On the propagation across the big bounce in an open quantum FRW cosmology

Corfu 2022: workshop on noncommutative and generalized geometry in string theory, gauge theory and related physical models

Emmanuele Battista

Department of Physics, University of Vienna

universität wien

OUTLINE

1. THE BACKGROUND GEOMETRY

2. CLASSICAL ANALYSIS OF THE FRW GEOMETRY: NULL AND TIMELIKE GEODESICS

3. QUANTUM ANALYSIS OF THE FRW GEOMETRY: SCALAR FIELD PROPAGATOR

4. CONCLUDING REMARKS

THE BACKGROUND GEOMETRY (1)

- **Framework: Ishibashi, Kawai, Kitazawa, Tsuchiya (IKKT) matrix theory.**
- **The background spacetime $\mathscr{M}^{3,1}$ can be described semi-classically as a projection of the fuzzy four-dimensional hyperboloid, which is obtained from five matrices X^a $(a = 0,\ldots,4)$ interpreted as quantizations of five embedding functions x^a**

$$x^a : \quad H^4 \hookrightarrow \mathbb{R}^{4,1}$$

Convenient parametrization of the four-dimensional hyperboloid

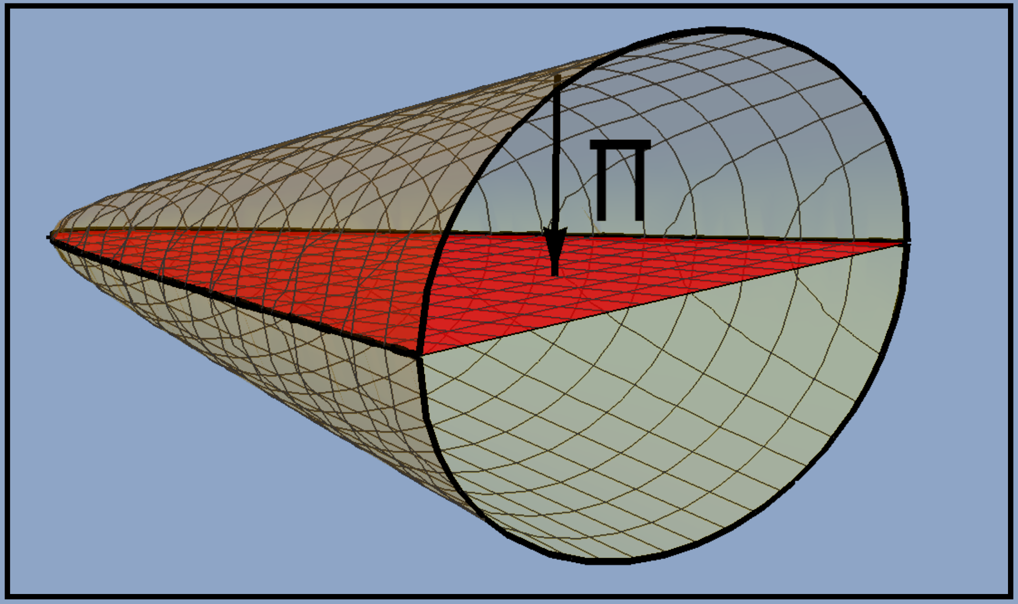
$$\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = R \begin{bmatrix} \cosh(\eta) \begin{pmatrix} \cosh(\chi) \\ \sinh(\chi)\sin(\theta)\cos(\varphi) \\ \sinh(\chi)\sin(\theta)\sin(\varphi) \\ \sinh(\chi)\cos(\theta) \end{pmatrix} \\ \sinh(\eta) \end{bmatrix},$$

$\eta, \chi \in \mathbb{R}$

THE BACKGROUND GEOMETRY (2)

- **The projection along the x^4 axis leads to a two-sheeted cover of the region $x_\mu x^\mu \leq R^2$, $(\mu = 0,\ldots,3)$.**
- **Parametrization of $\mathscr{M}^{3,1}$**

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = R\cosh(\eta) \begin{pmatrix} \cosh(\chi) \\ \sinh(\chi)\sin(\theta)\cos(\varphi) \\ \sinh(\chi)\sin(\theta)\sin(\varphi) \\ \sinh(\chi)\cos(\theta) \end{pmatrix} .$$



Projection Π from H^4 to $\mathscr{M}^{3,1}$

Pre-bounce: $\eta < 0$

Big bounce: $\eta = 0$

Post-bounce: $\eta > 0$

THE BACKGROUND GEOMETRY (3)

