CONSTRAINTS ON SCALAR-FERMION THEORIES

with Hugh Osborn, Colin Poole, Tom Steudtner

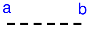
30th August 2022, Corfu Summer Institute

Introduction

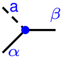
- Renormalisation group (RG) functions for scalar theories known to high orders
- Also RG functions for supersymmetric theories
- But β functions for general scalar-fermion theory only recently derived to 3 loops
- Use supersymmetric results to constrain general result
- $N = 1$ supersymmetry in $d = 4$ not very helpful
- Formally Gross-Neveu-Yukawa theory with N_Φ scalars and $N_f = \frac{1}{4}N_\Phi$ Dirac fermions has a supersymmetric fixed-point in the ϵ-expansion - “emergent supersymmetry” or “$N = \frac{1}{2}$ supersymmetry”. Can use this!
- This puts numerous constraints on RG functions.

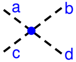
General fermion-scalar theory

Consider renormalisable 4-d fermion/scalar theory: Fields ϕ^a, ψ^α with Yukawa and quartic scalar interactions



$\equiv \delta^{ab}$,


$\equiv \delta^{\alpha\beta}$,


$\equiv y^{a\alpha\beta}$,


$\equiv \lambda^{abcd}$.

Yukawa β function can be written as

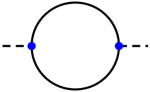
$$\beta_y^a = \tilde{\beta}_y^a + \gamma_\phi^{ab}\, y^b + \gamma_\psi\, y^a + y^a\, \gamma_\psi\,,$$

where $\tilde{\beta}_y^a$ is determined by 1PI diagrams and γ_ϕ, γ_ψ are anomalous dimensions. Similarly

$$\beta_\lambda^{abcd} = \tilde{\beta}_\lambda^{abcd} + \gamma_\phi^{e(a}\lambda^{bcd)e}$$

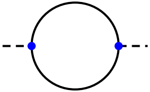
Use diagrammatic notation for tensor structures, e.g.

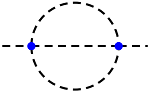
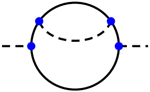
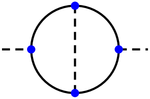
$$\gamma_\phi^{(1)} = \gamma_{\phi,1}\mathrm{tr}(y^a y^b) = \gamma_{\phi,1}$$

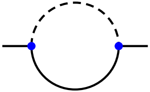


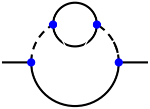
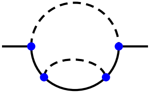
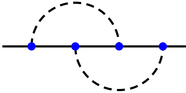
General fermion-scalar theory

Up to two loops

$\gamma_\phi = \gamma_{\phi,1}$ 

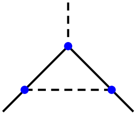
$+ \gamma_{\phi 2,a}$  $+ \gamma_{\phi,2b}$  $+ \gamma_{\phi 1,2c}$ 

$\gamma_\psi = \gamma_{\psi,1}$ 

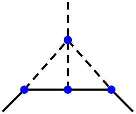
$+ \gamma_{\psi,2a}$  $+ \gamma_{\psi,2b}$  $+ \gamma_{\psi,2c}$ 

General fermion-scalar theory

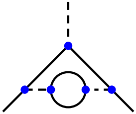
$\tilde{\beta}_y = \beta_{y,1}$



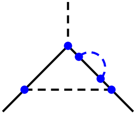
$+ \beta_{y,2a}$



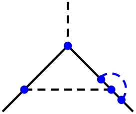
$+ \beta_{y,2b}$



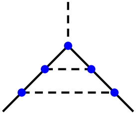
$+ \beta_{y,2c}$



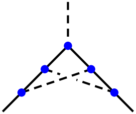
$+ \beta_{y,2d}$



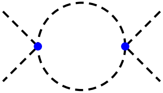
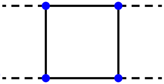
$+ \beta_{y,2e}$

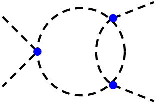
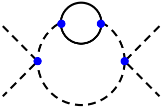
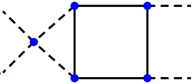


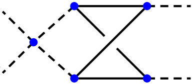
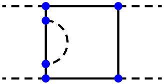
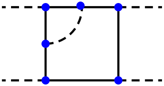
$+ \beta_{y,2f}$

