

Hartle ch 22

Einstein Equations

Carroll ch 4

Ferrari ch 6+7

- the dynamics

- the

"matter tells spacetime how to curve"

"curvature tells matter how to move"

equations

- difficult : highly non linear (unlike EM)

- can formulate initial value problem (with limitations...)

Newtonian Gravity

$$\frac{dv^i}{dt} = -\nabla^i \Phi \quad \left(\Phi = G \frac{M}{R} \right)$$

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$$\frac{dv^i}{dt} = -\nabla^i \Phi$$

↳ a force

$$\nabla^2 \Phi = 4\pi G \rho$$

↳ Gauss' law

Valid when:

- $v^i \ll 1$
- static field
- weak field

Einstein Gravity

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(Geometry) \leftrightarrow (matter)

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build using Minimal Coupling Principle: (MCP)

- small enough region described by SR physics

- we can't detect gravity with a local experiment

(\Rightarrow no direct coupling to curvature)

- laws of physics can be expressed in coordinate invariant form

(\Rightarrow tensorial equations)

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MCP is strong! why not $\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = \alpha \nabla_\nu R u^\nu u^\mu$?

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$$l_p \sim 1.6 \times 10^{-35} \text{ m} \sim (1.2 \times 10^{19} \text{ GeV})^{-1}$$

so is it because $\alpha \sim l_p^2$?

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$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0 \quad \begin{array}{c} \text{Carroll} \\ \rightarrow \\ \text{\S 4.1} \end{array} \quad \frac{dv^i}{dt} = \frac{1}{2} \partial^i h_{00}$$

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Carroll
§4.1

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Newton

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$$\Phi = -\frac{GM}{r}$$

$$h_{00} \approx -\frac{2GM}{r}$$

$$\Rightarrow g_{00} \approx -\left(1 + \frac{2GM}{r}\right)$$

Since $\nabla^2 \Phi = 4\pi\rho$

$\partial^2 g$ T_∞

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$$R_{\mu\nu} \stackrel{?}{=} k T_{\mu\nu}$$

No! $\nabla^\mu T_{\mu\nu} = 0$ and $\nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R \neq 0$ (Bianchi identity)

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$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

Good! $\nabla^\mu T_{\mu\nu} = 0$ $\nabla^\mu G_{\mu\nu} = 0$

Determine κ in weak+static field, slow moving matter:

$$T_{00} \approx \rho \quad R_{00} \approx -\frac{1}{2} \nabla^2 h_{00}$$

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$$\text{Then } R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \left(\frac{8\pi G}{\frac{n}{2}-1} T \right) = 8\pi G T_{\mu\nu}$$

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- 6 independent equations (Bianchi equation)
- 6 independent degrees of freedom ($g_{\mu\nu}$)
- can formulate initial value problem

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$T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$ Einstein equation in the vacuum (and $\Lambda = 0$)

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Cosmological Constant

If there is vacuum energy, must be Lorentz invariant

$$T_{\hat{\mu}\hat{\nu}}^{(\text{vac})} = -\rho_{\text{vac}} \eta_{\hat{\mu}\hat{\nu}} \quad \rightarrow \quad T_{\mu\nu}^{(\text{vac})} = -\rho_{\text{vac}} g_{\mu\nu} \quad \rho_{\text{vac}} = \text{constant}$$

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$p = w\rho$

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$$G_{\mu\nu} + (\delta n G \rho_{\text{vac}}) g_{\mu\nu} = \delta n G T_{\mu\nu}^M \Rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} = \delta n G T_{\mu\nu}^M$$

Cosmological Constant

$$\bullet |p_{vac}| \leq (10^{-12} \text{ GeV})^4 \ll (10^{18} \text{ GeV})^4$$

" "
E_p

• Λ has been introduced in cosmology by hand as a geometric term to produce static universe (Einstein's "biggest blunder")

In principle a free parameter to be determined by observation

$$G_{\mu\nu} + (\delta n G p_{vac}) g_{\mu\nu} = \delta n G T^{\mu}_{\nu} \Rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} = \delta n G T^{\mu}_{\nu}$$

Lagrangian Formulation

Derive Einstein equations from action principle

- explicit manifestation of symmetries
- can be used to build extensions to GR, or study connection of GR to other theories (e.g. strings)
- one road to (possible) quantum theory

