Non-commutative AdS_2/CFT_1 duality

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1. AdS, dS, H^d, S^d
2. Fuzzy spaces
3. ncEAdS$_2$
4. NC scalar field on ncAdS$_2$
5. Exact results
6. Conclusions

A. Stern, F. Lizzi, F. de Almeida, C. Xu

Maximally symmetric spaces

All are given as embeddings:

$$g_{\mu\nu} X^\mu X^\nu = L^2$$

1) $g_{\mu\nu} = \mathrm{diag}(+1, \dots, +1)$, $L^2 > 0$ — S^d

2) $g_{\mu\nu} = \mathrm{diag}(+1, \dots, +1, -1)$, $L^2 > 0$ — dS^d

Euclidean continuation

3) $g_{\mu\nu} = \mathrm{diag}(+1, \dots, -1, +1)$, $L^2 < 0$ — H^d

4) $g_{\mu\nu} = \mathrm{diag}(+1, \dots, -1, -1)$, $L^2 < 0$ — AdS^d

Euclidean continuation

$$dS^d \simeq S^{d-1} \times \mathbb{R} \quad \& \quad AdS^d \simeq \mathbb{R}^{d-1} \times S^1$$

Specifics of $d=2$

While in general case the isometries are:

$SO(d+1)$ for S^d
$SO(d,1)$ for H^d & dS^d
$SO(d-1,2)$ for AdS^d

when $d=2$, the isometry groups for H^d, dS^d and AdS^d are the same:

$$SO(2,1) \simeq SO(1,2)$$
$$so(2,1) \simeq so(1,2) \simeq su(1,1)$$

Geometry of AdS_2

Define a surface in $\mathbb{R}^{2,1}$:

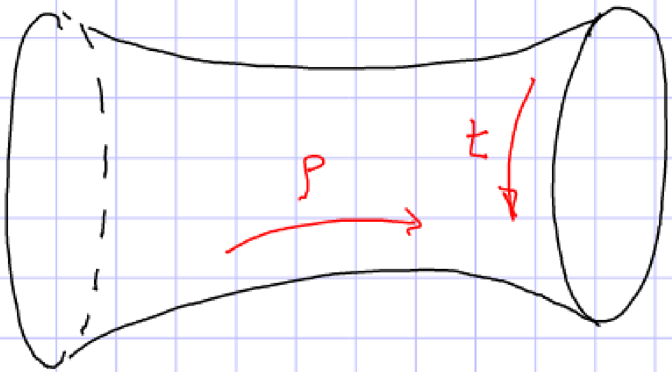
$$g_{\mu\nu} X^\mu X^\nu = -\ell^2, \quad \text{with } \ell^2 \in \mathbb{R}_+$$

Here $g_{\mu\nu} = \mathrm{diag}(-1, 1, -1)$

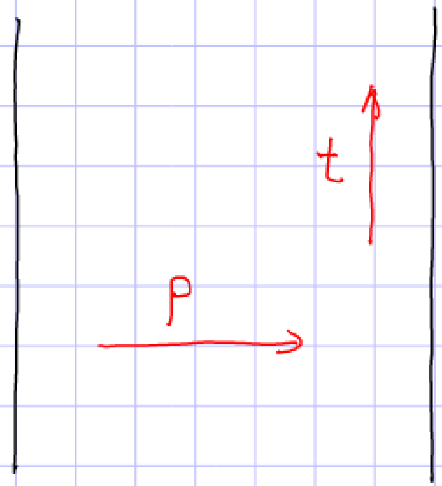
The obvious isometry: $SO(2,1)$

The induced metric in the global coordinates

$$ds^2\Big|_{AdS_2} = \ell^2\left[-\cosh^2\rho \, dt^2 + d\rho^2\right]$$



$\xRightarrow[\text{universal covering}]{CTC}$



Why do we need "fuzziness"?

