

# Strong homotopy algebras in conformal field theory and higher spin gravity

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### Main message: higher spin symmetry is new Virasoro



- Is there any symmetry behind (Chern-Simons) vector models like Ising model? **Slightly-broken higher spin symmetry** (Maldacena, Zhiboedov) is clearly not a usual Lie-type symmetry ...
- The structure behind is  $L_\infty$  strong homotopy algebra, behind it is a simple associative algebra of fuzzy sphere. Invariants = correlators and are uniquely fixed, which implies the **3d bosonization duality**
- On a different note: there has to be a closed subsector of vector models (including the Ising), which has a local  $AdS_4$  description as Chiral Higher Spin Gravity, yet unknown formality ...

## Symmetry

99.99%: **Lie group**  $G$ /**Lie algebra**  $\mathfrak{g}$  acting on some physical states.  
Group/Algebra = transformations without any info on what they act

**Yangian**: deformation of  $U(\mathfrak{g}[z])$  as a Hopf algebra. Spin-chains, planar  $\mathcal{N} = 4$  SYM and scattering amplitudes therein

**Strong homotopy algebras**: multi-linear products on graded spaces (Lie and associative algebras are examples). Nice organizing tool for  $QQ = 0$ : BV-BRST, string field theory, higher spin gravities, ...

**New**: (Chern-Simons) vector models (e.g.  $3d$  Ising, ...) have  $\infty$ -many almost conserved tensors  $\partial^m J_{ma_2 \dots a_s} \approx 0$  — **slightly-broken higher spin symmetry** (Maldacena, Zhiboedov). The right structure are certain  $L_\infty$ -algebras. **Symmetry gets entangled with its representation**. Explains  $3d$ -bosonization duality

- Higher spin symmetry: from canonical QFT/CFT to algebraic viewpoint on free/very large- $N$  CFT's
- Chern-Simons vector models and bosonization duality
- Slightly-broken higher spin symmetry as  $L_\infty$
- DQ of Poisson orbifolds, fuzzy sphere
- $3d$  bosonization duality via  $L_\infty$  glasses
- Chiral Higher Spin Gravity and yet unknown formality

Higher spin symmetry

## Unbroken higher spin symmetry: free CFT's

Let's take any free CFT, e.g. free boson or free fermion

$$\square\phi = 0$$

$$\not{\partial}\psi = 0$$

We find the stress-tensor  $J_{ab}$  and infinitely many (even spin) *higher spin conserved tensors*  $J_{a_1\dots a_s}$  (aka **higher spin currents**, old name — Zilch):

$$J_s = \phi\partial\dots\partial\phi + \dots$$

$$J_s = \bar{\psi}\gamma\partial\dots\partial\psi + \dots$$

They are quasi-primary at the unitarity bound and have  $\Delta = d + s - 2$ .

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Stress-tensor is responsible for conformal symmetry  $so(d, 2)$

$$Q_v = \int d^{d-1}x J_{0m}(x)v^m(x) \quad \partial^n v^m + \partial^m v^n \sim \eta^{mn}$$

where conformal killing vector carries adjoint of  $so(d, 2)$

$$v^a = \epsilon^a + x^a\epsilon + \epsilon^{a,b}x_b + \tilde{\epsilon}^a x^2 - 2x^a(\tilde{\epsilon} \cdot x)$$

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**What are higher spin currents responsible for?**



## Unbroken higher spin symmetry

Any symmetry is certainly useful unless too much ... 

Imagine a CFT  $d \geq 3$  with  $J_2 \equiv T_{ab}$  and  $J_s \equiv J_{a_1 \dots a_s}$ , all being traceless and conserved. Is it interesting?

One can show (Maldacena, Zhiboedov; Boulanger, Ponomarev, E.S, Taronna; Alba, Diab) that there are  $J_s$  with arbitrarily high spin (at least all even spins), and the correlators are

$$\langle J \dots J \rangle = \text{some free CFT}$$

In  $3d$  there are two choices: free boson  $\square\phi = 0$  and free fermion  $\not{\partial}\psi = 0$ .  
Note: they have different correlators of  $J$ !