- The **background** solution T^μ of the matrix model leads to the following **kinetic term**, valid for any field Φ

$$S[\Phi] = -\,\mathrm{Tr}[\mathrm{T}^\mu, \Phi][\mathrm{T}_\mu, \Phi]$$

Using the **semiclassical** relation $[T^\mu, .\,] \sim i\{t^\mu, .\,\}$, this can be written in the standard form

$$S[\Phi] = -\,\mathrm{Tr}[\mathrm{T}^\mu, \Phi][\mathrm{T}_\mu, \Phi] \sim \int \mathrm{d}^4\mathrm{x}\,\sqrt{|\mathrm{G}|}\,\mathrm{G}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$$

THE BACKGROUND GEOMETRY (4)

- $SO(3,1)$**-invariant FRW metric**

$$ds_G^2 = G_{\mu\nu}dx^\mu dx^\nu = -R^2|\sinh(\eta)|^3 d\eta^2 + R^2|\sinh(\eta)|\cosh^2(\eta)\, d\Sigma^2$$
$$= -dt^2 + a^2(t) d\Sigma^2 \, .$$

where

$$d\Sigma^2 = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2),$$

cosmic scale factor

$$|a(\eta)| = R\cosh(\eta)|\sinh(\eta)|^{1/2},$$

$$dt = R|\sinh(\eta)|^{3/2} d\eta.$$

CLASSICAL ANALYSIS (1)

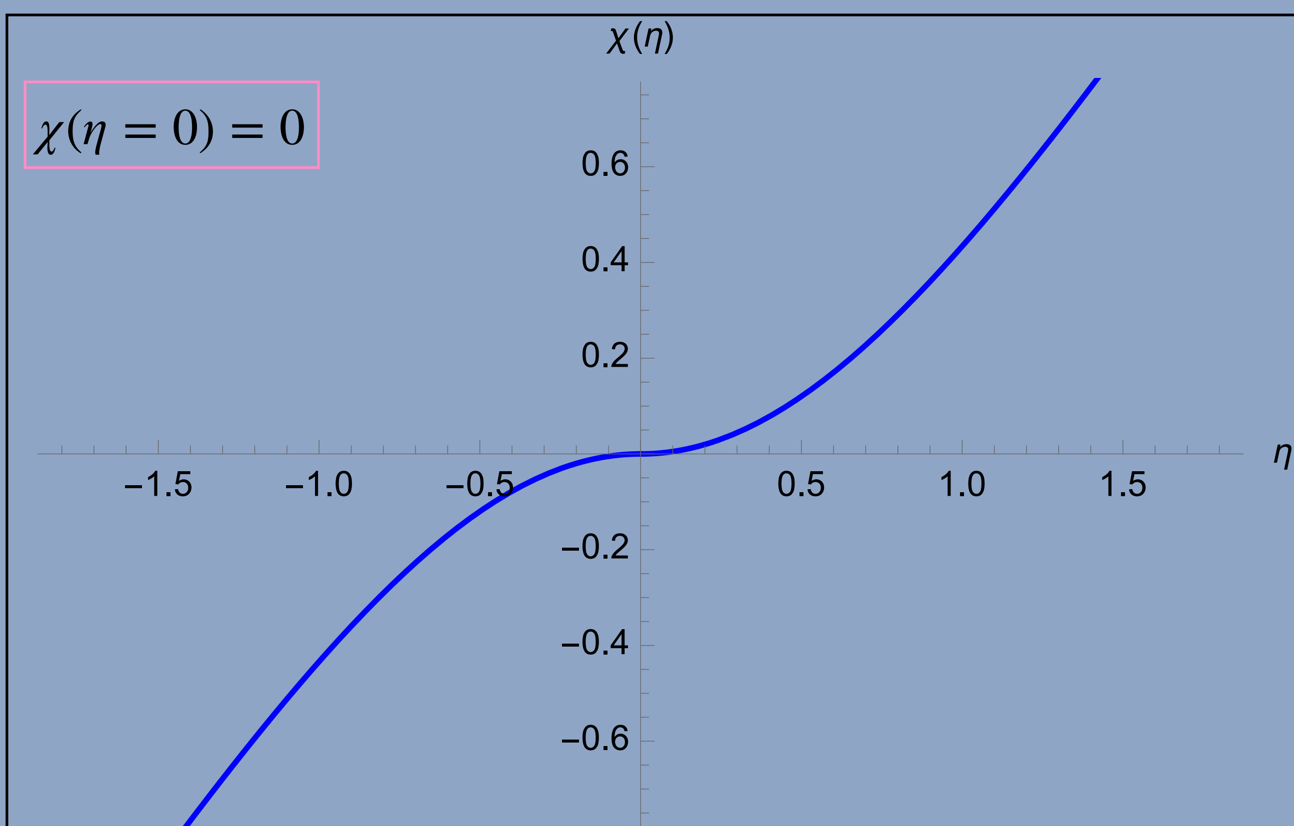
- **Null** and **timelike** geodesics of the FRW geometry having θ, φ constant.