- The general arguments (Fredenhagen et al 1995) as well as some specific models (strings, LQG) indicate that some kind of noncommutativity should describe (at least quasi-classical effects of) QG.
- Symmetries are very important, especially for AdS/CFT => would be nice to have NC spaces retaining as much symmetry as possible
- Fuzzy sphere is one of the most studied examples, preserving complete SU(2)

Doplicher, Fredenhagen & Roberts:

Algebra of functions (coord's) <=> space-time

i) How to "measure" a point (operational approach)

ii) Commutative limit: Minkowski as $\frac{\ell_p}{L} \to 0$ (possibly as a factor)

iii) Don't touch Poincaré!

$$\Downarrow$$

$$\Delta x_0 (\Delta x_1 + \Delta x_2 + \Delta x_3) \geq \ell_p^2$$

$$\Delta x_1 \Delta x_2 + \Delta x_2 \Delta x_3 + \Delta x_1 \Delta x_2 \geq \ell_p^2$$

Fuzzy Euclidean AdS_2 (Jurman & Steinaker, ...)

$X^\mu X_\mu = -\ell^2$ relation

$[K_\mu, K_\nu] = \epsilon_{\mu\nu\rho} K^\rho$ algebra of Killings

Introduce a Poisson structure:

$$\{X^\mu, X^\nu\} = \epsilon^{\mu\nu\rho} X_\rho$$

Then $K^\mu = \{X^\mu, \cdot\}$ and everything is compatible with AdS_2.

$(x, y)\,, \quad x, y \in \mathbb{R}$

$$\begin{cases} X^0 = -y \\ X^1 = -\frac{1}{2\ell} e^{-x} y^2 + \ell \sinh x \\ X^2 = -\frac{1}{2\ell} e^{-x} y^2 - \ell \cosh x \end{cases}$$

$$ds^2 = \ell^2 dx^2 + (dx - y\,dx)^2$$

$$\begin{cases} K^0 = \partial_x \\ K^1 = \frac{1}{\ell} e^{-x} y \partial_x - X^2 \partial_y \\ K^1 = \frac{1}{\ell} e^{-x} y \partial_x - X^1 \partial_y \end{cases}$$

$$\{x, y\} = 1$$

Fefferman-Graham

$(z, t) \quad z \in \mathbb{R}_+\,, \; t \in \mathbb{R}$

$$z = e^{-x}\,, \quad t = \frac{1}{\ell} e^{-x} y$$

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + dt^2\right)$$

$$\begin{cases} K^- = -\partial_t \\ K^0 = -t\partial_t - z\partial_z \\ K^+ = (z^2 - t^2)\partial_t - 2zt\partial_z \end{cases} \qquad K^\pm = K^2 \pm K^1$$

$$\{t, z\} = \frac{1}{\ell} z^2$$

- (x,y) has a very simple (canonical!) Poisson structure $\Rightarrow$ easy to quantize but the boundary corresponds to $x,y \to \infty$ $\Rightarrow$ non-trivial to analyze
- (z,t) Poisson structure is more complicated, but the boundary is nice: $z = +0$

In particular:

$$\begin{cases} K^- \to -\partial_t & \text{translations} \\ K^0 \to -t\partial_t & \text{dilatations} \\ K^+ \to -t^2\partial_t & \text{SCT} \end{cases}$$

IRR of SO(2,1)

- Specifics of 2^d case: the same isometry for both Minkowski & Euclidean cases.
- $\hat{X}^\mu \hat{X}_\mu = -\ell^2 \mathbb{1}$, $[\hat{X}^\mu, \hat{X}^\nu] = i\alpha \epsilon^{\mu\nu\rho} \hat{X}_\rho$; $(-1,1,-1)$ or $(1,1,-1)$

  $\downarrow$ Casimir of $so(2,1) \simeq su(1,1)$ $\qquad$ $\downarrow$ Mink. $\qquad$ $\downarrow$ Eucl.
- There are three different series of IRR

i) Principal	$k = -\frac{1}{2} - i\rho$	$(k+\frac{1}{2})^2 = \frac{1}{4} + \frac{\ell^2}{\alpha^2}$
ii) Supplemental	$-\frac{1}{2} < k < 0$	
iii) Discrete	$k < 0$	