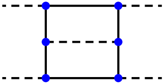


General fermion-scalar theory

$\tilde{\beta}_\lambda = \beta_{\lambda,1a}$  $+\ \beta_{\lambda,1b}$ 

$+\ \beta_{\lambda,2a}$  $+\ \beta_{\lambda,2b}$  $+\ \beta_{\lambda,2c}$ 

$+\ \beta_{\lambda,2d}$  $+\ \beta_{\lambda,2e}$  $+\ \beta_{\lambda,2f}$ 

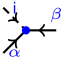
$+\ \beta_{\lambda,2g}$ 

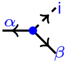
$U(1)$ theory

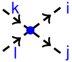
Complex fields with a $U(1)$ symmetry $\Rightarrow$ far fewer diagrams. Take

$$\Phi^a = (\varphi_i, \overline{\varphi}^i) \Rightarrow \Phi^a \Phi'^a = \varphi_i \overline{\varphi}'^i + \overline{\varphi}^i \varphi'_i,$$
$$\frac{1}{24}\lambda^{abcd}\,\Phi^a\Phi^b\Phi^c\Phi^d = \frac{1}{4}\lambda_{ij}{}^{kl}\,\overline{\varphi}^i\overline{\varphi}^j\varphi_k\varphi_l.$$

Now have directed lines with basic vertices now represented by

$\equiv \delta_i{}^j,$


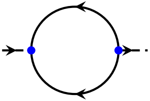
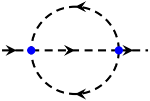
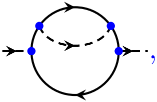
$\equiv y^{i\alpha\beta},$


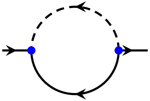
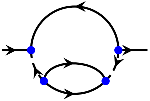
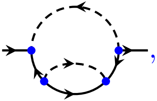
$\equiv \overline{y}_{i\alpha\beta},$


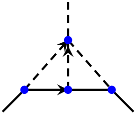
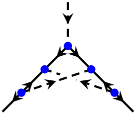
$\equiv \lambda_{ij}{}^{kl}.$

$U(1)$ theory

Now have just

$\gamma_\varphi = \gamma_{\phi,1}$  $+\ \gamma_{\phi,2a}$  $+\ \gamma_{\phi,2b}$  ,

$\gamma_\psi = \gamma_{\psi,1}$  $+\ \gamma_{\psi,2a}$  $+\ \gamma_{\psi,2b}$  ,

$\tilde{\beta}_y = \beta_{y,2a}$  $+\ \beta_{y,2f}$ 

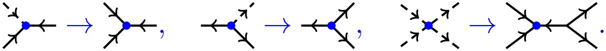
plus scalar diagrams (except $\beta_{\lambda,2f}$) – several ways to assign arrows.

Constraints from $N = 1$ supersymmetric theory

$N = 1$ supersymmetric theory for scalars and fermions obtained by equating numbers of scalar and fermion fields and restricting the couplings of the $U(1)$ theory so that

$$y^{i\alpha\beta} \to Y^{ijk} = Y^{(ijk)}\,, \qquad \bar{y}_{i\alpha\beta} \to \bar{Y}_{ijk} = \bar{Y}_{(ijk)}\,, \qquad \lambda_{ij}{}^{kl} \to \bar{Y}_{ijm} Y^{mkl}\,.$$

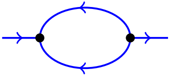
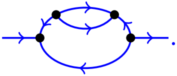
or



Supersymmetry, non-renormalisation theorem require

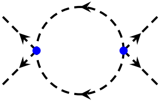
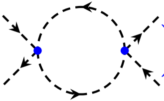
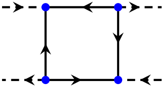
$$\gamma_{\psi}{}^{i}{}_{j} = \gamma_{\varphi}{}^{i}{}_{j} = \gamma_{S}{}^{i}{}_{j}\,, \qquad \tilde{\beta}_{Y}{}^{ijk} = 0\,, \qquad \tilde{\beta}_{\lambda\, ij}{}^{kl} = 2\,\bar{Y}_{ijm}\,\gamma^{m}{}_{n}\,Y^{nkl}\,.$$

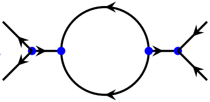
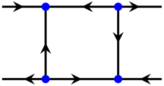
with

$\gamma_S^{(1)} = \gamma_{S,1}$  , $\qquad \gamma_S^{(2)} = \gamma_{S,2}$  .

