Twisted Differential Geometry and Dispersion Relations in κ-Deformed Cosmology

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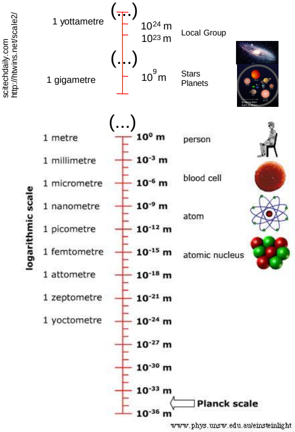
based on results developed in collaboration with
P. Aschieri and A. Pachoł
in J. High Energ. Phys. 2017, 152 (2017) [arXiv:1703.08726]
and JCAP 04 (2021) 025 [arXiv:2009.01051].

Plan:

1. Motivation and general framework
2. Part I: Flat spacetime
   Poincaré Casimir and twisted observables
   Twisted differential calculus
3. Part II: Noncommutative cosmology
   - Deformed wave equation in FLRW background
   - Dispersion relations

Quantum Gravity

gravitational interactions of matter and energy described by quantum theory



General Relativity

Quantum Mechanics

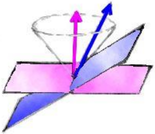
Quantum Gravity
= *quantum effects and gravitational interactions are equally strong*

It happens at the **Planck scale**
(unless extra dimensional theories are correct).

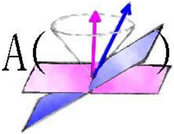
Noncommutative Geometry: origin of quantum space-times

points on manifold M ⟶ algebra of functions on M

http://visualrelativity.com



Minkowski spacetime (M, η) with the position of an event $p = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$



"Minkowski algebra" (abelian, associative,unital) with coordinate functions $x^\mu \in C^\infty(\mathbb{R}^4)$
$x^\mu : \mathbb{R}^4 \to \mathbb{C}$
$[x^\mu, x^\nu] = 0\,, \quad \mu, \nu = 0, 1, 2, 3.$

At the Planck scale $\quad x^\mu \to \hat{x}^\mu$

Example: **κ - Minkowski space-time**

$$[\hat{x}^0, \hat{x}^k] = \frac{i}{\kappa}\hat{x}^k \quad , \quad [\hat{x}^i, \hat{x}^k] = 0$$

The phase space of Quantum Mechanics

$$[x^\mu, P_\nu] = i\delta^\mu_\nu \hbar, \qquad [x^\mu, x^\nu] = 0, \qquad [P_\mu, P_\nu] = 0$$

Generated by the position $\boldsymbol{x^\mu}$ and momentum $\boldsymbol{P_\mu}$ generators
(the Heisenberg algebra)

admitting Hilbert space operator representation (CCR)

- Archetype of a **noncommutative space.**

Replacing 'space' by a **noncommutative algebra**.

- The Heisenberg uncertainty principle: $\Delta x^\mu \Delta P_\nu \geq \frac{\hbar}{2}$.

- Noncommutative geometry - generalised notion of geometry taking into account noncommuatative algebraic structure
- The deformation nature allows for obtaining quantum gravitational corrections to the classical (commutative) solutions.
- Can be helpful in providing the phenomenological models quantifying the effects of quantum gravity.
- One of the mostly studied possible phenomenological effects of quantum gravity is the **modification in wave dispersion**. Such investigations were inspired by the observations of gamma ray bursts (GRBs).

Quantum symmetries

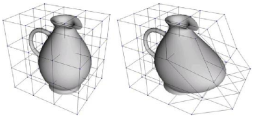
Deformed relativistic symmetries = **Hopf algebras**
quantum spacetimes = **Hopf module algebras**

- Hopf algebra $H(\mu, \eta, \Delta, \epsilon, S)$ is a structure composed by
  1. a (unital associative) algebra (H, μ, η)
  2. a (counital coassociative) coalgebra (H, Δ, ϵ)

  with $S : H \to H$ the antipode.

From any Lie algebra g one can make a Hopf algebra

$$H = (Ug, \Delta_0, S_0, \epsilon)$$



Quantum deformations

Classical $\xrightarrow[\mathcal{F} \in (Ug \otimes Ug)[[h]]]{\text{deformation = quantization}}$ Quantum

Space-time

$$\mathcal{A} = (C^\infty(M), \mu) \qquad [x^\mu, x^\nu] = 0$$

$$\xrightarrow{x^\mu \to \hat{x}^\mu}$$

$$\mathcal{A}^\mathcal{F} = (C^\infty(M)[[h]], \star)$$
$$x^\lambda \star x^\nu \doteq \mu \circ \mathcal{F}^{-1}(x^\lambda \otimes x^\nu)$$
$$[\hat{x}^\lambda, \hat{x}^\nu] = x^\lambda \star x^\nu - x^\nu \star x^\lambda$$

Symmetry

Lie Algebra

Hopf Algebra
(Quantum Group)

$$\mathfrak{g} \longrightarrow \mathcal{H} = (\mathcal{U}(\mathfrak{g}), \Delta_0, \epsilon, S_0)$$

$\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$

Poincare – Weyl '2009
$\mathcal{U}^\mathcal{F}_{\mathfrak{pw}} \quad (M_{\mu\nu}, P_\mu, D)$