$\{ |\epsilon_0, k, m\rangle \, , \, m \in \mathbb{Z} \}$ $\qquad$ $X^\mu X_\mu |\epsilon_0, k, m\rangle = -\alpha^2 k(k+1) |\epsilon_0, k, m\rangle$

i) Principal $\Rightarrow \ell^2 < 0 \Rightarrow$ Minkowski

ii) Supplemental $\Rightarrow \ell^2 < 0$, but k cannot go to ∞

$\Rightarrow$ extremely quantum geometry

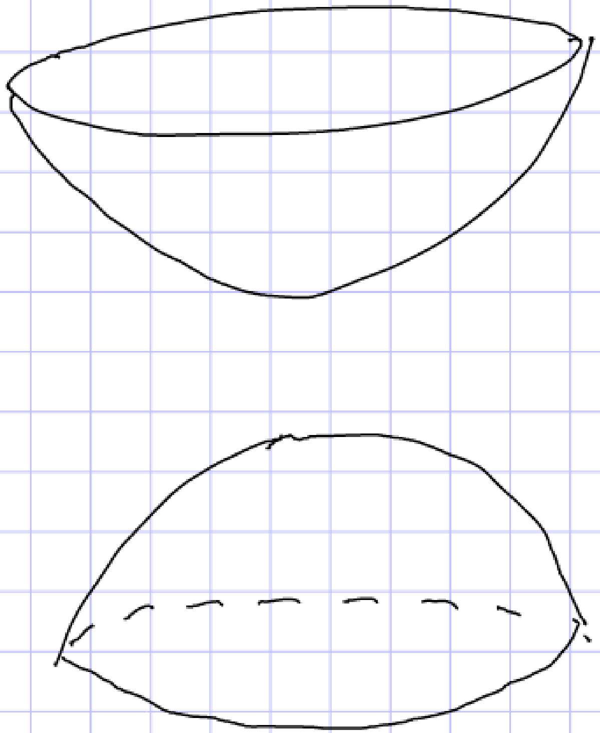
iii) Discrete:

a) $D^+(k)$, $\epsilon_0 = -k$

b) $D^-(k)$, $\epsilon_0 = +k$

$\Bigg\|$ $k < -1$ $\ell^2 > 0 \Rightarrow$ Euclidean

The cases (a) & (b) correspond to two halves of the $EAdS_2$



$D^+(k)$

$\hat{r} = X^1 - X^2 \quad \Rightarrow \quad \langle k,m|\hat{r}|k,m\rangle = \frac{\alpha}{\ell}(\varepsilon_0 + m)$

$\Rightarrow \quad m \to \infty$ is a "boundary"

Diagonalizing $\hat{r}$ one constructs a representation on $L_2(\mathbb{R}_+, dr)$:

$$\begin{cases} \pi^k(X^0) = i\alpha\left(r\frac{d}{dr} + \frac{1}{2}\right) \\ \pi^k(X^1) = \frac{\alpha^2}{2\ell}\left(\frac{d}{dr}r\frac{d}{dr} - \frac{(k+\frac{1}{2})^2}{r} + \frac{\ell^2}{\alpha^2}r\right) \\ \pi^k(X^2) = \frac{\alpha^2}{2\ell}\left(\frac{d}{dr}r\frac{d}{dr} - \frac{(k+\frac{1}{2})^2}{r} - \frac{\ell^2}{\alpha^2}r\right) \end{cases}$$

$$\psi_{k,m}(r) = \sqrt{\frac{m!}{(m-2k-1)!}}\; L_m^{-2k-1}\left(\frac{2\ell r}{\alpha}\right)$$

$$L_2(\mathbb{R}_+, dr) \to L_2(\mathbb{R}, dx) \quad , \; x := \ln r$$

This is the representation of $[x, y] = i\lambda \mathbb{1}$

It can be realized by the standard Moyal star

$$*_M = m \circ e^{\mathcal{F}} \quad , \; \mathcal{F} = \frac{i\lambda}{2} \epsilon^{ij} \partial_i \otimes \partial_j$$

Pros: Very simple to work with.