Constraints from $N = 1$ supersymmetric theory

At one loop this involves diagrammatically

$\beta_{\lambda,1a}\Big($  $+\,4$  $\Big) + \beta_{\lambda,1b}$ 

$\rightarrow \beta_{\lambda,1a}$  $+ (4\beta_{\lambda,1a} + \beta_{\lambda,1b})$ 

so we have

$$\gamma_{S,1} = \gamma_{\Psi,1} = \gamma_{\Phi,1}\,, \qquad 4\,\beta_{\lambda,1a} + \beta_{\lambda,1b} = 0\,, \qquad \beta_{\lambda,1a} = 2\,\gamma_{S,1}\,.$$

4 1-loop coefficients in the $U(1)$ theory and 3 $N = 1$ constraints
$\Rightarrow$ 1 $U(1)$ coefficient undetermined.

Constraints from $N = 1$ supersymmetric theory

At two loops the necessary conditions are

$$\gamma_{S,2} = \gamma_{\Psi,2a} + \gamma_{\Psi,2b} = 3\,\gamma_{\Phi,2a} + \gamma_{\Phi,2b} = \frac{1}{2}\beta_{\lambda,2b}\,,$$
$$\beta_{y,2a} + \beta_{y,2f} = 0\,,$$
$$2\,\beta_{\lambda,2a} + \beta_{\lambda,2d} = 4\,\beta_{\lambda,2a} + 2\,\beta_{\lambda,2c} + \beta_{\lambda,2g} = \beta_{\lambda,2a} + \beta_{\lambda,2b} + \beta_{\lambda,2e} = 0.$$

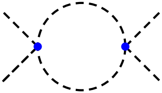
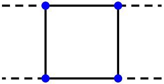
11 2-loop coefficients in the $U(1)$ theory and 6 $N = 1$ constraints $\Rightarrow$ 5 $U(1)$ coefficients undetermined.

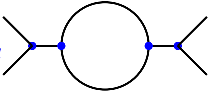
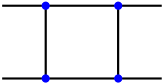
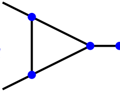
Constraints from $N=\frac{1}{2}$ supersymmetric theory

Special case of real general scalar-fermion theory[Fei,Giombi,Klebanov,Tarnopolsky; Liendo and Rong] with

$$(y^a)^{\alpha\beta} \rightarrow Y^{abc}, \quad \lambda^{abcd} \rightarrow Y^{abe}Y^{cde} + \text{perms}$$

For a 2-component Majorana fermion this is $N=1$-supersymmetric in $d=3$. Can access this in an ϵ expansion around $d=4$ by formally taking $N_f = \frac{1}{4}N_\phi$ Dirac fermions (modified fermion trace). Now $\tilde{\beta}_y \neq 0$ and one-loop constraints involve diagrammatically

$\beta_{\lambda,1a}$  $+\beta_{\lambda,1b}$  $\rightarrow$

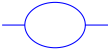
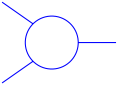
$\beta_{\lambda,1a}$  $+(4\beta_{\lambda,1a}+\beta_{\lambda,1b})$  $+4\beta_{\lambda,1a}$ 

Constraints from $N = \frac{1}{2}$ supersymmetric theory

leading to

$$
\begin{aligned}
\gamma_{S',1} &= \gamma_{\Phi,1} = \gamma_{\Psi,1}\,, \quad \beta_{Y,1} = \beta_{y,1}\,, \\
\beta_{\lambda,1a} &= 2\,\gamma_{S',1}\,, \quad 4\,\beta_{\lambda,1a} = 2\beta_{Y,1}\,, \\
4\,\beta_{\lambda,1a} + \beta_{\lambda,1b} &= 0,
\end{aligned}
$$

where

$\gamma^{(1)}_{S'} = \gamma_{S',1}$  , $\tilde{\beta}^{(1)}_{Y} = \beta_{Y,1}$  .

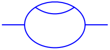
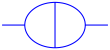
5 1-loop coefficients in the general scalar-fermion theory and 4 $N = \frac{1}{2}$ constraints $\Rightarrow$ 1 general coefficient undetermined.

