General $O(D)$-equivariant fuzzy hyperspheres via confining potentials and energy cutoffs

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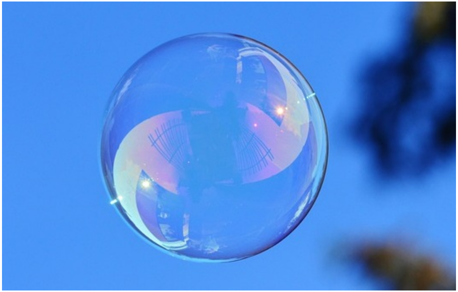

Workshop on Noncommutative and generalized geometry...
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Introduction

Some motivations for noncommutative (NC) space(time) algebras:

- To avoid UV divergences in QFT [Snyder 1947,...].
- As an arena for formulating QG compatible with $\Delta x \gtrsim L_p$ [Mead 1966, Doplicher et al 1994-95,...].
- As an arena for unifying interactions [Connes-Lott '92,...]

Given a quantum theory $\mathcal{T}$ on a commutative space how to find NC candidates $\overline{\mathcal{T}}$ approximating $\mathcal{T}$? One possible mechanism:

Let $\mathcal{H} \equiv$ Hilbert space of the system S, $\mathcal{A} \equiv \text{Lin}(\mathcal{H})$, $\overline{\mathcal{H}} \subset \mathcal{H}$ a subspace, $\overline{P} : \mathcal{H} \mapsto \overline{\mathcal{H}}$ its projection. Then

$$\overline{\mathcal{A}} \equiv \text{Lin}\left(\overline{\mathcal{H}}\right) = \{\overline{A} \equiv \overline{P}A\overline{P} \mid A \in \mathcal{A}\} \neq \mathcal{A}.$$

In particular, if $[x_i, x_j] = 0$, in general $[\overline{x_i}, \overline{x_j}] \neq 0$.

If $\overline{P}H = H\overline{P}$ ($H \equiv$ Hamiltonian of S) then no change in dynamics within $\overline{\mathcal{H}}$. If $\overline{\mathcal{H}} \equiv$ subspace with energies $E \leq \overline{E} \equiv$ cutoff, then $\overline{\mathcal{T}}$ is a low-energy effective approximation of $\mathcal{T}$.

Prototype: Landau model in D=2; $\overline{E} = E_0$ implies $[\overline{x_1}, \overline{x_2}] = \frac{i\hbar c}{ieB}$.

When may this be useful? E. g.:

- If $\overline{\mathcal{H}}^\perp$ is practically not accessible in preparing the initial state, nor through the interactions with the environment or the measurement apparatus, then $\overline{\mathcal{T}}$ on $\overline{\mathcal{H}}$ (smaller) is enough.
- If at $E > \overline{E}$ we expect new physics not accountable by $\mathcal{T}$, then $\overline{\mathcal{T}}$ may also help to figure out a new theory $\mathcal{T}'$ valid for all E.

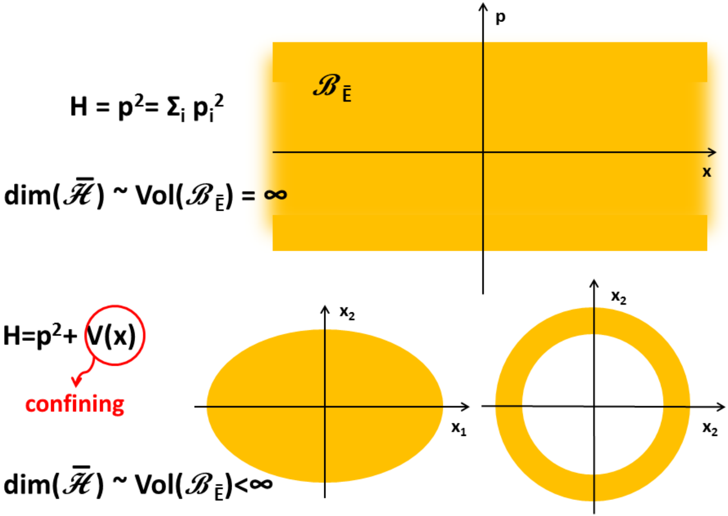
(Of course, the two may co-exist.)

If H is invariant under some group G, then $\overline{\mathcal{H}}, \overline{P}, \overline{\mathcal{T}}$ will be.

Consider quantum mechanics (QM) on $\mathbb{R}^D$, Hamiltonian $H(x,p)$.

$$\dim(\overline{\mathcal{H}}) \simeq \mathrm{Vol}(\mathcal{B}_{\overline{E}})/h^D,$$

$$\mathcal{B}_{\overline{E}} \equiv \{(x,p)\in\mathbb{R}^{2D} \mid H(x,p)\le\overline{E}\} = \text{classical phase space below } \overline{E}.$$



Adding a *'dimensional reduction' mechanism* we can obtain a NC, fuzzy approximation of QM on *submanifolds* of $\mathbb{R}^D$.
Here a sphere S^d, $d = D-1$ [GF, F. Pisacane 2017-19].

Consider a quantum particle in $\mathbb{R}^D$ configuration space with Hamiltonian

$$H = -\frac{1}{2}\Delta + V(r); \qquad (1)$$

we fix the minimum $V_0 = V(1)$ of the the confining potential $V(r)$ so that the ground state has energy $E_0 = 0$.

- Choose $V(r)$ and $\overline{E}$ fulfilling

$$V(r) \simeq V_0 + 2k\,(r-1)^2 \qquad (2)$$

if $V(r) \leq \overline{E}$; so that $V(r)$ has a harmonic behavior for $|r-1| \leq \sqrt{\frac{\overline{E}-V_0}{2k}}$.

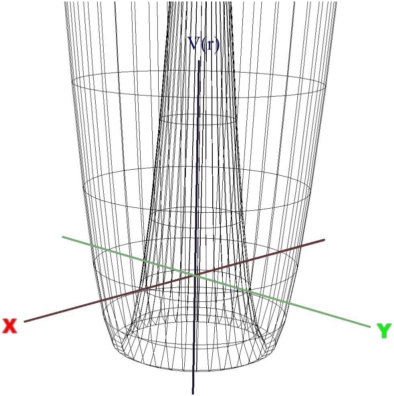


Figure 1: Three-dimensional plot of $V(r)$

- The minimum on the sphere $r=1$ is sharp if $V''(1)\equiv 4k \gg 0$.
- $\overline{E}$ low enough to *eliminate radial excitations* from Spectrum(H).