**When something is completely fixed, usually it is thanks to some symmetry. What is the symmetry behind?**

**Conserved tensor**  $\rightarrow$  **current**  $\rightarrow$  **charge**  $\rightarrow$  **symmetry**

$$j_m(v) = J_{ma_2 \dots a_s} v^{a_2 \dots a_s} \quad \partial^{(a_1} v^{a_2 \dots a_s)} = \eta^{(a_1 a_2} u^{a_3 \dots a_s)}$$

where  $v^{a_1 \dots a_{s-1}}$  is a conformal Killing tensor,  $\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \dots \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$  of  $so(d, 2)$

**Conserved tensor**  $\rightarrow$  **current**  $\rightarrow$  **charge**  $\rightarrow$  **higher-spin symmetry**

$$j_m(v) = J_{ma_2\dots a_s} v^{a_2\dots a_s} \quad \partial^{(a_1} v^{a_2\dots a_s)} = \eta^{(a_1 a_2} u^{a_3\dots a_s)}$$

where  $v^{a_1\dots a_{s-1}}$  is a conformal Killing tensor.

**Miracle 1:** the Lie algebra  $Q_s = \int J_s$  originates from an associative one

**Free CFT = Associative algebra**

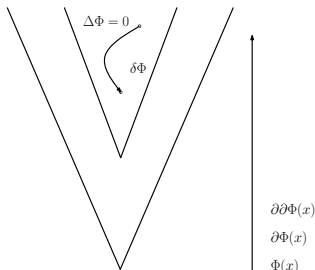
Indeed,  $\square\Phi = 0$  is  $so(d, 2)$ -invariant:

$$\delta_v\Phi = v^m \partial_m \Phi + \frac{d-2}{2d} (\partial_m v^m) \Phi$$

We can multiply such symmetries due to linearity of  $\square\Phi = 0$

$$\delta\Phi = \delta_{v_1\dots v_n}\Phi = \epsilon^{a_1\dots a_k} \partial_{a_1}\dots\partial_{a_k}\Phi + \dots$$

# Unbroken higher spin symmetry



Define  $|V\rangle$  as the space of one-particle states  $P_a \dots P_c |\phi\rangle$ , where  $|\phi\rangle \equiv \phi(0)|0\rangle$

Higher spin algebra  $\mathfrak{hs}$  is  $\mathfrak{gl}(V): V \rightarrow V$

$\mathfrak{hs}$  is DQ of the  $so(d, 2)$  coadjoint orbit  $\phi$

Higher spin currents are bilinear in  $\phi$  or  $\psi$

**Miracle 2:**  $J \leftrightarrow \mathfrak{hs}$  upon identifying  $|\phi\rangle|\phi\rangle$  with  $|\phi\rangle\langle\phi|$  by inversion  $R$

$$J \sim |V\rangle \otimes |V\rangle \quad \Longleftrightarrow \quad \mathfrak{hs} \sim |V\rangle \otimes \langle V|$$

## Unbroken higher spin symmetry: $3d$ specifics

In  $3d$  the module of one-particle states of free boson/fermion CFT's is just the  $2d$  harmonic oscillator (Dirac, 1963):

$$\begin{aligned}P_a \dots P_a |\phi\rangle &\sim a_\alpha^\dagger a_\beta^\dagger \dots a_\alpha^\dagger a_\beta^\dagger |0\rangle \\P_a \dots P_a |\psi\rangle &\sim a_\alpha^\dagger a_\beta^\dagger \dots a_\alpha^\dagger a_\beta^\dagger \mathbf{a}_\gamma^\dagger |0\rangle\end{aligned}$$

This is thanks to  $so(3, 2) \sim sp(4, \mathbb{R})$  and thanks to the oscillator realization of  $sp(2n)$ , e.g.  $P_m P^m \sim 0$ ,  $P_m = \sigma_m^{\alpha\beta} a_\alpha^\dagger a_\beta^\dagger$ ,  $\alpha, \beta, \dots = 1, 2$

Now it is obvious that  $\mathfrak{hs}$  is formed by even functions  $f(a^\dagger, a)$ . Formally, it is the even subalgebra of Weyl algebra  $A_2$ . Passing to  $p_i, q^j$  the product on  $\mathfrak{hs}$  is the familiar Moyal-Weyl star-product:

$$(f \star g)(q, p) = f(q, p) \exp \frac{i\hbar}{2} (\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q) g(q, p)$$