Conformal '2015
$\mathcal{U}^\mathcal{F}_{\mathfrak{so}(2,4)} \quad (M_{\mu\nu}, P_\mu, D, K_\mu)$

Inhom. Gen. Lin. '2009
$\mathcal{U}^\mathcal{F}_{\mathfrak{igl}(n)} \quad (L_{\mu\nu}, P_\mu)$

Lie algebra of vector fields as Hopf algebra

- **Deformations of spacetime symmetries** - Lie algebra g of vector fields ξ
- In the coordinate basis: $\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \partial_\mu$.
- This algebra generates the diffeomorphism symmetry; one can also consider subalgebras of g like Poincaré algebra or conformal algebra as symmetry.
- Universal enveloping algebra $U\Xi$ of vector fields includes linear differential operators.

Lie algebra of vectors fields as Hopf algebra

1. Ug as Hopf algebra $(Ug, \Delta_0, \epsilon, S_0)$, for $\xi \in g$ (in the coordinate basis : $\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \partial_\mu$):

$$
\begin{aligned}
[\xi, \eta] &= (\xi^\mu \partial_\mu \eta^\rho - \eta^\mu \partial_\mu \xi^\rho)\partial_\rho, \\
\Delta_0(\xi) &= \xi \otimes 1 + 1 \otimes \xi, \\
\varepsilon(\xi) &= 0, \quad S(\xi) = -\xi.
\end{aligned}
$$

Lie algebra of vector fields as Hopf algebra

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\varepsilon(\xi) &= 0, \quad S(\xi) = -\xi.
\end{aligned}$$

2. The **module algebra** $\mathcal{A} \ni x^\mu, x^\nu$ is an underlying spacetime of given symmetry:

$$\xi \triangleright (x^\mu \cdot x^\nu) = (\xi_1 \triangleright x^\mu) \cdot (\xi_2 \triangleright x^\nu)$$

where $\Delta(\xi) = \xi_1 \otimes \xi_2$ (Sweedler notation).

Drinfeld twisting techniques

provides quantized universal enveloping algebras (= Quantum Groups)

$$\begin{tikzcd} (Ug, \mathcal{A}) \arrow[r, bend left] & (Ug^{\mathcal{F}}, \mathcal{A}^{\mathcal{F}}) \arrow[l] \end{tikzcd}$$

The twist $\mathcal{F}$ is an invertible element of $(Ug \otimes Ug)[[h]]$.

$$\mathcal{F} = 1 \otimes 1 + \mathcal{O}(h), \qquad (\epsilon \otimes id)\mathcal{F} = 1 = (\epsilon \otimes id)\mathcal{F}$$

satisfying normalization and two-cocycle condition

$(\mathcal{F}_{12} = \mathcal{F} \otimes 1\,, \mathcal{F}_{23} = 1 \otimes \mathcal{F})$

$$\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F} \in (Ug \otimes Ug \otimes Ug)[[h]]\,.$$

It provides **quantum (triangular)** R-**matrix**: $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$; $\mathcal{R}^{-1} = \mathcal{R}_{21}$ satisfying the quantum Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\,.$$

Notation: $\mathcal{F} = \mathrm{f}^1 \otimes \mathrm{f}_2 = \sum_{\alpha=0}^{\infty}(\mathrm{f}^\alpha \otimes \mathrm{f}_\alpha)h^\alpha, \quad \mathcal{F}^{-1} = \bar{\mathrm{f}}^1 \otimes \bar{\mathrm{f}}_2 = \mathrm{f}_2 \otimes \mathrm{f}^1,$

(sum over $\alpha = 1, 2, \ldots$ assumed, in fact infinite formal power series in h)

$\mathrm{f}^\alpha, \bar{\mathrm{f}}^\alpha \in Ug$

Twist quantization can be applied to any **bigger Lie algebra** $g \subset g'$.

The twist changes the symmetry to twisted symmetry (as deformed Hopf algebra) $Ug^{\mathcal{F}}$

$$
\begin{aligned}
[\xi, \eta] &= (\xi^\mu \partial_\mu \eta^\rho - \eta^\mu \partial_\mu \xi^\rho)\partial_\rho, \\
\Delta^{\mathcal{F}}(\xi) &= \mathcal{F}\Delta_0(\xi)\mathcal{F}^{-1} \\
\varepsilon(\xi) &= 0, \quad S^{\mathcal{F}}(\xi) = \mathrm{f}^1 S_0(\mathrm{f}_2) S_0(\xi) S_0(\bar{\mathrm{f}}^{1'})\bar{\mathrm{f}}_{2'}
\end{aligned}
$$

- the algebra $([\cdot,\cdot])$ remains undeformed;
- the deformation depends on formal parameter h; which provides an undeformed case at the **zero-th order** in the deformation parameter h.

Coassociativity of the deformed coproduct and associativity of the star-multiplication is ensured by the two-cocycle condition.