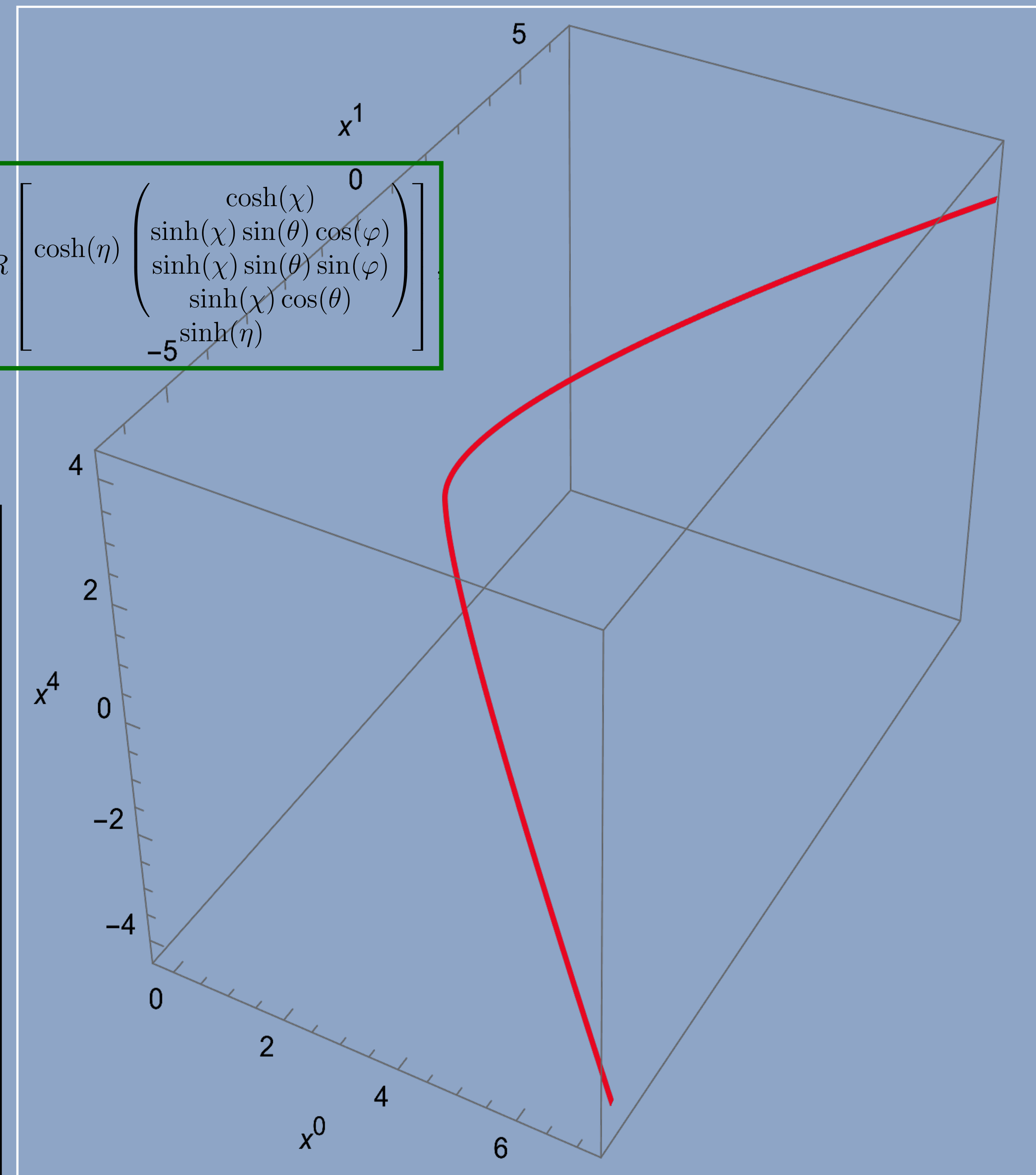
$$\frac{d\chi}{d\eta} = |\tanh \eta|,$$

Null geodesics

$$\chi(\eta) = \begin{cases} \log(\cosh \eta), & \eta \geqslant 0, \\ -\log(\cosh \eta), & \eta < 0. \end{cases}$$

$$\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = R \begin{bmatrix} \cosh(\eta) \begin{pmatrix} \cosh(\chi) \\ \sinh(\chi)\sin(\theta)\cos(\varphi) \\ \sinh(\chi)\sin(\theta)\sin(\varphi) \\ \sinh(\chi)\cos(\theta) \end{pmatrix} \\ \sinh(\eta) \end{bmatrix}$$