## Unbroken higher spin symmetry: correlators

**Conserved tensor**  $\rightarrow$  **current**  $\rightarrow$  **symmetry**  $\rightarrow$  **invariants=correlators**

**Is higher spin symmetry powerful enough to fix correlators?**

All correlators are invariants (Sundell, Colombo; Didenko, E.S.; ...)

$$\langle J \dots J \rangle = \text{Tr}(\Psi \star \dots \star \Psi) \quad \Psi \leftrightarrow J$$

$\Psi$  are coherent states representing  $J$  in the higher spin algebra. The correlators are invariant under full higher spin symmetry,  $\delta\Psi = [\Psi, \xi]_\star$ , which contains the conformal symmetry

**Easy to say, but can we compute them?**

yes, why not (Sundell, Colombo; Didenko, E.S.; ...)

$$\langle JJJJ \rangle_{F.B.} \sim \cos(Q_{13}^2 + Q_{24}^3 + Q_{31}^4 + Q_{43}^1) \cos(P_{12}) \cos(P_{23}) \cos(P_{34}) \cos(P_{41}) + \dots$$

## Unbroken higher spin symmetry: Summary

In every free CFT one finds  $\infty$ -many higher spin currents  $J_s \equiv J_{a_1 \dots a_s}$ , which generate HS-charges  $Q_s = \int J$ . By construction,  $Q$  generate an  $\infty$ -dim Lie algebra, an extension of  $so(3,2)$

$$[Q, Q] = Q \qquad [Q, J] = J \qquad [Q, \phi] = \phi$$

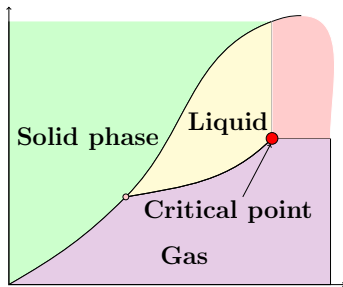
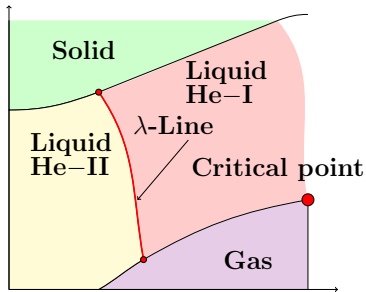
**Miracle 1:** the algebra originates from an associative HS-algebra  $\mathfrak{hs}$  via  $[a, b] = a \star b - b \star a$ . **Free CFT = associative algebra.** **Miracle 2:** HS-currents  $J$  are isomorphic to  $\mathfrak{hs}$ , representation twisted by inversion.

Correlators are invariants of this HS-algebra  $\mathfrak{hs}$

$$\langle J \dots J \rangle = \text{Tr}(\Psi \star \dots \star \Psi) \qquad \Psi \leftrightarrow J$$

**Important:** in  $3d$   $\mathfrak{hs}_{F.B.} \sim \mathfrak{hs}_{F.F.} \sim$  Weyl algebra of  $f(a_i^\dagger, a^j)$  and the invariants are the unique invariants of HS-algebra (Sharapov, E.S.)

# Chern-Simons Matter Theories and bosonization duality





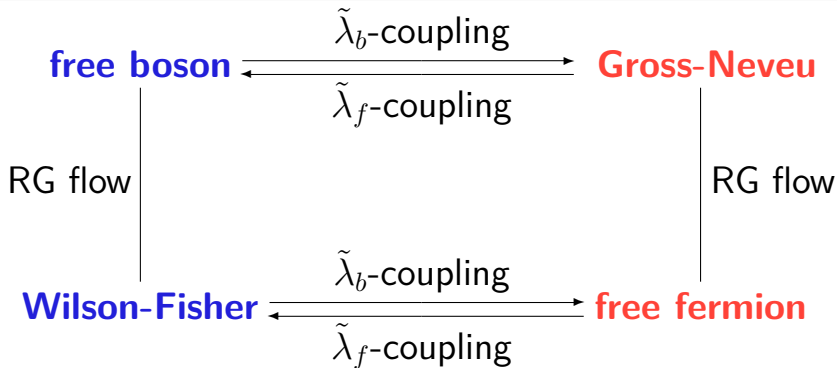
## Chern-Simons Matter theories and dualities