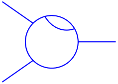
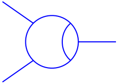
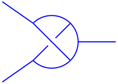
Constraints from $N = \frac{1}{2}$ supersymmetric theory

At two loops equality of $\gamma_\Phi,\ \gamma_\Psi$ and symmetry of β_y requires

$$\begin{aligned}
\gamma_{S',2A} &= 3\,\gamma_{\Phi,2a} + \gamma_{\Phi,2b} = \gamma_{\Psi,2a} + \gamma_{\Psi,2b} \\
\gamma_{S',2B} &= 6\,\gamma_{\Phi,2a} + \gamma_{\Phi,2c} = \gamma_{\Psi,2c}\,, \\
\beta_{Y,2A} &= \beta_{y,2b} = \beta_{y,2c} \\
\beta_{Y,2B} &= \beta_{y,2a} + \beta_{y,2d} = \beta_{y,2e} \\
\beta_{Y,2C} &= \beta_{y,2a} + \beta_{y,2f}\,,
\end{aligned}$$

where

$\gamma^{(2)}_{S'} = \gamma_{S',2A}$  $+\ \gamma_{S',2B}$  ,

$\tilde{\beta}^{(2)}_{Y} = \Big(\beta_{Y,2A}$  $+\ \beta_{Y,2B}$  + perms $\Big) + \beta_{Y,2C}$ 

Constraints from $N = \frac{1}{2}$ supersymmetric theory

Determining β_λ in terms of β_Y and $\gamma_{S'}$ imposes the restrictions

$$\beta_{\lambda,2a} = 2\,\gamma_{S',2A} = \beta_{Y,2A} = \tfrac{1}{2}\,\beta_{Y,2B}\,, \qquad 0 = \gamma_{S',2B} = \beta_{Y,2C}\,,$$
$$4\,\beta_{\lambda,2a} = 4\,\beta_{\lambda,2b} = -2\,\beta_{\lambda,2d} = -2\,\beta_{\lambda,2e} = -\beta_{\lambda,2f} = -\beta_{\lambda,2g},$$
$$\beta_{\lambda,2c} = 0\,.$$

19 2-loop coefficients in the general scalar-fermion theory and 15 $N = \frac{1}{2}$ constraints $\Rightarrow$ 4 general 2-loop coefficients undetermined.

a-function constraints)

- a-theorem $\Rightarrow$ existence of a-function which is monotonic between RG fixed points
- "Strong" a-theorem $\Rightarrow$ $\frac{\partial a}{\partial g} \propto \beta$
- Proposed explicit form for $N = 1$ supersymmetric a-function
- $\Rightarrow$ constraints on supersymmetric anomalous dimension

The a-function

For a general theory with couplings g^I can show [Osborn] there is a function a such that

$$\frac{\partial}{\partial g^I} a \equiv \partial_I a = T_{IJ} \beta^J(g), \quad \beta^J(g) = \mu \frac{d}{d\mu} g^J$$

$$\Rightarrow \mu \frac{d}{d\mu} a = \beta^I \partial_I a = G_{IJ} \beta^I \beta^J, \quad G_{IJ} = T_{(IJ)}.$$

The existence of the a-function also imposes constraints on the β-function coefficients[Antipin,Gillioz,Krog,Molgaard,Sannino; Poole, Thomsen; Steudtner]. For the general fermion-scalar theory there are no constraints at one loop and 4 at two loops. However only 2 are independent of those found already. Altogether we only have two 2-loop coefficients undetermined in the general theory!

Constraints *on* $N = 1$ β-functions

At three loops there are 143 coefficients, 106 $N = \frac{1}{2}$ conditions and 42 *a*-function conditions. However the “overlap” has not been determined $\Rightarrow$ don’t know how many coefficients are unconstrained.

