Energy-momentum tensor: EMT

- "Physical" or "metric" EMT: $T_{\mu\nu} = -\dfrac{2}{\sqrt{-g}} \dfrac{\delta S}{\delta g^{\mu\nu}}$ (1)

  - If we are in flat space time (SR), we write action with $\eta_{\mu\nu} \to g_{\mu\nu}$, $\partial_\mu \to \nabla_\mu$ etc (minimal coupling principle)

  - We vary the matter part of the action w.r.t. to $\delta g^{\mu\nu}$, including $\sqrt{-g}$

$$S = \int \sqrt{-g}\, \mathcal{L}(\phi, \nabla\phi) \to \delta S = \int \frac{\delta S}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = \int \sqrt{-g} \left(\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \right) \delta g^{\mu\nu}$$

  - We read off $T_{\mu\nu}$ from (1), and set back $g_{\mu\nu} \to \eta_{\mu\nu}$ (for SR)

- $$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$$

$\rightsquigarrow$ symmetric by construction

$\rightsquigarrow$ IF matter fields solutions to Einstein Eqs $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$, THEN $\nabla_\mu T^{\mu\nu} = 0$ is conserved (because $\nabla^\mu G_{\mu\nu} = 0$ from Bianchi identities). For each S that we construct, we must check if $\nabla_\mu T^{\mu\nu} = 0$

$\rightsquigarrow$ The canonical stress-energy tensor is constructed so that $\partial_\mu T^{\mu\nu} = 0$, using Noether's theorem. But it may not be equal to the physical EMT.

$\rightsquigarrow$ Notice that $G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \Rightarrow T = \frac{(1-n/2)}{8\pi G} R$ so $T = 0 \Rightarrow R = 0$

Scalar Field Theory

$$S = \int \sqrt{-g}\left[-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right] \qquad \mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)$$

Since $\partial_\mu\phi = \nabla_\mu\phi$ for a scalar field, equivalent forms of the action are:

$$S = \int\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu\phi - V(\phi)\right] = \int\sqrt{-g}\left[+\frac{1}{2}g^{\mu\nu}\phi\,\nabla_\mu\nabla_\nu\phi - V(\phi)\right]$$

$$= \int\sqrt{g}\left(\frac{1}{2}\phi\,\Box\phi - V(\phi)\right) \qquad \Box = \nabla_\mu\nabla^\mu \equiv \nabla^2$$

Sign of kinetic term depends on signature of $g_{\mu\nu}$. Here we have $(-+++)$, in QFT books $(+---)$ is used and $-\frac{1}{2}(\nabla\phi)^2 \to +\frac{1}{2}(\nabla\phi)^2$. On physical grounds, what matters is to keep $\mathcal{L} = T - V$, so the $(\partial_t\phi)^2$ or $\dot{\phi}^2$ term must be positive. In our case $-\frac{1}{2}(\nabla\phi)^2 = -\frac{1}{2}\eta^{00}(\partial_0\phi)^2 - \frac{1}{2}\eta^{ii}(\partial_i\phi)^2 = +\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\partial_i\phi)^2$

$$S = \int \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\mu \phi - V(\phi) \right]$$

$$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu} \quad , \qquad \delta g^{\mu\nu} = \delta g^{\nu\mu}$$

$$\sqrt{-g} \to \sqrt{-g} - \frac{1}{2} \sqrt{-g} \, g_{\mu\nu} \, \delta g^{\mu\nu}$$

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu \Rightarrow \delta g^{\mu\rho} g_{\rho\nu} + g^{\mu\rho} \delta g_{\rho\nu} = 0 \Rightarrow g_{\nu\rho} \delta g^{\rho\mu} = -g^{\mu\rho} \delta g_{\rho\nu}$$

$$S = \int \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\mu \phi - V(\phi)\right] = \int \sqrt{-g}\, \mathcal{L}$$

$$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu} \quad , \qquad \delta g^{\mu\nu} = \delta g^{\nu\mu}$$

$$\sqrt{-g} \to \sqrt{-g} - \frac{1}{2} \sqrt{-g}\, g_{\mu\nu} \, \delta g^{\mu\nu}$$

$$g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}{}_{\nu} \Rightarrow \delta g^{\mu\rho} g_{\rho\nu} + g^{\mu\rho} \delta g_{\rho\nu} = 0 \Rightarrow g_{\nu\rho} \delta g^{\rho\mu} = -g^{\mu\rho} \delta g_{\rho\nu}$$

$$\delta S = \int \delta \sqrt{-g}\, \mathcal{L} + \int \sqrt{-g}\, \delta \mathcal{L}$$

$$S = \int \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi\, \partial_\mu \phi - V(\phi)\right] = \int \sqrt{-g}\, \mathcal{L}$$

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$$g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}{}_{\nu} \Rightarrow \delta g^{\mu\rho} g_{\rho\nu} + g^{\mu\rho}\, \delta g_{\rho\nu} = 0 \Rightarrow g_{\nu\rho}\, \delta g^{\rho\mu} = -g^{\mu\rho}\, \delta g_{\rho\nu}$$

$$\delta S = \int \delta\sqrt{-g}\, \mathcal{L} + \int \sqrt{-g}\, \delta \mathcal{L}$$

$$= \int -\frac{1}{2} \sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}\, \mathcal{L} + \int \sqrt{-g}\, \delta\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi - V(\phi)\right]$$

$$S = \int \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\mu \phi - V(\phi)\right] = \int \sqrt{-g}\, \mathcal{L}$$

$$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu} \quad , \qquad \delta g^{\mu\nu} = \delta g^{\nu\mu}$$

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$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu \Rightarrow \delta g^{\mu\rho} g_{\rho\nu} + g^{\mu\rho}\delta g_{\rho\nu} = 0 \Rightarrow g_{\nu\rho}\, \delta g^{\rho\mu} = -g^{\mu\rho}\, \delta g_{\rho\nu}$$

$$\delta S = \int \delta\sqrt{-g}\, \mathcal{L} + \int \sqrt{-g}\, \delta \mathcal{L}$$

only metric dependence

$$= \int -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}\, \mathcal{L} + \int \sqrt{-g}\, \delta\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu\phi \partial_\nu \phi - \cancel{V(\phi)}\right]$$

$$= \int \sqrt{-g}\left(-\frac{1}{2} g_{\mu\nu} \mathcal{L}\right) \delta g^{\mu\nu} + \int \sqrt{-g}\left[-\frac{1}{2}\delta g^{\mu\nu}\, \partial_\mu \phi\, \partial_\nu \phi\right]$$

$$S = \int \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\mu \phi - V(\phi)\right] = \int \sqrt{-g}\, \mathcal{L}$$

$$= \int -\frac{1}{2}\sqrt{-g}\left[\partial_\mu \phi \, \partial_\nu \phi + g_{\mu\nu} \mathcal{L}\right] \delta g^{\mu\nu}$$

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only metric dependence

$$\begin{aligned}
\delta S &= \int \delta\sqrt{-g}\, \mathcal{L} + \int \sqrt{-g}\, \delta \mathcal{L} \\
&= \int -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu} \mathcal{L} + \int \sqrt{-g}\, \delta\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu\phi \partial_\nu \phi - \cancel{V(\phi)}\right] \\
&= \int \sqrt{-g}\left(-\frac{1}{2} g_{\mu\nu} \mathcal{L}\right) \delta g^{\mu\nu} + \int \sqrt{-g}\left[-\frac{1}{2}\delta g^{\mu\nu}\, \partial_\mu \phi \, \partial_\nu \phi\right]
\end{aligned}$$

$$S = \int \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi)\right] = \int \sqrt{-g}\, \mathcal{L}$$

$$= \underbrace{\int -\frac{1}{2}\sqrt{-g}\left[\partial_\mu \phi \, \partial_\nu \phi + g_{\mu\nu} \mathcal{L}\right]}_{\frac{\delta S}{\delta g^{\mu\nu}}} \delta g^{\mu\nu}$$

$$\delta S = \int \delta\sqrt{-g}\, \mathcal{L} + \int \sqrt{-g}\, \delta \mathcal{L}$$

only metric dependence

$$= \int -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu} \mathcal{L} + \int \sqrt{-g}\, \delta\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - \cancel{V(\phi)}\right]$$

