}^{\mathscr{I}}$ and ensure that the corresponding Hilbert space scatters into itself.
- Problem: Memory is not conserved so any construction that just "stitches" together these representations will not preserved under scattering.
- For example, one could consider the

$$\text{(uncountable) Direct sum: } \bigoplus_{\Delta} \mathscr{F}_{\Delta}^{\mathscr{I}} \text{ or a "Direct integral": } \int^{\oplus} d\mu_{\Delta} \mathscr{F}_{\Delta}^{\mathscr{I}}$$

but the scattering is still uncontrolled (i.e. non-vanishing "probability" to lie in a different representation.)

Does there exist a (separable) space of states which scatters into itself?

Massive QED - Faddeev-Kulish States

$$\mathcal{Q}_{i^0}(\lambda) = \mathcal{Q}_{i^-}(\lambda) - \frac{1}{4\pi}\int_{\mathbb{S}^2} \Delta_a^{\text{in}} \mathscr{D}^a \lambda$$

- Key Idea: The charge at spatial infinity is conserved. Therefore "in" Hilbert space of eigenstates of the charge $\boldsymbol{\mathcal{Q}}_{i^0}(\lambda)$ with eigenvalue $\mathcal{Q}_{i^0}(\lambda)$ will will map to an "out" Hilbert space of eigenstates with eigenvalue $\mathcal{Q}_{i^0}(\tilde{\lambda})$ [Faddeev & Kulish, '70]

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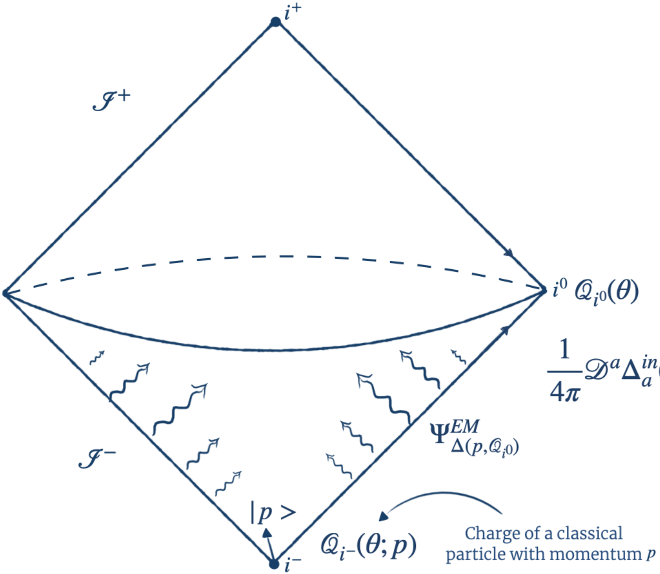

$$\frac{1}{4\pi}\mathscr{D}^a \Delta^{in}_a(\theta; p, \mathcal{Q}_{i^0}) = \mathcal{Q}_{i^-}(\theta; p) - \mathcal{Q}_{i^0}(\theta)$$

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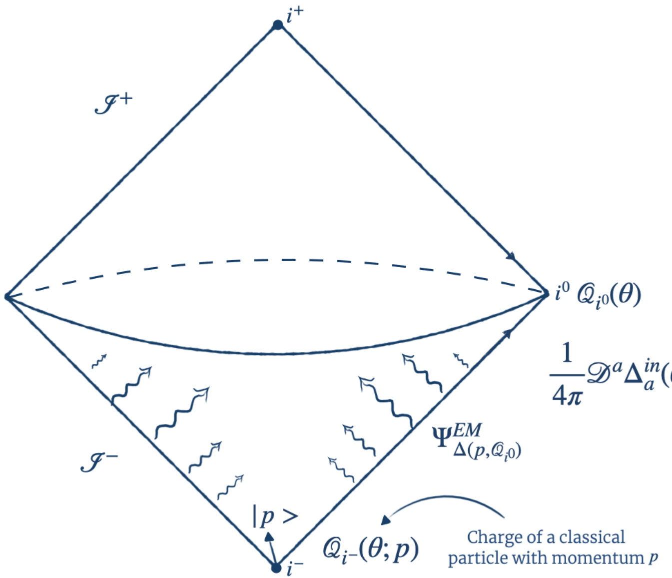