Then: $\overline{H}=\overline{L^2}$; the x_i generate all $\overline{\mathcal{A}}$, $[\overline{x_i},\overline{x_j}] \sim \frac{iL_{ij}}{k}$ *à la* Snyder.

- Choose $\overline{E}=\overline{E}(\Lambda)\equiv\Lambda(\Lambda+d-1)$, $k=k(\Lambda)\geq\Lambda^2(\Lambda+d-1)^2$; diverging with $\Lambda\in\mathbb{N}$. Renaming $\overline{\mathcal{H}},\overline{P},\overline{\mathcal{A}} \rightsquigarrow \mathcal{H}_\Lambda, P_\Lambda, \mathcal{A}_\Lambda$, we find

$$(\mathcal{H}_\Lambda,\mathcal{A}_\Lambda) \stackrel{\Lambda\to\infty}{\longrightarrow} (\mathcal{H},\mathcal{A}) \equiv \Big(\mathcal{L}^2(S^d),\, \mathrm{Lin}\Big(\mathcal{L}^2(S^d)\Big)\Big)$$

This is a $O(D)$-covariant fuzzy sphere $\{S^d_\Lambda\}_{\Lambda\in\mathbb{N}} \equiv \{(\mathcal{H}_\Lambda,\mathcal{A}_\Lambda)\}_{\Lambda\in\mathbb{N}}$, i.e. sequence of finite-dim approximations of ordinary QM on S^d![1]

$d=1,2$ in [GF, F. Pisacane 2018-20]; $d>2$ started in [F.Pis.20].
Here: its completion to all d, via simplification; as a complete set in $\mathcal{L}^2(S^d)$ we take polynomials in Cartesian coordinates t^i of $p\in S^d$, rather than spherical harmonics.
Finally, I will compare our S^d_Λ with other fuzzy spheres.

[1] A fuzzy space is a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ of *finite-dimensional* algebras such that $\mathcal{A}_n \stackrel{n\to\infty}{\longrightarrow} \mathcal{A} \equiv$algebra of regular functions on an ordinary manifold.

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Constructing $\mathcal{H}_\Lambda, P_\Lambda$

Let $\mathbf{x} := (x^1, ...x^D)$ be real Cartesian coordinates of $\mathbb{R}^D$; abbreviate $\partial_i \equiv \partial/\partial x^i$. Then $r^2 = \mathbf{x}^2 \equiv x^i x^i$, $\Delta = \partial_i \partial_i$. The self-adjoint operators

$$L_{ij} := i(x^j \partial_i - x^i \partial_j) \tag{3}$$

on $\mathcal{L}^2(\mathbb{R}^D)$ (angular momentum components) generate rotations of $\mathbb{R}^D$;

$$[iL_{ij}, v^h] = v^i \delta^h_j - v^j \delta^h_i \tag{4}$$

hold for all vectors $\mathbf{v}$, in particular $v^h = x^h, \partial_h$, and

$$[iL_{ij}, iL_{hk}] = i\left(L_{ik}\delta_{hj} - L_{jk}\delta_{hi} - L_{ih}\delta_{kj} + L_{jh}\delta_{ki}\right). \tag{5}$$

These are equivariant under $x^i \mapsto x'^i = Q^i_j x^j$, with $Q \in O(D)$. All scalars S, in particular $S = \Delta, r^2, V(r), H$, are invariant, whence $[S, L_{ij}] = 0$.

$$\Delta = \partial_r^2 + (D-1)\frac{1}{r}\partial_r - \frac{1}{r^2}\mathbf{L}^2, \tag{6}$$

where $\partial_r := \partial/\partial r$ and $\mathbf{L}^2 := L_{ij}L_{ij}/2$ is the quadratic Casimir of $Uso(D)$ and the Laplacian on the sphere S^d.

Solving the Schrödinger equation

The eigenvalues of $\mathbf{L}^2$ on $\mathcal{L}^2(\mathbb{R}^D)$, $\mathcal{L}^2(S^d)$ are $l\,(l+D-2)$, $l\in\mathbb{N}_0$.
Let V_D^l be the $\mathbf{L}^2=l(l+D-2)$ eigenspace within $\mathcal{L}^2(S^d)$.
Ansatz $\psi=T(\boldsymbol{\theta})\,f(r)$, $f(r)=r^{-d/2}g(r)$, $T\in V_D^l$ transforms the Schrödinger PDE $H\psi=E\psi$ into the ODE in the unknown $g(r)$

$$-g''(r)+\left[\frac{\left[D^2-4D+3+4l(l+D-2)\right]}{4r^2}+V(r)\right]g(r)=Eg(r). \quad (7)$$

Expanding [...] at lowest order in $(r-1)$ we get the harmonic oscillator eq.

$$-g''(r)+g(r)k_l\,(r-\widetilde{r}_l)^2=\widetilde{E}_l g(r), \quad (8)$$

which approximates well (7) in the spherical shell $V(r)\le\overline{E}$, because $V(r)$ has a sharp minimum at $r=1$. Here

$$\widetilde{r}_l:=1+\frac{b(l,D)}{3b(l,D)+2k},\quad \widetilde{E}_l:=E-V(1)-\frac{2b(l,D)[k+b(l,D)]}{3b(l,D)+2k},$$
$$k_l:=2k+3b(l,D),\quad b(l,D):=\frac{D^2-4D+3+4l(l+D-2)}{4}.$$

The (Hermite functions) square-integrable solutions of (8) are

$$g_{n,l}(r) = M_{n,l}\ e^{-\frac{\sqrt{k_l}}{2}(r-\widetilde{r}_l)^2} \cdot H_n\left((r-\widetilde{r}_l)\sqrt[4]{k_l}\right) \qquad \text{with } n \in \mathbb{N}_0$$

(here $M_{n,l}$ = normalization const., H_n = Hermite polynomials), whence

$$f_{n,l}(r) = \frac{M_{n,l}}{r^{\frac{D-1}{2}}}\ e^{-\frac{\sqrt{k_l}}{2}(r-\widetilde{r}_l)^2} \cdot H_n\left((r-\widetilde{r}_l)\sqrt[4]{k_l}\right) \qquad \text{with } n \in \mathbb{N}_0. \tag{9}$$