Star-product (deformation of module algebras)

$$A = (C^\infty(M), \cdot) \qquad \Longrightarrow \qquad A^{\mathcal{F}} = (C^\infty(M)[[h]], \star)$$

the algebra of smooth functions becomes a **noncommutative spacetime** with the twisted $\star$-product

$$x^\mu \star x^\nu \quad = \quad \cdot \quad \mathcal{F}^{-1}(x^\mu \otimes x^\nu) = \bar{\mathrm{f}}^1(x^\mu)\bar{\mathrm{f}}_2(x^\nu)$$

$x^\mu, x^\nu \in C^\infty(M)$.

- such $\star$-product is noncommutative and associative.
- $A^{\mathcal{F}}$ can be represented by deformed, $\star$-commutators of noncommutative coordinates:

$$[\hat{x}^\mu, \hat{x}^\nu] = [x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\mu \star x^\mu$$

Examples of quantum (noncommutative) spacetimes

1. Canonical (Moyal-Weyl) spacetime A_θ:
$$[\hat{x}^\mu, \hat{x}^\nu] = ih\theta^{\mu\nu}$$
with deformation parameter h of length^2 (L_P) dim.

S. Doplicher, K. Fredenhagen, J. E. Roberts, Commun. Math. Phys. 172 (1995), [arXiv:hep-th/0303037].

2. Lie-algebraic type spacetime:
$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{i}{\kappa}\theta^{\mu\nu}_\rho \hat{x}^\rho$$
with deformation parameter $\kappa = L_p^{-1}$ of mass (M_P) dim.

Special case: A_κ

$$[\hat{x}^0, \hat{x}^k] = \frac{i}{\kappa}\hat{x}^k \quad , \quad [\hat{x}^i, \hat{x}^k] = 0$$

- the so-called: κ-Minkowski spacetime.

S. Majid, H. Ruegg Phys.Lett. B334 (1994) [hep-th/9405107] ; S. Zakrzewski J. Phys. A 127 (1994).

Twisted generators

P. Aschieri, A. Schenkel, Adv. Theor. Math. Phys. 18 3 (2014), arXiv:1210.0241.

Within the Hopf algebra $\mathcal{H}^{\mathcal{F}} = \left(Ug^{\mathcal{F}}, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}}\right)$ we can introduce a notion of **quantum Lie algebra** g^F.

- g and of $g^{\mathcal{F}}$ are in 1-1 correspondence, for all $\chi \in g$ we have

$$\chi^{\mathcal{F}} = \bar{\mathrm{f}}^1(\chi)\,\bar{\mathrm{f}}_2 \in g^{\mathcal{F}}$$

  where $\xi(\chi) = \xi_1 \chi S(\xi_2)$ is the Ug adjoint action.

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  where $\xi(\chi) = \xi_1 \chi S(\xi_2)$ is the Ug adjoint action.
- the subspace $g^{\mathcal{F}}$ generates $Ug^{\mathcal{F}}$.
- has a deformed Lie bracket $[\ ,\ ]_{\mathcal{F}} : g^{\mathcal{F}} \otimes g^{\mathcal{F}} \to g^{\mathcal{F}}$ - given by the **adjoint action** of $Ug^{\mathcal{F}}$:

$$[\chi, \xi]_{\mathcal{F}} = \chi_{1^{\mathcal{F}}} \xi S^{\mathcal{F}}(\chi_{2^{\mathcal{F}}}) \in g^{\mathcal{F}}$$

  where $\Delta^{\mathcal{F}}(\chi) = \chi_{1^{\mathcal{F}}} \otimes \chi_{2^{\mathcal{F}}}$.

Twisted differential calculus - general framework

[S. Majid, R. Oeckl, Commun.Math.Phys. 205 (1999)
arXiv:math/9811054
P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess , Class.Quant.Grav. 23 (2006)
arXiv:hep-th/0510059]

The star-product between functions $g \in C^\infty(M)$ and 1-forms $\omega \in \Omega^r(M)$:

$$g \star \omega = \bar{\mathrm{f}}^1(g)\bar{\mathrm{f}}_2(\omega)$$

- the action of $\bar{\mathrm{f}}_\alpha$ - via the Lie derivative;
- Cartan's (magic) formula for the Lie derivative along the vector field ξ of an arbitrary form ω

  $$\mathcal{L}_\xi \omega = \mathrm{d} i_\xi \omega + i_\xi \mathrm{d}\omega.$$

  where d is the exterior derivative and i_ξ is the contraction along the vector field ξ.
- The $\star$-wedge product on two arbitrary forms ω and ω' is

  $$\omega \wedge_\star \omega' = \bar{\mathrm{f}}^1(\omega) \wedge \bar{\mathrm{f}}_2(\omega') = (-1)^{|\omega||\omega'|}\bar{\mathrm{r}}^1(\omega) \wedge_\star \bar{\mathrm{r}}_2(\omega')$$

In general, Cartan exterior differential $\mathrm{d}: A \to \Omega$

$$\mathrm{d}f = (\partial_\mu f)\mathrm{d}x^\mu$$

is consistent with the quantized wedge product $\wedge_\star$ for any Drinfeld twist and many formulas known from standard differential geometry can be generalized to the new setting, , e.g.

$$d(\omega \wedge_\star \omega') = d\omega \wedge_\star \omega' + (-1)^{|\omega|}\omega \wedge_\star d\omega'$$

- In particular

$$\begin{aligned} \mathrm{d}(f \star g) &= \mathrm{d}f \star g + f \star \mathrm{d}g, \\ \mathrm{d}^2 &= 0, \end{aligned}$$

- For twist quantization of other geometric objects on manifolds and their morphisms, see P. Aschieri et al.

