$\boxed{\text{Minkowski}}$

$$x(\lambda) = R\lambda \sinh\lambda \Rightarrow dx = R[\operatorname{sh}\lambda + \lambda \operatorname{ch}\lambda]\, d\lambda$$

$$t(\lambda) = R\lambda \cosh\lambda \Rightarrow dt = R[\operatorname{ch}\lambda + \lambda \operatorname{sh}\lambda]\, d\lambda$$

$$ds^2 = -dt^2 + dx^2 = R^2 d\lambda^2 [\underbrace{\operatorname{ch}^2 + \lambda^2 \operatorname{sh}^2 + 2\operatorname{sh}\operatorname{ch} - \operatorname{sh}^2}_{+1} - \lambda^2 \operatorname{ch}^2 - 2\lambda \operatorname{sh}\operatorname{ch}] = R^2(1-\lambda^2)\, d\lambda^2$$

$-1<\lambda<1 \Rightarrow 1-\lambda^2 > 0 \Rightarrow ds^2 > 0 \Rightarrow$ spacelike

$|\lambda|>1 \Rightarrow 1-\lambda^2 < 0 \Rightarrow ds^2 < 0 \Rightarrow$ timelike

$\lambda = \pm 1 \Rightarrow \qquad ds^2 = 0 \Rightarrow$ null

$$s = \int ds = \int_{-1}^{1} R\sqrt{1-\lambda^2}\, d\lambda = R\sqrt{\pi}\,\frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{1}{2}+\frac{3}{2})} = R\sqrt{\pi}\,\frac{\Gamma(\frac{3}{2})}{\Gamma(2)} = R\sqrt{\pi}\,\frac{\frac{\sqrt{\pi}}{2}}{1} = \frac{\pi R}{2}$$

$$v = \frac{d}{d\lambda} = \frac{dt}{d\lambda}\partial_t + \frac{dx}{d\lambda}\partial_x = R(\operatorname{ch}\lambda + \lambda\operatorname{sh}\lambda)\partial_t + R(\operatorname{sh}\lambda + \lambda\operatorname{ch}\lambda)\partial_x$$

ΕΜ ΠΕΔΙΟ ΣΤΗΝ ΕΘΣ

$$\epsilon_{ijk} B_k = \epsilon_{ijk} \tfrac{1}{2} \epsilon_{klm} F_{lm} = \epsilon_{kij} \epsilon_{klm} F_{lm}$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \tfrac{1}{2} F_{lm}$$

$$= \tfrac{1}{2}(F_{ij} - F_{ji}) = F_{ij}$$

$F_{i0} = \mathcal{E}_i$ $\quad F^{?}$

$F_{01} = -\mathcal{E}_1 \quad F_{02} = -\mathcal{E}_2 \quad F_{03} = -\mathcal{E}_3$

$F_{12} = \epsilon_{123} B_3 = B_3$

$F_{13} = \epsilon_{132} B_2 = -B_2$

$F_{23} = \epsilon_{231} B_1 = B_1$

$$\begin{pmatrix} 0 & -\mathcal{E}_1 & -\mathcal{E}_2 & -\mathcal{E}_3 \\ \mathcal{E}_1 & 0 & B_3 & -B_2 \\ \mathcal{E}_2 & -B_3 & 0 & B_1 \\ \mathcal{E}_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$\vec{\nabla}\times\vec{B} - \partial_t \vec{\mathcal{E}} = \vec{j} \Rightarrow \epsilon_{ijk}\partial_j B_k - \partial_0 \mathcal{E}_i = j_i$$

$$\epsilon_{ijk}\partial_j B_k = \tfrac{1}{2}\epsilon_{ijk}\epsilon_{klm}\partial_j F_{lm}$$

$$= \tfrac{1}{2}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j F_{lm}$$

$$= \tfrac{1}{2}(\partial_j F_{ij} - \partial_j F_{ji}) = \partial_j F_{ij}$$

$$= \partial_j F^{ij}$$

$$-\partial_t \mathcal{E}_i = -\partial_0 F_{i0} = +\partial_0 F^{i0}$$

we obtain $\partial_j F^{ij} + \partial_0 F^{i0} = j^i$ (1)

$$\vec{\nabla}\cdot\vec{\mathcal{E}} = \rho \Rightarrow \partial_j \mathcal{E}_j = j_0 \Rightarrow$$

$$\partial_j F_{j0} = j_0 \Rightarrow$$

$$-\partial_j F^{j0} = -j^0 \Rightarrow \partial_0 F^{00} + \partial_j F^{j0} = j^0 \ (2)$$

$$\Rightarrow (1)\ (2) \Leftrightarrow \partial_\mu F^{\mu\nu} = j^\nu$$

Εστω $\vec{\nabla}\cdot\vec{E} = \rho \Rightarrow$

$\partial_j E_j - \rho$

Υποθέτω:

$\partial_0(\partial_j E_j - \rho) =$

$\partial_j \partial_0 E_j - \partial_0 j^0$

But (4) $\Rightarrow \partial_0 E_j = (\nabla\times B)_j - j_j$

$= \epsilon_{jkm}\partial_k B_m - j_j$

So $\partial_0(\partial_j E_j - \rho) = \partial_j[\epsilon_{jkm}\partial_k B_m] - \partial_j j_j - \partial_0\rho$

αντισυμ

συμμ

$= -[\partial_j j^j + \partial_0 j^0] = 0$

$\Rightarrow \partial_j E_j - \rho =$ σταθ.

$\boxed{\text{Gödel}}$

1. $\partial_t\cdot\partial_t = g_{tt} = -1 < 0$ timelike

$\partial_r\cdot\partial_r = g_{rr} = \dfrac{1}{1+\left(\frac{r}{2a}\right)^2} > 0$ spacelike

$\partial_z\cdot\partial_z = g_{zz} = 1 > 0$ spacelike

$\partial_\phi\cdot\partial_\phi = g_{\phi\phi} = r^2\left[1-\left(\frac{r}{2a}\right)^2\right]$ → spacelike $r<2a$ → null $r=2a$ → timelike $r>2a$

2. $\partial_t g_{\mu\nu} = 0 \Rightarrow \partial_t$ KVF

$\partial_\phi g_{\mu\nu} = 0 \Rightarrow \partial_\phi$ KVF

$\partial_z g_{\mu\nu} = 0 \Rightarrow \partial_z$ KVF

geodesic $u^\mu = (\dot t, \dot r, \dot\phi, \dot z)$

$$k_0 = (\partial_t)^\mu u_\mu = g_{\mu\nu}(\partial_t)^\mu u^\nu$$
$$= g_{tt}(\partial_t)^t u^t + g_{t\phi}(\partial_t)^t u^\phi$$
$$= g_{tt}\,\dot{t} + g_{t\phi}\,\dot{\phi} =$$
$$= -\dot{t} - \frac{r^2}{\sqrt{2}a}\dot{\phi}$$

$$k_2 = (\partial_\phi)^\mu u_\mu = g_{\mu\nu}(\partial_\phi)^\mu u^\nu =$$
$$= g_{\phi t}(\partial_\phi)^\phi u^t + g_{\phi\phi}(\partial_\phi)^\phi u^\phi$$
$$= -\frac{r^2}{\sqrt{2}a}\dot{t} + r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\dot{\phi}$$

$$k_3 = (\partial_z)^\mu u_\mu = g_{\mu\nu}(\partial_z)^\mu u^\nu$$
$$= g_{zz}(\partial_z)^z u^z = \dot{z}$$

3. $$\left.\begin{aligned} k_0 &= -\dot{t} - \frac{r^2}{\sqrt{2}a}\dot{\phi} \\ k_2 &= -\frac{r^2}{\sqrt{2}a}\dot{t} + r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\dot{\phi} \end{aligned}\right\} \Rightarrow$$

$$\dot{t} = k_0 - \frac{1}{2a}\,\frac{4k_0 a - \sqrt{2}k_2}{1+\left(\frac{r}{2a}\right)^2}$$

$$\dot{\phi} = \frac{1}{2ar^2}\,\frac{2ak_2 - \sqrt{2}k_0 r^2}{1+\left(\frac{r}{2a}\right)^2}$$

$$\dot{z} = k_3$$

(3)

4. $u^\mu u_\mu = k \Rightarrow$

$$g_{tt}(\dot{t})^2 + g_{rr}(\dot{r})^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{\phi\phi}\dot{\phi}^2 + g_{zz}\dot{z}^2 = k$$

$\Rightarrow$

$$-\dot{t}^2 + \frac{1}{1+\left(\frac{r}{2a}\right)^2}\dot{r}^2 - \frac{2r^2}{\sqrt{2}a}\dot{t}\dot{\phi} + r^2\left(1-\left(\frac{r}{2a}\right)^2\right)\dot{\phi}^2 + \dot{z}^2 = k$$