The 'eigenvalues' in (8) are $\widetilde{\widetilde{E}}_{n,l} = (2n+1)\sqrt{k_l}$, whence the energies

$$E_{n,l} = (2n+1)\sqrt{k_l} + V(1) + \frac{2b(l,D)[k+b(l,D)]}{3b(l,D)+2k}.$$

We fix $V(1)$ requiring that $E_{0,0}=0$. Then, at leading order in k:

$$\begin{aligned} E_{n,l} &= 2n\sqrt{2k} + l(l+D-2) + O\left(k^{-\frac{1}{2}}\right), \\ \widetilde{r}_l &= 1 + \frac{b(l,D)}{2k} + O\left(k^{-2}\right). \end{aligned} \tag{10}$$

$E_{0,l} = l(l+D-2) =: E_l$ are the eigenvalues of the Laplacian $\mathbf{L}^2$ on S^2, while $E_{n,l} \to \infty$ as $k \to \infty$ if $n > 0$.

We can eliminate the latter (constrain $n = 0$) imposing a cutoff

$$\mathbf{E} \leq \Lambda(\Lambda+D-2) \equiv \overline{\mathbf{E}} < 2\sqrt{2\mathbf{k}}, \qquad \Lambda \in \mathbb{N} \tag{11}$$

$\mathcal{H}_\Lambda$ decomposes into eigenspaces of $H, \mathbf{L}^2$ (irreps of $O(D)$) as follows

$$\mathcal{H}_\Lambda = \bigoplus_{l=0}^{\Lambda} \mathcal{H}_\Lambda^l. \tag{12}$$

Given an orthonormal basis $\mathcal{B}_l \equiv \{Y_l^{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{I}_l}$ of V_D^l (e.g. spherical harmonics),an orthonormal basis of $\mathcal{H}_\Lambda^l$ consists of the

$$\psi_l^{\mathbf{m}} := f_l(r)\, Y_l^{\mathbf{m}}(\boldsymbol{\theta}) \tag{13}$$

with $f_l(r) \equiv f_{0,l}(r)$. The projection $\widetilde{P}_l : \mathcal{H} \to \mathcal{H}_\Lambda^l$ acts by

$$\left(\widetilde{P}_l\phi\right)(x) = \sum_{\mathbf{m}\in\mathbf{I}_l} \psi_l^{\mathbf{m}}(x) \int_{\mathbb{R}^D} d^D x' \psi_l^{\mathbf{m}*}(x')\phi(x').$$

If $\phi(r,\boldsymbol{\theta}) = \Theta_j(\boldsymbol{\theta})\,\phi(r)$ with $\Theta_j \in V_D^j$, then this simplifies to

$$\left(\widetilde{P}_l\phi\right)(r,\boldsymbol{\theta}) = \delta_{lj}\,\Theta_j(\boldsymbol{\theta})\, f_l(r) \int_0^\infty r'^d dr'\, f_l^*(r')\phi(r'). \tag{14}$$

Representations of $O(D)$ via polynomials in x^i, t^i

$\mathbb{C}[\mathbb{R}^D] \equiv \mathbb{C}[x^1, ...x^D] \equiv$ space of complex polynomial functions on $\mathbb{R}^D$,
$W_D^l \equiv$ subspace of homogeneous ones of degree $l \in \mathbb{N}_0$ carries a representation of $O(D)$ (and $Uso(D)$), which is reducible if $l \geq 2$: $r^2 W_D^{l-2} \subset W_D^l$ manifestly carries a smaller representation.

Let $\check{V}_D^l$ be the "trace-free" component of W_D^l: $W_D^l = \check{V}_D^l \oplus r^2 W_D^{l-2}$.

$$\dim(\check{V}_D^l) = \dim(W_D^l) - \dim(W_D^{l-2}) = \frac{(l+D-3)...(l+1)}{(D-2)!}(D+2l-2). \quad (15)$$

$\check{V}_D^l$ carries the irrep $\boldsymbol{\pi}_D^l$ of $Uso(D)$ and $O(D)$ within W_D^l characterized by the highest eigenvalue $\mathbf{L}^2 = l(l+D-2)$; it is also the subspace of W_D^l such that $\Delta \check{V}_D^l = 0$, i.e. of *harmonic* homogeneous polynomials. In fact, $X_{l,\pm}^{hk} \equiv (x^h \pm i x^k)^l \in W_D^l$ fulfill $\Delta X_{l,\pm}^{hk} = 0$, $\mathbf{L}^2 X_{l,\pm}^{hk} = l(l+D-2)\, X_{l,\pm}^{hk}$, $L_{hk} X_{l,\pm}^{hk} = \pm l\, X_{l,\pm}^{hk}$. $X_{l,+}^{hk}$, $X_{l,-}^{hk}$ as highest and lowest weight vectors.
A complete set in $\check{V}_D^l$ consists of trace-free homogeneous polynomials $X_l^{i_1 i_2 \dots i_l}$, which we obtain below applying the completely symmetric trace-free projector $\mathcal{P}^l$ to the $x^{i_1} x^{i_2} ... x^{i_l}$'s.

Enlarge $\mathbb{C}[\mathbb{R}^D]$ slightly by new coordinates r, r^{-1} subject to $r^2 = x^i x^i$, $rr^{-1} = 1$. The elements

$$t^i := x^i\, r^{-1} \tag{16}$$

can be regarded as coordinates of points of the unit sphere S^d, because

$$t^i t^i = 1. \tag{17}$$

The algebra Pol_D of complex polynomials in t^i, endowed with the scalar product $\langle T, T'\rangle := \int_{S^d} d\alpha\, T^* T'$ is a pre-Hilbert space dense in $\mathcal{L}^2(S^d)$.
$Pol_D^\Lambda \subset Pol_D \equiv$ subspace pol. of degree Λ, projection $P_\Lambda : Pol_D \to Pol_D^\Lambda$.
$Pol_D^\Lambda = W_D^\Lambda r^{-\Lambda} \oplus W_D^{\Lambda-1} r^{1-\Lambda}$ decomposes into irreps of $Uso(D)$ isomorphically to $W_D^\Lambda \oplus W_D^{\Lambda-1}$:

$$Pol_D^\Lambda = \bigoplus_{l=0}^{\Lambda} V_D^l, \qquad V_D^l = \check{V}_D^l r^{-l} \simeq \check{V}_D^l. \tag{18}$$

$$\dim\left(Pol_D^\Lambda\right) = \frac{(D+\Lambda-2)...(\Lambda+1)}{(D-1)!}\,(D+2\Lambda-1) =: N \;= \dim\left(V_{D+1}^\Lambda\right) \tag{19}$$