Cons: (x, y) are not very convenient to study the boundary, (z, t) are much nicer.

Correlators from AdS/CFT

In Fefferman-Graham coordinates $EAdS_2$ looks as

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + dt^2\right) \quad , \; z \in \mathbb{R}_+ \, , \; t \in \mathbb{R}$$

(The boundary is at $z \to 0$)

$$S[\Phi] = \frac{1}{2} \int_{\mathbb{R}\times\mathbb{R}_+} dt\, dz \left([\partial_z \Phi]^2 + [\partial_t \Phi]^2 + \left(\frac{m\ell}{z}\right)^2 \Phi^2 + \frac{2\lambda}{3z^2} \Phi^3 \right)$$

$$\left[\Box - \left(\frac{m\ell}{z}\right)^2\right] \Phi = \lambda \Phi^2 \quad \text{with} \quad \lim_{z\to 0} \phi(z,t) = z^{\Delta_-} \phi_0(t)$$

where $\Delta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + (m\ell)^2}$

and the Breitenlohner-Freedman is assumed

Using the bulk-2-boundary propagator:

$$K(z,t;t') = C_\Delta \left(\frac{z}{z^2+(t-t')^2} \right)^{\Delta_+}$$

and the bulk-2-bulk propagator:

$$G(\xi) = \frac{C_\Delta}{2\Delta_+ - 1} \left(\frac{\xi}{2} \right)^{\Delta} F\left(\frac{\Delta_+}{2}, \frac{\Delta_+}{2} + \frac{1}{2} ; \Delta_+ + \frac{1}{2} ; \xi^2 \right),$$

where $C_\Delta := \dfrac{\Gamma(\Delta_+)}{\sqrt{\pi}\,\Gamma(\Delta_+ - \frac{1}{2})}$, $\xi := \dfrac{2zz'}{z^2+z'^2+(t-t')^2}$

the solution is given by

$$\Phi_{sol}(z,t) = \int_{\mathbb{R}} dt' \, K(z,t;t')\, \phi_0(t') - \lambda \int \frac{dz' dt'}{z'^2} G(z,t;z',t') \times$$

$$\times \int dt_1 \int dt_2 \, K(z',t';t_1)\, K(z',t';t_2)\, \phi_0(t_1)\, \phi_0(t_2) + O(\lambda^2)$$

This gives the on-shell action:

$$S[\phi_0] = \underbrace{-\frac{1}{2}\int_{\mathbb{R}} dt\, \Phi_{sol}[\phi_0]\, \partial_z \Phi_{sol}[\phi_0]\Big|_{z=+0}}_{S^{bdy}[\phi_0]} + \underbrace{\frac{\lambda}{3}\int_{\mathbb{R}\times\mathbb{R}_+} \frac{dt\, dz}{z^2}\, \Phi^3_{sol}[\phi_0]}_{S^{blk}[\phi_0]}$$

Then

$$\langle \mathcal{O}(t_1)\mathcal{O}(t_2)\rangle = \frac{\delta^2 S[\Phi_{sol}]}{\delta\phi_0(t_1)\,\delta\phi_0(t_2)} \equiv \frac{\gamma(\Delta_+)}{|t-t'|^{2\Delta_+}}, \quad \gamma(\Delta_+) = -\frac{2\nu\,\Gamma(\Delta_+)}{\sqrt{\pi}\,\Gamma(\nu)}$$

$$\langle \mathcal{O}(t_1)\mathcal{O}(t_2)\mathcal{O}(t_3)\rangle = \lambda\left(\frac{3\Delta_+}{2\nu}+2\right)\frac{a_{\Delta_+}}{|t_1-t_2|^{\Delta_+}|t_2-t_3|^{\Delta_+}|t_3-t_1|^{\Delta_+}}$$

$$a_{\Delta_+} = \frac{\Gamma(\Delta_+/2)^3\,\Gamma((3\Delta_+-1)/2)}{2\pi\,\Gamma(\nu)^3}$$