$CFT_3$ : there exists a large class of models, vector models, Chern-Simons Matter theories (extends to ABJ(M))

$$\frac{k}{4\pi} S_{CS}(A) + \text{Matter} \begin{cases} (D\phi^i)^2 & \text{free boson} \\ (D\phi^i)^2 + g(\phi^i\phi^i)^2 & \text{Wilson-Fisher (Ising)} \\ \bar{\psi}\not{D}\psi & \text{free fermion} \\ \bar{\psi}\not{D}\psi + g(\bar{\psi}\psi)^2 & \text{Gross-Neveu} \end{cases}$$

- describe physics (Ising, quantum Hall, ...)
- break parity in general (Chern-Simons)
- two parameters  $\lambda = N/k$ ,  $1/N$  ( $\lambda$  continuous for  $N$  large)
- exhibit remarkable dualities, e.g. **3d bosonization duality** (Aharony, Alday, Bissi, Giombi, Karch, Maldacena, Minwalla, Prakash, Seiberg, Tong, Witten, Yacobi, Yin, Zhiboedov, ...)

## Chern-Simons Matter theories and dualities



The simplest gauge-invariant operators are higher spin currents

$$J_s = \phi D \dots D \phi$$

$$J_s = \bar{\psi} \gamma D \dots D \psi$$

which are AdS/CFT dual to higher spin fields

What is going on in Chern-Simons-matter theories?

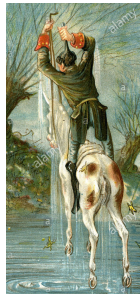
**HS-currents are responsible for their own non-conservation:**

$$\partial \cdot J_s = \sum_{s_1, s_2} C_{s, s_1, s_2}(\lambda) \frac{1}{N} [J_{s_1} J_{s_2}]$$

which is an exact non-perturbative quantum equation. In the large- $N$  we can use classical (representation theory) formulas for  $[JJ]$ .

The worst case  $\partial \cdot J =$  some other operator. The symmetry is gone, the charges are not conserved, do not form Lie algebra.

In our case the non-conservation operator  $[JJ]$  is made out of  $J$  themselves, but charges are still not conserved.



## Slightly-broken higher spin symmetry: what is it?

Initially we have well-defined charges and higher spin algebra  $\mathfrak{hs}$

$$\partial \cdot J_s = 0 \quad \Longrightarrow \quad Q_s = \int J_s \quad \Longrightarrow \quad [Q, Q] = Q \quad \& \quad [Q, J] = J \\ l(\xi_1, \xi_2) \quad \& \quad l(\xi, J)$$

The higher spin symmetry does not disappear completely:

$$\partial \cdot J = \frac{1}{N} [JJ] \quad [Q, J] = J + \frac{1}{N} [JJ]$$

**What is the right math?**

## Slightly-broken higher spin symmetry: what is it?

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**What is the right math?**

**We should deform the algebra together with its action on the module,**  
so that the module (currents) can 'backreact':

$$\delta_\xi J = l(\xi, J) + l(\xi, J, J) + \dots, \qquad [\delta_{\xi_1}, \delta_{\xi_2}] = \delta_\xi,$$

where  $\xi = l(\xi_1, \xi_2) + l(\xi_1, \xi_2, J) + \dots$

The consistency of such a structure leads to  **$L_\infty$** -algebras

## Strong homotopy algebras

Strong homotopy algebra is a graded space, e.g.  $V = V_{-1} \oplus V_0$  equipped with multilinear maps  $l_k(x_1, \dots, x_k)$  of degree-one. In our case

$$l_k(\xi, \xi, J, \dots, J) \qquad l_k(\xi, J, \dots, J)$$

that allow us to encode the deformed action

$$\delta_\xi J = l_2(\xi, J) + l_3(\xi, J, J) + \dots, \qquad [\delta_{\xi_1}, \delta_{\xi_2}] = \delta_\xi,$$

where  $\xi = l_2(\xi_1, \xi_2) + l_3(\xi_1, \xi_2, J) + \dots$ . The maps obey 'Jacobi' relations

$$\sum_{i+j=n} (\pm) l_i(l_j(x_{\sigma_1}, \dots, x_{\sigma_j}), x_{\sigma_{i+1}}, \dots, x_{\sigma_n}) = 0$$

$L_\infty$  originates from  $A_\infty$  constructed from a certain deformation of  $\mathfrak{hs}$ , which is related to para-statistics/fuzzy sphere (Sharapov, E.S.)