Part I
Flat spacetime (/background independent)

Symmetry

- Poincaré-Weyl-Lie algebra

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\lambda}] &= i(\eta_{\mu\lambda}M_{\nu\rho} - \eta_{\nu\lambda}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\rho}M_{\nu\lambda}), \\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) \quad , \qquad [P_\mu, P_\lambda] = 0, \\
[D, P_\mu] &= iP_\mu \quad , \quad [D, M_{\mu\nu}] = 0.
\end{aligned}
$$

The differential representation of the generators of Poincaré-Weyl algebra is

$$
P_\mu = -i\partial_\mu \quad ; \quad M_{\mu\nu} = -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \quad ; \quad D = -ix^\mu\partial_\mu
$$

Symmetry

- Poincaré-Weyl-Lie algebra

$$\begin{aligned}[M_{\mu\nu}, M_{\rho\lambda}] &= i(\eta_{\mu\lambda}M_{\nu\rho} - \eta_{\nu\lambda}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\rho}M_{\nu\lambda}),\\ [M_{\mu\nu}, P_\rho] &= i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) \quad , \qquad [P_\mu, P_\lambda] = 0,\\ [D, P_\mu] &= iP_\mu \quad , \quad [D, M_{\mu\nu}] = 0.\end{aligned}$$

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$$P_\mu = -i\partial_\mu \quad ; \quad M_{\mu\nu} = -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \quad ; \quad D = -ix^\mu\partial_\mu$$

Universal enveloping algebra of Poincaré-Weyl algebra - as Hopf algebra :

$$\Delta_0(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}$$

$$\Delta_0(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu \quad \text{and} \quad \Delta_0(D) = D \otimes 1 + 1 \otimes D$$

with antipodes

$$S(M_{\mu\nu}) = -M_{\mu\nu};\ S(P_\mu) = -P_\mu;\ S(D) = -D$$

and counits

$$\epsilon(M_{\mu\nu}) = \epsilon(P_\mu) = \epsilon(D) = 0$$

Jordanian twist $[D, P_0] = i\,P_0$

A.B., A.Pachoł, Phys.Rev.D79:045012 (2009) [arXiv:0812.0576].

For the deformation we can use Jordanian twist (with support in Poincaré-Weyl Hopf algebra)

- $$\mathcal{F} = \exp\left(-iD \otimes \sigma\right) \quad ; \quad \sigma = \ln\left(1 + \frac{1}{\kappa} P_0\right)$$

- κ - deformation parameter (classical limit when $\kappa \to \infty$)

- it provides

$$[x^0, x^k]_\star = x^0 \star x^k - x^k \star x^0 = \frac{i}{\kappa} x^k \quad , \quad [x^i, x^k]_\star = 0$$

κ-Minkowski spacetime

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For the deformation we can use Jordanian twist (with support in Poincaré-Weyl Hopf algebra)

- $$\mathcal{F} = \exp\left(-iD \otimes \sigma\right) \quad ; \quad \sigma = \ln\left(1 + \frac{1}{\kappa} P_0\right)$$

- $$\mathcal{F}^{-1} = 1 \otimes 1 + iD \otimes \frac{1}{\kappa} P_0 + \frac{1}{2} iD(iD - 1) \otimes \frac{1}{\kappa^2} P_0^2 \; + \ldots$$

- κ - deformation parameter (classical limit when $\kappa \to \infty$)

- it provides

$$[x^0, x^k]_\star = x^0 \star x^k - x^k \star x^0 = \frac{i}{\kappa} x^k \quad , \quad [x^i, x^k]_\star = 0$$

κ-Minkowski spacetime

Twisted generators

P. Aschieri, A.B., A.Pachoł,JHEP 152 (2017) [arXiv:1703.08726].

- Twisted generators of Poincaré-Weyl algebra:

$$P_\mu^{\mathcal{F}} = \bar{\mathrm{f}}^\alpha(P_\mu)\bar{\mathrm{f}}_\alpha = P_\mu \frac{1}{1+\frac{1}{\kappa}P_0}$$

$$M_{\mu\nu}^{\mathcal{F}} = M_{\mu\nu} \quad ; \quad D^{\mathcal{F}} = D$$

- Twisted Poincaré Casimir from $P_\mu^{\mathcal{F}}$

$$\Box^{\mathcal{F}} = P_\mu^{\mathcal{F}} P^{\mu\mathcal{F}} = P_\mu P^\mu \frac{1}{\left(1+\frac{1}{\kappa}P_0\right)^2}$$

- Twisted commutation relations

$$\begin{aligned}
\left[\Box^{\mathcal{F}}, P_\mu^{\mathcal{F}}\right]_{\mathcal{F}} &= 0 = \left[\Box^{\mathcal{F}}, M_{\mu\nu}^{\mathcal{F}}\right]_{\mathcal{F}} \\
\left[\Box^{\mathcal{F}}, D^{\mathcal{F}}\right]_{\mathcal{F}} &= -2i\Box^{\mathcal{F}}
\end{aligned}$$

Twisted Poincaré Casimir

- Poincaré Casimir $\Box = P_\mu P^\mu$ can be deformed through twist into:

$$\Box^{\mathcal{F}} = \frac{P_\mu P^\mu}{\left(1 + \frac{1}{\kappa} P_0\right)^2}$$

- This type of invariant on momentum space leading to deformed dispersion relation was already considered in DSR framework.