$$= \int \sqrt{-g}\left(-\frac{1}{2} g_{\mu\nu} \mathcal{L}\right) \delta g^{\mu\nu} + \int \sqrt{-g}\left[-\frac{1}{2}\delta g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi\right]$$

$$S = \int \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\mu \phi - V(\phi)\right] = \int \sqrt{-g}\, \mathcal{L}$$

$$= \int -\frac{1}{2}\sqrt{-g}\left[\partial_\mu \phi \, \partial_\nu \phi + g_{\mu\nu} \mathcal{L}\right] \delta g^{\mu\nu}$$

$$\Rightarrow \frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{g}\left[\partial_\mu \phi \, \partial_\nu \phi + g_{\mu\nu} \mathcal{L}\right] \Rightarrow T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu \phi \, \partial_\nu \phi + g_{\mu\nu} \mathcal{L}$$

$$\delta S = \int \delta \sqrt{-g}\, \mathcal{L} + \int \sqrt{-g}\, \delta \mathcal{L}$$

only metric dependence

$$= \int -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu} \mathcal{L} + \int \sqrt{-g}\, \delta\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - \cancel{V(\phi)}\right]$$

$$= \int \sqrt{-g}\left(-\frac{1}{2} g_{\mu\nu} \mathcal{L}\right) \delta g^{\mu\nu} + \int \sqrt{-g}\left[-\frac{1}{2} \delta g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi\right]$$

$$S = \int \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\mu \phi - V(\phi)\right] = \int \sqrt{-g}\, \mathcal{L}$$

$$= \int -\frac{1}{2}\sqrt{-g}\left[\partial_\mu \phi \, \partial_\nu \phi + g_{\mu\nu} \mathcal{L}\right] \delta g^{\mu\nu}$$

$$\Rightarrow \frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{g}\left[\partial_\mu \phi \, \partial_\nu \phi + g_{\mu\nu} \mathcal{L}\right] \Rightarrow T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu \phi \, \partial_\nu \phi + g_{\mu\nu} \mathcal{L}$$

In SR $g_{\mu\nu} = \eta_{\mu\nu}$, so $T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi + \eta_{\mu\nu} \mathcal{L}$, and

$$T_{00} = \partial_0 \phi \, \partial_0 \phi + \eta_{00}\left(-\frac{1}{2} \eta^{00} \partial_0 \phi \, \partial_0 \phi - \frac{1}{2} \eta^{ii} \partial_i \phi \, \partial_i \phi - V(\phi)\right)$$

$$= \dot{\phi}^2 - \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\partial_i \phi)^2 + V(\phi) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\partial_i \phi)^2 + V(\phi)$$

$$S = \int \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\mu\phi - V(\phi)\right] = \int \sqrt{-g}\,\mathcal{L}$$

$$= \int -\frac{1}{2}\sqrt{-g}\left[\partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\mathcal{L}\right]\delta g^{\mu\nu}$$

$$\Rightarrow \frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{g}\left[\partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\mathcal{L}\right] \Rightarrow T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\mathcal{L}$$

In SR $g_{\mu\nu} = \eta_{\mu\nu}$, so $T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi + \eta_{\mu\nu}\mathcal{L}$, and

$$T_{00} = \partial_0\phi\,\partial_0\phi + \eta_{00}\left(-\frac{1}{2}\eta^{00}\partial_0\phi\,\partial_0\phi - \frac{1}{2}\eta^{ii}\partial_i\phi\,\partial_i\phi - V(\phi)\right)$$

$$= \dot\phi^2 - \frac{1}{2}\dot\phi^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi) = \frac{1}{2}\dot\phi^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi) = \mathcal{H}$$

energy or Hamiltonian density

kinetic energy density

"elastic energy" density

Consider the equivalent action: $S = \int \sqrt{-g} \left(+ \frac{1}{2} \phi \nabla^2 \phi - V(\phi) \right)$

$$S = \int \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \phi \nabla_\mu \nabla_\nu \phi - V(\phi) \right)$$

$$\delta S = \int \delta \sqrt{-g} \, \mathcal{L} + \int \sqrt{-g} \, \frac{1}{2} \phi \, \delta \left(\nabla^2 \phi \right)$$

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$$S = \int \sqrt{-g}\left(\frac{1}{2} g^{\mu\nu} \phi \nabla_\mu \nabla_\nu \phi - V(\phi) \right)$$

$$\delta S = \int \delta \sqrt{-g}\, \mathcal{L} + \int \sqrt{-g}\, \frac{1}{2} \phi\, \delta\left(\nabla^2 \phi\right)$$

Compute: $\delta \nabla^2 \phi$:

$$\nabla^2 \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = g^{\mu\nu}\left(\partial_\mu \nabla_\nu \phi - \Gamma^\rho{}_{\mu\nu} \nabla_\rho \phi\right) = g^{\mu\nu}\left(\partial_\mu \partial_\nu \phi - \Gamma^\rho{}_{\mu\nu} \partial_\rho \phi\right)$$

Consider the equivalent action: $S = \int \sqrt{-g}\left(+ \frac{1}{2} \phi \nabla^2 \phi - V(\phi) \right)$

$$S = \int \sqrt{-g}\left(\frac{1}{2} g^{\mu\nu} \phi \nabla_\mu \nabla_\nu \phi - V(\phi) \right)$$

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Compute: $\delta \nabla^2 \phi$: $\quad$ (use $\nabla_\mu \phi = \partial_\mu \phi$)

$$\nabla^2 \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = g^{\mu\nu}\left(\partial_\mu \nabla_\nu \phi - \Gamma^\rho{}_{\mu\nu} \nabla_\rho \phi \right) = g^{\mu\nu}\left(\partial_\mu \partial_\nu \phi - \Gamma^\rho{}_{\mu\nu} \partial_\rho \phi \right)$$

$$\delta \nabla^2 \phi = \delta g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - g^{\mu\nu} \delta \Gamma^\rho{}_{\mu\nu} \nabla_\rho \phi$$

Consider the equivalent action: $S = \int \sqrt{-g}\left(+\frac{1}{2}\phi \nabla^2 \phi - V(\phi)\right)$

$$S = \int \sqrt{-g}\left(\frac{1}{2} g^{\mu\nu} \phi \nabla_\mu \nabla_\nu \phi - V(\phi)\right)$$

$$\delta S = \int \delta\sqrt{-g}\, \mathcal{L} + \int \sqrt{-g}\, \frac{1}{2} \phi\, \delta(\nabla^2 \phi)$$

Compute: $\delta \nabla^2 \phi$: $\quad$ (use $\nabla_\mu \phi = \partial_\mu \phi$)

$$\nabla^2 \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = g^{\mu\nu}(\partial_\mu \nabla_\nu \phi - \Gamma^\rho{}_{\mu\nu} \nabla_\rho \phi) = g^{\mu\nu}(\partial_\mu \partial_\nu \phi - \Gamma^\rho{}_{\mu\nu} \partial_\rho \phi)$$

$$\delta \nabla^2 \phi = \delta g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - g^{\mu\nu} \delta \Gamma^\rho{}_{\mu\nu} \nabla_\rho \phi$$

In class (see slides), we showed:

$$\delta \Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}\left[\nabla_\mu \delta g_{\nu\lambda} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu}\right]$$

Use $g^{\mu\lambda}\delta g_{\lambda\nu} = -\delta g^{\mu\lambda} g_{\lambda\nu}$, then (also $\nabla g = 0$, so $g\nabla(\dots) = \nabla(g\dots)$)

$$g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}\left[\nabla_\mu(g^{\mu\nu}\delta g_{\nu\lambda}) + \nabla_\nu(g^{\mu\nu}\delta g_{\mu\lambda}) - \nabla_\lambda(g^{\mu\nu}\delta g_{\mu\nu})\right]$$

$$= -\frac{1}{2} g^{\rho\lambda}\left[\nabla_\mu(\delta g^{\mu\nu} g_{\nu\lambda}) + \nabla_\nu(\delta g^{\mu\nu} g_{\mu\lambda}) - \nabla_\lambda(\delta g^{\mu\nu} g_{\mu\nu})\right]$$

$$= -\frac{1}{2} g^{\rho\lambda}\left[g_{\nu\lambda}\nabla_\mu\delta g^{\mu\nu} + g_{\mu\lambda}\nabla_\nu\delta g^{\mu\nu} - g_{\mu\nu}\nabla_\lambda\delta g^{\mu\nu}\right]$$

$$= -\frac{1}{2}\left[g^{\rho\lambda}g_{\nu\lambda}\nabla_\mu\delta g^{\mu\nu} + g^{\rho\lambda}g_{\mu\lambda}\nabla_\nu\delta g^{\mu\nu} - g^{\rho\lambda}g_{\mu\nu}\nabla_\lambda\delta g^{\mu\nu}\right]$$

$$= -\frac{1}{2}\left[\delta^\rho_\nu\nabla_\mu\delta g^{\mu\nu} + \delta^\rho_\mu\nabla_\nu\delta g^{\mu\nu} - g_{\mu\nu}\nabla^\rho\delta g^{\mu\nu}\right]$$

$$= -\frac{1}{2}\left[2\delta^\rho_\nu\nabla_\mu\delta g^{\mu\nu} - g_{\mu\nu}\nabla^\rho\delta g^{\mu\nu}\right] \qquad (\text{use } \delta g^{\mu\nu} = \delta g^{\nu\mu})$$