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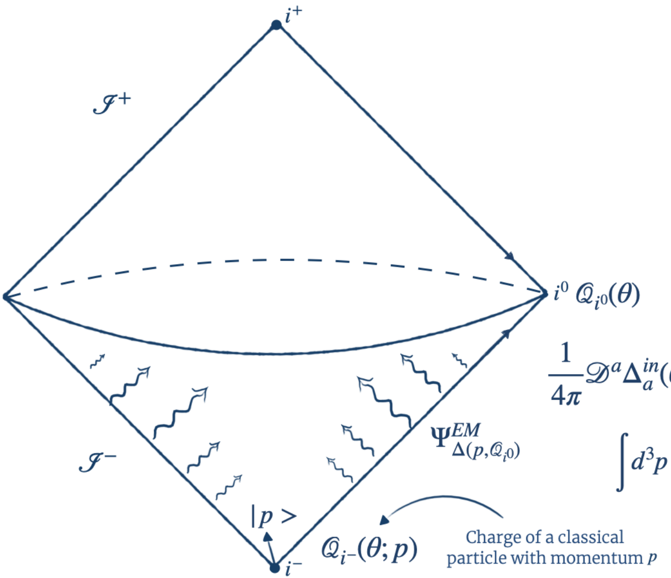

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  The corresponding Hilbert space of dressed electrons is $\mathscr{H}_{\mathcal{Q}_{i^0}}$
- $\mathcal{Q}_{i^0}(\lambda)$ is *not* Lorentz invariant unless $\mathcal{Q}_{i^0} = 0$. Lorentz boosts cannot act on $\mathscr{H}_{\mathcal{Q}_{i^0}}$ unless $\mathcal{Q}_{i^0} = 0$ [Frohlich, Morchio & Strocchi, '79].

Massive QED - Faddeev-Kulish States

- The angular momentum is undefined for all states in $\mathscr{H}_{\mathcal{Q}_{i^0}}$ unless $\mathcal{Q}_{i^0} = 0$. This includes the total electric charge! Therefore, in order for this to work, one must also put any extra charges "behind the moon" [Frohlich, Morchio & Strocchi, '79].

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- The Hilbert spaces $\mathscr{H}^{\text{in}}_{\mathcal{Q}_{i0}=0}$ and $\mathscr{H}^{\text{out}}_{\mathcal{Q}_{i0}=0}$
  1. constitute states of finite energy-momentum and angular momentum
  2. contains all "hard" scattering processes (since radiation field can have arbitrarily low frequencies and all extra charges are behind the moon)
  3. is separable (admits a countable basis)

Consequently, there is a well-defined unitary S-matrix in massive QED:

$$S : \mathscr{H}^{\text{in}}_{\mathcal{Q}_{i0}=0} \rightarrow \mathscr{H}^{\text{out}}_{\mathcal{Q}_{i0}=0}$$

Failure of FK: Massless QED and Linearized Gravity

$$\mathcal{Q}_{i^0}(\lambda) = \mathcal{J}(\lambda) - \frac{1}{4\pi}\int_{\mathbb{S}^2} \Delta_a^{\text{in}} \mathscr{D}^a \lambda$$

- In massless QED, the analogous construction is to pair eigenstates of the incoming charge-current flux with memory. However, the eigenvalue is now a δ-function on $\mathbb{S}^2$. The required "dressings" have "collinear divergences" and therefore have infinite energy! All "FK states" are unphysical except the vacuum

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$$\mathcal{Q}^{\text{GR}}_{i^0}(f) = -\frac{1}{8\pi}\int_{\mathbb{S}^2} \Delta^{\text{in}}_{ab}\mathscr{D}^a\mathscr{D}^b f(\theta) + \int_{\mathscr{I}^-} f(\theta) T_{vv}(v,\theta)$$

- In linearized quantum gravity one can again repeat the FK construction. [Akhoury & Choi, 2017] In this case there are no collinear divergences so the "dressings" are not singular. However, we cannot set $\mathcal{Q}^{\text{GR}}_{i^0} = 0$ since this would set the total four-momenum to zero! (*Can't hide mass behind the moon!*) All "FK states" have undefined angular momentum except the vacuum

Vacuum Gravity - Failure of Faddeev-Kulish Hilbert Space

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Theorem

The unique eigenstate of $\boldsymbol{\mathcal{Q}}_{i^0}^{GR}(f)$ is the vacuum state with vanishing eigenvalue.