Scalar field on $ncAdS_2$

$$S_{NC} = -\frac{1}{2\ell}\,\mathrm{Tr}\left([X^\mu,\Phi][X_\mu,\Phi] - (\alpha\ell m)^2\Phi^2 - \frac{2}{3}\alpha^2\lambda\Phi^3\right) \overset{*_{z,t}}{\Longrightarrow}$$

$$S_{NC} = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}_+} dz\,dt\ \frac{1}{z^2}\left(\frac{\ell t}{z}\Delta_t\phi - (t\partial_t + z\partial_z)S_t\phi\right)^2 + \left(\frac{\alpha^2}{4}+\ell^2\right)\Delta_t\phi^2\Big)$$

$$+\left(\frac{m\ell}{z}\right)^2 \Phi *_{z,t}\Phi + \frac{2\lambda}{3z^2}\Phi *_{z,t}\Phi *_{z,t}\Phi \equiv$$

$$\equiv \frac{1}{2}\int\frac{dz\,dt}{z^2}(K_*^\mu\Phi)(K_{*\mu}\Phi) + \dots$$

$$K_*^- = -\frac{\ell}{z}\Delta_t \qquad\qquad K_*^- = -\partial_t$$

$$K_*^0 = -t\partial_t - z\partial_z \qquad \overset{z\to 0}{\Longrightarrow} \qquad K_*^0 = -t\partial_t$$

$$K_*^+ = -2t(t\partial_t + z\partial_z)S_t + \frac{\ell}{z}(\dots \qquad\qquad K_*^+ = -t^2\partial_t$$

Free case

E.O.M. $\frac{\alpha^2}{2}[\hat{X}_a,[X^a,\Phi]] =: \frac{\alpha^2}{2}\Delta\Phi = \hat{X}_a \Phi \hat{X}^a + \Phi = \frac{\alpha^2 m^2}{2}\Phi$

Defining (somewhat formally):

$\hat{z} := (\hat{X}^2 - \hat{X}^1)^{-1}, \quad \hat{t} := -\frac{1}{2}(\hat{z}\hat{X}^1 + \hat{X}^1\hat{z}) \qquad (\hat{r} := \hat{z}^{-1})$

$\Rightarrow \hat{X}^0 = -\frac{1}{2}(\varkappa^2\hat{z} + \hat{t}\,\hat{z}^{-1}\hat{t} + \hat{z}^{-1}), \quad \hat{X}^1 = -\frac{1}{2}(\hat{z}^{-1}\hat{t} + \hat{t}\,\hat{z}^{-1})$

$\hat{X}^2 = -\frac{1}{2}(\varkappa^2\hat{z} + \hat{t}\,\hat{z}^{-1}\hat{t} - \hat{z}^{-1})$

$\varkappa^2 = 1 + \frac{\alpha^2}{4}$

$$-\hat{r}\left[\hat{t},\hat{r}\left[\hat{t},\Phi\right]\hat{r}\right]\hat{r}-\varkappa^2\left[\hat{r},\left[\hat{r},\Phi\right]\right]=\alpha^2 m^2\hat{r}\Phi\hat{r}$$

$$\Phi=T(\hat{t})\,R(\hat{r})\,T(\hat{t})\quad \text{with}\quad \dot{T}=i\beta T\quad\Rightarrow$$

$$\left[(\alpha^2\beta^2-\hat{r}^2)R'\right]'+\left[m^2-\frac{4\varkappa^2\beta^2}{\alpha^2\beta^2-\hat{r}^2}\right]R=0\qquad \text{Legendre}$$