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- one road to (possible) quantum theory

Simple:
$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) + S^M$$

$$\delta S = 0 \Rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Functional Differentiation

$$F: \mathbb{R}^n \rightarrow \mathbb{R} \quad q_i \rightarrow F(q_i) \in \mathbb{R}$$

$$S: F(\mathbb{R}^n) \rightarrow \mathbb{R} \quad \phi(x) \rightarrow S[\phi(x)] \in \mathbb{R}$$

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Diagrammatic annotations:
- A vertical double-headed red arrow connects q_i and $\phi(x)$.
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$$dF = \sum_i \frac{\partial F}{\partial q_i} dq_i = F(q_i + dq_i) - F(q_i) + \mathcal{O}(dq_i^2)$$

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$$F: \mathbb{R}^n \rightarrow \mathbb{R} \quad q_i \rightarrow F(q_i) \in \mathbb{R}$$

$$S: F(\mathbb{R}^n) \rightarrow \mathbb{R} \quad \phi(x) \rightarrow S[\phi(x)] \in \mathbb{R}$$

(Note: In the original image, red arrows indicate the mapping from q_i to $\phi(x)$ and from $\phi(x)$ to $F(q_i)$, with a double-headed arrow between q_i and x .)

$$dF = \sum_i \frac{\partial F}{\partial q_i} dq_i = F(q_i + dq_i) - F(q_i) + \mathcal{O}(dq_i^2)$$

$$\delta S = \sum_x \frac{\delta S}{\delta \phi(x)} \delta \phi(x)$$

Functional Differentiation

$$F: \mathbb{R}^n \rightarrow \mathbb{R} \quad q_i \rightarrow F(q_i) \in \mathbb{R}$$

$$S: F(\mathbb{R}^n) \rightarrow \mathbb{R} \quad \phi(x) \rightarrow S[\phi(x)] \in \mathbb{R}$$

Diagrammatic notes: A double-headed red arrow connects q_i and $\phi(x)$. A red arrow points from $\phi(x)$ to $F(q_i)$. A red arrow points from $F(q_i)$ to $S[\phi(x)]$. A red double-headed arrow connects i and x .

$$dF = \sum_i \frac{\partial F}{\partial q_i} dq_i = F(q_i + dq_i) - F(q_i) + \mathcal{O}(dq_i^2)$$

$$\delta S = \sum_x \frac{\delta S}{\delta \phi(x)} \delta \phi(x)$$

Diagrammatic notes: A red arrow points from $\delta \phi(x)$ to δS .

Functional Differentiation

$$F: \mathbb{R}^n \rightarrow \mathbb{R} \quad q_i \rightarrow F(q_i) \in \mathbb{R}$$

$$S: F(\mathbb{R}^n) \rightarrow \mathbb{R} \quad \phi(x) \rightarrow S[\phi(x)] \in \mathbb{R}$$

(Note: Red arrows in the original image indicate a mapping from q_i to x and from x to $\phi(x)$, with a double-headed arrow between q_i and x .)

$$dF = \sum_i \frac{\partial F}{\partial q_i} dq_i = F(q_i + dq_i) - F(q_i) + \mathcal{O}(dq_i^2)$$

$$\delta S = \int dx \frac{\delta S}{\delta \phi(x)} \delta \phi(x) = S[\phi(x) + \delta \phi(x)] - S[\phi(x)] + \mathcal{O}(\delta \phi^2)$$

Functional Differentiation

$$F: \mathbb{R}^n \rightarrow \mathbb{R} \quad q_i \rightarrow F(q_i) \in \mathbb{R}$$

$$S: F(\mathbb{R}^n) \rightarrow \mathbb{R} \quad \phi(x) \rightarrow S[\phi(x)] \in \mathbb{R}$$

(Note: Red arrows in the original image indicate that q_i corresponds to x and $F(q_i)$ corresponds to $S[\phi(x)]$.)

$$dF = \sum_i \frac{\partial F}{\partial q_i} dq_i = F(q_i + dq_i) - F(q_i) + \mathcal{O}(dq_i^2)$$

(Note: Green text in the original image indicates that $\frac{\partial F}{\partial q_i}$ is the partial derivative wrt q_i at i .)

$$\delta S = \int dx \frac{\delta S}{\delta \phi(x)} \delta \phi(x) = S[\phi(x) + \delta \phi(x)] - S[\phi(x)] + \mathcal{O}(\delta \phi^2)$$

(Note: Green text in the original image indicates that $\frac{\delta S}{\delta \phi(x)}$ is the functional derivative wrt $\phi(x)$ at x .)