$\Rightarrow$

$$-\dot{t}^2 + \frac{1}{1+\left(\frac{r}{2a}\right)^2}\dot{r}^2 - \frac{2r^2}{\sqrt{2}a}\left(k_0 - \frac{1}{2a}\frac{4k_0a-\sqrt{2}k_2}{1+\left(\frac{r}{2a}\right)^2}\right)\left(\frac{1}{2ar^2}\frac{2ak_2-\sqrt{2}k_0r^2}{1+\left(\frac{r}{2a}\right)^2}\right)$$

$$+r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\left[\frac{1}{2ar^2}\frac{2ak_2-\sqrt{2}k_0r^2}{1+\left(\frac{r}{2a}\right)^2}\right] + k_3^2 = k \Rightarrow$$

$$\dot{r}^2 = \underbrace{\left[k_0^2 - k_3^2 + \frac{\sqrt{2}k_0k_2}{a} + k\right]}_{2\mathcal{E}} - \underbrace{\frac{k_2^2}{r}}_{\frac{L^2}{2}} - \underbrace{\frac{k_0^2+k_3^2-k}{4a^2}}_{A^2} r^2 \Rightarrow$$

$$\mathcal{E} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}A^2r^2 + \frac{L^2}{2r^2}$$

$$\mathcal{E} = \frac{1}{2}\left(k_0^2 - k_3^2 + \frac{\sqrt{2}k_0k_2}{a} + k\right)$$

$$A^2 = \frac{1}{2a^2}\left(k_0^2 + k_3^2 - k\right) \qquad L = \sqrt{2}k_2$$

$t=0 \quad r=R \quad \phi=\omega t \quad z=0$

$u^t=\dot{t}=0 \quad u^r=\dot{r}=0 \quad u^\phi=\dot{\phi}=\omega \quad u^z=\dot{z}=0$

$$u_\mu u^\mu = g_{\mu\nu} u^\mu u^\nu = g_{\phi\phi} u^\phi u^\phi$$

$$= R^2\left[1-\left(\frac{R}{2a}\right)^2\right]\omega^2$$

For $R>2a$ $u^\mu u_\mu < 0$ timelike

$$u^\mu u_\mu = -1 \Rightarrow R^2\left[1-\left(\frac{R}{2a}\right)^2\right]\omega^2 = -1$$

$$\Rightarrow \omega^2 = \frac{1}{R^2\left[\left(\frac{R}{2a}\right)^2-1\right]} \Rightarrow \omega = \frac{1}{R\sqrt{\left(\frac{R}{2a}\right)^2-1}}$$

6. $\ddot{x}^\mu = 0 \quad \forall\mu$

$\dot{\phi} = \omega \neq 0$

$$a^\mu = \ddot{x}^\mu + \Gamma^\mu{}_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma$$

$$= 0 + \Gamma^\mu{}_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma$$

$$= \Gamma^\mu{}_{\phi\phi}\dot{\phi}\dot{\phi}$$

$$= \Gamma^\mu{}_{\phi\phi}\omega^2$$

only $\Gamma^r{}_{\phi\phi} = r\left[1+\left(\frac{r}{2a}\right)^2\right]\left[2\left(\frac{r}{2a}\right)^2-1\right] \neq 0$

$$\rightsquigarrow a^r = R\left[1+\left(\frac{R}{2a}\right)^2\right]\left[2\left(\frac{R}{2a}\right)^2-1\right]\omega^2$$

Stress Energy Tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \qquad S = \int \sqrt{-g}\left[-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right]$$

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)$$

$$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu} \qquad \delta g^{\mu\nu} = \delta g^{\nu\mu} \qquad \delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

$$\delta S = \int \delta\sqrt{-g}\,\mathcal{L} + \int\sqrt{-g}\,\delta\mathcal{L} = \int -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}\,\mathcal{L} + \int\sqrt{-g}\,\delta\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \cancel{V(\phi)}^{\,0}\right]$$

$$= \int\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\mathcal{L}\right)\delta g^{\mu\nu} + \int\sqrt{-g}\left[-\frac{1}{2}\delta g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\right]$$

$$= \int \underbrace{-\frac{1}{2}\sqrt{-g}\left[\partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\mathcal{L}\right]}_{\delta S/\delta g^{\mu\nu}}\delta g^{\mu\nu} \quad \Rightarrow \quad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\mathcal{L}$$