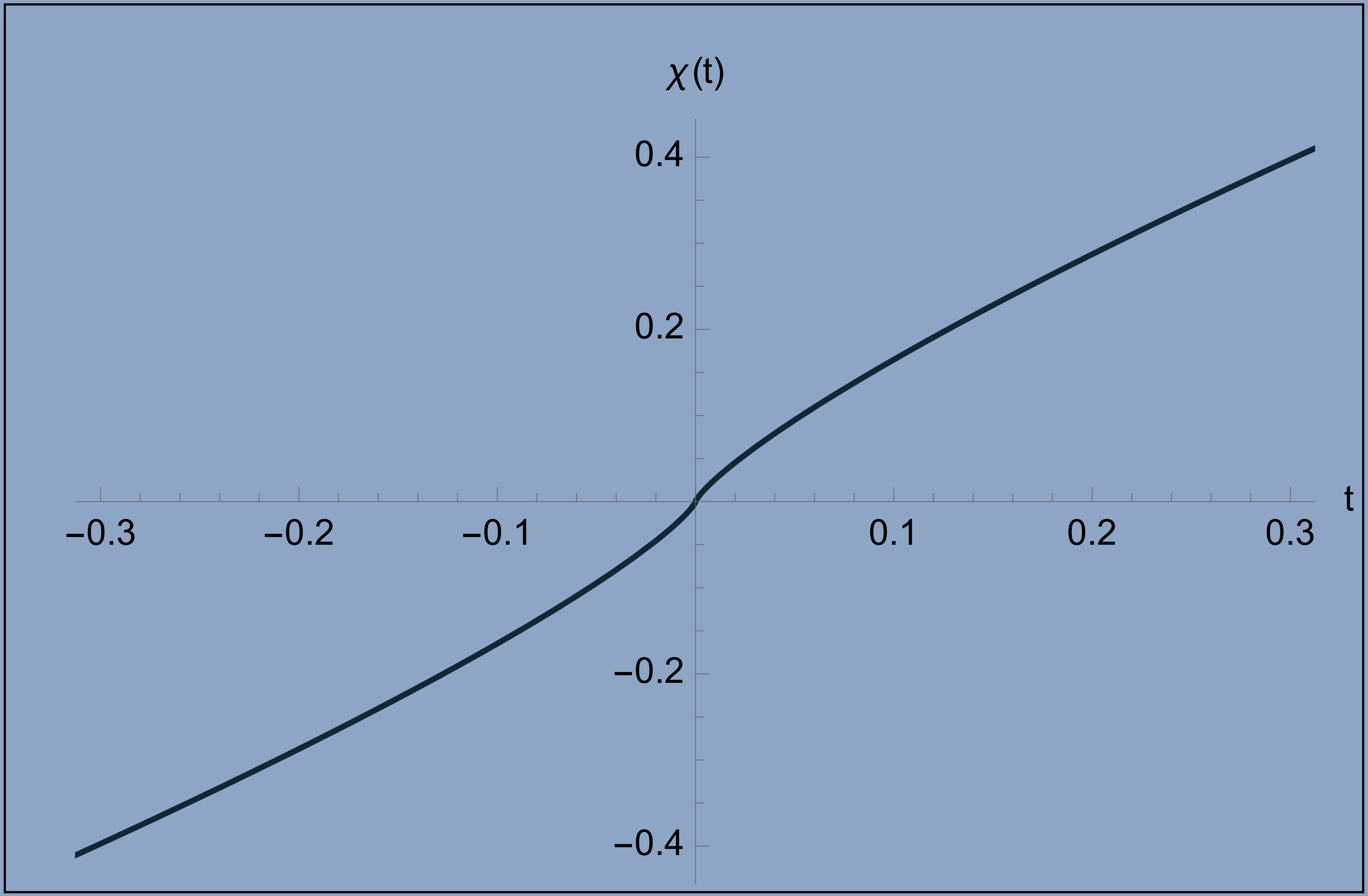


CLASSICAL ANALYSIS (2)

Early-time ($t \to t_0$) solution $\chi(t)$ for null geodesics

$$\chi(t) \overset{t\to t_0}{\sim} \begin{cases} \frac{1}{2}\left(\frac{5}{2R}\right)^{4/5}(t-t_0)^{4/5}, & t \geqslant t_0, \\ -\frac{1}{2}\left(\frac{5}{2R}\right)^{4/5}(t-t_0)^{4/5}, & t < t_0, \end{cases}$$