Action Principle

$$S[q_i(t)] = \int dt L(q_i, \dot{q}_i)$$

function of q_i, \dot{q}_i

↑ ↑
independent
variables

⇒ it makes sense to calculate

$$\frac{\partial L}{\partial q_i}, \quad \frac{\partial L}{\partial \dot{q}_i}$$

Action Principle

$$S[q_i(t)] = \int dt L(q_i, \dot{q}_i)$$



functional of
 $q_i(t)$

$$q_i(t) \rightarrow S[q_i(t)] \in \mathbb{R}$$

Action Principle

$$S[q_i(t)] = \int dt L(q_i, \dot{q}_i)$$

↓
functional of $q_i(t)$

↳ number $S[q_i(t)]$ calculated by substituting $q_i(t)$. Then $\dot{q}_i = \frac{dq_i(t)}{dt}$

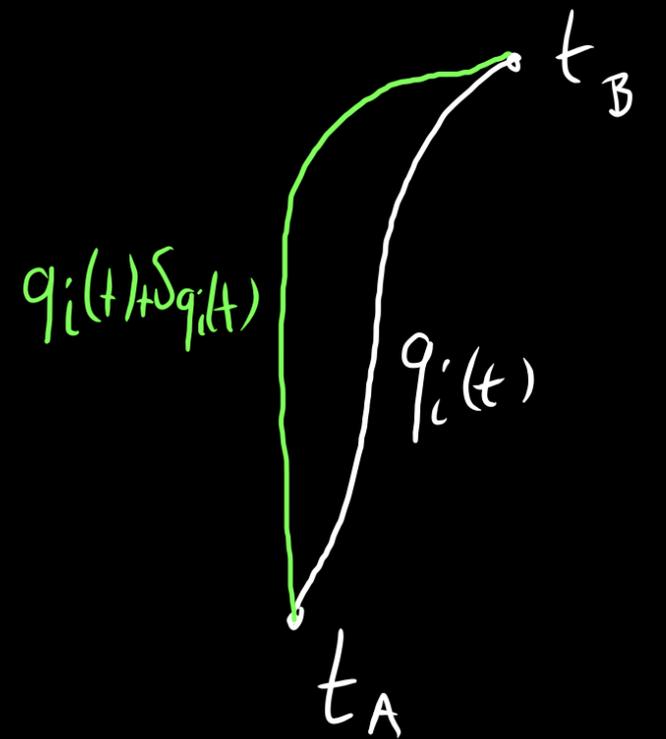
$$q_i(t) \rightarrow S[q_i(t)] \in \mathbb{R}$$

