Workshop on the Standard Model and beyond
Corfu Summer Institute
28 August – 9 September

Dark Matter from sterile-sterile neutrino mixing

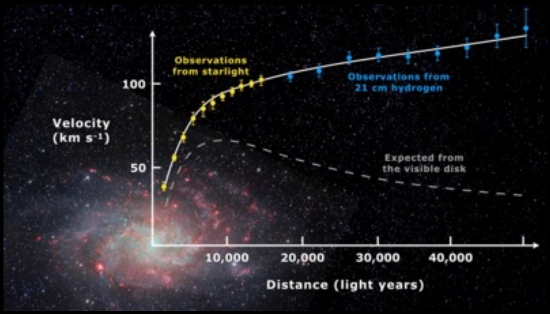
Pasquale Di Bari
(University of Southampton)


© Cesar Vega Tole

Dark Matter

At the present time dark matter acts as a cosmic glue keeping together

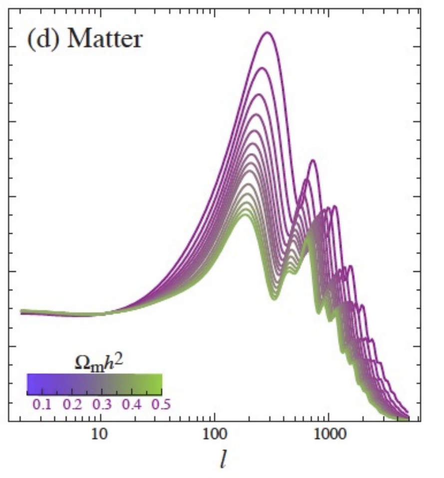
Stars in galaxies....



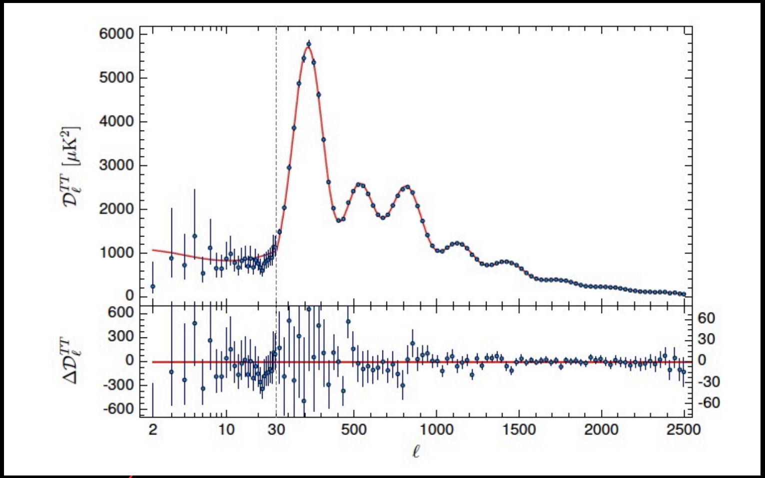
... and galaxies in clusters of galaxies (such as in Coma cluster)

...but it also has to be primordial and BSM* to understand structure formation and CMB anisotropies

(Hu, Dodelson, astro-ph/0110414)



(Planck 2018, 1807.06209)

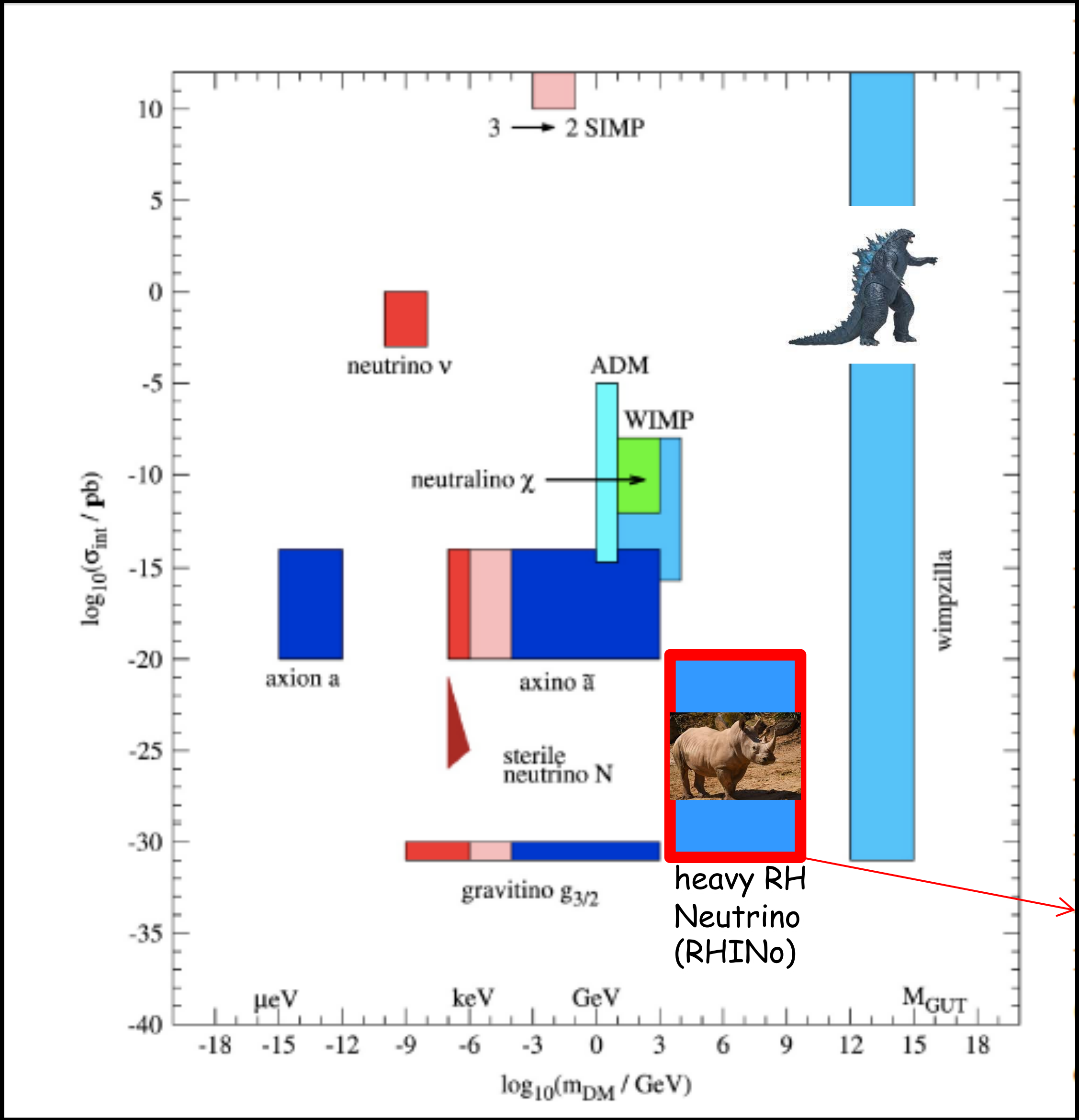


(CMB + BAO)

$$\boxed{\Omega_{CDM,0}h^2 = 0.11933 \pm 0.0009 \sim 5\Omega_{B,0}h^2}$$

Beyond the WIMP paradigm

(from Baer et al.1407.0017)



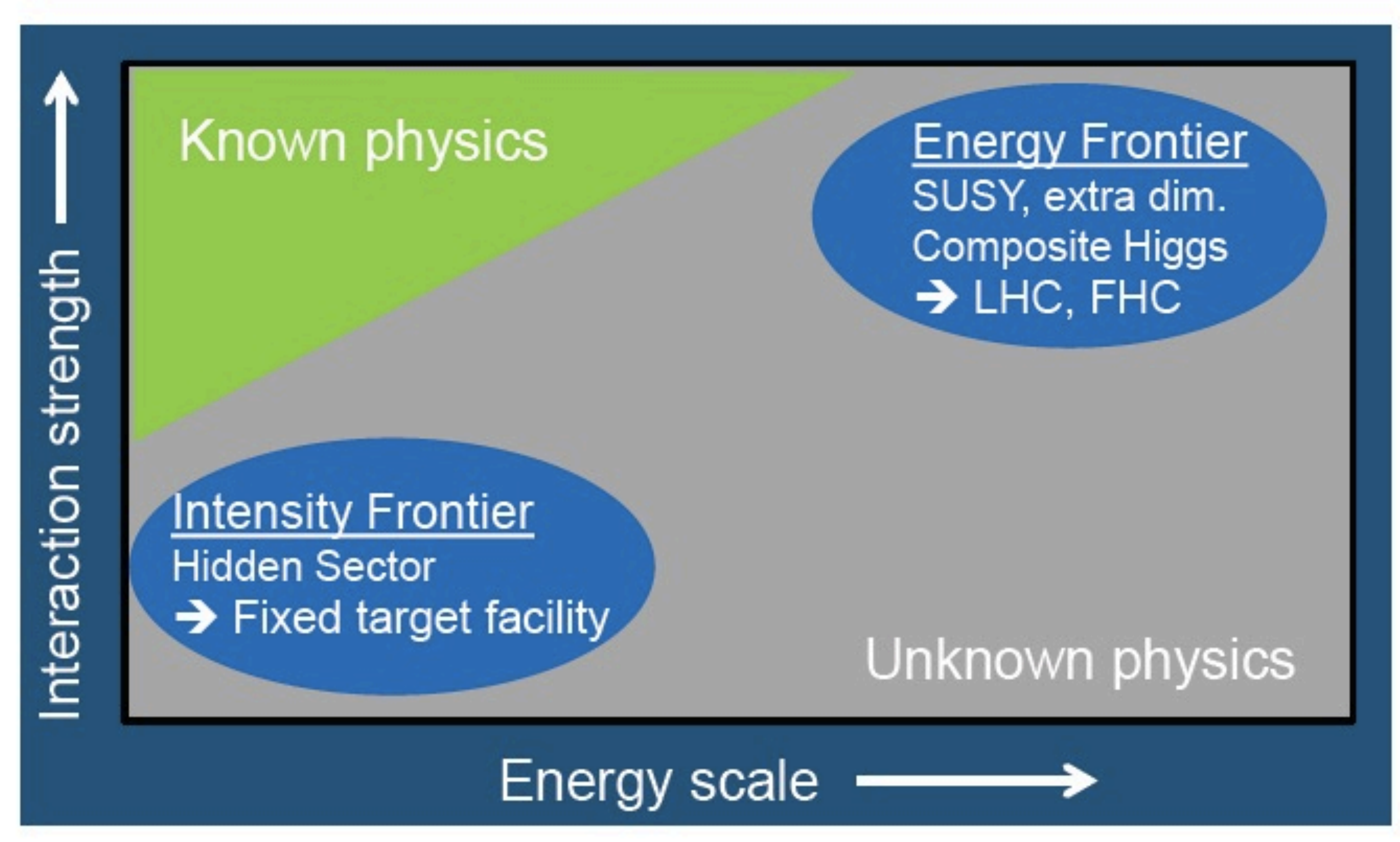
heavy RH Neutrino (RHINo)

(PDB, Anisimov '08)

The more we know the less we understand?