In class (see slides), we showed:

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}\left[\nabla_\mu\delta g_{\nu\lambda} + \nabla_\nu\delta g_{\mu\lambda} - \nabla_\lambda\delta g_{\mu\nu}\right]$$

$$\delta S = \int \sqrt{-g}\, \mathcal{L} + \frac{1}{2}\int \sqrt{-g}\, \phi\, \delta(\nabla^2 \phi) =$$

$$= \int -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}\, \mathcal{L} + \int \sqrt{-g}\, \frac{1}{2}\, \delta g^{\mu\nu}\, \phi\, \nabla_\mu \nabla_\nu \phi + \int \sqrt{-g} \left(-\frac{1}{2}\, g^{\mu\nu}\, \delta \Gamma^\rho{}_{\mu\nu} \nabla_\rho \phi \right)$$

$$= -\frac{1}{2}\int \sqrt{-g}\, (g_{\mu\nu}\, \mathcal{L})\, \delta g^{\mu\nu} + \frac{1}{2}\int \sqrt{-g}\, (\phi \nabla_\mu \nabla_\nu \phi)\, \delta g^{\mu\nu} + \frac{1}{2}\int \sqrt{-g}\, \frac{1}{2}\left[2\, \delta^\rho{}_\nu\, \nabla_\mu \delta g^{\mu\nu} - g_{\mu\nu} \nabla^\rho \delta g^{\mu\nu} \right] \phi \nabla_\rho \phi$$

$$g^{\mu\nu}\, \delta \Gamma^\rho{}_{\mu\nu} = -\frac{1}{2}\left[2\, \delta^\rho{}_\nu\, \nabla_\mu \delta g^{\mu\nu} - g_{\mu\nu} \nabla^\rho \delta g^{\mu\nu} \right]$$

$$\delta S = \int \sqrt{-g}\, \mathcal{L} + \frac{1}{2}\int \sqrt{-g}\, \phi\, \delta(\nabla^2 \phi) =$$

$$= \int -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}\, \mathcal{L} + \int \sqrt{-g}\, \frac{1}{2}\, \delta g^{\mu\nu}\, \phi\, \nabla_\mu \nabla_\nu \phi + \int \sqrt{-g}\left(-\frac{1}{2}\, g^{\mu\nu}\, \delta\Gamma^\rho{}_{\mu\nu}\, \nabla_\rho \phi\right)$$

$$= -\frac{1}{2}\int \sqrt{-g}\,(g_{\mu\nu}\mathcal{L})\,\delta g^{\mu\nu} + \frac{1}{2}\int \sqrt{-g}\,(\phi\, \nabla_\mu \nabla_\nu \phi)\,\delta g^{\mu\nu} + \frac{1}{2}\int \sqrt{-g}\,\frac{1}{2}\left[2\delta^\rho{}_\nu\, \nabla_\mu \delta g^{\mu\nu} - g_{\mu\nu}\nabla^\rho \delta g^{\mu\nu}\right]\phi \nabla_\rho \phi$$

$$= -\frac{1}{2}\int \sqrt{-g}\,(g_{\mu\nu}\mathcal{L})\,\delta g^{\mu\nu} + \frac{1}{2}\int \sqrt{-g}\,(\phi\, \nabla_\mu \nabla_\nu \phi)\,\delta g^{\mu\nu} + \frac{1}{2}\int \sqrt{-g}\,\delta^\rho{}_\nu\left(\nabla_\mu \delta g^{\mu\nu}\right)\left(\phi\, \nabla_\rho \phi\right)$$

$$+ \frac{1}{2}\int \sqrt{-g}\left(-\frac{1}{2}\right) g_{\mu\nu}\left(\nabla^\rho \delta g^{\mu\nu}\right)\left(\phi\, \nabla_\rho \phi\right)$$

integration by parts

$$= -\frac{1}{2}\int \sqrt{-g}\,(g_{\mu\nu}\mathcal{L})\,\delta g^{\mu\nu} + \frac{1}{2}\int \sqrt{-g}\,(\phi\, \nabla_\mu \nabla_\nu \phi)\,\delta g^{\mu\nu} - \frac{1}{2}\int \sqrt{-g}\,\delta g^{\mu\nu}\left(\nabla_\mu(\phi\, \nabla_\nu \phi)\right)$$

$$- \frac{1}{2}\int \sqrt{-g}\left(-\frac{1}{2}\right) g_{\mu\nu}\,\delta g^{\mu\nu}\left(\nabla^\rho(\phi\, \nabla_\rho \phi)\right)$$

$$g^{\mu\nu}\,\delta\Gamma^\rho{}_{\mu\nu} = -\frac{1}{2}\left[2\,\delta^\rho{}_\nu\, \nabla_\mu \delta g^{\mu\nu} - g_{\mu\nu}\nabla^\rho \delta g^{\mu\nu}\right]$$

$$= -\frac{1}{2}\int\sqrt{-g}\left(+\frac{1}{2}\phi\nabla^2\phi - V(\phi)\right)g_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}\int\sqrt{-g}\left(\phi\nabla_\mu\nabla_\nu\phi\right)\delta g^{\mu\nu}$$

$$-\frac{1}{2}\int\sqrt{-g}\ \nabla_\mu\phi\nabla_\nu\phi\ \delta g^{\mu\nu} \qquad - \frac{1}{2}\int\sqrt{-g}\ \ \phi\ \nabla_\mu\nabla_\nu\phi\ \delta g^{\mu\nu}$$

$$+\frac{1}{2}\int\sqrt{-g}\ \frac{1}{2}\nabla^\rho\phi\nabla_\rho\phi\ g_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}\int\sqrt{-g}\ \ \phi\ \nabla^2\phi\ \ g_{\mu\nu}\delta g^{\mu\nu}$$

$$= -\frac{1}{2}\int\sqrt{-g}\left(g_{\mu\nu}\mathcal{L}\right)\delta g^{\mu\nu} + \frac{1}{2}\int\sqrt{-g}\left(\phi\nabla_\mu\nabla_\nu\phi\right)\delta g^{\mu\nu} - \frac{1}{2}\int\sqrt{-g}\quad \delta g^{\mu\nu}\left(\nabla_\mu\left(\phi\nabla_\nu\phi\right)\right)$$

$$-\frac{1}{2}\int\sqrt{-g}\left(-\frac{1}{2}\right)g_{\mu\nu}\delta g^{\mu\nu}\left(\nabla^\rho\left(\phi\nabla_\rho\phi\right)\right)$$

$$= -\frac{1}{2}\int \sqrt{-g}\left(+\frac{1}{2}\phi \cancel{\nabla^2}\phi - V(\phi)\right) g_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}\int \sqrt{-g}\left(\phi \cancel{\nabla_\mu \nabla_\nu}\phi\right)\delta g^{\mu\nu}$$

$$-\frac{1}{2}\int \sqrt{-g}\ \nabla_\mu \phi \nabla_\nu \phi\ \delta g^{\mu\nu} \qquad -\frac{1}{2}\int \sqrt{-g}\ \phi \cancel{\nabla_\mu \nabla_\nu}\phi\ \delta g^{\mu\nu}$$

$$+\frac{1}{2}\int \sqrt{-g}\ \frac{1}{2}\nabla^\rho \phi \nabla_\rho \phi\ g_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}\int \sqrt{-g}\ \phi \cancel{\nabla^2}\phi\ g_{\mu\nu}\delta g^{\mu\nu}$$