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- Intuition: Memory and Energy flux are not independent! In gravity, the gravitational radiation "sources" (i.e. via energy flux) its own memory. Matching the memory to the energy flux introduces more radiation! This introduces more energy flux and so on...

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There is no "preferred" Hilbert space for scattering in quantum gravity ("Non-Faddeev-Kulish" representations also fail)

Algebraic Scattering Theory

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- Any state in a Hilbert space can be expressed as a list of correlation functions. Conversely, given a list of correlation functions (satisfying commutation relations, positivity, ...) one can construct (by GNS) a Hilbert space where this list of correlation functions is packaged as a vector in the Hilbert space. Thus viewing a state as a list of correlation functions or as a vector in a Hilbert space are essentially equivalent. [Witten, 2022],[Hollands & Wald, 2014]

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- However, by considering states as lists of correlation functions one is now freed from choosing in advance a particular Hilbert space. Starting with some "in" set of correlation functions (with whatever memory or charges one wants) one should then be able to calculation the "out" correlation functions by the same sort of LSZ perturbative methods used in S-matrix calculations.

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It would be interesting to further develop such an (IR-finite) scattering theory!

Summary

- IR divergences arise from sticking a state in a Hilbert space to which it doesn't belong.
- In massive QED the Faddeev-Kulish representation is a preferred representation but, as opposed to a "proof of principle" it is actually a "fluke"!
- Non-Faddeev-Kulish representations do not work
- A well-defined (IR-finite) scattering theory can be, in principal, constructed by simply evolving "in" correlation functions to "out" correlation functions.

Yang Mills

$$\mathcal{Q}^{\mathrm{YM}}_{i^0}(\lambda) = -\frac{1}{4\pi}\int_{\mathbb{S}^2} \Delta^{\mathrm{YM,in}}_{i,a}\mathscr{D}^a\lambda^i + \frac{1}{2\pi}\int_{\mathscr{I}^+} q^{ab}\lambda_i(\theta)[A_a, E_b]^i$$

- In Yang Mills theories, $\mathcal{Q}^{\mathrm{YM}}_{i^0}(\lambda)$ is determined by the incoming gluon color-flux as well as the incoming memory of the gluon field.
- The dressing procedure again introduces severe "collinear divergences" in the dressing. Due to the nonlinearities of Yang Mills, the dressing further contributes to the charge. However the color-flux of the dressing is *infinite* and so this dressing procedure fails.
- One could consider some other procedure other than dressing. However, eigenstates of the large gauge charges correspond to Casimirs of the Lie-algebra. Therefore, for example,

$$\langle \boldsymbol{E}_{i,a}(x)\rangle = 0 \ , \ \langle \boldsymbol{E}_{i,a}(x_1)\boldsymbol{E}_{j,b}(x_1)\rangle = k_{ij}W_{ab}(x_1,x_2)\ \ldots$$

There are insufficiently many states to do scattering theory!

Algebraic Scattering Theory

- A state $\omega : \mathscr{A} \to \mathbb{C}$ on the algebra $\mathscr{A}$ is equivalent to specifying a list of (positive) n-point correlation functions $\omega(\boldsymbol{E}_{a_1}(x_1)\dots\boldsymbol{E}_{a_n}(x_n))$
- Given $\mathscr{A}_{\text{in/out}}$ the "superscattering matrix" \$ is defined as a map from "in" algebraic states to "out" algebraic states

$$\omega_{\text{out}} = \$\omega_{\text{in}}$$

- Conservation of Probability: If ω_{in} is any normalised state (i.e. $\omega_{\text{in}}(\mathbf{1}) = 1$) then $\omega_{\text{out}} = \$\omega_{\text{in}}$ satisfies $\omega_{\text{out}}(\mathbf{1}) = 1$.
- Pure to Pure evolution: If ω_{in} is pure (i.e. cannot be expressed as the (convex) sum of other states) then ω_{out} is also pure.
- Probability of measuring any observable: A state ω specifies the expected value of all powers of any smeared observable $\boldsymbol{E}(s)$. These moments uniquely determine a probability distribution of measuring the field observable.