Right-handed neutrino laboratory searches

(SHIP proposal, 1504.04855)



Known physics

Energy Frontier
SUSY, extra dim.
Composite Higgs
→ LHC, FHC

Intensity Frontier
Hidden Sector
→ Fixed target facility

Unknown physics

Dark matter from active-sterile neutrino mixing

(Dodelson Widrow '94; Shi, Fuller '99; Dolgov and Hansen '00; Asaka, Blanchet,Shaposhnikov '05)

- Type-I seesaw Lagrangian $-\mathcal{L}^{\nu}_{mass} = \overline{\nu}_L m_D \nu_R + \frac{1}{2}\overline{\nu^c_R} M \nu_R + h.c. = -\frac{1}{2}(\overline{\nu_L}\ \overline{\nu^c_R}) \begin{pmatrix} 0 & m_D^T \\ m_D & M \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu^c_R \end{pmatrix} + h.c.$

$$\nu_{1L} \simeq U^{\dagger}_{1\alpha}\left(\nu_{L\alpha} - \frac{m_{D\alpha 1}}{M_1}\nu^c_{R1}\right)$$

- LH-RH neutrino mixing

$$N_{1R} \simeq \nu_{1R} + \frac{m_{D\alpha 1}}{M_1}\nu^c_{L\alpha}$$

- For $M_1 \ll m_e$ $\Rightarrow$ $\tau_1 = 5\times 10^{26}\,s\left(\frac{M_1}{keV}\right)^{-5}\left(\frac{10^{-8}}{\theta^2}\right) \gg t_0$ $\qquad \theta^2 \equiv \frac{\sum_\alpha |m_{D\alpha 1}|^2}{M_1^2}$

- Solving Boltzmann equations an abundance is produced at T~100 MeV:

$$\Omega_{N_1} h^2 \sim 0.1\,\frac{\theta^2}{10^{-8}}\left(\frac{M_1}{\mathrm{keV}}\right)^2 \sim \Omega_{DM,0}h^2$$

- The lightest neutrino mass $m_1 \lesssim 10^{-5}$ eV $\Rightarrow$ hierarchical limit
- The N_1's also radiatively decay and this produces constraints from X-rays (or opportunities to observe it).
- Considering also structure formation constraints, one is forced to consider a resonant production induced by a large lepton asymmetry
- $L \sim 10^{-4}$ (3.5 keV line?). (Horiuchi et al. '14; Bulbul at al. '14; Abazajian '14)

Heavy RH neutrino as dark matter ?

(Anisimov,PDB '08)

What production mechanism? For high masses just a tiny abundance is needed:

$$N_{DM} \simeq 10^{-9}(\Omega_{DM,0}h^2)N_\gamma(t_{prod})\frac{\mathrm{TeV}}{M_{DM}}$$

Suppose a RH neutrino has tiny Yukawa couplings (e.g., proportional to a small symmetry breaking parameter):

$$m_D \simeq \begin{pmatrix} \varepsilon_{e1} & m_{De2} & m_{De3} \\ \varepsilon_{\mu 1} & m_{D\mu 2} & m_{D\mu 3} \\ \varepsilon_{\tau 1} & m_{D\tau 2} & m_{D\tau 3} \end{pmatrix} \text{ or } m_D \simeq \begin{pmatrix} m_{De1} & \varepsilon_{e2} & m_{De3} \\ m_{D\mu 1} & \varepsilon_{\mu 2} & m_{D\mu 3} \\ m_{D\tau 1} & \varepsilon_{\tau 2} & m_{D\tau 3} \end{pmatrix} \text{ or } m_D \simeq \begin{pmatrix} m_{De1} & m_{De2} & \varepsilon_{e3} \\ m_{D\mu 1} & m_{D\mu 2} & \varepsilon_{\mu 3} \\ m_{D\tau 1} & m_{D\tau 2} & \varepsilon_{\tau 3} \end{pmatrix}$$

$$m_D = V_L^\dagger D_{m_D} U_R \qquad D_{m_D} \equiv v\, diag(h_A, h_B, h_C) \text{ with } h_A \leq h_B \leq h_C$$

$$\boxed{\tau_{DM} = \frac{4\pi}{h_A^2\, M_{DM}} = 0.87 h_A^2\, 10^{-26}\frac{\mathrm{TeV}}{M_{DM}} s} \quad \Rightarrow \quad \boxed{\tau_{DM} > \tau_{DM}^{\min} \simeq 10^{28} s \Rightarrow h_A < 10^{-27}\sqrt{\frac{\mathrm{TeV}}{M_{DM}} \times \frac{10^{28}\mathrm{s}}{\tau_{DM}^{\min}}}}$$

Too small to reproduce the correct abundance with any production mechanism within a minimal type-I seesaw extension

Many proposed production mechanisms

Recently many production mechanisms have been proposed especially to address IceCube initially seemingly anomalous PeV neutrino events:

- from $SU(2)_R$ extra-gauge interactions (LRSM) (Fornengo,Niro, Fiorentin);

- from inflaton decays (Anisimov,PDB'08; Higaki, Kitano, Sato '14);

- from resonant annihilations through SU(2)' extra-gauge interactions
  (Dev, Kazanas,Mohapatra,Teplitz, Zhang '16);

- From new $U(1)_Y$ interactions connecting DM to SM (Dev, Mohapatra,Zhang '16);

- From $U(1)_{B-L}$ interactions (Okada, Orikasa '12);

- ………………

In all these models IceCube data are fitted through fine tuning of parameters responsible for decays (they are post-dictive)

A 5-dimensional Higgs portal operator as a way out

(Anisimov hep-ph/0612024, Anisimov,PDB 0812.5085)

$$\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}^{\nu}_{Y+M} + \mathcal{L}_{A}$$

Type-I
Seesaw
Lagrangian

$$-\mathcal{L}^{\nu}_{Y+M} = \overline{L}_{\alpha} h_{\alpha I} N_I \tilde{\phi} + \frac{1}{2}\overline{N^c_I} M_I N_I + h.c.$$

Anisimov
operator(s)

$$\mathcal{L}_A = \sum_{I,J} \frac{\lambda_{IJ}}{\Lambda} \phi^{\dagger}\phi\, \overline{N^c_I} N_J + h.c.$$

$$= \boxed{\frac{\lambda_{DS}}{\Lambda}\phi^{\dagger}\phi\, \overline{N^c_D} N_S} + \frac{\lambda_{SS}}{\Lambda}\phi^{\dagger}\phi\, \overline{N^c_S} N_S + \frac{\lambda_{DD}}{\Lambda}\phi^{\dagger}\phi\, \overline{N^c_D} N_D + h.c. \;\; (N_D = N_3; N_S = N_2)$$

Remarks:

- from SMEFT to νSMEFT (talks by C. DeGrande and Jim Talbert)
- They are kind of Weinberg operators, a further step up
- They extend Higgs portal renormalizable operator (Patt,Wilczek hep-ph/0605188)

DM from Higgs induced neutrino mixing

(Anisimov '06, Anisimov,PDB '08)

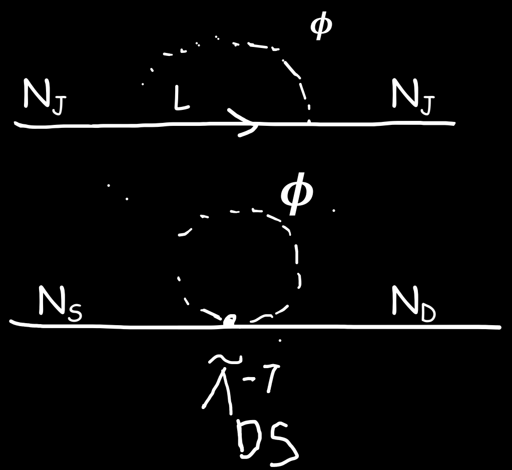
Assume new (5-dim) interactions with the standard Higgs:

$$\mathcal{L}_A = \frac{\lambda_{DS}}{\Lambda}\phi^\dagger\phi\,\overline{N_D^c}N_S$$

In general they are non-diagonal in the Yukawa basis: this generates a RH neutrino mixing. Consider a 2 RH neutrino mixing for simplicity. Interactions generate effective potentials from self-energies

From Yukawa interactions $V_S^Y = \frac{T^2}{8p}h_S^2$

From mixing $V_{DS}^\Lambda = \frac{T^2}{12\Lambda}\lambda_{DS}$



$$\tilde{\Lambda}_{DS} = \Lambda / \lambda_{DS}$$

Effective mixing Hamiltonian :

$$\Delta H \simeq \begin{pmatrix} -\frac{\Delta M^2}{4p} - \frac{T^2}{16p}h_S^2 & \frac{T^2}{12\tilde{\Lambda}_{DS}} \\ \frac{T^2}{12\tilde{\Lambda}_{DS}} & \frac{\Delta M^2}{4p} + \frac{T^2}{16p}h_S^2 \end{pmatrix}$$

$$\Delta M^2 \equiv M_S^2 - M_D^2$$

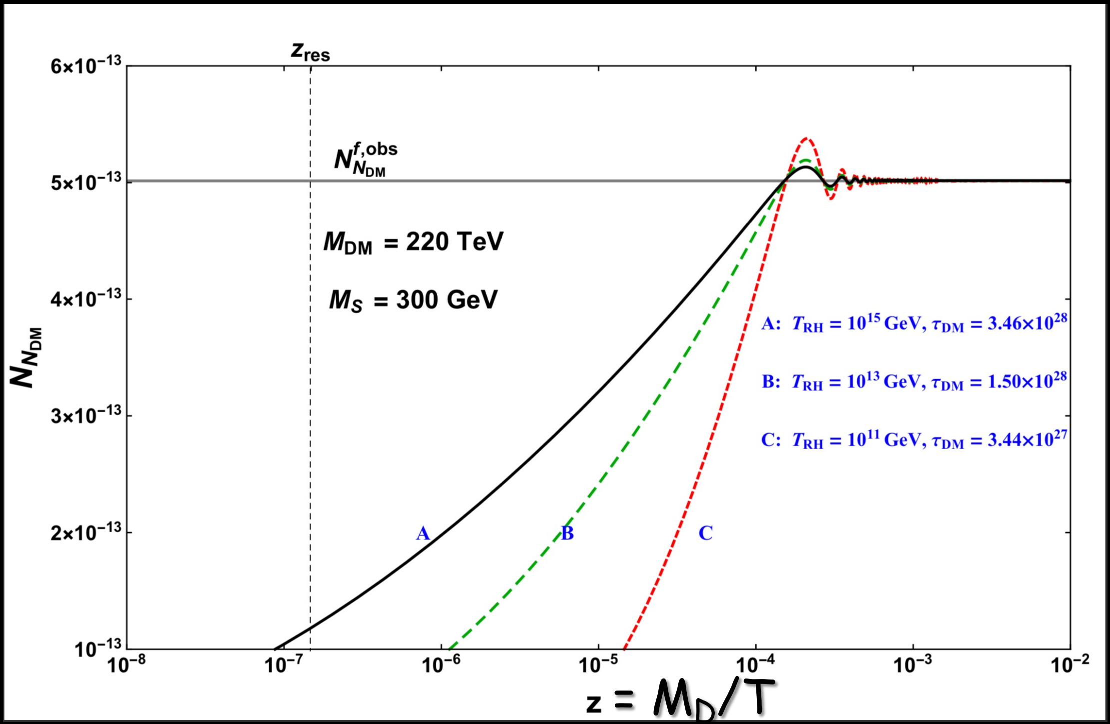
Density matrix calculation of the relic abundance