$O(D)$-irreps via trace-free completely symmetric projectors

Let $(\pi, \mathcal{E})$ be the D-dimensional irreducible unitary representation (irrep) of $Uso(D)$ and $O(D)$; $V_D^1 \simeq \mathcal{E}$. Under $\pi \otimes \pi$,

$$\mathcal{E} \otimes \mathcal{E} \;=\; \underbrace{\mathcal{P}^-\big(\mathcal{E} \otimes \mathcal{E}\big)}_{antisym.\ irrep} \oplus \underbrace{\mathcal{P}\big(\mathcal{E} \otimes \mathcal{E}\big) \oplus \mathcal{P}^t\big(\mathcal{E} \otimes \mathcal{E}\big)}_{sym.\ red.\ rep.\ \mathcal{P}^+\left(\mathcal{E}\otimes\mathcal{E}\right)} \tag{20}$$

$\mathcal{P}^t \equiv$ 1-dim trace projector. In an orthonormal basis of $\mathcal{E}$ $g_{ij} = g^{ij} = \delta_{ij}$,

$$\mathcal{P}^{t\,ij}_{\;\;kl} = \frac{1}{D}\,\delta^{ij}\delta_{kl}. \tag{21}$$

The $\frac{1}{2}(D-1)(D+2)$-dim trace-free symmetric projector $\mathcal{P}$ is given by

$$\mathcal{P} := \mathcal{P}^+ - \mathcal{P}^t = \frac{1}{2}\left(\mathrm{id}_{D^2} + \mathrm{P}\right) - \mathcal{P}^t \tag{22}$$

($\mathrm{P} \equiv$ permutator, $\mathrm{id}_{D^n} \equiv$ identity operator on $\mathcal{E}^{\otimes^n}$). $\mathcal{P}\mathcal{P}^- = 0 = \mathcal{P}^-\mathcal{P}$,

$$\mathcal{P}\mathcal{P}^t = 0 = \mathcal{P}^t\mathcal{P} \quad \Leftrightarrow \quad \mathcal{P}^{ij}_{hk}\delta^{hk} = 0 = \delta_{ij}\mathcal{P}^{ij}_{hk} \tag{23}$$

The generalizations $\mathcal{P}^l$ of $\mathcal{P}$ acting on $\mathcal{E}^{\otimes^l}$ (trace-free completely symmetric projectors) are uniquely determined by requiring $\left(\mathcal{P}^l\right)^2 = \mathcal{P}^l$,

$$\begin{aligned} \mathcal{P}^l\mathcal{P}^-_{n(n+1)} &= 0, \quad \mathcal{P}^l\mathcal{P}^t_{n(n+1)} = 0, \\ \mathcal{P}^-_{n(n+1)}\mathcal{P}^l &= 0, \quad \mathcal{P}^t_{n(n+1)}\mathcal{P}^l = 0, \end{aligned} \qquad n = 1, ..., l-1 \quad (24)$$

As a bonus $\mathrm{tr}_{1\ldots l}\left(\mathcal{P}^l\right) = \dim(V_D^l)$. The right relations in (24) amount to

$$\mathcal{P}^{l\,i_1\ldots i_l}_{\;j_1\ldots j_l}\delta^{j_nj_{n+1}} = 0, \quad \delta_{i_ni_{n+1}}\mathcal{P}^{l\,i_1\ldots i_l}_{\;j_1\ldots j_l} = 0, \qquad n = 1, ..., l-1. \qquad (25)$$

Proposition 1 $\mathcal{P}^{l+1}$ can be recursively expressed as a polynomial in the permutators $\mathrm{P}_{12}, ..., \mathrm{P}_{(l-1)l}$ and trace projectors $\mathcal{P}^t_{12}, ..., \mathcal{P}^t_{(l-1)l}$ via

$$\mathcal{P}^{l+1} = \mathcal{P}^l_{12\ldots l} M_{l(l+1)} \mathcal{P}^l_{12\ldots l}, \qquad (26)$$

$$= \mathcal{P}^l_{2\ldots(l+1)} M_{12} \mathcal{P}^l_{2\ldots(l+1)}, \qquad (27)$$

where $$M \equiv M(l+1) = \frac{1}{l+1}\left[\mathrm{id}_{D^2} + l\,\mathrm{P} - \frac{2Dl}{D+2l-2}\mathcal{P}^t\right] \qquad (28)$$

As a consequence, the $\mathcal{P}^l$ are symmetric, $(\mathcal{P}^l)^T = \mathcal{P}^l$.

The homogeneous polynomials

$$X_l^{i_1\dots i_l} := \mathcal{P}^{l\,i_1\dots i_l}_{\;\;j_1\dots j_l} x^{j_1}\dots x^{j_l} \tag{29}$$

are harmonic, i.e. satisfy $\Delta X_l^{i_1\dots i_l} = 0$, and are eigenvectors of $\mathbf{L}^2$:

$$\mathbf{L}^2\, X_l^{i_1\dots i_l} = l(D-2+l)\, X_l^{i_1\dots i_l}. \tag{30}$$

They make up a complete set in $\check{V}_D^l$, but are not all independent: they are completely symmetric and trace-free, i.e. fulfill

$$\delta_{i_n i_{n+1}} X_l^{i_1\dots i_l} = 0, \qquad\qquad n = 1, \dots, l-1. \tag{31}$$

Proposition 2 The maps $L_{hk} : \check{V}_D^l \to \check{V}_D^l$ explicitly act as follows:

$$iL_{hk} X_l^{i_1\dots i_l} \;=\; l\,\mathcal{P}^{l\,i_1\dots i_l}_{\;\;j_1\dots j_l} \left(\delta^{kj_1} X_l^{hj_2\dots j_l} - \delta^{hj_1} X_l^{kj_2\dots j_l}\right). \tag{32}$$