$$= -\frac{1}{2}\int \sqrt{-g}\left(\nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu}\left[-\frac{1}{2}\nabla^\rho \phi \nabla_\rho \phi - V(\phi)\right]\right)\delta g^{\mu\nu}$$

$$= -\frac{1}{2}\int \sqrt{-g}\left(\nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu}\mathcal{L}\right)\delta g^{\mu\nu}$$

$$= -\frac{1}{2}\int\sqrt{-g}\left(+\frac{1}{2}\phi\cancel{\nabla^2}\phi - V(\phi)\right)g_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}\int\sqrt{-g}\left(\phi\cancel{\nabla_\mu\nabla_\nu}\phi\right)\delta g^{\mu\nu}$$

$$-\frac{1}{2}\int\sqrt{-g}\;\nabla_\mu\phi\nabla_\nu\phi\;\delta g^{\mu\nu} \qquad -\frac{1}{2}\int\sqrt{-g}\;\phi\cancel{\nabla_\mu\nabla_\nu}\phi\;\delta g^{\mu\nu}$$

$$+\frac{1}{2}\int\sqrt{-g}\;\frac{1}{2}\nabla^\rho\phi\nabla_\rho\phi\;g_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}\int\sqrt{-g}\;\phi\cancel{\nabla^2}\phi\;g_{\mu\nu}\delta g^{\mu\nu}$$

$$= -\frac{1}{2}\int\sqrt{-g}\left(\nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}\left[-\frac{1}{2}\nabla^\rho\phi\nabla_\rho\phi - V(\phi)\right]\right)\delta g^{\mu\nu}$$

$$= -\frac{1}{2}\int\sqrt{-g}\left(\nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}\mathcal{L}\right)\delta g^{\mu\nu}$$

Therefore $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}\mathcal{L}$ is the same for both actions

$$S = \int\sqrt{-g}\left(-\frac{1}{2}(\nabla\phi)^2 - V(\phi)\right) = \int\sqrt{-g}\left(+\frac{1}{2}\phi\nabla^2\phi - V(\phi)\right)$$

Electromagnetic Field (no sources: $J^\mu = 0$)

$$S = \int \sqrt{-g}\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) = \int \sqrt{-g}\left(-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\right) = \int \sqrt{-g}\left(-\frac{1}{4} F^2\right)$$

$$F^2 \equiv F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = 2 \nabla_{[\mu} A_{\nu]} = (dA)_{\mu\nu}$$

vary $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$

Electromagnetic Field (no sources: $J^\mu = 0$)

$$S = \int \sqrt{-g}\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) = \int \sqrt{-g}\left(-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\right) = \int \sqrt{-g}\left(-\frac{1}{4} F^2\right)$$

$$F^2 \equiv F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = 2 \nabla_{[\mu} A_{\nu]} = (dA)_{\mu\nu}$$

vary $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho$$

$$\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\rho_{\nu\mu} A_\rho$$

Electromagnetic Field (no sources: $J^\mu = 0$)

$$S = \int \sqrt{-g}\left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\right) = \int \sqrt{-g}\left(-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\right) = \int \sqrt{-g}\left(-\frac{1}{4} F^2\right)$$

$$F^2 \equiv F_{\mu\nu}F^{\mu\nu}$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = 2\,\nabla_{[\mu}A_{\nu]} = (dA)_{\mu\nu}$$

vary $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$

$$\left.\begin{aligned} \nabla_\mu A_\nu &= \partial_\mu A_\nu - \cancel{\Gamma^\rho_{\mu\nu} A_\rho} \\ \nabla_\nu A_\mu &= \partial_\nu A_\mu - \cancel{\Gamma^\rho_{\nu\mu} A_\rho} \end{aligned}\right\} \Rightarrow \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} \qquad \text{(torsion free)}$$

Electromagnetic Field (no sources: $J^\mu = 0$)

$$S = \int \sqrt{-g}\left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\right) = \int \sqrt{-g}\left(-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\right) = \int \sqrt{-g}\left(-\frac{1}{4} F^2\right)$$

$$F^2 \equiv F_{\mu\nu}F^{\mu\nu}$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = 2\nabla_{[\mu}A_{\nu]} = (dA)_{\mu\nu}$$

vary $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$

$$\left.\begin{aligned} \nabla_\mu A_\nu &= \partial_\mu A_\nu - \cancel{\Gamma^\rho_{\mu\nu}} A_\rho \\ \nabla_\nu A_\mu &= \partial_\nu A_\mu - \cancel{\Gamma^\rho_{\nu\mu}} A_\rho \end{aligned}\right\} \Rightarrow \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} \qquad \text{(torsion free)}$$

Therefore, $\delta F_{\mu\nu} = 0$

Electromagnetic Field (no sources: $J^\mu = 0$)

$$S = \int \sqrt{-g}\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) = \int \sqrt{-g}\left(-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\right) = \int \sqrt{-g}\left(-\frac{1}{4} F^2\right)$$

$$\delta S = -\frac{1}{4}\int \delta\sqrt{-g}\, F^2 - \frac{1}{4}\int \sqrt{-g}\, \delta g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} - \frac{1}{4}\int \sqrt{-g}\, g^{\mu\rho}\, \delta g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$$

Electromagnetic Field (no sources: $J^\mu = 0$)

$$S = \int \sqrt{-g}\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) = \int \sqrt{-g}\left(-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\right) = \int \sqrt{-g}\left(-\frac{1}{4} F^2\right)$$

$$\delta S = -\frac{1}{4}\int \delta\sqrt{-g}\, F^2 - \frac{1}{4}\int \sqrt{-g}\, \underset{\rho\to\nu}{\delta g^{\mu\rho}}\, \underset{\nu\to\rho}{g^{\nu\sigma}} F_{\mu\nu} F_{\rho\sigma} - \frac{1}{4}\int \sqrt{-g}\, \underset{\sigma\to\mu}{g^{\mu\rho}}\, \underset{\mu\to\sigma}{\delta g^{\nu\sigma}} F_{\mu\nu} F_{\rho\sigma}$$

$$= -\frac{1}{4}\int \sqrt{-g}\left(-\frac{1}{2} g_{\mu\nu}\delta g^{\mu\nu}\right) - \frac{1}{4}\int \sqrt{-g}\, \delta g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4}\int \sqrt{-g}\, g^{\sigma\rho}\, \delta g^{\nu\mu} \underset{-F_{\nu\sigma}}{\underset{\shortparallel}{F_{\sigma\nu}}}\, \underset{-F_{\mu\rho}}{\underset{\shortparallel}{F_{\rho\mu}}}$$

$$= -\frac{1}{2}\int \sqrt{-g}\left[g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F^2\right]\delta g^{\mu\nu}$$

Electromagnetic Field (no sources: $J^\mu = 0$)

$$S = \int \sqrt{-g}\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) = \int \sqrt{-g}\left(-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\right) = \int \sqrt{-g}\left(-\frac{1}{4} F^2\right)$$

$$\delta S = -\frac{1}{4}\int \delta\sqrt{-g}\, F^2 - \frac{1}{4}\int \sqrt{-g}\, \delta g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} - \frac{1}{4}\int \sqrt{-g}\, g^{\mu\rho} \delta g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$\rho \to \nu \qquad \nu \to \rho \qquad\qquad \sigma \to \mu \qquad \mu \to \sigma$

$$= -\frac{1}{4}\int \sqrt{-g}\left(-\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu}\right) - \frac{1}{4}\int \sqrt{-g}\, \delta g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4}\int \sqrt{-g}\, g^{\sigma\rho} \delta g^{\nu\mu} \underbrace{F_{\sigma\nu}}_{= -F_{\nu\sigma}} \underbrace{F_{\rho\mu}}_{= -F_{\mu\rho}}$$

$$= -\frac{1}{2}\int \sqrt{-g}\left[g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F^2\right] \delta g^{\mu\nu}$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F^2$$