(P.Di Bari, K. Farrag, R. Samanta, Y. Zhou, 1908.00521)

Density matrix equation for the DM-source RH neutrino system (using a monocromatic approximation p~3T)

$$\frac{dN_{IJ}}{dt} = -i\,[\Delta H, N]_{IJ} - \begin{pmatrix} 0 & \frac{1}{2}(\Gamma_D+\Gamma_S)N_{DS} \\ \frac{1}{2}(\Gamma_D+\Gamma_S)N_{SD} & (\Gamma_D+\Gamma_S)(N_{N_S}-N_{N_S}^{eq}) \end{pmatrix}$$

Example for initial N_S thermal abundance

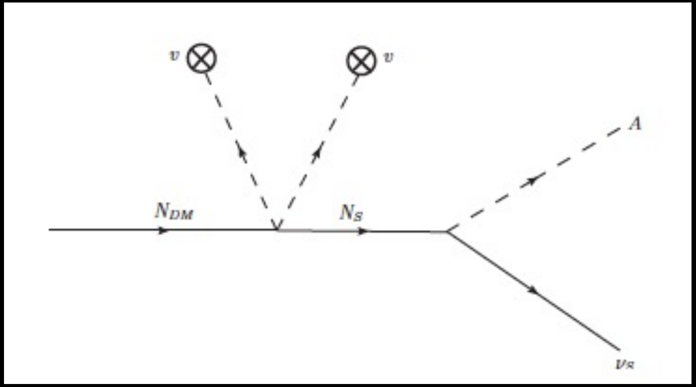


Constraints from decays

(Anisimov,PDB '08; Anisimov,PDB'10; P.Ludl.PDB,S.Palomarez-Ruiz'16)

2 body decays ($M_S > M_W$)

DM neutrinos unavoidably decay today into A+leptons (A=H,Z,W) through the same mixing that produced them in the very early Universe



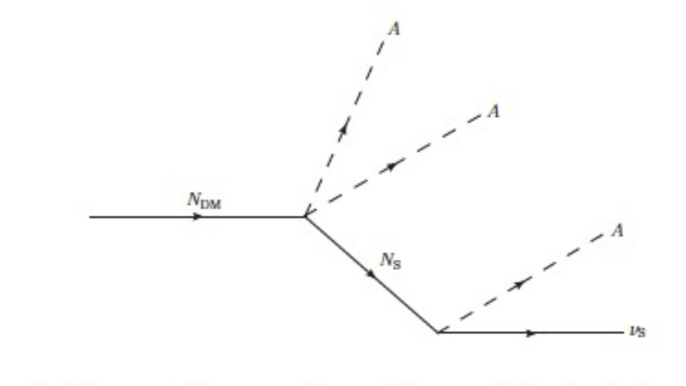
$$\theta_{\Lambda 0} = \frac{2\, v^2/\widetilde{\Lambda}_{\rm DS}}{M_{\rm D}\,(1 - M_{\rm S}/M_{\rm D})}$$

mixing angle today (for $\theta_{\Lambda 0} \ll 1$)

$$\Gamma_{{\rm D}\to A+\ell_{\rm S}} = \frac{h_{\rm S}^2}{\pi}\left(\frac{v^2}{\widetilde{\Lambda}}\right)^2 \frac{M_{\rm D}}{(M_{\rm D} - M_{\rm S})^2}\,.$$

⇒ Lower bound on M_{DM}

4 body decays



$$N_{\rm DM} \to 2\,A + N_{\rm S} \to 3\,A + \nu_{\rm S}\ (A = W^{\pm}, Z, H).$$

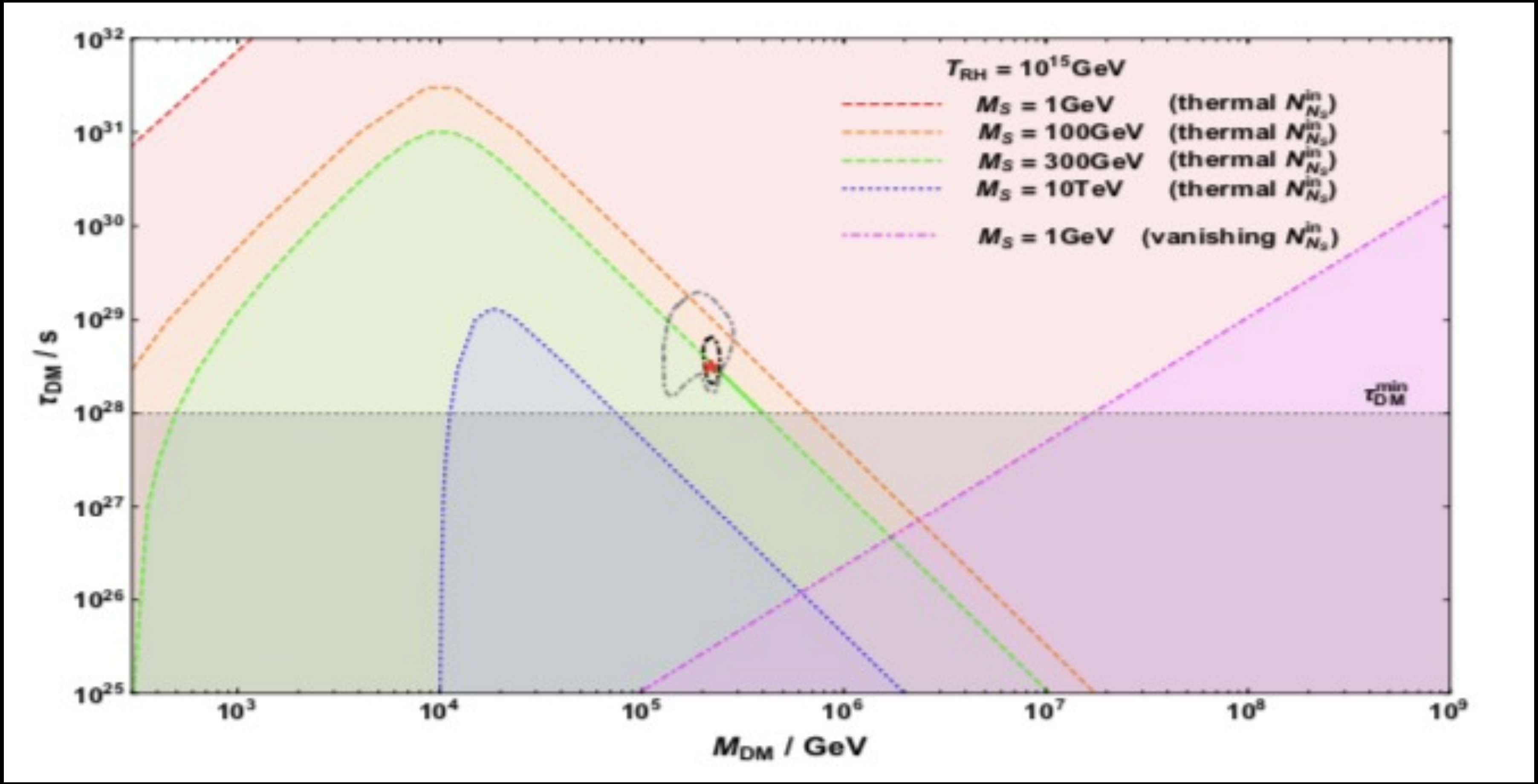
$$\Gamma_{{\rm D}\to 3A+\ell_{\rm S}} = \frac{\Gamma_{\rm S}}{15\cdot 2^{11}\cdot \pi^4}\,\frac{M_{\rm D}}{M_{\rm S}}\left(\frac{M_{\rm D}}{\widetilde{\Lambda}_{\rm DS}}\right)^2$$

⇒ Upper bound on M_{DM}

3 body decays and annihilations also can occur but yield weaker constraints

DM lifetime vs. mass plane: allowed regions

(P.Di Bari, K. Farrag, R. Samanta, Y. Zhou, 1908.00521)



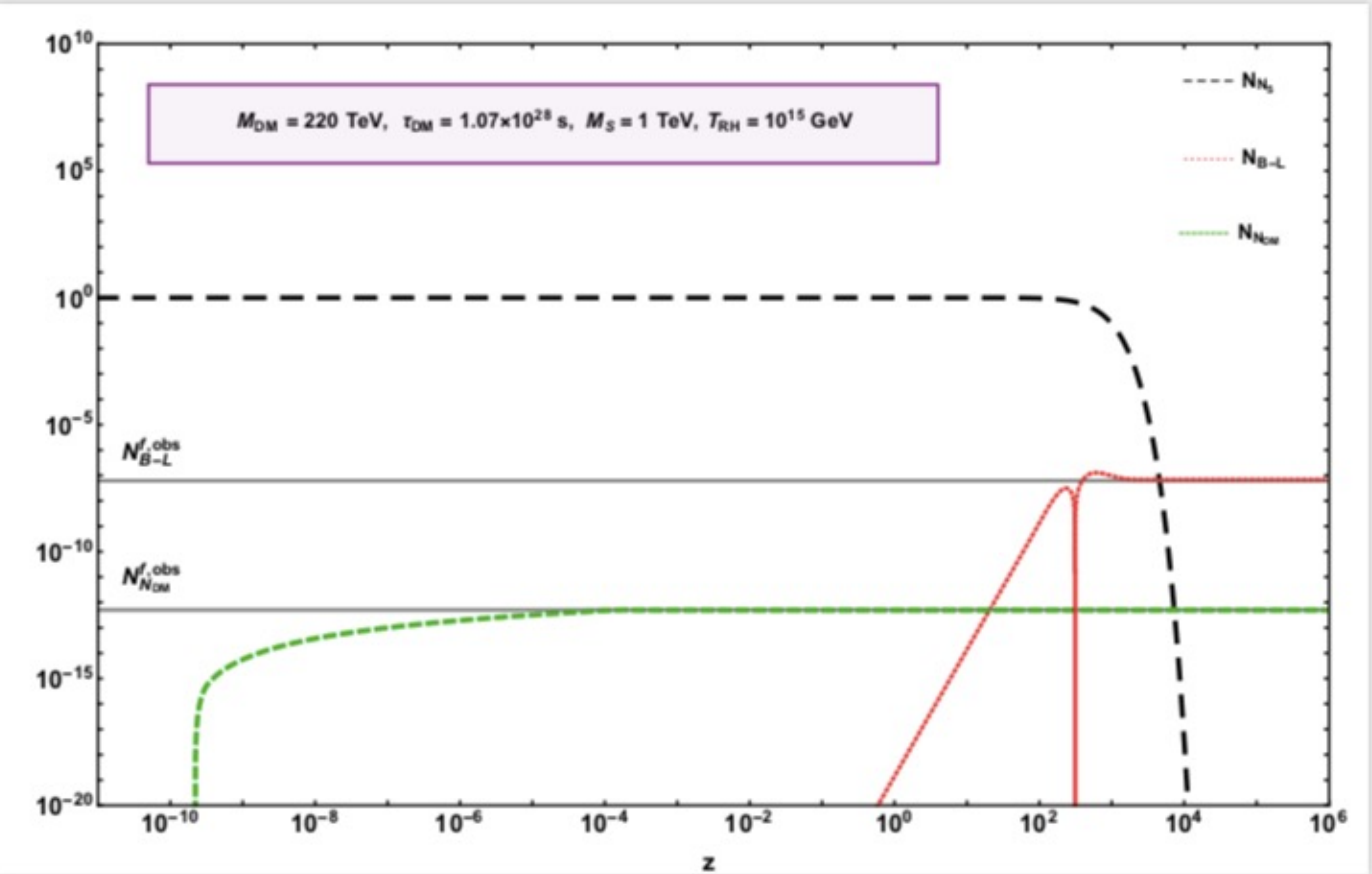
Solutions only for initial thermal N_S abundance, unless M_S~ 1 GeV and $M_D \gtrsim 10^7$ GeV