## Deformations of Poisson Orbifold: Weyl Algebra

Everyone knows that the Weyl algebra  $A_1$  is rigid

$$[q, p] = i\hbar \quad \text{no deformation of} \quad f(q, p) \star g(q, p)$$

Suppose that  $Rf(q, p) = f(-q, -p)$ , i.e. we can realize it as

$$R^2 = 1 \quad RqR = -q \quad RpR = -p$$

The crossed-product algebra  $A_1 \rtimes \mathbb{Z}_2$  is soft (Wigner; Yang; Mukunda; ...):

$$[q, p] = i\hbar + i\nu R$$

Also known as para-bose oscillators. Even  $R(f) = f$  lead to  $gl_\lambda = U(sp_2)/(C_2 - \lambda(\lambda-1))$  (Feigin), also (Madore; Bieliavsky et al) as fuzzy-sphere, NC hyperboloid, also (Plyushchay et al) as anyons.

Orbifold  $\mathbb{R}^2/\mathbb{Z}_2$  admits 'second' quantization on top of the Moyal-Weyl  $\star$ -product, (Pope et al; Joung, Mrtchyan; Korybut; Basile et al; Sharapov, E.S., Sukhanov)

## Slightly broken higher spin symmetry: summary

- necessary for bosonization:  $\mathfrak{hs}$  (boson)  $\sim \mathfrak{hs}$  (fermion) (Dirac, 1963)
- there exist exactly one invariant,  $\text{Tr}(\Psi \star \dots \star \Psi)$ , to serve as  $n$ -point correlator  $\langle J \dots J \rangle$  for free/large- $N$  limit
- $L_\infty$  depends on two pheno parameters, to be related to  $k, N$
- invariants are unobstructed and have a quasi-free form

$$\text{Tr}_o \log_o[1 - \Psi] \quad \text{mod irr}, \quad a \circ b = a \star b + \phi_1(a, b) \mathbf{R} + \dots$$

- a simple consequence is that correlators are very special

$$\langle J \dots J \rangle = \sum \langle \text{fixed} \rangle_i \times \text{params}$$

**This implies 3d bosonization since  $\langle J \dots J \rangle$  know everything** and it does not matter what matter  $J$  are made of,  $\phi$  or  $\psi$



Chiral higher spin gravity

## Remarks on Higher Spin Gravity

AdS/CFT duals of (Chern-Simons) vector models are HiSGRA since conserved tensor  $J_s$  is dual to (massless) gauge field in  $AdS_4$  (Sundborg; Klebanov, Polyakov; Sezgin Sundell; Leight, Petkou; Giombi, Yin, ... )

$$\partial^m J_{ma_2\dots a_s} = 0 \quad \iff \quad \delta\Phi_{\mu_1\dots\mu_s} = \nabla_{(\mu_1}\xi_{\mu_2\dots\mu_s)}$$

Instead of tedious quantum calculations in Chern-Simons matter one could do the standard holographic computation in the HiSGRA dual.

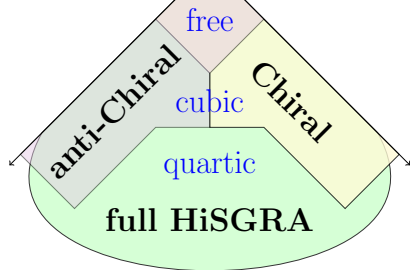
**No free lunch:** generic HiSGRA do not (want to 😊) exist/too nonlocal (untamed); nice exceptions:  $3d$ , conformal, IKKT (Steinacker, ..., Tran), Chiral

Chiral HiSGRA (Metsaev; Ponomarev, E.S.): all spins  $s = 0, (1), 2, (3), \dots$ ; at least one-loop UV-finite (E.S., Tran, Tsulaia); any sign of the cosmological constant is allowed, including zero; the smallest and the only local HiSGRA with propagating massless fields; specific predictions for higher derivative modifications of gravity; some kind of self-dual theory (Ponomarev); trivial  $S$ -matrix in flat space, but not in  $AdS$  ...