Compute components: For $\eta_{\mu\nu} = (- + + +)$

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 \\ -\mathcal{E}_1 & 0 & B_3 & -B_2 \\ -\mathcal{E}_2 & -B_3 & 0 & B_1 \\ -\mathcal{E}_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad (F_{\mu\nu}) = \begin{pmatrix} 0 & -\mathcal{E}_1 & -\mathcal{E}_2 & -\mathcal{E}_3 \\ \mathcal{E}_1 & 0 & B_3 & -B_2 \\ \mathcal{E}_2 & -B_3 & 0 & B_1 \\ \mathcal{E}_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$F^{0i} = -F^{i0} = \mathcal{E}^i \qquad F_{0i} = -F_{i0} = -\mathcal{E}_i$$

$$F_{ij} = \epsilon_{ijk} B_k = F^{ij} \qquad B_k = \frac{1}{2}\epsilon_{kij} F_{ij}$$

$$A_\mu = (-\phi, \vec{A}) \qquad A^\mu = (\phi, \vec{A})$$

$$\mathcal{E}_i = -F_{0i} = -(\partial_0 A_i - \partial_i A_0) = -\frac{\partial A_i}{\partial t} + \frac{\partial A_0}{\partial x^i} = -\frac{\partial A_i}{\partial t} - \frac{\partial \phi}{\partial x^i} = -(\vec{\nabla}\phi)_i - \frac{\partial A_i}{\partial t}$$

$$B_k = \frac{1}{2}\epsilon_{kij} F_{ij} = \frac{1}{2}\epsilon_{kij}(\partial_i A_j - \partial_j A_i) = \frac{1}{2}\epsilon_{kij}\, 2\partial_i A_j = (\vec{\nabla}\times\vec{A})_k$$

$$F^2 = F_{\mu\nu} F^{\mu\nu} = F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij} = 2 F_{0i} F^{0i} + F_{ij} F^{ij}$$

$$F^2 = F_{\mu\nu} F^{\mu\nu} = F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij} = 2 F_{0i} F^{0i} + F_{ij} F^{ij}$$

$$= 2(-\mathcal{E}_i)(\mathcal{E}_i) + \epsilon_{ijk} B_k \, \epsilon_{ijl} B_l$$

$$= -2\mathcal{E}^2 + (\delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj}) B_k B_l$$

$$F^2 = F_{\mu\nu} F^{\mu\nu} = F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij} = 2 F_{0i} F^{0i} + F_{ij} F^{ij}$$

$$= 2(-\mathcal{E}_i)(\mathcal{E}_i) + \epsilon_{ijk} B_k \, \epsilon_{ijl} B_l$$

$$= -2\mathcal{E}^2 + (\delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj}) B_k B_l$$

$$= -2\mathcal{E}^2 + 3\,\delta_{kl} B_k B_l - \delta_{jl}\delta_{kj} B_k B_l$$

$$= -2\mathcal{E}^2 + 3 B_k B_k - B_k B_k$$

$$= -2(\mathcal{E}^2 - B^2)$$

$$F^2 = F_{\mu\nu} F^{\mu\nu} = F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij} = 2 F_{0i} F^{0i} + F_{ij} F^{ij} = -2(\mathcal{E}^2 - B^2)$$

$$T_{00} = F_{0\rho} F_0{}^{\rho} - \frac{1}{4} \eta_{00} F^2 = F_{0i} F_{0i} + \frac{1}{4} F^2 = (-\mathcal{E}_i)(-\mathcal{E}_i) - \frac{1}{2} \mathcal{E}^2 + \frac{1}{2} B^2 = \frac{\mathcal{E}^2 + B^2}{2}$$

$$F^2 = F_{\mu\nu}F^{\mu\nu} = F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} = 2F_{0i}F^{0i} + F_{ij}F^{ij} = -2(\mathcal{E}^2 - B^2)$$

$$T_{00} = \frac{1}{2}(\mathcal{E}^2 + B^2) = T^{00}$$

$$T_{0i} = F_{0\rho}F_i{}^{\rho} - \frac{1}{4}\cancelto{0}{\eta_{0i}}F^2 = F_{0j}F_{ij} = (-\mathcal{E}_j)\epsilon_{ijk}B_k = -\epsilon_{ijk}\mathcal{E}_j B_k = -(\vec{\mathcal{E}}\times\vec{B})_i$$

$$T^{0i} = -T_{0i} = (\vec{\mathcal{E}}\times\vec{B})_i$$

$$F^2 = F_{\mu\nu} F^{\mu\nu} = F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij} = 2 F_{0i} F^{0i} + F_{ij} F^{ij} = -2(\mathcal{E}^2 - B^2)$$

$$T_{00} = \frac{1}{2}(\mathcal{E}^2 + B^2) = T^{00}$$

$$T_{0i} = F_{0\rho} F_i{}^{\rho} - \frac{1}{4} \cancelto{0}{\eta_{0i}} F^2 = F_{0j} F_{ij} = (-\mathcal{E}_j)\epsilon_{ijk} B_k = -\epsilon_{ijk} \mathcal{E}_j B_k = -(\vec{\mathcal{E}} \times \vec{B})_i$$

$$T^{0i} = -T_{0i} = (\vec{\mathcal{E}} \times \vec{B})_i$$

$$T_{ij} = F_{i\rho} F_j{}^{\rho} - \frac{1}{4} \eta_{ij} F^2 = F_{i0} F_j{}^0 + F_{ik} F_j{}^k - \frac{1}{4} \delta_{ij} F^2$$

$$F_{i0} F_j{}^0 = -F_{i0} F_{j0} = -\mathcal{E}_i \mathcal{E}_j$$

$$F_{ik} F_j{}^k = F_{ik} F_{jk} = \epsilon_{ikl} B_l \epsilon_{jkm} B_m = \epsilon_{kli} \epsilon_{kmj} B_l B_m$$

$$= (\delta_{lm} \delta_{ij} - \delta_{lj} \delta_{im}) B_l B_m = \delta_{ij} B^2 - B_i B_j$$

$$F^2 = F_{\mu\nu}F^{\mu\nu} = F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} = 2F_{0i}F^{0i} + F_{ij}F^{ij} = -2(\mathcal{E}^2 - \mathcal{B}^2)$$

$$T_{00} = \frac{1}{2}(\mathcal{E}^2 + \mathcal{B}^2) = T^{00}$$

$$T_{0i} = F_{0\rho}F_i{}^\rho - \frac{1}{4}\cancel{\eta_{0i}}^{\,0} F^2 = F_{0j}F_{ij} = (-\mathcal{E}_j)\epsilon_{ijk}\mathcal{B}_k = -\epsilon_{ijk}\mathcal{E}_j\mathcal{B}_k = -(\vec{\mathcal{E}}\times\vec{\mathcal{B}})_i$$

$$T^{0i} = -T_{0i} = (\vec{\mathcal{E}}\times\vec{\mathcal{B}})_i$$

$$T_{ij} = F_{i\rho}F_j{}^\rho - \frac{1}{4}\eta_{ij}F^2 = F_{i0}F_j{}^0 + F_{ik}F_j{}^k - \frac{1}{4}\delta_{ij}F^2$$

$$= -\mathcal{E}_i\mathcal{E}_j - \mathcal{B}_i\mathcal{B}_j + \delta_{ij}\mathcal{B}^2 + \frac{1}{4}\delta_{ij}\,2(\mathcal{E}^2 - \mathcal{B}^2)$$

$$= \left(-\mathcal{E}_i\mathcal{E}_j + \frac{1}{2}\delta_{ij}\mathcal{E}^2\right) + \left(-\mathcal{B}_i\mathcal{B}_j + \frac{1}{2}\delta_{ij}\mathcal{B}^2\right)$$

Prove $\partial_\mu T^{\mu\nu} = 0$ (on shell)

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left(F^{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta^\mu{}_\nu F^2 \right) =$$

$$= \partial_\mu F^{\mu\rho} \, F_{\nu\rho} + F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} \delta^\mu{}_\nu \, F^{\rho\sigma} \partial_\mu F_{\rho\sigma}$$

Prove $\partial_\mu T^{\mu\nu} = 0$ (on shell)

on shell (Maxwell's eq)
$\partial_\mu F^{\mu\rho} = 0$

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left(F^{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta^\mu{}_\nu F^2 \right) =$$

$$= \cancel{\partial_\mu F^{\mu\rho}} F_{\nu\rho} + F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} \delta^\mu{}_\nu F^{\rho\sigma} \partial_\mu F_{\rho\sigma}$$

$$= F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} F^{\rho\sigma} \partial_\nu F_{\rho\sigma}$$