Can one think of processes able to thermalize the
N_S abundance prior to the oscillations?
Two good motivations

Unifying Leptogenesis and Dark Matter

(PDB, K. Farrag, R. Samanta, Y. Zhou, 1908.00521)

A solution for initial thermal N_S abundance:

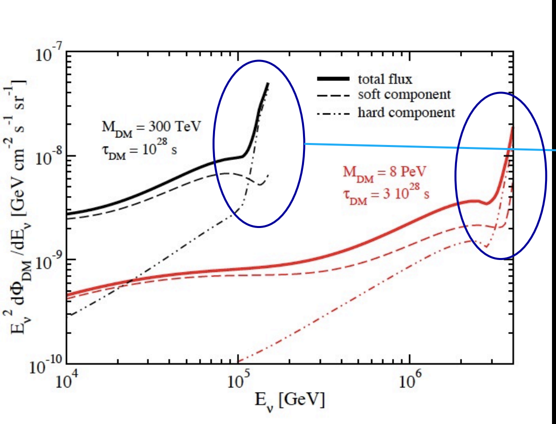


Very high energy neutrinos from decays

(Anisimov,PDB,0812.5085;PDB, P.Ludl,S. Palomarez-Ruiz 1606.06238)

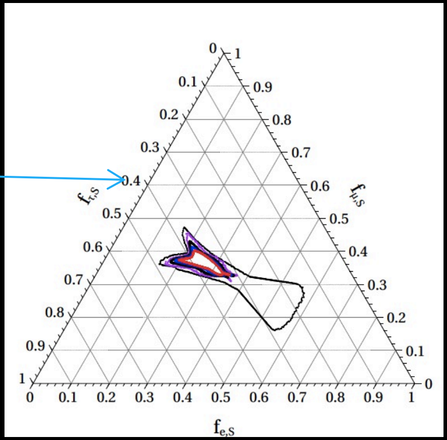
- DM neutrinos unavoidably decay today into A+leptons (A=H,Z,W) through the same mixing that produced them in the very early Universe
- Potentially testable high energy neutrino contribution

Energy neutrino flux



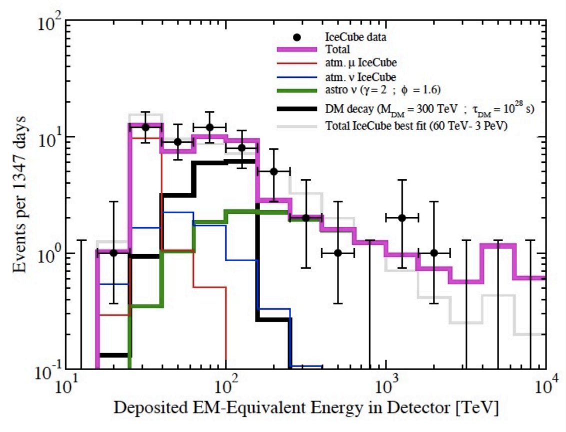
Flavour composition at the detector

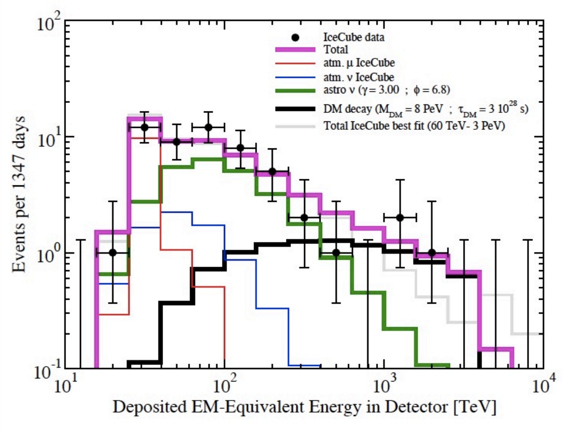
Hard component



Neutrino events at IceCube: 2 examples

M_{DM}=300TeV

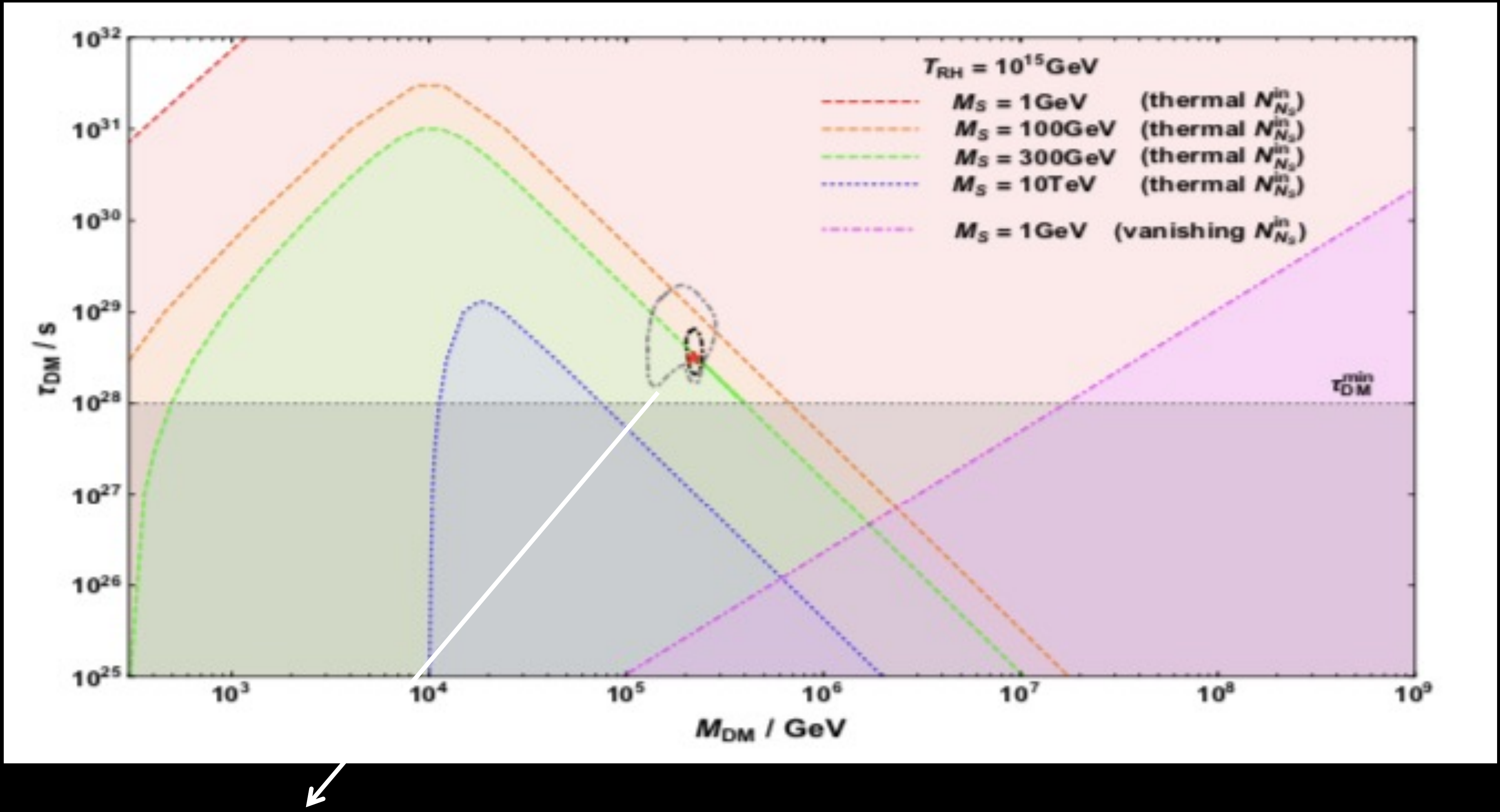




M_{DM}=8 PeV

DM lifetime vs. mass plane: allowed regions

(P.Di Bari, K. Farrag, R. Samanta, Y. Zhou, 1908.00521)



95% C.L. region where neutrinophilic DM decays well fit an excess in the neutrino flux at ~100 TeV energies in addition to an astrophysical component (Chianese et al. 1808.02486)

Including Higgs portal interactions for N_S

(PDB, A. Murphy, arXiv 2209.xxxx)

$$\mathcal{L}_A = \frac{\lambda_{DS}}{\Lambda}\phi^\dagger\phi\,\overline{N^c_{DM}}N_S + \frac{\lambda_{SS}}{\Lambda}\phi^\dagger\phi\,\overline{N^c_S}N_S$$

Can these interactions thermalise the source neutrinos prior to the mixing?