# Chiral HiSGRA and Chern-Simons Matter

Chern-Simons Matter Theories

AdS/CFT



Chiral HiSGRA is a local HiSGRA;

It has the right spectrum to be dual to Chern-Simons matter, but it cannot be;

Its very existence implies Chern-Simons matter have 2 closed subsectors;

Its very existence supports  $3d$  bosonization duality;

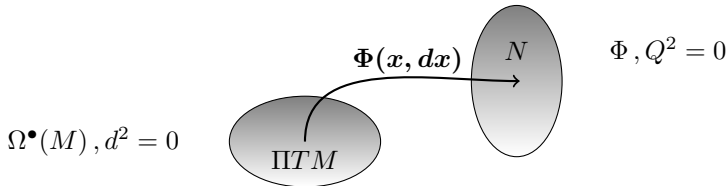
**Bosonization is manifest!**

(anti)-Chiral Theories provide a complete base for 3-pt amplitudes

$$V_3 = V_{chiral} \oplus \bar{V}_{chiral} \leftrightarrow \langle JJJ \rangle$$

## Chiral HiSGRA and Chern-Simons Matter

Let us be given a  $Q$ -manifold (better an  $L_\infty$ -algebra)



then we can always write a sigma-model:

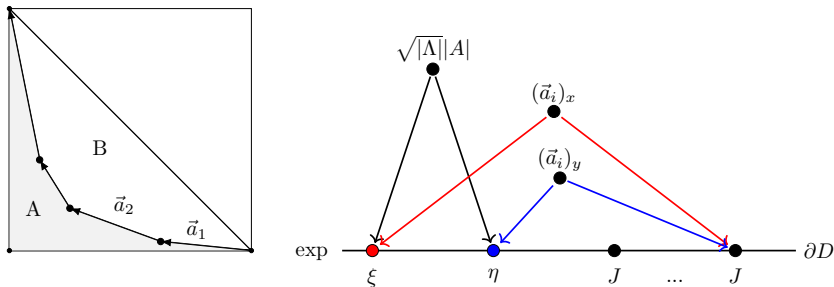
$$d\Phi = Q(\Phi) \equiv l_2(\Phi, \Phi) + l_3(\Phi, \Phi, \Phi) + \dots$$

**Any PDE can be cast into such a form ...** (Barnich, Grigoriev)

Other names: Free Differential Algebras (Sullivan), in physics: (van Nieuwenhuizen; Fre, D' Auria); FDA=unfolding (Vasiliev), AKSZ (AKSZ), gauged-PDE (Grigoriev, Kotov)

## A piece of a yet unknown formality

Moyal-Weyl, **hs**,  $2d$  Poisson/Symplectic  $\pi^{AB} = \text{const}$ , only boundary graphs survive in (Shoikhet-Tsygan)-Kontsevich formality. The first maps:  $m(\xi, \eta) = \xi \star \eta$ ,  $m(\xi, \eta, J) \sim \phi_1(\xi, \eta) \star JR$ , but we need locality! Result for  $m(\xi, \eta, J, \dots, J)$ :



(Sharapov, E.S., Van Dongen) the configuration space is of convex polygons  $B$  or swallowtails  $A$ , related to Grassmannian. The maps exponentiate like for Moyal-Weyl.

Our  $A_\infty$  is a pre-Calabi-Yau algebra (Kontsevich, ...). **Formality? TFT?**

- **Higher spin symmetry is new** ( $d = 3, \dots$ ) **Virasoro** 😊
- Slightly-broken symmetry should be understood as  $L_\infty$
- Uniqueness of  $L_\infty$ -invariants implies the  $3d$  bosonization duality
- **New type of a physical symmetry** where transformations (algebra) and the object (module) deform together
- Anomalous dimensions of HS-currents are small even for Ising model,  $N = 1$ , e.g.  $\Delta(J_4) = 5.02$  instead of 5
- Chiral Higher Spin Gravity is dual to a closed subsector of (Chern-Simons) vector models. How to find it? It should extend to small  $N$  due to integrability, implications for Ising?
- Locality implies specific  $A_\infty$ -maps that have (Shoikhet-Tsygan)-Kontsevich as first two layers. Bigger Formality? TFT?
- **More:** Snowmass paper, ArXiv: 2205.01567

Thank you for your attention!