Proposition 3 The completely symmetric and trace-free polynomials

$$T_l^{i_1 i_2 \dots i_l} := \frac{1}{r^l} X_l^{i_1 i_2 \dots i_l} = \mathcal{P}^{l\, i_1 \dots i_l}_{\;\;j_1 \dots j_l} t^{j_1} \dots t^{j_l} \tag{33}$$

make up a complete set $\mathcal{T}_l$ in V_D^l (but not a basis: $\delta_{i_1 i_2} T_l^{i_1 \dots i_l} = 0$, etc.). The actions of the operators $\mathbf{L}^2$, iL_{hk}, $t^h\cdot$ on the $T_l^{i_1 \dots i_l}$ explicitly read

$$\mathbf{L}^2\, T_l^{i_1 \dots i_l} = l(l+D-2)\, T_l^{i_1 \dots i_l}, \tag{34}$$

$$iL_{hk}\, T_l^{i_1 \dots i_l} \;\; = \;\; l\, \mathcal{P}^{l\, i_1 \dots i_l}_{\;\;j_1 \dots j_l} \left(\delta^{kj_1}\, T_l^{hj_2 \dots j_l} - \delta^{hj_1}\, T_l^{kj_2 \dots j_l}\right). \tag{35}$$

$$t^h\, T_l^{i_1 \dots i_l} = T_{l+1}^{hi_1 \dots i_l} + \frac{l}{D+2l-2}\, \mathcal{P}^{l\, i_1 i_2 \dots i_l}_{\;\;hj_2 \dots j_l}\, T_{l-1}^{j_2 \dots j_l} \;\in\; V_D^{l+1} \oplus V_D^{l-1}, \tag{36}$$

$$t^i\, T_l^{ii_2 \dots i_l} = \frac{1}{D+2l-2} \left[D+l-1-\frac{2l-2}{D+2l-4}\right] T_{l-1}^{i_2 \dots i_l} \;\in\; V_D^{l-1}, \tag{37}$$

Embedding $\mathbb{R}^D \hookrightarrow \mathbb{R}^{D+1}$, *isomorphism* $Pol_D^\Lambda \simeq V_{D+1}^\Lambda$

Abbreviate $\mathbf{D} \equiv D+1$. To avoid confusion, in dimension $\mathbf{D}$ we add it as a subscript , e.g. the distance $r_\mathbf{D}$ from the origin in $\mathbb{R}^\mathbf{D}$, vs. $r \equiv r_D$ in $\mathbb{R}^D$. Henceforth (x^i), (x^I) stand resp. for real Cartesian coordinates of $\mathbb{R}^D, \mathbb{R}^\mathbf{D}$: $h,i,j \in \{1,...,D\}$, $H,I,J \in \{1,...,\mathbf{D}\}$; we naturally embed $\mathbb{C}[\mathbb{R}^D] \hookrightarrow \mathbb{C}[\mathbb{R}^\mathbf{D}]$. We naturally embed $O(D) \hookrightarrow SO(\mathbf{D})$ identifying the subgroup $O(D)$ as the little group of the $\mathbf{D}$-th axis; its Lie algebra is generated by the L_{hk}.

$$T_{\mathbf{D},\Lambda}^{I_1...I_\Lambda} := \mathcal{P}^{\Lambda\, I_1...I_\Lambda}_{\mathbf{D}\, J_1...J_\Lambda}\, t^{J_1}...t^{J_\Lambda} \in V_\mathbf{D}^\Lambda \tag{38}$$

make up a complete set in $V_\mathbf{D}^\Lambda$. Defining

$$F_{\mathbf{D},\Lambda}^{i_1...i_l} := \mathcal{P}^{l\, i_1...i_l}_{\ \ j_1...j_l}\, T_{\mathbf{D},\Lambda}^{j_1...j_l\mathbf{D}...\mathbf{D}}, \tag{39}$$

one finds that these factorize as

$$F_{\mathbf{D},\Lambda}^{i_1...i_l} = p_{\Lambda,l}\left(t^\mathbf{D}\right)\ T_l^{i_1...i_l}, \tag{40}$$

$$p_{\Lambda,l} = \left(t^\mathbf{D}\right)^{\Lambda-l} + \left(t^\mathbf{D}\right)^{\Lambda-l-2} b_{\Lambda,l+2} + \left(t^\mathbf{D}\right)^{\Lambda-l-4} b_{\Lambda,l+4} + ... \tag{41}$$

Proposition 4 $V_{\mathbf{D}}^{\Lambda}$ decomposes into the following irreps under $Uso(D)$:

$$V_{\mathbf{D}}^{\Lambda} = \bigoplus_{l=0}^{\Lambda} V_{D,\Lambda}^{l}, \tag{42}$$

where $V_{D,\Lambda}^{l} \simeq V_{D}^{l}$ is spanned by the $F_{\mathbf{D},\Lambda}^{i_1\dots i_l}$. These are eigenvectors of $\mathbf{L}^2$,

$$\mathbf{L}^2 F_{\mathbf{D},\Lambda}^{i_1\dots i_l} = l(l+D-2) F_{\mathbf{D},\Lambda}^{i_1\dots i_l}, \tag{43}$$

transform under L_{hk} as the $T_l^{i_1\dots i_l}$, and under $L_{h\mathbf{D}}$ as follows:

$$iL_{h\mathbf{D}} F_{\mathbf{D},\Lambda}^{i_1\dots i_l} = (\Lambda - l)\, F_{\mathbf{D},\Lambda}^{hi_1\dots i_l} - \frac{l(\Lambda+l+D-2)}{D+2l-2}\, \mathcal{P}^{l\,i_1 i_2\dots i_l}_{\;\;hj_2\dots j_l}\, F_{\mathbf{D},\Lambda}^{j_2\dots j_l}\,. \tag{44}$$

Action of $\overline{x}^i, \overline{L}_{hk}$, and relations among them

$$\psi_l^{i_1 i_2 \dots i_l} := T_l^{i_1 i_2 \dots i_l} f_l(r) \qquad (45)$$

($i_h \in \{1, ..., D\}$ for $h \in \{1, ..., l\}$) make up a complete set $\mathcal{S}^l_{D,\Lambda}$ in the eigenspace $\mathcal{H}^l_\Lambda$ of H, $\mathbf{L}^2$, with eigenvalues E_l, $l(l+D-2)$. They are completely symmetric under permutation of the indices and fulfill

$$\delta_{i_n i_{n+1}} \psi_l^{i_1 \dots i_l} = 0, \qquad n = 1, ..., l-1. \qquad (46)$$