Action Principle

$$S[q_i(t)] = \int dt L(q_i, \dot{q}_i)$$

Consider variations $q_i(t) \rightarrow q_i(t) + \delta q_i(t)$

$$\text{s.t. } \delta q_i(t_A) = \delta q_i(t_B) = 0$$



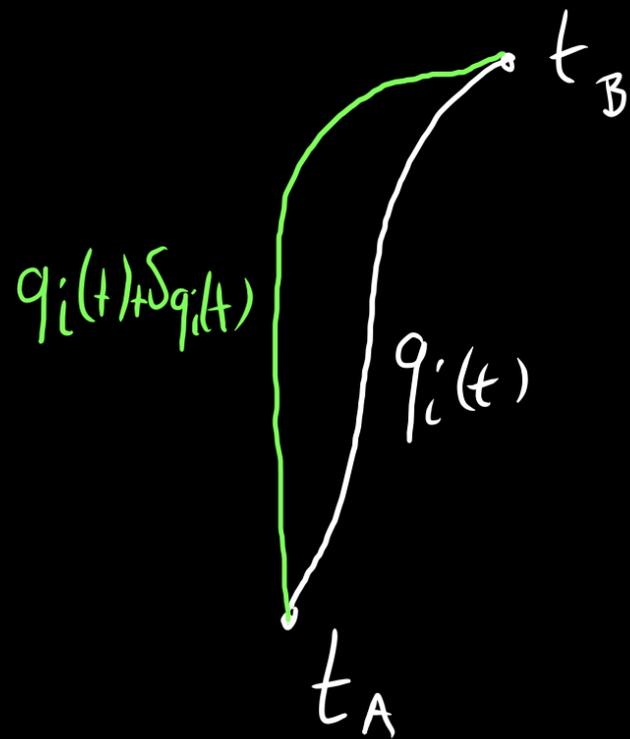
Action Principle

$$S[q_i(t)] = \int dt L(q_i, \dot{q}_i)$$

Consider variations $q_i(t) \rightarrow q_i(t) + \delta q_i(t)$

s.t. $\delta q_i(t_A) = \delta q_i(t_B) = 0$, then

$$\delta S = 0 \Rightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \text{e.o.m}$$



Action Principle

Field Theory:

$$S[\phi(x)] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

Action Principle

Field Theory:

$$S[\phi(x)] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

$$S[\phi(x) + \delta\phi(x)] = \int d^4x \mathcal{L}(\phi + \delta\phi, \partial_\mu \phi + \delta\partial_\mu \phi)$$

Action Principle

Field Theory:

$$S[\phi(x)] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

$$S[\phi(x) + \delta\phi(x)] = \int d^4x \mathcal{L}(\phi + \delta\phi, \partial_\mu \phi + \delta\partial_\mu \phi)$$

$$= \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) + \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta(\partial_\mu \phi(x)) \right] + \mathcal{O}(\delta^2)$$

Action Principle

Field Theory:

$$S[\phi(x)] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

$$S[\phi(x) + \delta\phi(x)] = \int d^4x \mathcal{L}(\phi + \delta\phi, \partial_\mu \phi + \delta\partial_\mu \phi)$$

$$= \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) + \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta(\partial_\mu \phi(x)) \right] + \mathcal{O}(\delta^2)$$

$$= S[\phi(x)] + \int d^4x \frac{\delta S}{\delta \phi(x)} \delta\phi(x)$$

$$\underbrace{\int d^4x \frac{\delta S}{\delta \phi(x)} \delta\phi(x)}_{= \delta S}$$

Action Principle

$$\delta S = \int d^4x \frac{\delta S}{\delta \phi(x)} \delta \phi(x)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta (\partial_\mu \phi(x)) \right]$$

get rid of: trade for $\delta \phi(x)$
to calculate $\frac{\delta S}{\delta \phi}$

$$\underbrace{\int d^4x \frac{\delta S}{\delta \phi(x)} \delta \phi(x)}_{= \delta S}$$

Action Principle

$$\delta S = \int d^4x \frac{\delta S}{\delta \phi(x)} \delta \phi(x)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta (\partial_\mu \phi(x)) \right]$$

$$\phi(x) \rightarrow \phi(x) + \delta \phi(x) \Rightarrow \partial_\mu \phi(x) \rightarrow \partial_\mu \phi(x) + \partial_\mu (\delta \phi(x))$$

Action Principle

$$\delta S = \int d^4x \frac{\delta S}{\delta \phi(x)} \delta \phi(x)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta (\partial_\mu \phi(x)) \right]$$

$$\phi(x) \rightarrow \phi(x) + \delta \phi(x) \Rightarrow \partial_\mu \phi(x) \rightarrow \partial_\mu \phi(x) + \underbrace{\partial_\mu (\delta \phi(x))}_{\delta (\partial_\mu \phi(x))}$$

$$\Rightarrow \delta (\partial_\mu \phi(x)) = \partial_\mu (\delta \phi(x))$$

Action Principle

$$\delta S = \int d^4x \frac{\delta S}{\delta \phi(x)} \delta \phi(x)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta (\partial_\mu \phi(x)) \right]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \partial_\mu (\delta \phi(x)) \right]$$

$$\Rightarrow \delta (\partial_\mu \phi(x)) = \partial_\mu (\delta \phi(x))$$

Action Principle

$$\delta S = \int_V d^4x \frac{\delta S}{\delta \phi(x)} \delta \phi(x)$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta (\partial_\mu \phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \partial_\mu (\delta \phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \right] \delta \phi(x) \right]$$

$$+ \int_V d^4x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta \phi(x) \right]$$