$$R_{1\pm}=P^{\pm\frac{2}{\alpha}\varkappa}_{\nu-\frac{1}{2}}\left(\frac{\hat{r}}{\alpha\beta}\right)\qquad R_{2\pm}=Q^{\pm\frac{2}{\alpha}\varkappa}_{\nu-\frac{1}{2}}\left(\frac{\hat{r}}{\alpha\beta}\right)$$

$$R_{3\pm}=P^{\pm\frac{2}{\alpha}\varkappa}_{-\nu-\frac{1}{2}}\left(\frac{\hat{r}}{\alpha\beta}\right)\qquad R_{4\pm}=Q^{\pm\frac{2}{\alpha}\varkappa}_{-\nu-\frac{1}{2}}\left(\frac{\hat{r}}{\alpha\beta}\right)$$

$$\nu=\sqrt{m^2+\frac{1}{4}}$$

• Commutative limit, $\alpha \to 0$

$$\alpha \equiv 0 \qquad z^2 \frac{d^2}{dz^2}\tilde{R} + z\frac{d}{dz}\tilde{R} - \left(m^2 + \frac{1}{4} + 4\beta^2 z^2\right)\tilde{R} = 0$$

The solution regular at $r \to 0$ $(z \to \infty)$

$$\tilde{R} = \sqrt{z}\, K_\nu(2\beta z)$$

Taking the limit is trickier, the result being:

$$\phi = \Gamma\left(\frac{2}{\alpha}\varkappa + 1\right) e^{i\beta\hat{t}}\, P_{\nu-\frac{1}{2}}^{-\frac{2}{\alpha}\varkappa}\left(\frac{\hat{r}}{\alpha\beta}\right) e^{i\beta\hat{t}}$$

$$P_{-\Delta_-}^{-\frac{2}{\alpha}\varkappa} \xrightarrow[z \to +0]{} \frac{2^{-\Delta_-}\,\Gamma\left(\frac{1}{2} - \Delta_-\right)}{\sqrt{\pi}\;\Gamma\left(\Delta_+ + \frac{2}{\alpha}\varkappa\right)}(\alpha\beta z)^{\Delta_-} + \frac{2^{-\Delta_+}\,\Gamma\left(\Delta_- - \frac{1}{2}\right)}{\sqrt{\pi}\;\Gamma\left(\Delta_- + \frac{2}{\alpha}\varkappa\right)}(\alpha\beta z)^{\Delta_+}$$

- Massless limit

When $m=0$ $\quad \Phi_\pm = e^{i\beta\hat{t}}\left(\frac{1\pm\alpha\beta\hat{z}}{1\mp\alpha\beta\hat{z}}\right)^{\frac{\varkappa}{\alpha}} e^{i\beta\hat{t}}$

In (AP, A. Sterm 2021) we got

$$\Phi = F_+(\Xi_+) + F_-(\Xi_-), \quad \Xi_\pm := \pm\varkappa\hat{z} + i\hat{t}$$

HWK: Show that $([\hat{t},\hat{z}] = i\alpha\hat{z}^2)$

$$e^{i\beta\hat{t}}\left(\frac{1\pm\alpha\beta\hat{z}}{1\mp\alpha\beta\hat{z}}\right)^{\frac{\varkappa}{\alpha}} e^{i\beta\hat{t}} = e^{2\beta\,\Xi_\pm}.$$

AdS/CFT

Back to the symbols and $\ast$-product.

$$S_{\text{on-shell}} = -\frac{1}{2}\int_{\mathbb{R}} dt \, \hat{\Phi}\, \partial_z \hat{\Phi}\Big|_{z\to+0} ,$$

$$\hat{\Phi} = \frac{1}{2\pi}\int d\omega \; a(\omega)\; P_{-\Delta_-}^{-\frac{2\gamma}{\alpha}}\left(\frac{2}{|\omega|\alpha z}\right) e^{i\omega t}$$

