On supersymmetric sigma models and the AKSZ construction

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Goal

- Study the super AKSZ formalism using integral forms.
  inspiration: [Grassi–Maccaferri '16]

- Derive the 2-dimensional $\mathcal{N} = (1,1)$ sigma model on a boundary of a topological model, following Ševera '16.

AKSZ formalism

[Aleksandrov, Kontsevich, Schwarz, Zaboronski '95] [Roytenberg '06]

Ingredients

- ordinary oriented 3-dimensional manifold M
- dg symplectic manifold X of degree 2:
  - space described using coordinates y^a with some $\mathbb{Z}$-degree

$$y^a y^b = (-1)^{\deg y^a \cdot \deg y^b} y^b y^a$$

  - degree 2 symplectic form $\omega = d\alpha$, $\alpha = \alpha_a(y) dy^a$
  - degree 3 function $\Theta(y)$ satisfying $\{\Theta, \Theta\} = 0$

Classical field theory in the Batalin–Vilkovisky formulation

- Fields: differential forms y^a on M

$$S(y) = \int_M \alpha_a(y) dy^a - \Theta(y) \qquad \{S, S\} = 0$$

Example

Data:

- Lie algebra $\mathfrak{g}$ with an invariant inner product
  - choosing a basis e_a we have $[e_a, e_b] = f^c{}_{ab} e_c$ and $\langle e_a, e_b \rangle = t_{ab}$

dg symplectic manifold

- $X = \mathfrak{g}[1]$ with coordinates e^a od degree 1 and

$$\omega = \tfrac{1}{2} t_{ab}\, de^a de^b, \qquad \Theta = -\tfrac{1}{6} f_{abc} e^a e^b e^c$$

Resulting theory

- Chern–Simons in its full BV description

$$S(e) = \int_M \tfrac{1}{2} t_{ab} e^a de^b + \tfrac{1}{6} f_{abc} e^a e^b e^c, \qquad e = c + A + A^+ + c^+$$

Supermanifolds and integral forms

If M is a supermanifold (of dimension $1|1$):

- Coordinates: σ (even), ϑ (odd)
- Differential forms: generated by $d\sigma$ (odd), $d\vartheta$ (even)
- No integration of differential forms: $(d\vartheta)^2 \neq 0$

Integral forms [Bernstein, Leites '77]

- Distribution in σ, ϑ, $d\sigma$, $d\vartheta$, supported at $d\vartheta = 0$

$$\mu = \underbrace{a(\sigma,\vartheta)d\sigma\delta(d\vartheta)}_{\text{codimension } 0} + \underbrace{b(\sigma,\vartheta)\delta(d\vartheta) + c(\sigma,\vartheta)d\sigma\delta'(d\vartheta)}_{\text{codimension } 1} + \dots$$

- differential form $\times$ integral form $=$ integral form
- Note that $d\vartheta\,\delta(d\vartheta) = 0$, $d\vartheta\,\delta'(d\vartheta) = -\delta(d\vartheta)$, ...

$$\int_M \mu = \int_{\mathrm{Ber}} a(\sigma,\vartheta)$$

Towards the $\mathcal{N} = (1,1)$ model, following [Ševera '16]

Target (dg symplectic manifold)

exact Courant algebroid with generalised metric R

$X = T^*[2]T[1]N$, $\Theta = d - H$ (H closed 3-form on N)

x^i on $N \rightsquigarrow \underbrace{x^i}_{\deg 0}, \underbrace{\xi^i}_{\deg 1}, \underbrace{\pi_i}_{\deg 1}, \underbrace{p_i}_{\deg 2}$ on X

$$\omega = dp_i dx^i + d\pi_i d\xi^i, \qquad \Theta = p_i \xi^i - \tfrac{1}{6} H_{ijk}(x)\xi^i \xi^j \xi^k$$

$$R\colon \qquad x'^i = x^i \qquad \xi'^i = g^{ij}(x)\pi_j \qquad \pi'_i = g_{ij}(x)\xi^j \qquad p'_i = p_i$$

Source

M of dimension $3|2$, with $d\mu = 0$ and $\partial M =$ superstring worldsheet

∂M: σ, $\bar\sigma$ (even), ϑ, $\bar\vartheta$ (odd), $D = \frac{\partial}{\partial\vartheta} + \vartheta\frac{\partial}{\partial\sigma}$, $\bar D = \frac{\partial}{\partial\bar\vartheta} + \bar\vartheta\frac{\partial}{\partial\bar\sigma}$

canonical integral form $\mu_\partial = (d\sigma - \vartheta d\vartheta)(d\bar\sigma - \bar\vartheta d\bar\vartheta)\delta'(d\vartheta)\delta'(d\bar\vartheta)$

$$\star\colon \qquad d\sigma \mapsto d\sigma \qquad d\vartheta \mapsto d\vartheta \qquad d\bar\sigma \mapsto -d\bar\sigma \qquad d\bar\vartheta \mapsto -d\bar\vartheta$$

Towards the $\mathcal{N} = (1,1)$ model

Boundary conditions: ghostless and given by generalised metric and $\star$

$$
\begin{aligned}
x^i &= \bar{x}^i + x_1^i + x_2^i + x_3^i + \dots \\
\xi^i &= \qquad \bar{\xi}^i + \xi_2^i + \xi_3^i + \dots \\
\pi_i &= \qquad \bar{\pi}_i + \pi_{2i} + \pi_{3i} + \dots \\
p_i &= \qquad\qquad \bar{p}_i + p_{3i} + \dots
\end{aligned}
$$

$\bar{x}^i$ arbitrary $\qquad \bar{\pi}_i = \star g_{ij}(\bar{x})\bar{\xi}^j \qquad \bar{\xi}^i = \star g^{ij}(\bar{x})\bar{\pi}_j \qquad \bar{p}_i = \dots$

Action:

$$
S = \int_M \mu \left(p_i dx^i + \tfrac{1}{2}\pi_i d\xi^i + \tfrac{1}{2}\xi^i d\pi_i - p_i\xi^i + \tfrac{1}{6}H_{ijk}(x)\xi^i\xi^j\xi^k\right)
$$

The $\mathcal{N} = (1,1)$ model

Integration by parts

$$S = \int_M \mu \left(p_i(dx^i - \xi^i) + \pi_i d\xi^i + \tfrac{1}{6} H_{ijk}(x)\xi^i\xi^j\xi^k\right) + \tfrac{1}{2}\int_{\partial M} \mu_\partial \, \pi_i \xi^i$$

Integrating over p_i in the bulk: $\quad \xi^i = dx^i + \Xi, \qquad \mu\,\Xi = 0$

Applying boundary condition

$$S(\bar{x}) = \int_{\partial M} \mu_\partial \, g_{ij}(\bar{x}) d\bar{x}^i \star d\bar{x}^j + \tfrac{1}{6}\int_M \mu \, H_{ijk}(\bar{x}) d\bar{x}^i d\bar{x}^j d\bar{x}^k$$

$$\boxed{S(y) = \int_{\mathrm{Ber}(\partial M)} g_{ij}(y) D y^i \bar{D} y^j + \int_M \mu \, y^* H \qquad y\colon \partial M \to (N, g, H)}$$

if $H = dB \quad \rightsquigarrow \quad S(y) = \int_{\mathrm{Ber}(\partial M)} (g_{ij} + B_{ij}) D y^i \bar{D} y^j$ [Gates, Hull, Roček '84]