Big Bounce: $t = t_0$



$t_0 = 0$

Big Bounce: $t = 0$

$\chi(t = 0) = 0$

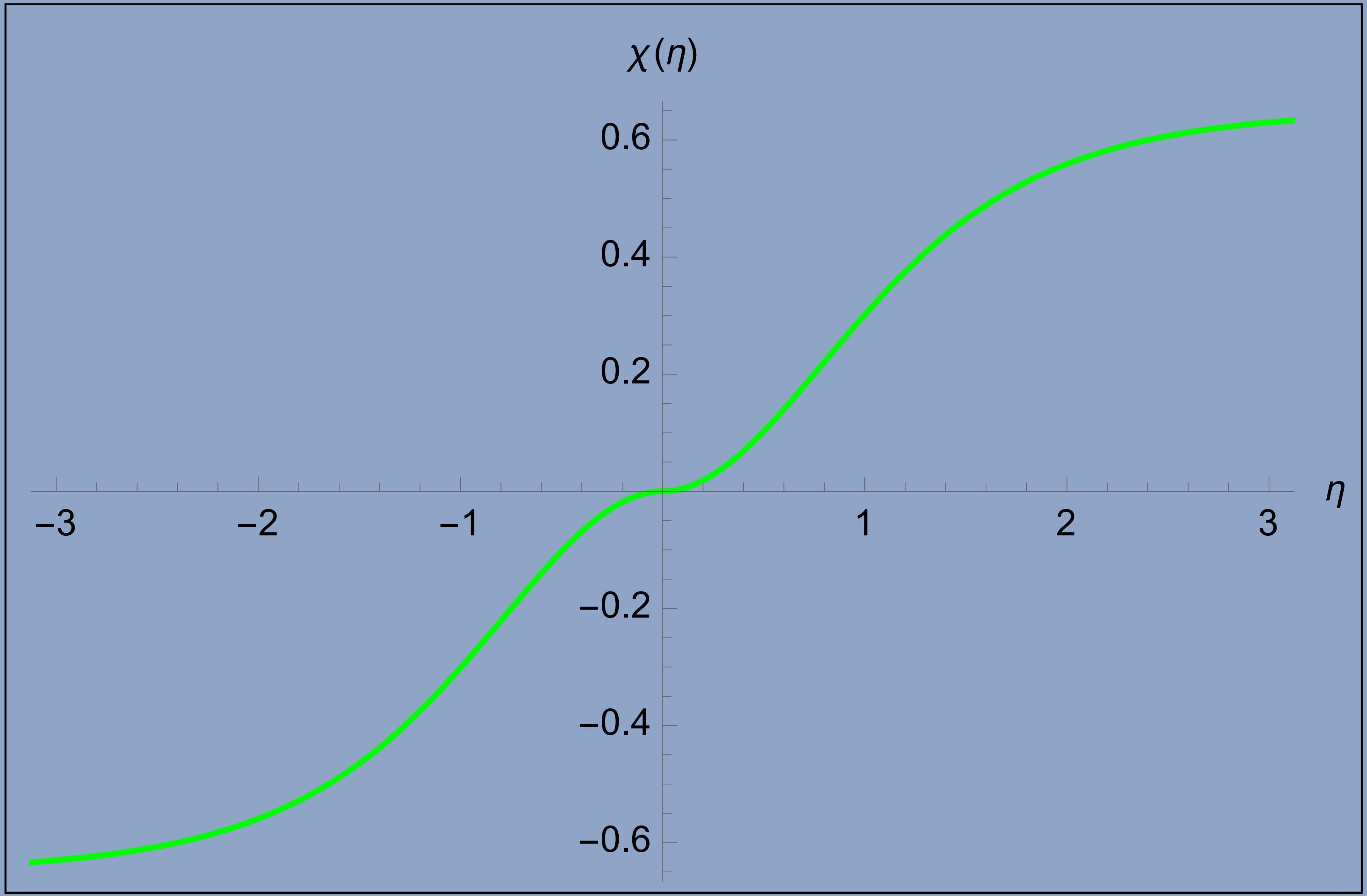
CLASSICAL ANALYSIS (3)

Timelike geodesics of a non-comoving observer (constant θ, φ)

$$\frac{d\chi(\eta)}{d\eta} = \frac{|\tanh \eta|}{\sqrt{1 + a^2(\eta)/\Pi^2}} \, .$$