Let us modify the kinetic equations including these processes:

$$\frac{dN_{IJ}}{dt} = -i\,[\Delta H, N]_{IJ} - \begin{pmatrix} 0 & \frac{1}{2}(\Gamma_D+\Gamma_S)N_{DS} \\ \frac{1}{2}(\Gamma_D+\Gamma_S)N_{SD} & (\Gamma_D+\Gamma_S)(N_{N_S}-N^{eq}_{N_S}) + \frac{\langle\sigma_{\phi\phi\to N_S N_S}v\rangle}{R^3}(N^2_{N_S}-N^{eq\,2}_{N_S}) \end{pmatrix}$$

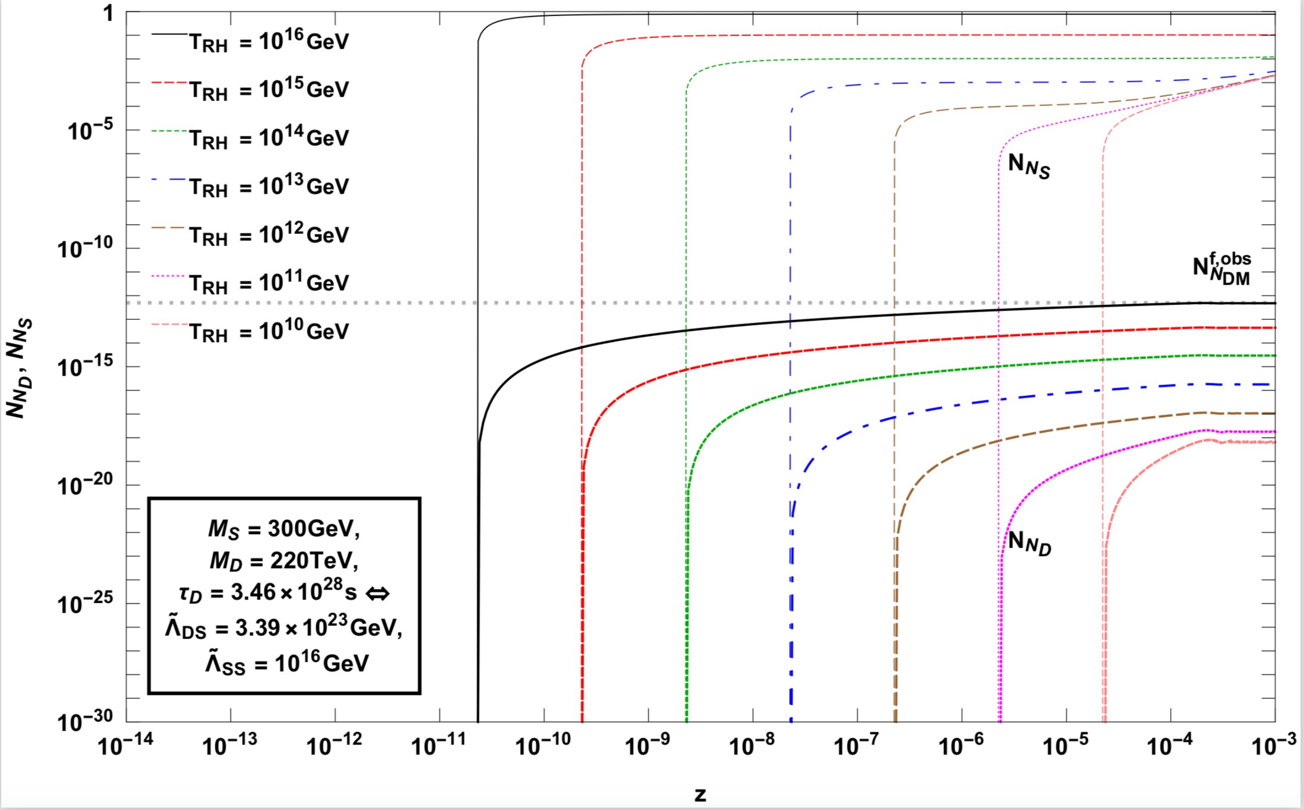
$$A(z) \equiv \frac{\langle\sigma_{\phi\phi\to N_S N_S}v\rangle}{R^3 H z} = \frac{A(z=1)}{z^2}; \qquad \langle\sigma_{\phi\phi\to N_S N_S}v\rangle_{T\gg M_S} \simeq \frac{1}{4\pi\tilde{\Lambda}^2_{SS}}$$

$$\Rightarrow A(z=1) \simeq g_N\,\frac{3}{16}\,\frac{\xi(3)}{\pi^3}\sqrt{\frac{90}{8\pi^3 g_R}}\,\frac{M_D M_{Pl}}{\tilde{\Lambda}^2_{SS}}$$

Condition for the thermalisation of the N_S abundance

(PDB, A. Murphy, arXiv 2209.xxxx)

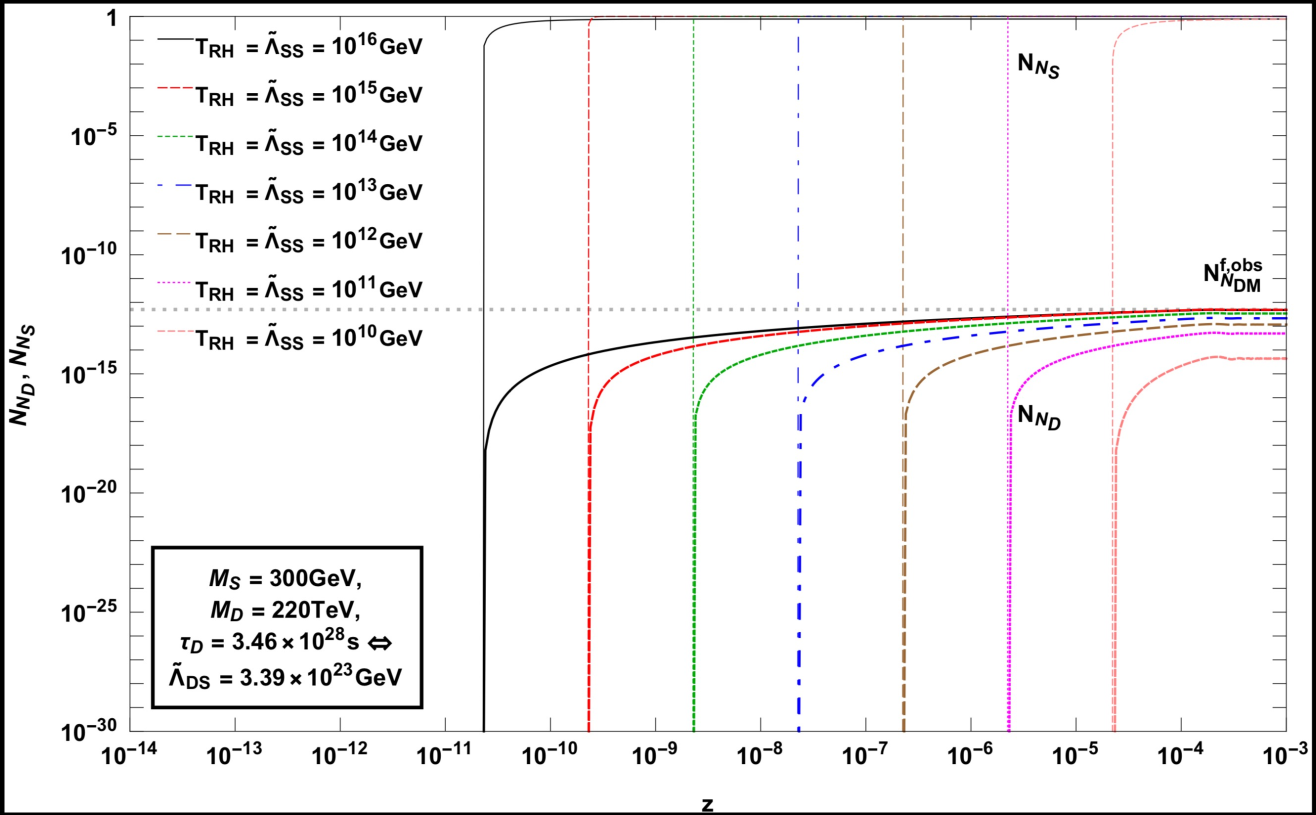
$$\Rightarrow N_{N_S}(z_{in} << z << 1) - N_{N_S}(z_{in}) \simeq \frac{A_1}{z_{in}} \simeq 1.0 \times \left(\frac{T_{in}}{10^{16}\,GeV}\right)\left(\frac{10^{16}\,GeV}{\tilde{\Lambda}_{SS}}\right)^2 \simeq 1$$



Condition for the thermalisation of the N_S abundance

(PDB, A. Murphy, arXiv 2209.xxxx)

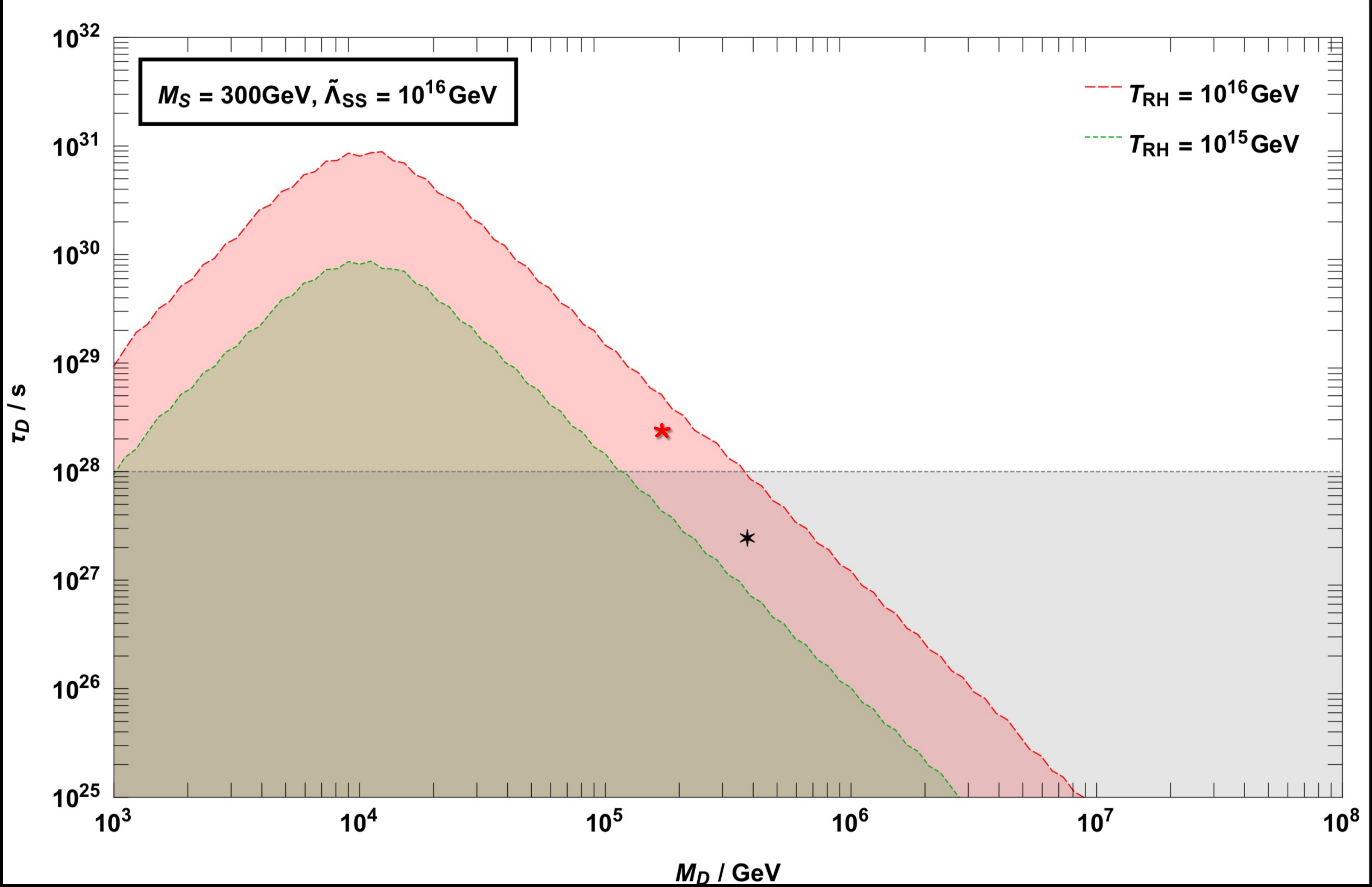
$$\Rightarrow N_{N_S}(z_{in} \ll z \ll 1) - N_{N_S}(z_{in}) \simeq \frac{A_1}{z_{in}} \simeq 1.0 \times \left(\frac{T_{in}}{10^{16}\,\mathrm{GeV}}\right)\left(\frac{10^{16}\,\mathrm{GeV}}{\tilde{\Lambda}_{SS}}\right)^2 \simeq 1$$