$\mathcal{S}_{D,\Lambda} := \cup_{l=0}^{\Lambda} \mathcal{S}^l_{D,\Lambda}$ is a complete set in $\mathcal{H}_\Lambda$. By (45), (35), (14), (36),

$$i\overline{L}_{hk}\, \psi_l^{i_1 i_2 \dots i_l} = l\, \mathcal{P}^{l\, i_1 \dots i_l}_{\ \ j_1 \dots j_l} \left(\delta^{kj_1} \psi_l^{hj_2 \dots j_l} - \delta^{hj_1} \psi_l^{kj_2 \dots j_l} \right). \qquad (47)$$

$$\overline{x}^i\, \psi_l^{i_1 i_2 \dots i_l} = c_{l+1}\, \psi_{l+1}^{ii_1 \dots i_l} + \frac{c_l\, l}{D+2l-2}\, \mathcal{P}^{l\, i_1 i_2 \dots i_l}_{\ \ ij_2 \dots j_l}\, \psi_{l-1}^{j_2 \dots j_l}. \qquad (48)$$

where $\quad c_l \quad := \begin{cases} \sqrt{1 + \frac{(2D-5)(D-1)}{2k} + \frac{(l-1)(l+D-2)}{k}} & \text{if } 1 \leq l \leq \Lambda, \\ 0 & \text{otherwise,} \end{cases}$

up to $O\left(k^{-3/2}\right)$ corrections that can be set to equal to zero by a suitable choice of V. Henceforth we adopt (47-48) as exact *definitions* of $\overline{L}_{hk}, \overline{x}^i$.

Proposition. *The $\overline{x}^i, \overline{L}_{hk} \in \mathcal{A}_\Lambda$ are self-adjoint operators fulfilling*

$$[i\overline{L}_{ij}, \overline{x}^h] = \overline{x}^i \delta^h_j - \overline{x}^j \delta^h_i, \qquad \varepsilon^{i_1 i_2 i_3 \ldots i_D} \overline{x}^{i_1} \overline{L}_{i_2 i_3} = 0, \tag{49}$$

$$[i\overline{L}_{ij}, i\overline{L}_{hk}] = i\left(\overline{L}_{ik}\delta^j_h - \overline{L}_{jk}\delta^i_h - \overline{L}_{ih}\delta^j_k + \overline{L}_{jh}\delta^i_k\right), \tag{50}$$

$$(\overline{x}^h \pm i\overline{x}^k)^{2\Lambda+1} = 0, \quad (\overline{L}^{hj} + i\overline{L}^{kj})^{2\Lambda+1} = 0, \quad \textit{if } h, i, j \textit{ distinct}, \tag{51}$$

$$\left[\overline{x}^i, \overline{x}^j\right] = \underbrace{\left(-\frac{I}{k} + K\widetilde{P}^\Lambda\right) i\overline{L}_{ij}}_{Snyder-like}, \qquad K := \frac{1}{k} + \frac{1+B/k+(\Lambda-1)(\Lambda+D-2)/k}{D+2\Lambda-2}, \tag{52}$$

$$\overline{\mathbf{x}}^2 = 1 + \frac{\overline{\mathbf{L}}^2}{k} + \frac{B}{k} - \frac{\Lambda+D-2}{2\Lambda+D-2}\left[1 + \frac{B}{k} + \frac{\Lambda(\Lambda+d)}{k}\right]\widetilde{P}^\Lambda =: \chi(\mathbf{L}^2). \tag{53}$$

here $\widetilde{P}^l \equiv$ projection on $\mathbf{L}^2 = l(l+D-2)$ eigenspace, $B := d(2d-3)/2$.

Generalizes Proposition 4.1 in [GF, F. Pisacane18]. We obtain S^d_Λ choosing $k = k(\Lambda)$ respecting (**??**); the commutative limit is $\Lambda \to \infty$.

We remark that:

- $[\overline{x}^i, \overline{x}^j]$ depend only on the angular momentum (also $\widetilde{P}^\Lambda$ can be expressed as a function of $\overline{\mathbf{L}}^2$) and are Snyder-like, i.e. are $\propto \overline{L}_{ij}$.
- The ordered monomials in $\overline{x^i}, \overline{L}_{hk}$ make up a basis of the N^2-dim vector space $\mathcal{A}_\Lambda$: also the $\widetilde{P}^l$ can be expressed as polynomials in $\overline{\mathbf{L}}^2$.
- Actually, $\overline{x}^i$ *generate* the $*$-algebra $\mathcal{A}_\Lambda \equiv End(\mathcal{H}_\Lambda)$: express also $\overline{L}_{hk}$ as non-ordered polynomials in $\overline{x}^i$ via (52).
- Eq. (49-53) are fully $O(D)$-equivariant, also under one or more $\overline{x}^i \mapsto -\overline{x}^i$ (e.g. parity), contrary to Madore-Hoppe FS.
- $\overline{\mathbf{x}}^2 = \chi(\overline{\mathbf{L}}^2) \neq$ const, eigenspaces $\mathcal{H}_\Lambda^l$. All eigenvalues r_l^2, except r_Λ^2, are close to 1, slightly grow with l and collapse to 1 as $\Lambda \to \infty$.

We slightly enlarge $Uso(D)$ by introducing the new generator

$$\lambda = \frac{1}{2}\left[\sqrt{(D-2)^2 + 4\mathbf{L}^2} - D + 2\right] \tag{54}$$

fulfilling $\lambda(\lambda+D-2) = \mathbf{L}^2$, so that V_D^l is a $\lambda = l$ eigenspace.