Π: conserved momentum

$\chi(\eta = 0) = 0$
$\Pi = 1$



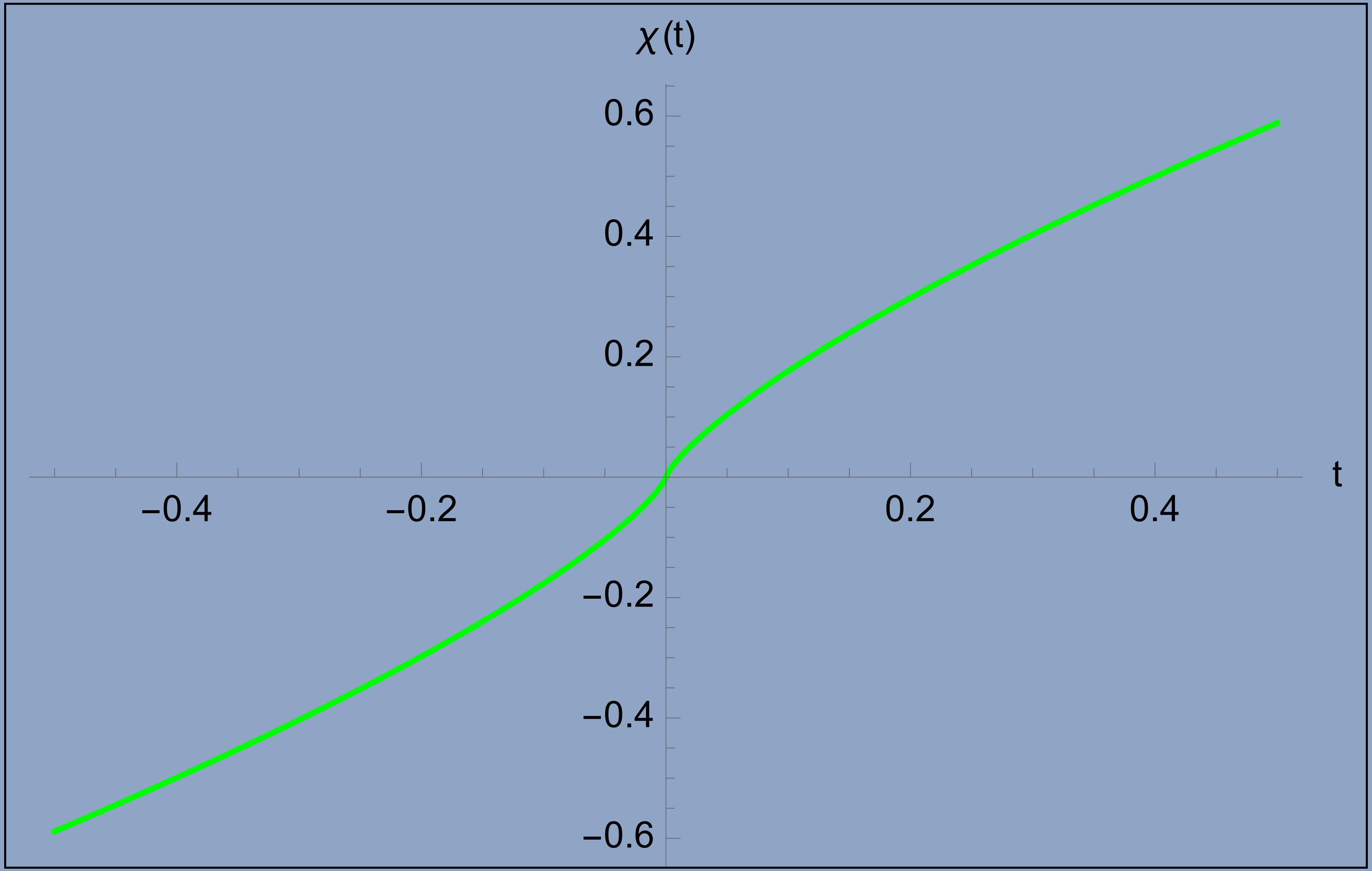
CLASSICAL ANALYSIS (4)

Early-time timelike geodesics

$$\chi(t) \overset{t\to 0}{\sim} \begin{cases} \frac{5}{3}\left[2 - 2\sqrt{1+t^{2/5}} + t^{2/5}\sqrt{1+t^{2/5}}\right], & t \geqslant 0, \\ -\frac{5}{3}\left[2 - 2\sqrt{1+t^{2/5}} + t^{2/5}\sqrt{1+t^{2/5}}\right], & t < 0. \end{cases}$$

Big Bounce: $t = 0$

$\chi(t=0) = 0$
$\Pi = 1$



QUANTUM ANALYSIS (1)

- **Propagator of a scalar field ϕ having mass m on the background FRW geometry**

Matrix d'Alembertian $\Box$ governing the propagation of ϕ

$$\Box = [T^\mu, [T_\mu, .]] \sim -\{t^\mu, \{t_\mu, .\}\},$$

semi-classical limit

$$\Box\phi = \frac{1}{R^2}\left[3\tanh(\eta)\partial_\eta + \partial_\eta^2 - \tanh^2\eta\left(\frac{2}{\tanh\chi}\partial_\chi + \partial_\chi^2\right) - \frac{\tanh^2\eta}{\sinh^2\chi}\left(\frac{1}{\tan\theta}\partial_\theta + \partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\varphi^2\right)\right]\phi.$$

QUANTUM ANALYSIS (2)

Action for the field ϕ (in the semi-classical limit)

$$S_\varepsilon\left[\phi\right] = \int \Omega \phi^*(x)\left(-\Box - m^2 + i\varepsilon\right)\phi(x),$$

$\varepsilon > 0$

where

$$\Omega = \frac{1}{|\sinh(\eta)|} d^4x = \cosh^3(\eta) d\eta \sinh^2(\chi) d\chi \sin(\theta) d\theta d\varphi$$

Ω **: SO(4,1)-invariant**
volume form

QUANTUM ANALYSIS (3)