[J. Magueijo and L. Smolin in Phys.Rev.Lett.88 (2002), hep-th/0112090; and in Phys.Rev.D67 (2003), gr-qc/0207085.]

Twisted observables

Twisted generators $X^{\mathcal{F}} \in g^{\mathcal{F}}$ as the observables.

- given $\mathcal{F} \longrightarrow$ unique $g^{\mathcal{F}}$;
- $X^{\mathcal{F}}$ act on fields as **quantum infinitesimal transformations**;

- they are the generators of the twisted Lie algebra $g^{\mathcal{F}}$ and are closed under the twisted commutator $[\cdot,\cdot]_{\mathcal{F}}$:

$$\begin{aligned}
\left[M^{\mathcal{F}}_{\rho\lambda}, M^{\mathcal{F}}_{\mu\nu}\right]_{\mathcal{F}} &= -i(\eta_{\mu\lambda}M^{\mathcal{F}}_{\nu\rho} - \eta_{\nu\lambda}M^{\mathcal{F}}_{\mu\rho} + \eta_{\nu\rho}M^{\mathcal{F}}_{\mu\lambda} - \eta_{\mu\rho}M^{\mathcal{F}}_{\nu\lambda}),\\
\left[M^{\mathcal{F}}_{\mu\nu}, P^{\mathcal{F}}_{\rho}\right]_{\mathcal{F}} &= i(\eta_{\nu\rho}P^{\mathcal{F}}_{\mu} - \eta_{\mu\rho}P^{\mathcal{F}}_{\nu}),\\
[P^{\mathcal{F}}_{\mu}, P^{\mathcal{F}}_{\lambda}]_{\mathcal{F}} &= 0 \quad , \quad \left[D^{\mathcal{F}}, P^{\mathcal{F}}_{\lambda}\right]_{\mathcal{F}} = iP^{\mathcal{F}}_{\lambda}.
\end{aligned}$$

- $P^{\mathcal{F}}_{\mu}$ are Hermitean.

Twisted observables

- $P_\mu^{\mathcal{F}}$ - have the interpretations as the generators of infinitesimal (deformed) translations;
- we confirm this by recalling their associated differential geometry (Part II);
- they allow us to define the appropriate (Poincaré) Casimir operator in the twisted Lie algebra:

$$\left[\Box^{\mathcal{F}}, P_\mu^{\mathcal{F}}\right]_{\mathcal{F}} = 0 = \left[\Box^{\mathcal{F}}, M_{\mu\nu}^{\mathcal{F}}\right]_{\mathcal{F}} \quad , \quad \left[\Box^{\mathcal{F}}, D^{\mathcal{F}}\right]_{\mathcal{F}} = -2i\Box^{\mathcal{F}}$$

Dispersion relation: Flat spacetime

- Deformed wave equation: $\square^{\mathcal{F}}\phi = P^{\mathcal{F}}_{\mu} P^{\mu\mathcal{F}}\phi$ for massless particles is equivalent to $\square\phi = 0$.
- The energy-momentum dispersion relations $P^{\mathcal{F}}_{\mu} P^{\mu\mathcal{F}} = 0$ are undeformed.
- The group velocity $v_g = \frac{d\omega}{dk} = c$ is as in the classical case due to the fact that the (usual) plane waves are the 'eigenvectors' of the twisted observables.
- $P^{\mathcal{F}}_{\mu} e^{ik_{\mu}x^{\mu}} = k^{\mathcal{F}}_{\mu} e^{ik_{\mu}x^{\mu}}$ evaluation of the energy momentum operator on the monochromatic wave leads to modified Einstein -Planck relatons:

$$E^{\mathcal{F}} = \omega^{\mathcal{F}} = \frac{\omega}{1 - \frac{i}{\kappa}\omega} \quad \text{and} \quad \mathbf{p}^{\mathcal{F}} = \frac{\mathbf{k}}{1 - \frac{i}{\kappa}\omega}$$

$\kappa \to E_p$ (Planck energy).

Differential calculus deformed with Jordanian twist

P. Aschieri, A.B., A.Pachoł,JHEP 152 (2017) [arXiv:1703.08726].

- For the **twisted differential calculus** we use the coordinate basis where the basis 1-forms are denoted as $\mathrm{d}x^{\mu}$.

- The action of a vector fields in the twist is via Lie derivative:

$$\mathcal{L}_{P_{\mu}}(\mathrm{d}x^{\nu}) \;=\; 0, \quad \mathcal{L}_{D}(\mathrm{d}x^{\mu}) = -i\mathrm{d}x^{\mu}$$

  since $d(\partial_0 x^{\mu}) = 0$.