Theorem. There are a $O(D)$-module isomorphism $\varkappa_\Lambda : \mathcal{H}_\Lambda \to V_{\mathbf{D}}^\Lambda$ and a $O(D)$-equivariant algebra map $\kappa_\Lambda : \mathcal{A}_\Lambda \equiv \mathrm{End}(\mathcal{H}_\Lambda) \to \boldsymbol{\pi}_{\mathbf{D}}^\Lambda\big[Uso(\mathbf{D})\big]$, s. t.

$$\varkappa_\Lambda(a\psi) = \kappa_\Lambda(a)\varkappa_\Lambda(\psi), \qquad \forall\, \psi \in \mathcal{H}_\Lambda, \quad a \in \mathcal{A}_\Lambda\,. \tag{55}$$

On the $\psi_l^{i_1\dots i_l}$ (spanning $\mathcal{H}_\Lambda$) and on generators $L_{hi}, \overline{x}^i\cdot$ of $\mathcal{A}_\Lambda$:

$$\varkappa_\Lambda\left(\psi_l^{i_1\dots i_l}\right) := a_{\Lambda,l} F_{\mathbf{D},\Lambda}^{i_1\dots i_l} = a_{\Lambda,l}\, p_{\Lambda,l}\, T_l^{i_1\dots i_l}, \quad l = 0,1,...,\Lambda, \tag{56}$$

$$\kappa_\Lambda\left(\overline{L}_{hi}\right) := \boldsymbol{\pi}_{\mathbf{D}}^\Lambda(L_{hi})\,, \qquad \kappa_\Lambda\left(\overline{x}^i\right) := \boldsymbol{\pi}_{\mathbf{D}}^\Lambda\big[m_\Lambda^*(\lambda) L_{\mathbf{D}i}\, m_\Lambda(\lambda)\big]\,, \tag{57}$$

where $p_{\Lambda,l}$ are the polynomials in $t^{\mathbf{D}}$ of degree $\Lambda-l$ given in (41), and

$$a_{\Lambda,l} = a_{\Lambda,0}\; i^l \sqrt{\frac{\Lambda(\Lambda-1)...(\Lambda-l+1)}{(\Lambda+D-1)(\Lambda+D)...(\Lambda+l+D-2)}},$$

$$m_\Lambda(s) = \sqrt{\frac{\Gamma\left(\frac{\Lambda+s+d}{2}\right)\,\Gamma\left(\frac{\Lambda-s+1}{2}\right)\,\Gamma\left(\frac{s+1+d/2+iA}{2}\right)\,\Gamma\left(\frac{s+1+d/2-iA}{2}\right)}{\Gamma\left(\frac{\Lambda+s+D}{2}\right)\,\Gamma\left(\frac{\Lambda-s}{2}+1\right)\,\Gamma\left(\frac{s+d/2+iA}{2}\right)\,\Gamma\left(\frac{s+d/2-iA}{2}\right)\,\sqrt{k}}};$$

here $A := \sqrt{k+(D-1)(D-3)3/4}$, and Γ is Euler gamma function.

As known, the group of $*$-automorphisms of $\mathcal{A}_\Lambda \simeq M_N(\mathbb{C})$ is

$$b \to gbg^{-1}, \qquad b \in \mathcal{A}_\Lambda, \quad g \in SU(N).$$

We can identify a subgroup $\simeq SO(\mathbf{D})$, acting via the N-dim irrep $(V_{\mathbf{D}}^\Lambda, \boldsymbol{\pi}_{\mathbf{D}}^\Lambda)$; it consists of matrices $g = \boldsymbol{\pi}_{\mathbf{D}}^\Lambda \left[e^{i\alpha}\right]$, where $\alpha \in so(\mathbf{D})$.

$O(D) \subset SO(\mathbf{D}) \subset SU(N)$ plays the role of isometry subgroup of our fuzzy sphere. It includes parity.

$SO(D)$ (rotations of $\mathbb{R}^D$): choosing $\alpha = \alpha^{ij} L_{ij} \in so(D)$;
$O(D)$ with determinant -1, (e.g. inverting one axis of $\mathbb{R}^D$):
$\alpha = \alpha^{ij} L_{ij} + \beta^i L_{i\mathbf{D}}$.

Identifying $\mathcal{A}_\Lambda$ as a fuzzy quantization of a coadjoint orbit of $O(D+1)$

If $G =$ compact semisimple Lie group, and $\mathfrak{g}\equiv$Lie(G), $\boldsymbol{\lambda} \in \mathfrak{g}^* \simeq \mathfrak{g}$, the coadjoint orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ may be defined as

$$\mathcal{O}_{\boldsymbol{\lambda}} := G/G_{\boldsymbol{\lambda}} \quad \text{where} \quad G_{\boldsymbol{\lambda}} := \left\{g \in G \mid g\boldsymbol{\lambda}g^{-1} = \boldsymbol{\lambda}\right\}. \tag{58}$$

Clearly $G_{\Lambda\boldsymbol{\lambda}} = G_{\boldsymbol{\lambda}}$ if $\Lambda \neq 0$. If $\mathcal{H}_{\boldsymbol{\lambda}}$ carries irrep with highest weight $\boldsymbol{\lambda}$, setting $\mathcal{A}_\Lambda \equiv \text{End}\left(\mathcal{H}_{\Lambda\boldsymbol{\lambda}}\right)$, one can regard [Hawkins 99] the sequence $\{\mathcal{A}_\Lambda\}_{\Lambda\in\mathbb{N}}$ as a fuzzy quantization of the symplectic space $\mathcal{O}_{\boldsymbol{\lambda}}$.

Set $G = SO(\mathbf{D})$. Cartan subalgebra $\mathfrak{h} \subset so(\mathbf{D})$ with basis

$$H_\sigma \equiv L_{D\mathbf{D}}, \quad H_{\sigma-1} \equiv L_{(d-1)d}, \quad ..., \quad H_1 = \begin{cases} L_{12} & \text{if } \mathbf{D} = 2\sigma, \\ L_{23} & \text{if } \mathbf{D} = 2\sigma+1. \end{cases} \tag{59}$$

As a highest weight vector of the irrep $\boldsymbol{\pi}_{\mathbf{D}}^{\Lambda}$ of $Uso(\mathbf{D})$ on $V_{\mathbf{D}}^{\Lambda} \simeq \mathcal{H}_\Lambda$ we choose $\Omega_{\mathbf{D}}^{\Lambda} \equiv (t^D + it^{\mathbf{D}})^\Lambda$. The associated weight, i.e. the joint spectrum of $H \equiv (H_1, ..., H_\sigma)$, is $\Lambda\boldsymbol{\lambda}$, where $\boldsymbol{\lambda} = (0, ..., 0, 1) \in \mathfrak{h}^*$. Identifying $\boldsymbol{\lambda}$ with $H_{\boldsymbol{\lambda}} \propto H_\sigma \in \mathfrak{h}$ via the Killing form, we find $G_{\boldsymbol{\lambda}} = SO(2) \times SO(d)$, with $so(2)$ spanned by H_σ, $so(d)$ spanned by the L_{ij} with $i, j < D$. Hence $\mathcal{O}_{\boldsymbol{\lambda}} = SO(\mathbf{D})/\big(SO(2) \times SO(d)\big)$ has dimension

$$\frac{D(D+1)}{2} - 1 - \frac{d(d-1)}{2} = 2d,$$

exactly as the dimension of the cotangent space T^*S^d, i.e. the classical phase space over the sphere S^d! Consistent with interpreting $\mathcal{A}_\Lambda$ as the algebra of observables (quantized phase space) on the fuzzy sphere S^d_Λ.