Defining $\hat{\phi}_\epsilon(t) := \frac{1}{2\pi}\int d\omega \; a(\omega)\; P_{-\Delta_-}^{-\frac{2\gamma}{\alpha}}\left(\frac{2}{|\omega|\alpha\epsilon}\right) e^{i\omega t}$

and $\varphi(\omega) := a(\omega)\; P_{-\Delta_-}^{-\frac{2\gamma}{\alpha}}\left(\frac{2}{|\omega|\alpha\epsilon}\right)$

$$\Rightarrow \quad \hat{\Phi} = \frac{1}{2\pi}\int d\omega \; \varphi(\omega) \frac{P_{-\Delta_-}^{-\frac{2\varkappa}{\alpha}}\left(\frac{2}{|\omega|\alpha z}\right)}{P_{-\Delta_-}^{-\frac{2\varkappa}{\alpha}}\left(\frac{2}{|\omega|\alpha \epsilon}\right)} e^{i\omega t}$$

$$S^{NC}_{\text{on-shell}} = -\left(\frac{\alpha}{2}\right)^{2\nu} \frac{\nu}{\sqrt{\pi}} \frac{\Gamma(\nu+\frac{1}{2})\,\Gamma(\nu+\frac{2\varkappa}{\alpha}+\frac{1}{2})}{\Gamma(\nu)\,\Gamma(-\nu+\frac{2\varkappa}{\alpha}+\frac{1}{2})} \int dt \int dt' \frac{\varphi_0(t)\,\varphi_0(t')}{|t-t'|^{2\nu+1}}$$

$$S^{com}_{\text{on-shell}} = -\frac{\nu}{\sqrt{\pi}} \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)} \int dt \int dt' \frac{\varphi_0(t)\,\varphi_0(t')}{|t-t'|^{2\nu+1}}$$

$$\frac{\langle \mathcal{O}(t_1)\,\mathcal{O}(t_2)\rangle_{NC}}{\langle \mathcal{O}(t_1)\,\mathcal{O}(t_2)\rangle} = \left(\frac{\alpha}{2}\right)^{2\nu} \frac{\Gamma(\nu+\frac{2\varkappa}{\alpha}+\frac{1}{2})}{\Gamma(-\nu+\frac{2\varkappa}{\alpha}+\frac{1}{2})} \longrightarrow 1 + \underbrace{\frac{\nu}{12}\left(\frac{13}{4}-\nu^2\right)\alpha^2}_{\text{AP, Stern, Almeida 2019}} + O(\alpha^4)$$

Interacting Case

E.O.M. $\quad -\frac{1}{\alpha^2}\left[\hat{X}^\mu,\left[\hat{X}_\mu,\Phi\right]\right]-(\ell m)^2\Phi=\lambda\Phi^2$

Can be solved perturbatively w.r.t. α,
then used to calculate $S_{NC}[\Phi_{sol}[\phi_0]]$
leading only to a partial result

$$\langle\mathcal{O}(t_1)\mathcal{O}(t_2)\mathcal{O}(t_3)\rangle_{NC}=\left(1+c\,\frac{\alpha^2}{\ell^2}\right)\langle\mathcal{O}(t_1)\mathcal{O}(t_2)\mathcal{O}(t_3)\rangle+\mathcal{O}\left(\frac{\alpha^4}{\ell^4}\right)$$

To make a further progress one needs
a bette understanding of a fully NC
"Green's function".

Conclusions

- It seems that AdS/CFT continues to work in noncommutative setting.
- What is the other (CFT) side?
  - truncating the representation ≡ making approximately ncAdS...
- Is it possible to find the exact result? ✓
- Generalizations:
  - other types of fields ✓ (AP, Stern 2021)
  - higher dimensions (AP, A. Stern & F. Lizzi, in progress) ✓ (AP, Lizzi, Stern, Xu 2020)

- Green's function $\Rightarrow$ a non-perturbative (in α) control over the interacting case.
- Relation to SYK? To JT gravity?