For the validity of the EFT: $T_{RH} \lesssim \tilde{\Lambda}_{SS}$

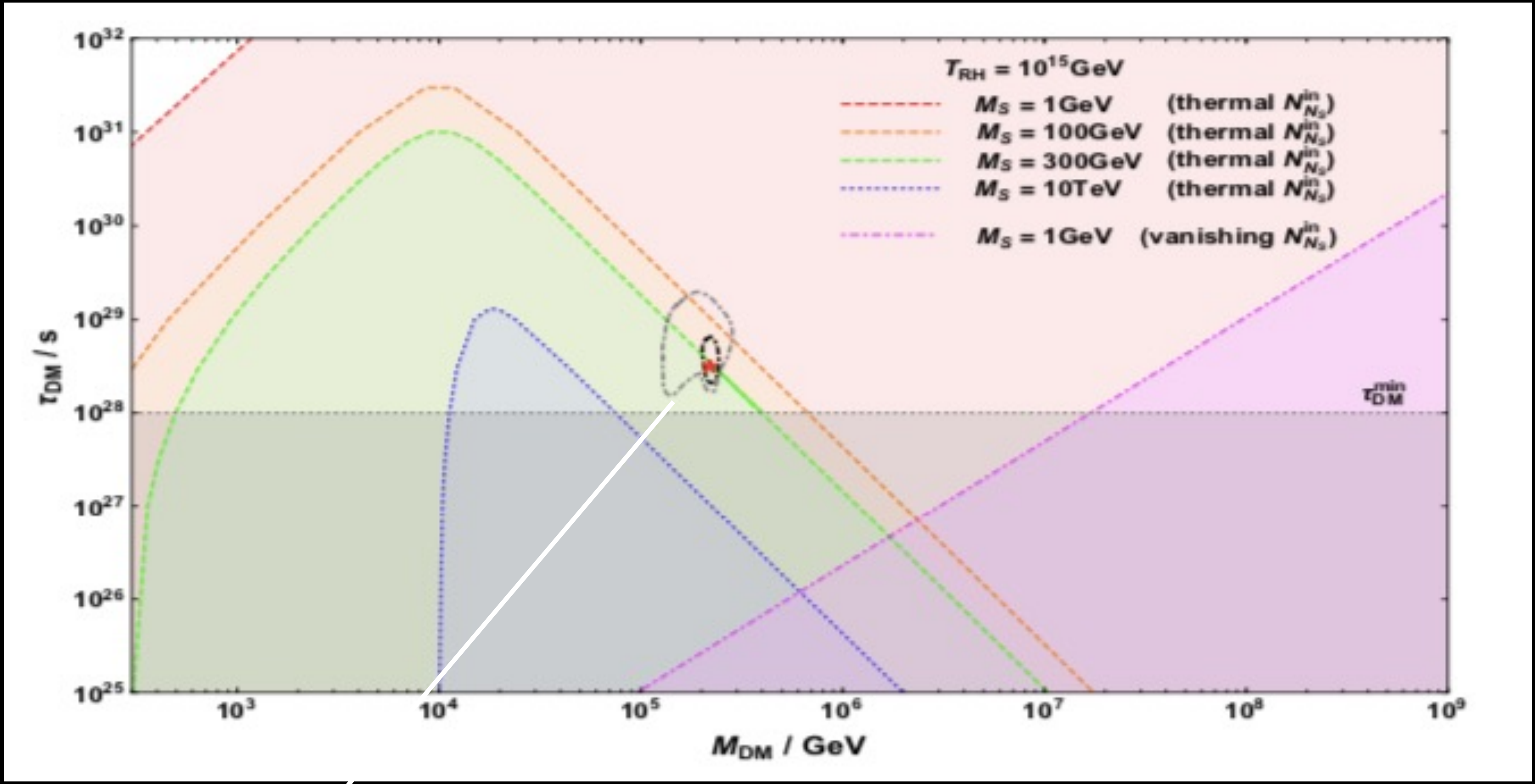
DM lifetime vs. mass plane: allowed regions

(PDB, A. Murphy, arXiv 2209.xxxx)



DM lifetime vs. mass plane: allowed regions

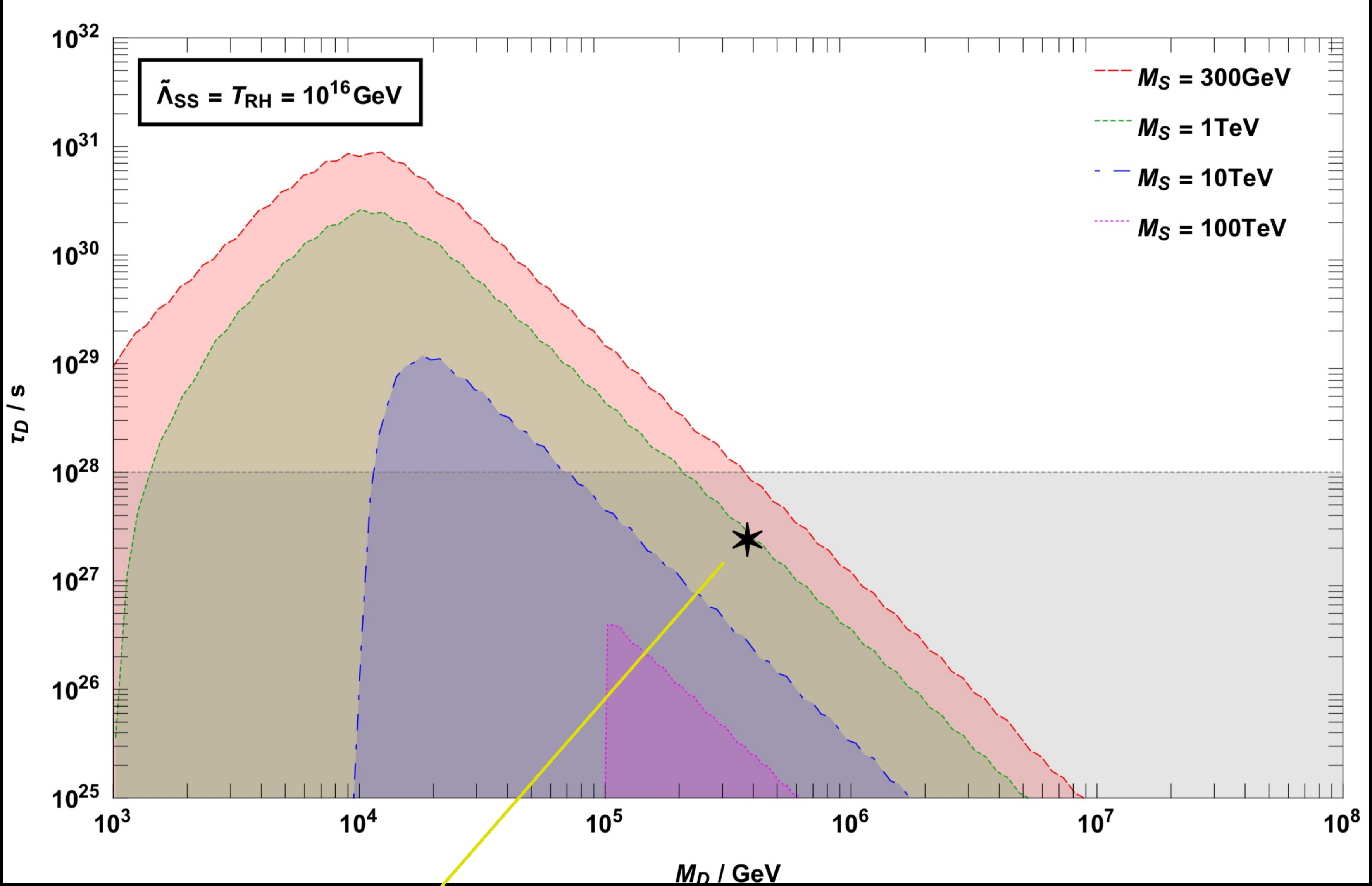
(P.Di Bari, K. Farrag, R. Samanta, Y. Zhou, 1908.00521)



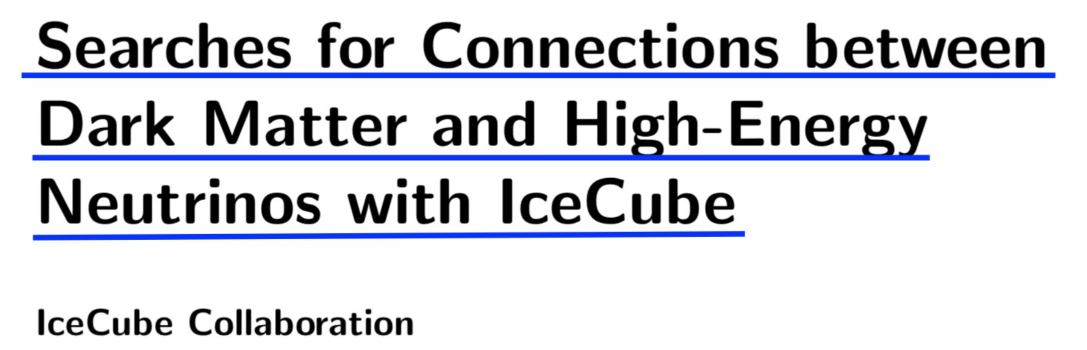
95% C.L. region where neutrinophilic DM decays well fit an excess in the neutrino flux at ~100 TeV energies in addition to an astrophysical component (Chianese et al. 1808.02486)

DM lifetime vs. mass plane: allowed regions

(PDB, A. Murphy, arXiv 2209.xxxx)