- Using these relations one can show that the basis 1-forms anticommute:

$$\mathrm{d}x^{\mu} \wedge_{\star} \mathrm{d}x^{\nu} = \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}$$

  Therefore we have:

$$\mathrm{d}x^{\mu} \wedge_{\star} \mathrm{d}x^{\nu} = \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} = -\mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu} = -\mathrm{d}x^{\nu} \wedge_{\star} \mathrm{d}x^{\mu}$$

In our particular model, one gets

$$\begin{aligned} f \star \mathrm{d}x^\mu &= f \mathrm{d}x^\mu \\ \mathrm{d}x^\mu \star f &= \mathrm{d}x^\mu (1 + \frac{1}{\kappa} P_0) f \end{aligned}$$

Therefore:

$$[f, \mathrm{d}x^\mu]_\star = \frac{i}{\kappa} \mathrm{d}x^\mu \partial_0 f$$

Part II
Noncommutative cosmology

- Noncommutative differential geometry approach is based on Drinfeld twist ($\mathcal{F}$) deformation.

- Can be implemented for any twist ($\mathcal{F}$) and any curved background (g).

- Toy model:
  Jordanian twist - giving κ-Minkowski spacetime in flat space ($[x^0, x^k]_\star = x^0 \star x^k - x^k \star x^0 = \frac{i}{\kappa} x^k$)
  - in the presence of a Friedman-Lemaitre-Robertson-Walker (FLRW) cosmological background (in 2D).

Wave equation in curved spacetime

- The wave equation in curved spacetime is governed by the Laplace-Beltrami operator (for Lorenzian even dimensional manifolds):

$$\Box_{LB}\varphi = *\mathrm{d} * \mathrm{d}\varphi\,, \quad (+\mathrm{d} * \mathrm{d} * \varphi = 0)$$

- The Laplace-Beltrami operator is a generalization to curved spacetime of the D'Alembert operator and on a scalar field φ we have (using local coordinates)

$$\Box_{LB}\varphi = *\mathrm{d} * \mathrm{d}\varphi = \frac{1}{\sqrt{g}}\partial_\nu\left[\sqrt{g}\,g^{\nu\mu}\partial_\mu\varphi\right]$$

Hodge star deformed

A linear map $* : \Omega^r(M) \to \Omega^{n-r}(M)$. In local coordinates for an r-form is given by

$$*\omega = \frac{\sqrt{g}}{r!\,(n-r)!}\omega_{\mu_1 \dots \mu_r}\epsilon^{\mu_1 \dots \mu_r}{}_{\nu_{r+1} \dots \dots \nu_n}\; dx^{\nu_{r+1}} \wedge \dots \; dx^{\nu_n}$$

where $\sqrt{g}$ is the square root of the absolute value of the determinant of the metric, the completely antisymmetric tensor $\epsilon_{\nu_1 \dots \nu_n}$ is normalized to $\epsilon_{1 \dots n} = 1$ and indices are lowered and raised with the metric g and its inverse.

- The deformation of the Hodge $*$ operation is explicitly dependent on the twist form:

$$*^{\mathcal{F}} = \bar{\mathrm{f}}^{1}_{(1)} \triangleright \circ * \circ S\left(\bar{\mathrm{f}}^{1}_{(2)}\right) \triangleright \circ \bar{\mathrm{f}}_{2} \triangleright$$

- For Jordanian twist ($\mathcal{L}_{P_\nu}(\mathrm{d}x^\mu) = 0$) the non vanishing is only the zero-th order:

$$*^{\mathcal{F}}\left(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_s}\right) = *\left(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_s}\right)$$

Deformed Laplace-Beltrami operator

P. Aschieri, A. B., A.Pachoł, [arXiv:2009.01051].

- Deformation of the Laplace-Beltrami operator for any twist:

$$\Box^{\mathcal{F}}_{LB}\varphi = *^{\mathcal{F}} d *^{\mathcal{F}} d\varphi$$

- The wave equation for the scalar field in terms of twisted momenta for deformed LB op. with Jordanian twist:

$$\Box^{\mathcal{F}}_{LB}\varphi = \frac{1}{\sqrt{g}} \star \frac{\partial^{\mathcal{F}}_{\rho}}{\left(1+\frac{i}{\kappa}\partial^{\mathcal{F}}_{0}\right)^{1-n}} \left((\sqrt{g}g^{\mu\rho}) \star \frac{\partial^{\mathcal{F}}_{\mu}}{\left(1+\frac{i}{\kappa}\partial^{\mathcal{F}}_{0}\right)^{n-1}}\varphi \right) = 0$$

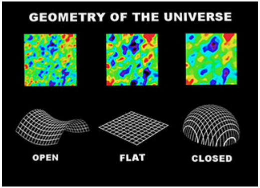
Solutions of deformed wave eq. for FRWL metric

Friedman-Robertson-Walker-Lemaitre (FRWL) metric

(for simplicity in 2 dimensions)

$$g = -dt^2 + a^2(t)\,dx^2$$

where $a(t)$ - scale factor



2-dim twisted wave equation

$$-a \star \partial_0^2 \varphi - (\partial_0 a) \star \left(1 - \frac{i}{\kappa}\partial_0\right) \partial_0 \varphi + a^{-1} \star \partial_x^2 \varphi = 0$$