By other generic irrep of $Uso(\mathbf{D})$ the coadjoint orbit would have been some other equivariant bundle over S^d.

Discussion and conclusions

We have built a sequence $(\mathcal{A}_\Lambda, \mathcal{H}_\Lambda)_{\Lambda\in\mathbb{N}}$ of finite-dim, $O(D)$-equivariant $(D=d+1)$ approximations of QM of a zero-spin particle on the sphere S^d by imposing $E \leq \Lambda(\Lambda+d-1)$ on QM of the particle in $\mathbb{R}^D$ subject to a potential $V_\Lambda(r)$ with a sharp minimum on S^d.

$\mathcal{A}_\Lambda$ are fuzzy approximations of the *whole algebra of observables* of the particle on S^d (phase space algebra); $\mathbf{x}^2 \gtrsim 1$ collapses to 1 as $\Lambda \to \infty$.

$\mathcal{A}_\Lambda \simeq \pi_{\mathbf{D}}^\Lambda[Uso(\mathbf{D})]$, with the irrep $V_{\mathbf{D}}^\Lambda, \pi_{\mathbf{D}}^\Lambda$ of $Uso(\mathbf{D})$, $\mathbf{D} \equiv D+1$. Can be regarded as a fuzzy quantization of a coadjoint orbit of $O(\mathbf{D}) \simeq T^*S^d$.

$\mathcal{H}_\Lambda \simeq V_{\mathbf{D}}^\Lambda$ and the subspace $\mathcal{C}_\Lambda \subset \mathcal{A}_\Lambda$ of trace-free, completely symmetrized polynomials in the $\overline{x}^i$ carry *reducible* representations of $O(D)$:

$$\mathcal{H}_\Lambda = \bigoplus_{l=0}^{\Lambda} \mathcal{H}_\Lambda^l \simeq V_{\mathbf{D}}^\Lambda = \bigoplus_{l=0}^{\Lambda} V_D^l, \qquad \mathcal{C}_\Lambda \simeq \bigoplus_{l=0}^{2\Lambda} V_D^l \tag{60}$$

As $\Lambda \to \infty$ these become the decompositions of $\mathcal{L}^2(S^d)$ and of $C(S^d)$,

$$\mathcal{L}^2(S^d) = \bigoplus_{l=0}^{\infty} V_D^l, \qquad C(S^d) = \bigoplus_{l=0}^{\infty} V_D^l. \tag{61}$$

S^2_Λ vs. Madore-Hoppe Fuzzy Sphere S^2_n (seminal fuzzy space):
$\mathcal{A}_n \simeq M_n(\mathbb{C})$, generated by coordinates x^i $(i=1,2,3)$ fulfilling

$$[x^i, x^j] = \frac{2i}{\sqrt{n^2-1}}\varepsilon^{ijk}x^k, \quad r^2 := x^i x^i = 1, \qquad n \in \mathbb{N}\setminus\{1\}; \qquad (62)$$

(62) are $SO(3)$-equivariant, not $O(3)$-, e.g. not under parity $x^i \mapsto -x^i$.
In fact $L^i = x^i\sqrt{n^2-1}/2$ make up the standard basis of $so(3)$ in the irrep (π_3^l, V_3^l) characterized by $L^iL^i = l(l+1)$, $n = 2l+1$.
Moreover, the irreducible carrier space V_3^l does not go to $\mathcal{L}^2(S^2)$, cf (60).

S^2_Λ vs. fuzzy S^d with $d=4$ [Grosse, Klimcik, Presnajder 1996], $d \geq 3$ [Ramgoolam 2001, Dolan, O'Connor 2003, ...]: based on $End(V)$ where V carries an irrep of $Uso(\mathbf{D})$ that is also an *irrep* of $Uso(D)$.

$X^i \equiv L_{i\mathbf{D}}$ play the role of fuzzy Cartesian coordinates.

$\mathbf{X}^2 := X^iX^i = \mathbf{L}^2_{\mathbf{D}} - \mathbf{L}^2$ is central, can be set=1.

Also Snyder-like commutation relations, hence $O(D)$-equivariant.
However, the irreducible carrier space V_D^l does not go to $\mathcal{L}^2(S^d)$, cf (60).

In [Steinacker et al. 2016-17] fuzzy 4-spheres S^4_N through reducible repr. of $Uso(5)$ obtained decomposing irreps $\rho_{N,0,n}$ of $Uso(6)$ with suitable highest weights $(N, 0, n)$; so $End(V) \simeq \rho_{N,0,n}[Uso(6)]$. Again the X^i play the role of fuzzy Cartesian coordinates.
The $O(5)$-scalar $\mathbf{X}^2 = X^i X^i$ is no longer central, but its spectrum is still very close to 1 *only if* $N \gg n$, in analogy with our result.
Again, the decomposition of V into irreps of $Uso(D)$ does not go to (60).
If $n = 0$ then $\mathbf{X}^2 \equiv 1$ ($\Rightarrow$ irrep), and one recovers the fuzzy 4-sphere of [Grosse, Klimcik, Presnajder 1996].

$\mathbf{x}^2 \simeq 1$ in our S^d_Λ is guaranteed by adopting $x^i = g(L^2) X^i g(L^2)$ rather than X^i as noncommutative cartesian coordinates, and $\mathbf{x}^2 = x^i x^i$.

Many directions for future investigations. In particular, it will be interesting to find optimally localized and coherent states on these S^d_Λ for general d, mimicking what has been done for $d = 1, 2$ [GF, F. Pisacane 2020]. More importantly, use them for QFT.

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