Eigenfunctions of the d'Alembertian operator

$$\Box\phi = \lambda\phi,$$

Separation *ansatz*

$$\phi(\eta,\chi,\theta,\varphi) = \tilde{\phi}(\eta,\chi)Y_l^m(\theta,\varphi),$$
$$\tilde{\phi}(\eta,\chi) = f(\eta)g(\chi),$$

$Y_l^m(\theta,\varphi)$ **: spherical harmonics**
degree l, order m

The functions $f(\eta), g(\chi)$ satisfy the following ordinary differential equations

$$\left(\partial_\eta^2 + 3\tanh(\eta)\partial_\eta - \beta\tanh^2\eta - \lambda R^2\right) f(\eta) = 0,$$
$$\left(\partial_\chi^2 + \frac{2}{\tanh\chi}\partial_\chi - \frac{l(l+1)}{\sinh^2\chi} - \beta\right) g(\chi) = 0,$$

$\beta \in \mathbb{R}$

QUANTUM ANALYSIS (4)

Eigenmodes of d'Alembertian operator ($q \in \mathbb{R}, s > 0, \chi > 0$)

$$\Upsilon^{s_\pm,q}_{l,m}(\eta,\chi,\theta,\varphi) := \frac{1}{\sqrt{\cosh^3\eta\,\sinh\chi}} \mathsf{P}^{\pm is}_{\nu}(\tanh\eta)\, \mathcal{Q}^{iq}_{l}(\coth\chi)\, Y^m_l(\theta,\varphi),$$

$\mathsf{P}^{\pm is}_{\nu}(\tanh\eta)$: **Legendre functions of first kind** ($\tanh(\eta) \in (-1,1)$).
$\pm is$: **order**, ν : **degree**

$$\nu = \frac{1}{2}\left(2\sqrt{1+\beta}-1\right) = -\frac{1}{2} + i|q|,$$
$$s = \sqrt{-\left(\frac{9}{4}+\beta+\lambda R^2\right)} > 0.$$

$\mathcal{Q}^{iq}_{l}(\coth\chi)$: **Legendre functions of second kind** ($\coth\chi \in (1,+\infty)$).
iq : **order**, l : **degree**

$$q = \pm\sqrt{-(1+\beta)}$$

$$\beta < -1$$

Eigenvalue: $\lambda = (q^2 - s^2 - 5/4)R^{-2}$

QUANTUM ANALYSIS (5)

Consider the following "flat" regime (FR)

FR: $\chi < 1, \qquad q \gg l$

In FR we have

$$\mathrm{Re}\left[\frac{\mathcal{Q}_l^{iq}(\coth\chi)}{\sinh\chi}\frac{q^l}{e^{-\pi q}\Gamma(iq+l+1)}\right] \underset{q\gg l}{\overset{\chi<1}{\sim}} j_l(q\chi)$$

$$\mathrm{Im}\left[\frac{\mathcal{Q}_l^{iq}(\coth\chi)}{\sinh\chi}\frac{q^l}{e^{-\pi q}\Gamma(iq+l+1)}\right] \overset{\chi<1}{\sim} 0$$

$\Gamma(x)$: Euler gamma function

$j_l(q\chi)$: spherical Bessel functions

$$\Upsilon^{s_\pm,q}_{l,m}(\eta,\chi,\theta,\varphi) \underset{\mathrm{FR}}{\overset{\eta\to+\infty}{\sim}} \frac{1}{\sqrt{\cosh^3\eta}}\frac{e^{-\pi q}j_l\left(q\chi\right)\Gamma(iq+l+1)}{q^l}\frac{e^{\pm i\eta s}}{\Gamma(1\mp is)}Y_l^m(\theta,\varphi).$$

$$\Upsilon^{s_\pm,q}_{l,m}(\eta,\chi,\theta,\varphi) \underset{\mathrm{FR}}{\overset{\eta\to 0}{\sim}} \frac{\sqrt{\pi}\,2^{\pm is}\,e^{\pm i\eta s}}{\Gamma\left(\frac{3}{4}-\frac{iq}{2}\mp\frac{is}{2}\right)\Gamma\left(\frac{3}{4}+\frac{iq}{2}\mp\frac{is}{2}\right)}\frac{e^{-\pi q}j_l\left(q\chi\right)\Gamma(iq+l+1)}{q^l}Y_l^m\left(\theta,\varphi\right).$$

QUANTUM ANALYSIS (6)

- **Late-time ($\eta \to +\infty$) local propagator in position space**

$$\langle \phi(x)\phi^*(x')\rangle \overset{\eta\to+\infty}{\underset{\mathrm{FR}}{\sim}} \langle \phi(x)\phi^*(x')\rangle_{\mathrm{L}}^{\eta\to+\infty,\mathrm{FR}} + \langle \phi(x)\phi^*(x')\rangle_{\mathrm{SL}}^{\eta\to+\infty,\mathrm{FR}},$$