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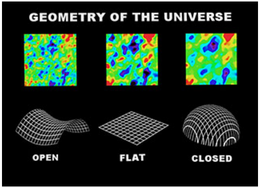

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In the classical limit it reduces to:

$$-a\partial_0^2\varphi - \dot{a}\partial_0\varphi + \tfrac{1}{a}\partial_i^2\varphi = 0 \text{ where } \dot{a} = \partial_0 a(t)$$

Classical version of equation

$$-a\partial_0^2\varphi - \dot{a}\partial_0\varphi + \frac{1}{a}\partial_i^2\varphi = 0$$

- separation of variables: $\varphi = \lambda\,(t)\,e^{-ikx}$
- $$a\ddot{\lambda} + \dot{\lambda}\dot{a} + k^2\lambda\frac{1}{a} = 0$$
- it corresponds (in conformal time $d\eta = \frac{dt}{a}$ to harmonic oscillator type equation

  $$(\partial_\eta^2 + k^2)\lambda = 0$$

  which has the well known solution $\lambda = \exp i\omega\eta$: $\omega^2 = k^2$.

$$\phi = A(k)\exp i(k\eta(t) - kx)\,, \quad v_p = v_g = \frac{d\eta}{dt} = \frac{1}{a}\,.$$

$$v_{ph} = av_g = 1 = c$$

Twisted wave equation

$$a \star \partial_0^2 \varphi + (\partial_0 a) \star \left(1 - \frac{i}{\kappa}\partial_0\right) \partial_0 \varphi - a^{-1} \star \partial_x^2 \varphi = 0$$

- In the noncommutative case in 2 dimensions we consider the solution of the form: $\varphi = \lambda(t) \star e^{-ikx} = \lambda(t)\, e^{-ikx}$

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We simplify the equation as:

$$a \star \partial_0^2 \lambda + \partial_0 (a) \star \left(1 - \frac{i}{\kappa}\partial_0\right) \partial_0 \lambda + a^{-1} \star k^2 \lambda = 0$$

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Expand star-product in the first order of $\frac{1}{\kappa}$

$$a\partial_0^2\lambda + \partial_0(a)\left(1 - \frac{i}{\kappa}\partial_0\right)\partial_0\lambda + a^{-1}k^2\lambda - \frac{i}{\kappa}t\left(\partial_0 a\, \partial_0^3\lambda + \partial_0^2 a\, \partial_0^2\lambda + k^2\partial_0 a^{-1}\partial_0\lambda\right) = 0$$

Conformal time - classical case strategy

- As in the classical case - change the coordinates into conformal time η, and $' = \partial_\eta$
- Introduce simplified notation $s = \ln a; s' = \frac{a'}{a}; \frac{a''}{a} = s'' + (s')^2;$

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- Introduce simplified notation $s = \ln a; s' = \frac{a'}{a}; \frac{a''}{a} = s'' + (s')^2$;
- Look for the (perturbative) solution of the type:

$$\lambda = \exp\left(i\omega\eta + \frac{i}{\kappa}F\right)$$

- Classical part (at 0-th order) remains:

$$\left(\omega^2 - k^2\right)\lambda = 0$$

- And equation on $F(\eta)$ becomes:

(using the zero-th order solution $\omega = k$),

$$F'' + 2ikF' = \frac{ikt(\eta)}{a^2}\left(2\left(s'\right)^3 - 2s's'' - 2k^2 s' + ik\left(s'' - 3\left(s'\right)^2\right)\right) - \frac{ik}{a}s'\left(s' - ik\right) .$$

Group velocity for the wave

Starting from

$$\varphi_k(x,t) = \lambda(t)\star e^{-ikx} = \lambda(t)e^{-ikx} = \exp\left(ik\eta + \frac{i}{\kappa}F\right)e^{-ikx} = e^{i(f_k(t)-kx)}$$

we get:

$$f_k\left(t\right) = \left(k\eta + \frac{1}{\kappa}F\right)(t)$$

Group velocity expression

$$v_g = \frac{\partial x}{\partial t} = \frac{\partial}{\partial k}\frac{\partial f_k(t)}{\partial t}$$

$\Longrightarrow$ we need to compute $\dot{F} = \partial F/\partial t$.

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- easily obtained from the differential equation for F in the physical regime we are interested in: cosmic time related to large scale structure formation, and high frequency waves.
- There are three frequency parameters in the differential equation on F: $\omega = k$, t^{-1} and the Hubble parameter H;
- we obviously have $\omega >> t^{-1}$ for the present cosmic time as well as the cosmic time of emission of the travelling γ-ray, typically at redshift below $z = 10$.
- Similarly $\omega >> H \sim t^{-1}$