Constraints on $N = 1$ β-functions

Now look at constraints imposed by a-function on β-functions of $N = 1$ supersymmetry. In fact, up to three loops there are no constraints from the mere existence of a. However, a proposed explicit all-orders result for the $N = 1$ supersymmetric a-function is

$$a = \tfrac{1}{12}\, n_C - \frac{1}{2}\,\mathrm{tr}(\gamma^2) + \tfrac{1}{3}\,\mathrm{tr}(\gamma^3) + \Lambda \circ \beta_{\bar{Y}} + \beta_Y \circ H \circ \beta_{\bar{Y}}$$

[Barnes, Intriligator, Wecht, Wright; Kutasov and Schwimmer; Osborn, Freedman] where

$$Y \circ \bar{Y} = Y^{ijk}\,\bar{Y}_{ijk}$$

Constraints on $N = 1$ β-functions

a-function constraint can be written

$$d_Y a \equiv dY \circ \frac{\partial}{\partial Y} a = dY \circ T \circ \beta_{\bar{Y}}$$

Assume $H = 0$. We have

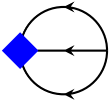
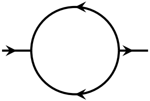
$$d_Y a = \mathrm{tr}[d_Y \gamma((\bar{Y}\Lambda) - \gamma + \gamma^2)] + d_Y \Lambda \circ \beta_{\bar{Y}},$$

where $(\bar{Y}\Lambda)^i{}_j = \bar{Y}_{klj}\Lambda^{(klj)}$. This will satisfy the consistency condition if

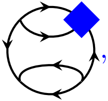
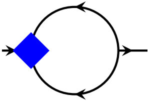
$$(\bar{Y}\Lambda) = \gamma - \gamma^2 + \Theta \circ \beta_{\bar{Y}}$$

for some Θ (sufficient but not necessary). Call this the "Λ-equation".

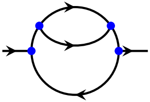
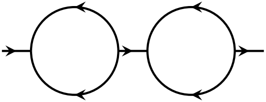
Constraints on $N = 1$ β-functions

$\Lambda^{(1)} \circ \beta_{\bar{Y}} = \lambda^{(1)}$ , $\Rightarrow (\bar{Y}\Lambda)^{(1)} = 3\lambda^{(1)}$ 

Λ-equation $\Rightarrow (\bar{Y}\Lambda)^{(1)} = (\gamma^S)^{(1)} \Rightarrow \lambda^{(1)} = \frac{1}{6}$.

$\Lambda^{(2)} \circ \beta_{\bar{Y}} = \lambda^{(2)}$ , $\Theta^{(1)} \circ \beta_{\bar{Y}} = \theta^{(1)}$ 

Λ-equation $\Rightarrow (\bar{Y}\Lambda)^{(2)} - \Theta^{(1)} \circ \beta^{(1)}_{\bar{Y}}$

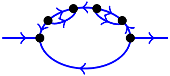
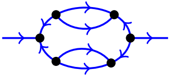
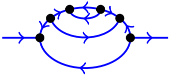
$= (\lambda^{(2)} - \frac{1}{2}\theta^{(1)})\Big(2$  $+$  $\Big)$

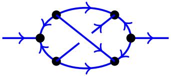
$= \gamma^{S(2)} - (\gamma^{S(1)})^2$

$\Rightarrow \lambda^{(2)} - \frac{1}{2}\theta^{(1)} = -\frac{1}{4} \Rightarrow \gamma^S_2 = -\frac{1}{2}$.

Constraints *on* $N = 1$ β-functions

At three loops

$\gamma^{S(3)} = \gamma^S_{3A}$  $+ \gamma^S_{3B}$  $+ \gamma^S_{3C}$ 

$+ \gamma^S_{3D}$  ,

Constraints on $N = 1$ β-functions

The Λ-equation requires

$$\gamma^S_{3A} - 2\gamma^S_{3B} - \frac{1}{2}\gamma^S_{3C} = -\frac{1}{2},$$

which is satisfied by the DRED results

$$\gamma^S_{3A} = -\frac{1}{4}, \quad \gamma^S_{3B} = -\frac{1}{8}, \quad \gamma^S_{3C} = 1, \quad \gamma^S_{3D} = \frac{3}{2}\zeta(3).$$

At four loops the Λ equation imposes 3 constraints, which are all satisfied.

Conclusions

- The existence of $N = 1$ and even more $N = \frac{1}{2}$ supersymmetry can be used to determine a majority of the coefficients in the RG functions; especially when augmented with the existence of the a-function.
- The proposed explicit form for the $N = 1$ a-function is consistent with RG results up to four loops.
- Recently the $N = 1$ anomalous dimension has been computed to 5 loops for the Wess-Zumino model[Gracey], providing further data for testing this result.