"L": leading **"SL": subleading**

Leading contribution

$$\langle\phi(x)\phi^*(x')\rangle_{\mathrm{L}}^{\eta\to+\infty,\mathrm{FR}} = \frac{4R^2}{\pi^2}\sum_{l,m}\frac{Y_l^m(\theta,\varphi)\left[Y_l^m(\theta',\varphi')\right]^*}{\sqrt{\left(\cosh^3\eta\right)\left(\cosh^3\eta'\right)}}\int_{-\infty}^{+\infty} ds\, e^{is(\eta-\eta')}\int_0^{+\infty}\frac{dq\, q^2 j_l(q\chi)\, j_l(q\chi')}{\left(s^2-q^2+\frac{5}{4}-m^2R^2+i\varepsilon\right)},$$

This resembles the standard Feynman propagator on a flat Minkowski space recalling that

$$e^{i\vec{q}\cdot\vec{x}} = 4\pi\sum_{l,m} i^l j_l(qr) Y_l^m(\hat{q})\left[Y_l^m(\hat{x})\right]^*,$$

Rayleigh equation

$q=|\vec{q}|,\ r=|\vec{x}|, \hat{x}=\vec{x}/r, \hat{q}=\vec{q}/q$

$$\int d^3q\, \frac{e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}}{s^2-q^2-M^2} = (4\pi)^2\sum_{l,m}\left[Y_l^m(\hat{x})\right]^* Y_l^m(\hat{x}')\int_0^{+\infty} dq\frac{q^2}{s^2-q^2-M^2} j_l(qr) j_l(qr').$$

QUANTUM ANALYSIS (7)

Subleading contribution to the late-time propagator

$$\langle \phi(x)\phi^*(x')\rangle_{\mathrm{SL}}^{\eta\to+\infty,\mathrm{FR}} = \frac{4R^2}{\pi^4}\sum_{l,m}\frac{Y_l^m(\theta,\varphi)\left[Y_l^m(\theta',\varphi')\right]^*}{\sqrt{\left(\cosh^3\eta\right)\left(\cosh^3\eta'\right)}}\int_{-\infty}^{+\infty} ds\, e^{is\left(\eta+\eta'\right)}$$

$$\times\int_0^{+\infty} dq\frac{j_l\left(q\chi\right)j_l\left(q\chi'\right)q^2 s\cosh(\pi q)\sinh(\pi s)}{\left(s^2-q^2+\frac{5}{4}-m^2R^2+i\varepsilon\right)}\Gamma\left(\frac{1}{2}-iq-is\right)\Gamma\left(\frac{1}{2}+iq-is\right)\Gamma^2\left(is\right).$$

QUANTUM ANALYSIS (8)

- **Local propagator in position space near the big bounce ($\eta, \eta' \to 0$)**

$$\langle \phi(x)\phi^*(x')\rangle \underset{\text{FR}}{\overset{\eta,\eta'\to 0}{\sim}} 4R^2 \sum_{l,m} Y_l^m(\theta,\varphi)\,[Y_l^m(\theta',\varphi')]^* \int_{-\infty}^{+\infty} ds \int_0^{+\infty} dq\, \frac{q^2 j_l\,(q\chi)\, j_l\,(q\chi')\, e^{is(\eta-\eta')}}{\left(s^2 - q^2 + \frac{5}{4} - m^2R^2 + i\varepsilon\right)}$$
$$\times \frac{-is}{[\cosh(\pi q) - i\sinh(\pi s)]} \frac{1}{\left|\Gamma\left(\frac{3}{4}+\frac{iq}{2}+\frac{is}{2}\right)\Gamma\left(\frac{3}{4}+\frac{iq}{2}-\frac{is}{2}\right)\right|^2}.$$

The early-time propagator:

- **bounded and well-defined ;**
- **Standard correlation between points x, x' lying on the same sheet of $\mathscr{M}^{3,1}$;**
- **Propagation between points located on opposite sheets of $\mathscr{M}^{3,1}$ near the big bounce.**

CONCLUDING REMARKS (1)

- Classical analysis of FRW geometry: general-relativity tools; investigation of null and timelike geodesics;

- Quantum analysis of FRW geometry: quantum-field-theory techniques; evaluation of the scalar field propagator.

- Future perspectives and open problems:

1. Our analysis is restricted to non-interacting test particles. For a physical particle near the big bounce, there would be infinitely strong interactions with the infinite matter density;
2. Investigation of the correlation between pre-big bounce and post-big bounce eras;
3. Propagation of fermions

CONCLUDING REMARKS (2)

Further details can be found in

Emmanuele Battista and Harold C. Steinacker,
"On the propagation across the big bounce in an open quantum FLRW cosmology",
arXiv: 2207.01295 [gr-qc] (2022).