In this regime equation for F simplifies to

$$2ikF' = -\frac{2ik^3 ts'}{a^2}$$

$$\dot{F} = -\frac{k^2 t \dot{a}}{a^3} \ .$$

- The group velocity, at the first order in the $\frac{1}{\kappa}$ deformation, results

$$v_g = \frac{\partial x}{\partial t} = \frac{\partial}{\partial k}\frac{\partial f_k(t)}{\partial t} = \frac{1}{a} + \frac{1}{\kappa}\frac{\partial \dot{F}}{\partial k} = \frac{1}{a}\left(1 - \frac{2}{\kappa}\frac{kt\dot{a}}{a^2}\right) = \frac{1}{a}\left(1 - \frac{2}{\kappa}\frac{\omega t\dot{a}}{a^2}\right)$$

- Taking into account the $\frac{1}{a}$ factor due to the comoving coordinates and inserting the flat spacetime speed of light c we see that κ-spacetime noncommutativity in the presence of a FLRW metric leads to a velocity of photons $v_{ph} = v_g a$ given by

$$v_{ph} = c\left(1 - \frac{2}{\kappa}\frac{\omega t\dot{a}}{a^2}\right) .$$

- If we define (as usual) the energy where classical Lorentz violation (in our case Lorentz deformation) is manifested $E_{LV} := |\kappa|\hbar$.
- The variation of the speed of light v_{ph} with respect to the usual one c (of photons in flat spacetime, or of low energetic photons) is then given by

$$|1 - v_{ph}/c| \sim \frac{E_{ph}}{E_{LV}} \frac{2t\dot{a}}{a^2} \quad .$$

Comments on the results

$$v_{ph} = c(1 - \frac{2}{\kappa}\frac{\omega t \dot{a}}{a^2}) \ .$$

- The **combined effects of noncommutativity and gravity** affect the velocity of light by a term linearly dependent on the frequency ω, the cosmic time t, the Hubble parameter $H = \dot{a}/a$ and inversely proportional to the scale factor.
- We have $v_{ph} < c$ for $\frac{1}{\kappa}$ a positive time (as it is usually considered, and in an expansion phase of the universe $\dot{a} > 0$).
- In flat spacetime ($\dot{a} = 0$) as well as in commutative spacetime ($\kappa \to \infty$) there are no modified dispersion relations.
- This result offers an explicit cosmological correction to the usually considered models, which assume as the leading power for the correction to the light speed the expression $v_{ph} \ \sim \ c(1 - \frac{E_{ph}}{E_{LV}})$.
- one can actually estimate the order of magnitude of the variation of the speed of light.

Comments on the results

- We can also study the time lag Δt between the arrival of a low energetic and a high energetic photon emitted simultaneously during a gamma ray burst.
- the comoving distance between the gamma ray burst and the observer is the same for both photons;
- for the high energy photon it reads $\int_{t_{em}}^{t_0+\Delta t} v_g \, dt$
- for the low energy one it reduces to $\int_{t_{em}}^{t_0} \frac{c}{a} \, dt$.
- Equating these distances, and considering only first order corrections we obtain that the time delay Δt is given by

$$\Delta t = \frac{2E_{ph}}{E_{LV}} \int_{t_{em}}^{t_0} \frac{t\dot{a}}{a^3} dt = \frac{2E_{ph}}{E_{LV}} \int_0^z t\,(1+z')dz' \ .$$

- For the range of redshifts we are interested into (up to $z \sim 10$) we can use the analytic solution $a(t) = (1+z)^{-1} = (\frac{\Omega_m}{\Omega_\Lambda})^{1/3} \sinh^{2/3}(t/t_\Lambda)$, $t_\Lambda = \frac{2}{3H_0\sqrt{\Omega_\Lambda}}$ and obtain the time lag

$$\Delta t = 2\frac{E_{ph}}{E_{LV}} t_\Lambda \int_0^z \operatorname{arcsinh}\sqrt{\frac{\Omega_\Lambda}{\Omega_m}(1+z')^{-3}}\ (1+z')dz' \ .$$

- Our model gives a time lag that is ~ 3 times the ones considered in the typical 'Lorentz invariance violation' literature.

Comments on the results

- In the present work, as a first approximation, we have considered a commutative gravity background, hence noncommutativity affects only propagation of light.

- In a noncommutative theory of gravity consistently coupled to light, one should consider the backreaction effects of turning on noncommutativity also on the gravitational field.

Summary

- Quantum spacetimes - motivated by the Planck scale effects.
- Twist deformation and Noncommutative Geometry - allow for obtaining quantum corrections to the classical solutions. Twisted generators as observables.
- Framework is valid not only for the flat spacetimes, but allows for more general **curved background** as well.
- The result that the combined effects of noncommutativity and curvature produce modified dispersion relations is expected to be a **general feature** of wave equations in noncommutative curved spacetime.

Summary

- we used a top-down approach that complements the bottom-up one of phenomenological models.
- we applied noncommutative differential geometry to derive the propagation of waves in noncommutative cosmology.
- we studied a **noncommutative deformation of the wave equation in curved background** and we discuss the modification of dispersion relations due to the presence of both noncommutativity and curvature of spacetime.
- as a first approximation we turn on noncommutativity in the usual (classical) homogeneous and isotropic gravity solution given by FLRW spacetime, and derive the wave equation for massless particles in this context.
- This is a first step toward a more comprehensive approach that encompasses both the dynamics of light and of gravity in a noncommutative spacetime. We have considered a classical gravity background.

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Thank you for your attention!