Prove $\partial_\mu T^{\mu\nu} = 0$ (on shell)

on shell (Maxwell's eq)
$\partial_\mu F^{\mu\rho} = 0$

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left(F^{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta^\mu{}_\nu F^2 \right) =$$

$$= \cancel{\partial_\mu F^{\mu\rho}} F_{\nu\rho} + F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} \delta^\mu{}_\nu F^{\rho\sigma} \partial_\mu F_{\rho\sigma}$$

$$= F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} F^{\rho\sigma} \partial_\nu F_{\rho\sigma}$$

Bianchi identities (Maxwell's eqs)

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0$$

Prove $\partial_\mu T^{\mu\nu} = 0$ (on shell)

on shell (Maxwell's eq)

$\partial_\mu F^{\mu\rho} = 0$

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left(F^{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta^\mu{}_\nu F^2 \right) =$$

$$= \cancel{\partial_\mu F^{\mu\rho}} F_{\nu\rho} + F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} \delta^\mu{}_\nu F^{\rho\sigma} \partial_\mu F_{\rho\sigma}$$

$$= F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} F^{\rho\sigma} \partial_\nu F_{\rho\sigma}$$

Bianchi identities (Maxwell's eqs)

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0 \Rightarrow$$

$$F^{\mu\rho} \left(\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} \right) = 0 \Rightarrow F^{\mu\rho} \partial_\mu F_{\nu\rho} + F^{\mu\rho} \underbrace{\partial_\rho F_{\mu\nu}}_{\mu \leftrightarrow \rho} + F^{\mu\rho} \underbrace{\partial_\nu F_{\rho\mu}}_{F_{\rho\mu} = -F_{\mu\rho}} = 0$$

$$F^{\mu\rho} \partial_\mu F_{\nu\rho} + F^{\rho\mu} \partial_\mu F_{\rho\nu} - F^{\mu\rho} \partial_\nu F_{\mu\rho} = 0$$

Prove $\partial_\mu T^{\mu\nu} = 0$ (on shell)

on shell (Maxwell's eq)
$\partial_\mu F^{\mu\rho} = 0$

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left(F^{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta^\mu{}_\nu F^2 \right) =$$

$$= \cancel{\partial_\mu F^{\mu\rho}} F_{\nu\rho} + F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} \delta^\mu{}_\nu F^{\rho\sigma} \partial_\mu F_{\rho\sigma}$$

$$= F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} F^{\rho\sigma} \partial_\nu F_{\rho\sigma}$$

Bianchi identities (Maxwell's eqs)

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0 \Rightarrow$$

$$F^{\mu\rho} \left(\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} \right) = 0 \Rightarrow F^{\mu\rho} \partial_\mu F_{\nu\rho} + F^{\mu\rho} \underbrace{\partial_\rho F_{\mu\nu}}_{\mu \leftrightarrow \rho} + F^{\mu\rho} \underbrace{\partial_\nu F_{\rho\mu}}_{F_{\rho\mu} = -F_{\mu\rho}} = 0$$

$$F^{\mu\rho} \partial_\mu F_{\nu\rho} + \underbrace{F^{\rho\mu}}_{F^{\rho\mu} = -F^{\mu\rho}} \underbrace{\partial_\mu F_{\rho\nu}}_{F_{\rho\nu} = -F_{\nu\rho}} - F^{\mu\rho} \partial_\nu F_{\mu\rho} = 0$$

$$\Rightarrow 2 F^{\mu\rho} \partial_\mu F_{\nu\rho} - F^{\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$$

Prove $\partial_\mu T^{\mu\nu} = 0$ (on shell)

on shell (Maxwell's eq)

$$\partial_\mu F^{\mu\rho} = 0$$

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left(F^{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta^\mu{}_\nu F^2 \right) =$$

$$= \cancel{\partial_\mu F^{\mu\rho}} F_{\nu\rho} + F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} \delta^\mu{}_\nu F^{\rho\sigma} \partial_\mu F_{\rho\sigma}$$

$$= F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} F^{\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$$

Bianchi identities (Maxwell's eqs)

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0 \Rightarrow$$

$$F^{\mu\rho} \left(\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} \right) = 0 \Rightarrow F^{\mu\rho} \partial_\mu F_{\nu\rho} + F^{\mu\rho} \underbrace{\partial_\rho F_{\mu\nu}}_{\mu \leftrightarrow \rho} + F^{\mu\rho} \partial_\nu \underbrace{F_{\rho\mu}}_{F_{\rho\mu} = -F_{\mu\rho}} = 0$$

$$F^{\mu\rho} \partial_\mu F_{\nu\rho} + \underbrace{F^{\rho\mu}}_{F^{\rho\mu} = -F^{\mu\rho}} \partial_\mu \underbrace{F_{\rho\nu}}_{F_{\rho\nu} = -F_{\nu\rho}} - F^{\mu\rho} \partial_\nu F_{\mu\rho} = 0$$

$$\Rightarrow F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} F^{\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$$

Diffeomorphism invariance of S^{EH} and Bianchi identities

$$S^{EH} = \int \sqrt{-g}\, R \quad \Rightarrow \quad \delta S^{EH} = \int \sqrt{-g}\, G_{\mu\nu}\, \delta g^{\mu\nu} + \underset{\text{- ignore -}}{(\text{surface terms})}$$

If ξ^μ generates a diffeo, then $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ (see lectures)

$$\delta_\xi S^{EH} = 0 \Rightarrow \int \sqrt{-g}\, G_{\mu\nu} \left(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu \right) = 0 \Rightarrow$$

$$- \int \sqrt{-g} \left(\nabla^\mu G_{\mu\nu}\, \xi^\nu + \nabla^\nu G_{\mu\nu}\, \xi^\mu \right) = 0 \Rightarrow \text{(integrate by parts - ignore B.T.s)}$$

$$- \int \sqrt{-g} \left(\nabla^\mu G_{\mu\nu}\, \xi^\nu + \nabla^\mu G_{\nu\mu}\, \xi^\nu \right) = 0 \Rightarrow \quad (G_{\mu\nu} = +G_{\nu\mu})$$

$$- 2 \int \sqrt{-g}\, \nabla^\mu G_{\mu\nu}\, \xi^\nu = 0 \quad \forall \xi^\nu \quad \Rightarrow \quad \nabla^\mu G_{\mu\nu} = 0$$

Canonical EMT (flat spacetime - SR)

Let $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$, $S = \int d^4x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x))$

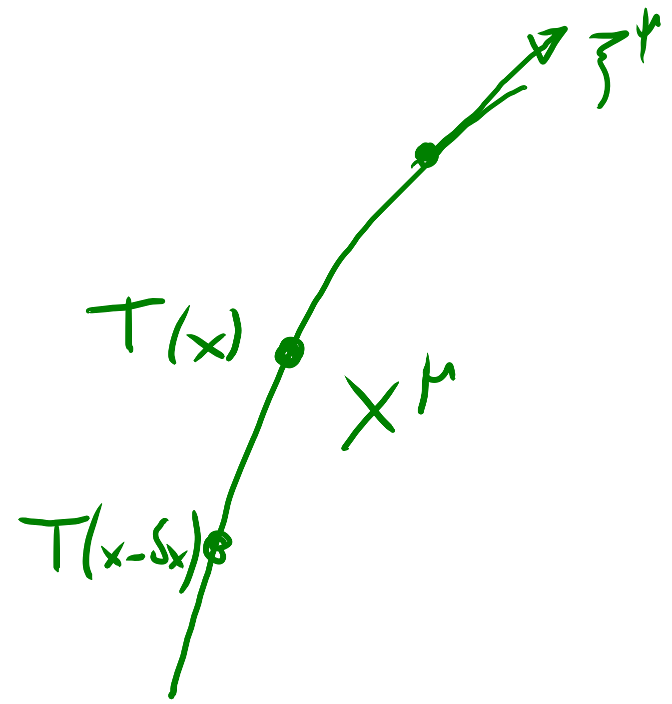
and $x^\mu \to x^{\mu\prime} = x^\mu + \xi^\mu$ $\quad \phi(x) \to \phi'(x) = \phi(x) + \delta_\xi \phi(x) = \phi(x) - \mathcal{L}_\xi \phi(x)$

(for scalars $\mathcal{L}_\xi \phi = \xi^\mu \partial_\mu \phi$)