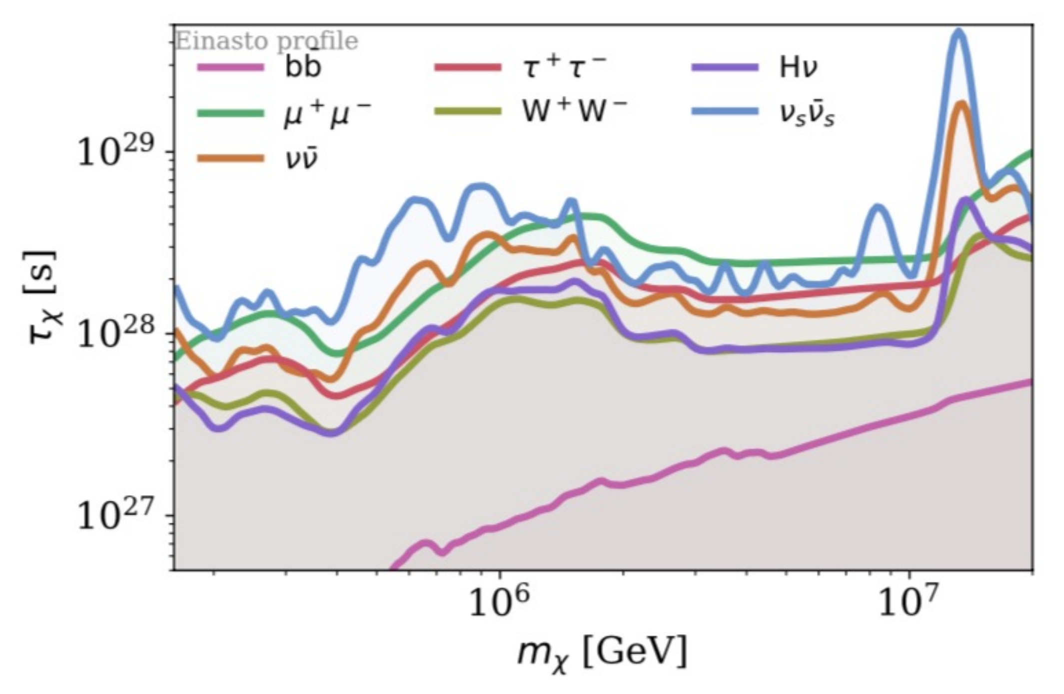
(IceCube collaboration 2205.12950)



Searches for Connections between Dark Matter and High-Energy Neutrinos with IceCube

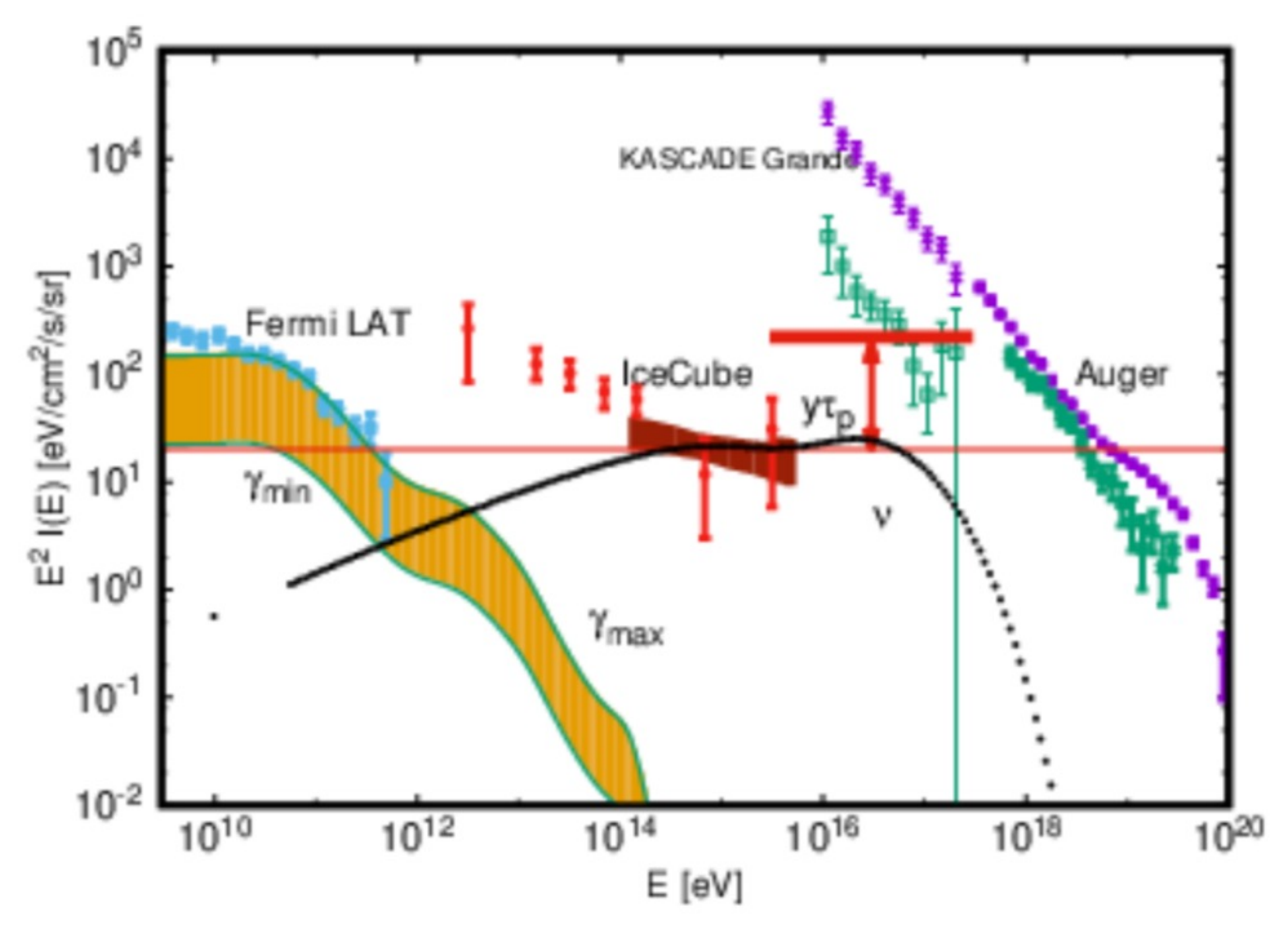
IceCube Collaboration

(2205.12950)



2.5σ significance when compared to the null hypothesis
best fit point: m_D=386 TeV, $\tau_D=2.8\times10^{27}$ s

Multimessenger analysis

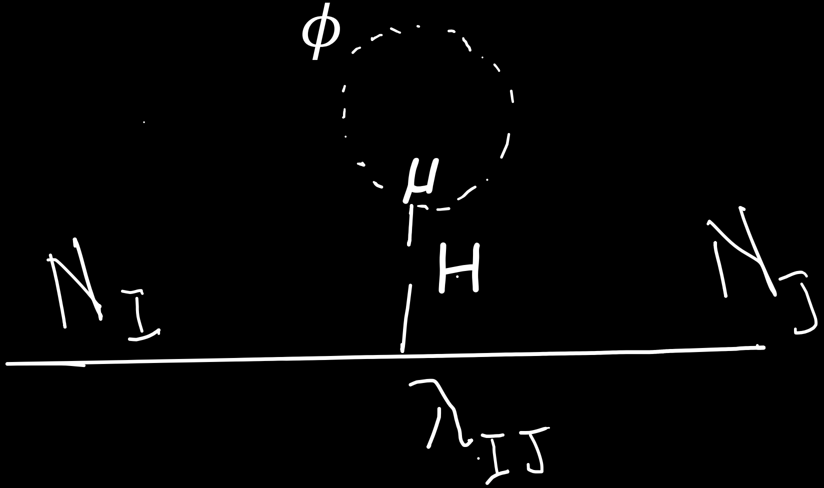


From Kachelriess 2201.04535 (IceCube 3 year data)

A possible GUT origin ? (1)

(**Anisimov,PDB, 2008;** P.Ludl.PDB,S.Palomarez-Ruiz'16; Kolb and Long 1708.04293)

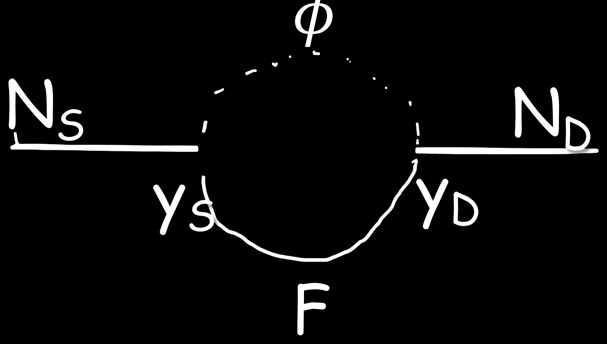
$$M_H \sim M_{GUT} \Rightarrow \tilde{\Lambda}_{DS} = \frac{1}{\lambda_{IJ}} \underbrace{\boxed{\frac{M_{GUT}^2}{\mu}}}_{= \Lambda}$$

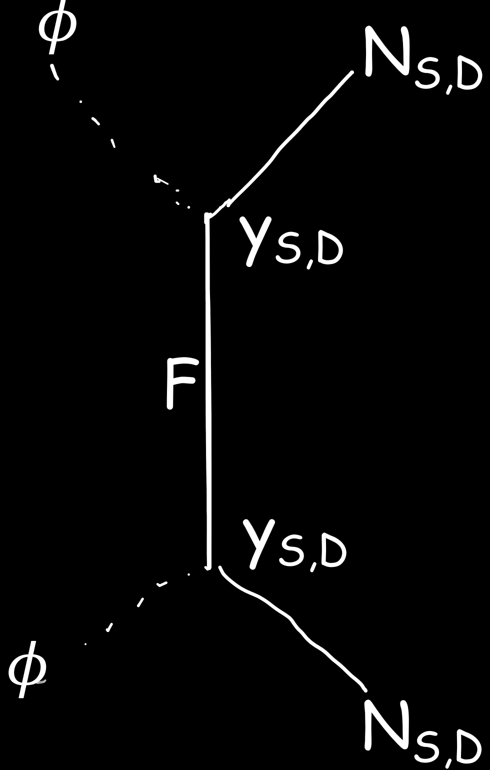


For $\mu \sim 10^9$GeV one can have $\tilde{\Lambda}_{DS} \sim 10^{23}$ GeV and $\lambda_{DS} \sim O(1)$ but one cannot reproduce simultaneously $\tilde{\Lambda}_{SS} \sim 10^{16}$GeV with the same scale Λ

A possible GUT origin (2)?

(Anisimov,PDB, 2008; PDM, A. Murphy 2209.xxxx)





This time one can have one scale $\Lambda = M_F \sim M_{GUT}$ and for $y_S \sim 1$ and $y_D \sim 10^{-7}$:

$$\tilde{\Lambda}_{DS} = \frac{\Lambda}{y_D y_S} \sim 10^{23}\,\text{GeV} \qquad \tilde{\Lambda}_{SS} = \frac{\Lambda}{y_S y_S} \sim \Lambda \sim 10^{16}\,\text{GeV} \qquad \tilde{\Lambda}_{DD} = \frac{\Lambda}{y_D y_D} \sim 10^{30}\,\text{GeV}$$

$y_D \sim 10^{-7}$ could be understood as a small symmetry (e.g. Z_2) breaking parameter

Summary

- The DM puzzle might have a solution at higher scales than those usually explored and….

- ….neutrino physics is a good place where to look for such a solution. A high scale RH neutrino playing the role of DM requires an extension of the usual type-I seesaw Lagrangian (able already to explain neutrino masses and mixing and the matter-antmatter asymmetry via leptogenesis).

- Higgs induced sterile-sterile neutrino mixing provides not only a way to produce dark neutrinos with the right abundance and….also to make them shining.

- Higgs portal interactions for the seesaw (source) neutrino enhance the dark neutrino production and allow to lift the scale of leptogenesis certainly above 300 GeV but even higher, how much higher exactly? We will soon address this question.

- Interestingly, the IceCube collaboration find an excess in the neutrino flux at energies well explained by a RHINo DM (~100 TeV) and further support (or constraints) might come relatively soon from γ-ray experiments and therefore……looking forward to hearing from next speaker!
  - THANK YOU!