May the higher spin symmetry  
be with you

## Two Poisson Brackets

Consider functions  $f(q, p, \mathbf{R})$  on the phase space

**1st Poisson bracket:**

$$\{f, g\} = m \pi_1(f \otimes g) \qquad \pi_1 = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p$$

**2nd Poisson bracket:**

$$\{f, g\} = m \pi_1 \int f(q(1-2u_1), p(1-2u_1)) \otimes g(q(1-2u_2), p(1-2u_2)) \mathbf{R}$$

the integral over simplex  $0 < u_1 < u_2 < 1$  is a leftover of the Kontsevich-Shoikhet graph. It can be obtained (Feigin, Felder, Shoikhet) from Kontsevich-Shoikhet-Tsygan Formality

In the general case of  $A_n \rtimes \Gamma$  the complete deformation can be obtained (Sharapov, E.S., Sukhanov) via Homological Perturbation Theory. There is a relation to Dunkl derivative for  $\hbar = 0$  (Tang)



## $L_\infty$ from $A_\infty$ ; $A_\infty$ from soft associative algebras

We need to construct  $L_\infty$  that 'deforms' our initial data = algebra + module, both originating from an associative algebra  $A = \mathfrak{hs} \rtimes \mathbb{Z}_2$ .

One can show (Sharapov, E.S.) that such  $L_\infty$  can be constructed as long as  $A$  is soft, i.e. can be deformed as an associative algebra:

$$a \circ (b \circ c) = (a \circ b) \circ c \qquad a \circ b = a \star b + \sum_{k=1} \phi_k(a, b) \hbar^k$$

The maps can be obtained from an auxiliary  $A_\infty$

$$\begin{aligned} m_3(a, b, u) &= \phi_1(a, b) \star u \quad \rightarrow \quad l_3 \\ m_4(a, b, u, v) &= \phi_2(a, b) \star u \star v + \phi_1(\phi_1(a, b), u) \star v \quad \rightarrow \quad l_4 \end{aligned}$$

Our algebra can be deformed thanks to para-statistics/anyons ... otherwise, any soft associative algebra leads to a certain  $A_\infty$

## Empty Flat Space

**Any PDE can be cast into such a form ...**

Flat space is still a flat connection  $dA = \frac{1}{2}[A, A]$

$$de^a = \omega^{a,b} \wedge e^b \quad \text{no torsion}$$

$$d\omega^{a,b} = \omega^{a,c} \wedge \omega^{c,b} \quad \text{Riemann vanishes}$$

vielbein  $e^a \equiv e^a_\mu dx^\mu$ ; spin-connection  $\omega^{a,b} \equiv \omega^{a,b}_\mu dx^\mu$

**Minkowski space = flat Poincare algebra connection\***

**How to add gravitatonal waves?**

Any PDE can be cast into such a form ... but in practice cannot

Einstein equations are already painful enough

$$\begin{aligned} de^a &= \omega^{a,b} \wedge e^b && \text{no torsion} \\ d\omega^{a,b} &= \omega^{a,c} \wedge \omega^{c,b} + e_m \wedge e_n C^{ab,mn} && \text{Einstein is here!} \end{aligned}$$

vielbein  $e^a \equiv e^a_\mu dx^\mu$ ; spin-connection  $\omega^{a,b} \equiv \omega^{a,b}_\mu dx^\mu$ ; Weyl tensor  $C^{ab,cd}$

# Einstein Equations

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$$d\omega^{a,b} = \omega^{a,c} \wedge \omega^{c,b} + e_m \wedge e_n C^{ab,mn} \quad \text{Einstein is here!}$$

$$dC^{ab,mn} = \omega C^{ab,cd} + e_k C^{ab,mn,k} \quad \text{Bianchi for Weyl}$$

vielbein  $e^a \equiv e^a_\mu dx^\mu$ ; spin-connection  $\omega^{a,b} \equiv \omega^{a,b}_\mu dx^\mu$ ; Weyl tensor  $C^{ab,cd}$ ; **and more  $C^{ab,cd,k}$ , ... and nonlinear equations for them**

We have to introduce auxiliary fields to parameterize the on-shell jet:

$$R_{ab,mn} = C_{ab,mn} \quad (\text{Riemann} = \text{its Weyl}) \sim \text{Einstein}$$

Closed-form for SDYM and SDGR (E.S., Van Dongen),  $l_2, l_3$