About the sign: $\delta_\xi \phi = -\mathcal{L}_\xi \phi$

- We transport fields along integral curves of ξ^μ
- what we call variation

$$T(x) \to T'(x) = T(x) + \delta_\xi T(x)$$



is computed by comparing the new field which is pushed forwarded from the position $x - \delta x$ to x. Ie. we compare the new field to the old at x. Then $T'(x) = \phi_{*\xi} T(x) = -\phi^*_\xi T(x)$

ϕ_* : push forward
ϕ^* : pull back

But $\mathcal{L}_\xi$ is defined by pull back $\mathcal{L}_\xi T = \lim_{\xi^\mu \to 0} [\phi^*_\xi T - T]$, so $T'(x) = T(x) - \mathcal{L}_\xi T(x)$

Canonical EMT (flat spacetime - SR)

Let $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$, $S = \int d^4x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x))$

and $x^\mu \to x^{\mu'} = x^\mu + \xi^\mu$ $\quad \phi(x) \to \phi(x) + \delta_\xi \phi(x) = \phi(x) - \mathcal{L}_\xi \phi(x)$

(for scalars $\mathcal{L}_\xi \phi = \xi^\mu \partial_\mu \phi$)

If the action remains invariant:

$$S' = \int d^4x' \, \mathcal{L}(\phi'(x'), \partial_\mu \phi'(x')) = \int d^4x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = S$$

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$$d^4x' = (1 + \partial_\mu \xi^\mu) d^4x \quad , \quad \mathcal{L}(\phi(x'), \partial_\mu \phi(x')) = \mathcal{L}(\phi(x), \partial_\mu \phi(x)) + \xi^\mu \partial_\mu \mathcal{L} + \cdots$$

$$\mathcal{L}(\phi', \partial_\mu \phi') = \mathcal{L}(\phi + \delta_\xi \phi, \partial_\mu \phi + \partial_\mu \delta_\xi \phi) = \mathcal{L}(\phi, \partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\xi \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta_\xi \phi + \cdots$$

$$S' = \int d^4x (1 + \partial_\mu \xi^\mu) \left(\mathcal{L} + \xi^\mu \partial_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\xi \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta_\xi \phi + \cdots \right)$$

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$$S' = \int d^4x (1 + \partial_\mu \xi^\mu)\left(\mathcal{L} + \xi^\mu \partial_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\xi \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta_\xi \phi + \cdots \right)$$

$$= \int d^4x \left(\mathcal{L} + (\partial_\mu \xi^\mu)\mathcal{L} + \xi^\mu (\partial_\mu \mathcal{L}) + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\xi \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta_\xi \phi + \cdots \right)$$

$$= \int d^4x\, \mathcal{L} + \int d^4x \left(\partial_\mu (\xi^\mu \mathcal{L}) + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi} \delta_\xi \phi - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right]}_{\text{equations of motion } = 0} \delta_\xi \phi + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\xi \phi \right] \right)$$

$$= S + \int d^4x\, \partial_\mu \left[\xi^\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \xi^\nu \delta_\nu \phi \right]$$

$$= S + \int d^4x\, \partial_\mu \left[\left(\delta^\mu{}_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\nu \phi \right) \xi^\nu \right]$$

Define $\delta_\nu \xi$ by

$$\delta_\xi \phi = - \mathcal{L}_\xi \phi = - \xi^\nu \delta_\nu \phi$$

For scalar $\delta_\nu \phi = \partial_\nu \phi$

$$S' = \int d^4x \left(\mathcal{L} + (\partial_\mu \xi^\mu) \mathcal{L} + \xi^\mu (\partial_\mu \mathcal{L}) + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\xi \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta_\xi \phi + \cdots \right)$$

$$= \int d^4x\, \mathcal{L} + \int d^4x \Big(\partial_\mu (\xi^\mu \mathcal{L}) + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi} \delta_\xi \phi - \partial_\mu \Big[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Big]}_{\text{equations of motion } = 0} \delta_\xi \phi + \partial_\mu \Big[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\xi \phi \Big] \Big)$$

$$= S + \int d^4x\, \partial_\mu \Big[\xi^\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \xi^\nu \delta_\nu \phi \Big]$$

$$= S + \int d^4x\, \partial_\mu \Big[\Big(\delta^\mu{}_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\nu \phi \Big) \xi^\nu \Big]$$

Therefore if $\delta S = 0$ we have the conserved current

$$J^\mu = \Big(\delta^\mu{}_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\nu \phi \Big) \xi^\nu$$

$$= \int d^4x\, \mathcal{L} + \int d^4x \Big(\partial_\mu (\xi^\mu \mathcal{L}) + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi} \delta_\xi \phi - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right]}_{\text{equations of motion } = 0} \delta_\xi \phi + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\xi \phi \right] \Big)$$

$$= S + \int d^4x\, \partial_\mu \left[\xi^\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \xi^\nu \delta_\nu \phi \right]$$

$$= S + \int d^4x\, \partial_\mu \left[\left(\delta^\mu{}_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\nu \phi \right) \xi^\nu \right]$$

Therefore if $\delta S = 0$ and $\xi^\nu = \text{const}$ (translations), then

$$T^\mu{}_\nu = - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\nu \phi + \delta^\mu{}_\nu \mathcal{L}$$

(overall sign chosen so that $T^{00} = \rho > 0$)

is conserved: $\partial_\mu T^\mu{}_\nu = 0$

If we have many fields ϕ_n, then we sum over:

$$T^\mu{}_\nu = -\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \delta_\nu \phi_n + \delta^\mu{}_\nu \mathcal{L}$$

e.g. $$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \delta_\nu A_\rho - \delta^\mu{}_\nu \mathcal{L}$$ for EM

$$T^\mu{}_\nu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\nu \phi + \delta^\mu{}_\nu \mathcal{L}$$

is conserved: $\partial_\mu T^\mu{}_\nu = 0$

Real Scalar Field

$$S = \int d^4x \left(-\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi) \right) = \int d^4x \left(-\frac{1}{2} \eta^{\lambda\nu} \partial_\lambda \phi \, \partial_\nu \phi - V(\phi) \right)$$

$$T^\mu{}_\nu = - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\nu \phi + \delta^\mu{}_\nu \mathcal{L} = - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \delta^\mu{}_\nu \mathcal{L} \quad , \quad \delta_\nu \phi = \partial_\nu \phi$$

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$$T^\mu{}_\nu = - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\nu \phi + \delta^\mu{}_\nu \mathcal{L} = - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \delta^\mu{}_\nu \mathcal{L} \quad , \quad \delta_\nu \phi = \partial_\nu \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = - \frac{1}{2} \eta^{\lambda\nu} \delta^\mu{}_\lambda \partial_\nu \phi - \frac{1}{2} \eta^{\lambda\nu} \partial_\lambda \phi \, \delta^\mu{}_\nu$$

$$= - \frac{1}{2} \eta^{\mu\nu} \partial_\nu \phi - \frac{1}{2} \eta^{\lambda\mu} \partial_\lambda \phi$$

$$= - \partial^\mu \phi$$

Real Scalar Field

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$$T^\mu{}_\nu = - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\nu \phi + \delta^\mu{}_\nu \mathcal{L} = - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \delta^\mu{}_\nu \mathcal{L} \quad , \quad \delta_\nu \phi = \partial_\nu \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = - \partial^\mu \phi$$

$$\Rightarrow \quad T^\mu{}_\nu = \partial^\mu \phi \, \partial_\nu \phi + \delta^\mu{}_\nu \mathcal{L} \quad \Rightarrow \quad T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi + \eta_{\mu\nu} \mathcal{L}$$

- same as the physical/metric EMT

EM field ($J^\mu = 0$)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \eta^{\mu\lambda} \eta^{\nu\sigma} F_{\mu\nu} F_{\lambda\sigma}$$

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$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \eta^{\mu\lambda} \eta^{\nu\sigma} F_{\mu\nu} F_{\lambda\sigma}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)} = -\frac{1}{4} \cdot 2 \; \eta^{\mu\lambda} \eta^{\nu\sigma} F_{\mu\nu} \frac{\partial F_{\lambda\sigma}}{\partial(\partial_a A_\gamma)}$$

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$$\frac{\partial F_{\lambda\sigma}}{\partial(\partial_\alpha A_\gamma)} = \frac{\partial}{\partial(\partial_\alpha A_\gamma)} \left(\partial_\lambda A_\sigma - \partial_\sigma A_\lambda \right)$$

$$= \delta^\alpha{}_\lambda \delta^\gamma{}_\sigma - \delta^\alpha{}_\sigma \delta^\gamma{}_\lambda$$