First order phase transition associated to Majorana mass generation in the Majoron model

(PDB, D. Marfatia, YL. Zhou 2106.00025)

$$-\mathcal{L}_{N_I+\sigma} = \overline{L_\alpha}\, h_{\alpha I}\, N_I\, \tilde{\Phi} + \frac{\lambda_I}{2}\sigma\, \overline{N_I^c}\, N_I + V_0(\sigma) + h.c. \quad (respecting\ U_L(1)\ symmetry)$$

$$\sigma = \frac{1}{\sqrt{2}}(\sigma_1 + i\sigma_2)\,, \qquad <\sigma> = \frac{v_T}{\sqrt{2}}$$

At the end of the σ-phase transition, after SB, L is violated and

$$\sigma = \frac{e^{i\theta}}{\sqrt{2}}(v_0 + S + iJ) \qquad M_I = \lambda_I \frac{v_0}{\sqrt{2}} \sim M \ \ (seesaw\ scale)$$

Dirac neutrino mass matrix $m_D = v_{ew}\, h/\sqrt{2}$ generated after EWSB

At the moment let us assume $T_\star > v_{ew}$ (high scale scenarios)

After both symmetry breakings: $m_\nu = -\dfrac{v_{ew}^2}{2}\dfrac{h_{\alpha I} h_{\beta I}}{M_I}$

Given the measured values of the neutrino oscillation mass scales, RH neutrinos thermalise prior to the phase transition and contribute to the thermal potential

DARK SECTOR ≡ N_I 's + J + S   VISIBLE SECTOR ≡ SM particles

The minimal model

$$V_0(\sigma) = -\mu^2 |\sigma|^2 + \lambda |\sigma|^4 \;\Rightarrow\; v_0 = \sqrt{\mu^2/\lambda} \qquad (\lambda, \mu^2 > 0)$$

In the broken phase: $\sigma = \frac{e^{i\theta}}{\sqrt{2}}(v_0 + S + iJ)$

J is a massless Majoron and S has a mass $m_S=(2\lambda)^{1/2}v_0$

For the one-loop finite temperature effective potential one finds a polynomial

$$V_{\rm eff}^T(\sigma_1) \simeq D\,(T^2 - T_0^2)\sigma_1^2 - A\,T\,\sigma_1^3 + \frac{1}{4}\lambda_T\,\sigma_1^4\,,$$

The minimal model

$$2D\underbrace{T_0}^2 = \lambda v_0^2 + \frac{N}{8\pi^2}\frac{M^4}{v_0^2} - \frac{3}{8\pi^2}\lambda^2 v_0^2$$

↳ destabilization temperature ("end" of phase transition)

$$D = \frac{\lambda}{8} + \frac{N}{24}\frac{M^2}{v_0^2}\,; \quad A = \frac{(3\lambda)^{3/2}}{12\pi}$$

Finally:

$$\lambda_T = \lambda \boxed{-} \frac{NM^4}{8\pi^2 v_0^2}\log\frac{a_F T^2}{e^{3/2}M^2} + \frac{9\lambda^2}{16\pi^2}\log\frac{a_B T^2}{e^{3/2}m_s^2}$$

↳ You can't increase N arbitrarily!

Adding an auxiliary scalar

(Espinosa, Quirós'07)
(Kehayias, Profumo'09)

very heavy → real scalar

$$V_0(\sigma,\eta)=V_0(\sigma)+\underbrace{V_{\eta\sigma}(\sigma,\eta)+V_\eta(\eta)}_{\text{new terms}}$$

the most important term is contained in $V_{\eta\sigma}$:

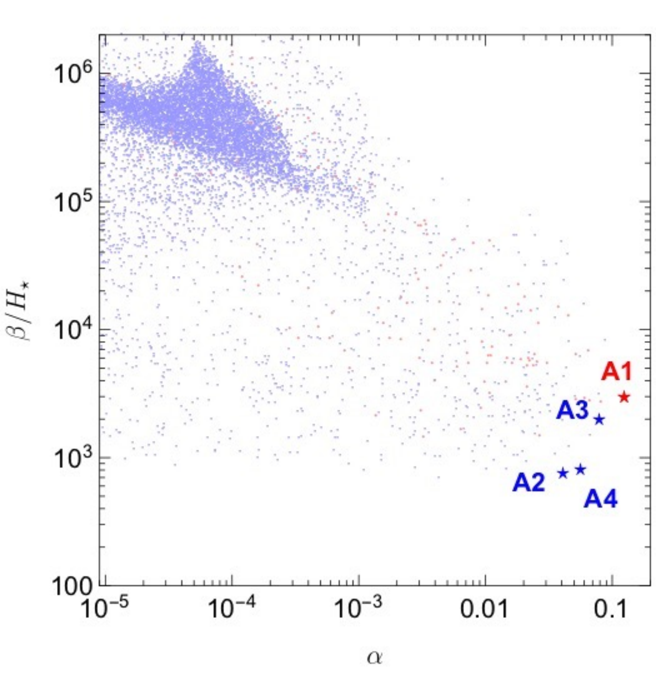
$$V_{\eta\sigma}(\eta,\sigma)=\frac{\delta_1}{2}|\sigma|^2\eta+\frac{\delta_2}{2}|\sigma|^2\eta^2$$

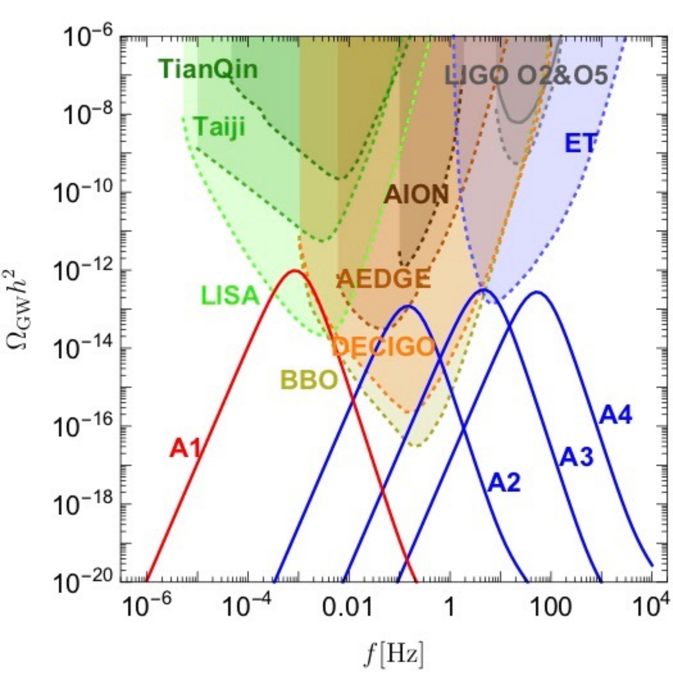
η undergoes a PT settling to its true vacuum prior to the σ-PT

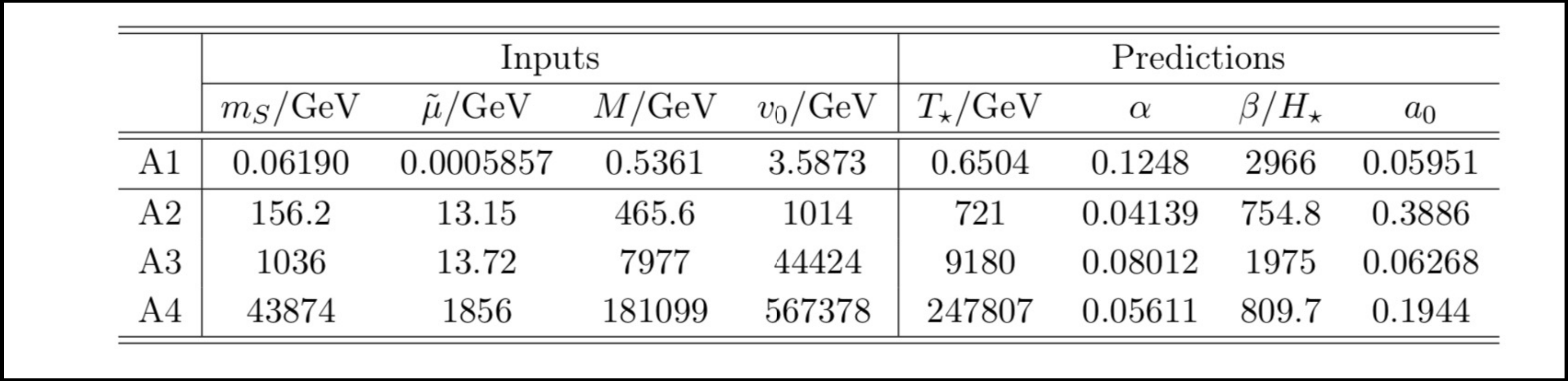
$$\Rightarrow V^T_{eff}(\sigma_1,\tilde{u})=\frac{1}{2}\tilde{M}_T^2\sigma_1^2-(AT+\tilde{\mu})\sigma_1^3+\frac{1}{4}\lambda_T\sigma_1^4$$

$\tilde{u}\propto\delta_2$

Adding an auxiliary scalar: GW spectrum







	Inputs				Predictions			
	m_S/GeV	$\tilde{\mu}/\mathrm{GeV}$	M/GeV	v_0/GeV	$T_\star/\mathrm{GeV}$	α	$\beta/H_\star$	a_0
A1	0.06190	0.0005857	0.5361	3.5873	0.6504	0.1248	2966	0.05951
A2	156.2	13.15	465.6	1014	721	0.04139	754.8	0.3886
A3	1036	13.72	7977	44424	9180	0.08012	1975	0.06268
A4	43874	1856	181099	567378	247807	0.05611	809.7	0.1944

Mass varying source right-handed neutrino

(PDB, D. Marfatia, YL. Zhou 2001.07637)

$$
\begin{aligned}
-\mathcal{L}_\lambda \;=\; & \frac{1}{2}\, M_{\rm DM}\, \overline{N^c_{\rm DM}}\, N_{\rm DM} + \frac{1}{2}\, M_{\rm D}\, \overline{N^c_{\rm D}}\, N_{\rm D} + \frac{\lambda_{\rm S}}{2}\, \eta\, \overline{N^c_{\rm S}}\, N_{\rm S} \\
& + \frac{1}{\tilde{\Lambda}}\, \Phi^\dagger\, \Phi\, \overline{N^c_{\rm D}}\, N_{\rm S} + \frac{1}{\tilde{\Lambda}}\, \Phi^\dagger\, \Phi\, \overline{N^c_{\rm DM}}\, N_{\rm D} + {\rm h.c.}\,. \qquad (1)
\end{aligned}
$$

The scalar field η acquires a vev v_η during a first order phase transition and accordingly N_S acquires a space-time dependent mass:

$$M_S(x,t) = \lambda_S\, v_\eta(x,t)$$

The bubble wall profile is well described by a kink solution found in the thin wall approximation:

$$v_\eta(r,t) = \frac{1}{2}\,\bar{v}_\eta\, \left[1 - \tanh\left(\frac{r - v_{\rm w}\,(t - t_\star)}{\Delta_{\rm w}}\right)\right]\,,$$