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$$\frac{\partial F_{\lambda\sigma}}{\partial(\partial_\alpha A_\gamma)} = \frac{\partial}{\partial(\partial_\alpha A_\gamma)} \left(\partial_\lambda A_\sigma - \partial_\sigma A_\lambda \right)$$

$$= \delta^\alpha{}_\lambda \delta^\gamma{}_\sigma - \delta^\alpha{}_\sigma \delta^\gamma{}_\lambda \qquad \Rightarrow$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)} = -\frac{1}{2} \eta^{\mu\lambda} \eta^{\nu\sigma} F_{\mu\nu} \delta^\alpha{}_\lambda \delta^\gamma{}_\sigma + \frac{1}{2} \eta^{\mu\lambda} \eta^{\nu\sigma} F_{\mu\nu} \delta^\alpha{}_\sigma \delta^\gamma{}_\lambda$$

$$= -\frac{1}{2} \eta^{\mu\alpha} \eta^{\nu\gamma} F_{\mu\nu} + \frac{1}{2} \eta^{\mu\gamma} \eta^{\nu\alpha} F_{\mu\nu} = -\frac{1}{2} F^{\alpha\gamma} + \frac{1}{2} F^{\gamma\alpha} = -F^{\alpha\gamma}$$

EM field ($J^\mu = 0$)

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$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)} = - F^{\alpha\gamma}$$

If we use $\delta_\mu A_\nu = \partial_\mu A_\nu$, then

$$T^\alpha{}_\beta = - \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)} \partial_\beta A_\gamma + \delta^\alpha{}_\beta \mathcal{L} = + F^{\alpha\gamma} \partial_\beta A_\gamma - \frac{1}{4} \delta^\alpha{}_\beta F^2$$

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compare with physical.

$$T^\alpha{}_\beta = F^{\alpha\gamma} F_{\beta\gamma} - \frac{1}{4} \delta^\alpha{}_\beta F^2$$

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compare with physical:

$$T^\alpha{}_\beta = F^{\alpha\gamma} \underline{F_{\beta\gamma}} - \frac{1}{4} \delta^\alpha{}_\beta F^2$$

<u>not the same!</u>

$$T_{\text{phys}}{}^{\alpha}{}_{\beta} = F^{\alpha\gamma} F_{\beta\gamma} - \frac{1}{4}\delta^{\alpha}{}_{\beta} F^2 = F^{\alpha\gamma}(\partial_\beta A_\gamma - \partial_\gamma A_\beta) - \frac{1}{4}\delta^{\alpha}{}_{\beta} F^2$$

$$= F^{\alpha\gamma}\partial_\beta A_\gamma - F^{\alpha\gamma}\partial_\gamma A_\beta - \frac{1}{4}\delta^{\alpha}{}_{\beta} F^2$$

$$= T_{\text{canonical}}{}^{\alpha}{}_{\beta} - F^{\alpha\gamma}\partial_\gamma A_\beta$$

$$T^{\alpha}{}_{\beta} = -\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)}\partial_\beta A_\gamma + \delta^{\alpha}{}_{\beta}\mathcal{L} = + F^{\alpha\gamma}\underline{\partial_\beta A_\gamma} - \frac{1}{4}\delta^{\alpha}{}_{\beta} F^2$$

Compare with physical:

$$T^{\alpha}{}_{\beta} = F^{\alpha\gamma}\underline{F_{\beta\gamma}} - \frac{1}{4}\delta^{\alpha}{}_{\beta} F^2$$

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Bessen-Hagen procedure:

We have not used full invariance of action:

1. translations $x^\mu \to x^\mu + \xi^\mu$

2. $A_\mu \to \frac{\partial x^{\mu'}}{\partial x^\mu} A_{\mu'}$

3. gauge invariance $A_\mu \to A_\mu + \partial_\mu \phi$

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1+2 encoded in $\delta_\xi A_\nu = -\mathcal{L}_\xi A_\nu = -\xi^\mu \partial_\mu A_\nu - A_\mu \partial_\nu \xi^\mu$

3 " " $\delta_\phi A_\nu = \partial_\nu \phi$

Take $\phi = A_\mu \xi^\mu$

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Take $\phi = A_\mu \xi^\mu \Rightarrow \partial_\nu \phi = \partial_\nu A_\mu \xi^\mu + A_\mu \partial_\nu \xi^\mu$

$$\delta_\xi A_\nu = -\xi^\mu \partial_\mu A_\nu - \cancel{A_\mu \partial_\nu \xi^\mu} + \partial_\nu A_\mu \xi^\mu + \cancel{A_\mu \partial_\nu \xi^\mu} = -(\partial_\mu A_\nu - \partial_\nu A_\mu)\xi^\mu$$

$$\left.\begin{array}{l} \Rightarrow \quad \delta_\xi A_\nu = - F_{\mu\nu}\,\xi^\mu \\ \text{But } \delta_\xi A_\nu = - \delta_\mu A_\nu\,\xi^\mu \end{array}\right\} \Rightarrow \quad \delta_\mu A_\nu = F_{\mu\nu}$$

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Take $\phi = A_\mu \xi^\mu \Rightarrow \partial_\nu \phi = \partial_\nu A_\mu\, \xi^\mu + A_\mu \partial_\nu \xi^\mu$

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$$\Rightarrow \quad \delta_\xi A_\nu = - F_{\mu\nu}\,\xi^\mu$$

$$\text{But } \delta_\xi A_\nu = - \delta_\mu A_\nu\,\xi^\mu \quad \Bigg\} \Rightarrow \quad \delta_\mu A_\nu = F_{\mu\nu}$$

Then

$$T^\alpha{}_\beta = - \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)}\,\delta_\beta A_\gamma + \delta^\alpha{}_\beta \mathcal{L}$$

$$= - \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)}\,F_{\beta\gamma} + \delta^\alpha{}_\beta \mathcal{L}$$

$$= + \; F^{\alpha\gamma} F_{\beta\gamma} - \frac{1}{4}\delta^\alpha{}_\beta F^2 \quad = \quad T_{phys}{}^\alpha{}_\beta \quad !$$

Belinfante tensor:

If $T^{\mu\nu}$ is conserved $\partial_\mu T^{\mu\nu} = 0$, then so is

$$T_B^{\mu\nu} = T^{\mu\nu} + \partial_\rho \chi^{\rho\mu\nu} \quad \text{, with} \quad \chi^{\rho\mu\nu} = -\chi^{\mu\rho\nu}$$

$\chi^{\rho\mu\nu}$ = "the superpotential"

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Indeed $\partial_\mu T_B^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu \partial_\rho \chi^{\rho\mu\nu}$

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antisymmetric

Indeed $\quad \partial_\mu T_B^{\mu\nu} = \cancel{\partial_\mu T^{\mu\nu}} + \cancel{\partial_\mu \partial_\rho \chi^{\rho\mu\nu}} = 0$

by assumption

symmetric

Belinfante tensor:

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$\chi^{\rho\mu\nu}$ = "the superpotential"

$$\partial_\mu T_B^{\mu\nu} = 0$$

Choose $\chi^{\rho\mu\nu} = -F^{\rho\mu} A^\nu \Rightarrow \partial_\rho \chi^{\rho\mu\nu} = -\partial_\rho F^{\rho\mu} \cdot A^\nu - F^{\rho\mu} \partial_\rho A^\nu$

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But from eq. of motion (on-shell) $\partial_\rho F^{\rho\mu} = 0 \Rightarrow \partial_\rho \chi^{\rho\mu\nu} = -F^{\rho\mu}\partial_\rho A^\nu$

and $T_B^{\mu\nu} = T^{\mu\nu} - F^{\rho\mu}\partial_\rho A^\nu = T^{\mu\nu}_{\text{physical}}$

Bibliography:

[1] M. Montesinos + E. Flores, Symmetric energy-momentum tensor in Maxwell, Yang-Mills, and Proca theories obtained using only Noether's theorem, Revista Mexicana de Física 52 (2006) 29-36

(use google scholar to download scholar.google.com)

[2] From arXiv.org: 2107.10329 Baker, Linnemann
1807.04671 Ilin, Paston

[3] Wikipedia: - Stress-Energy Tensor
- Belinfante-Rosenfeld stress-energy tensor