Action Principle

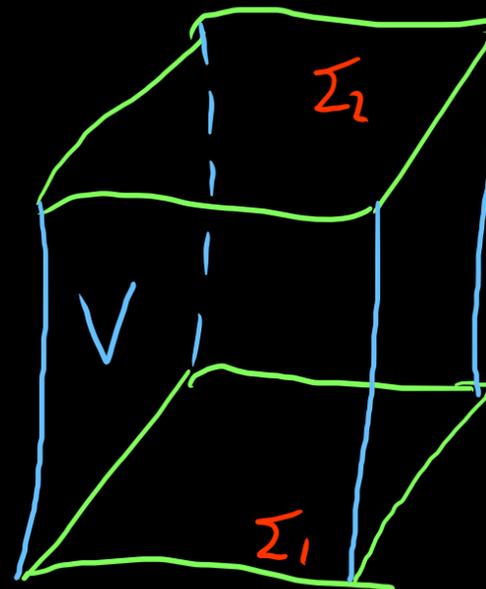
$$\delta S = \int_V d^4x \frac{\delta S}{\delta \phi(x)} \delta \phi(x)$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta (\partial_\mu \phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \partial_\mu (\delta \phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \right] \delta \phi(x) \right]$$

$$+ \int_{\partial V} d^3y \eta^\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta \phi(x) \right]$$



Action Principle

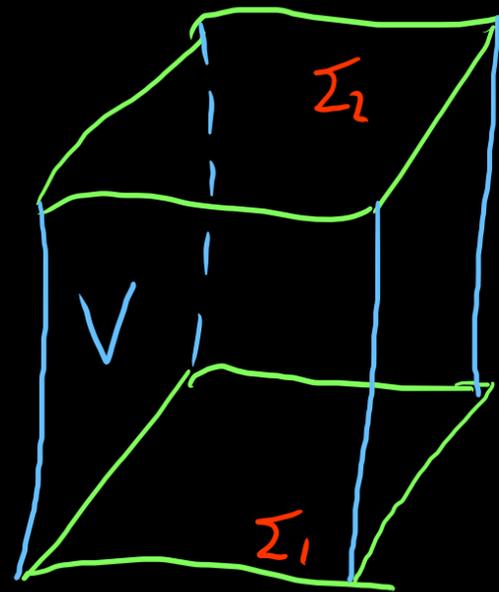
$$\delta S = \int_V d^4x \frac{\delta S}{\delta \phi(x)} \delta \phi(x)$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta (\partial_\mu \phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \partial_\mu (\delta \phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \right] \delta \phi(x) \right]$$

$$+ \int_{\partial V} d^3y \eta^\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta \phi(x) \right]$$



$$\delta \phi|_{\Sigma_1} = \delta \phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi|_{\partial V} = 0$$

Choose $\delta \phi$ to make this zero

Action Principle

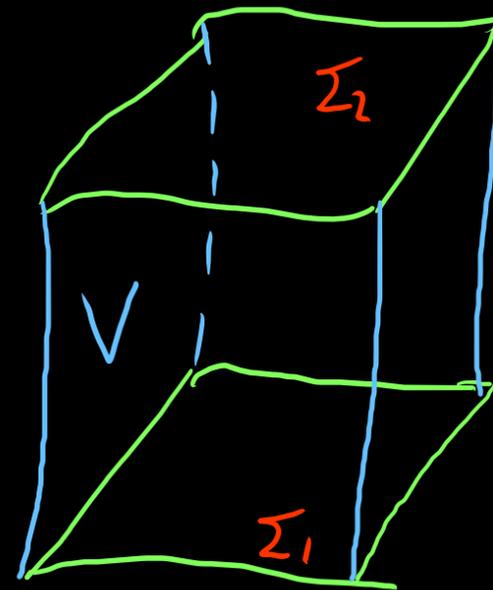
$$\delta S = \int_V d^4x \frac{\delta S}{\delta \phi(x)} \delta \phi(x)$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta (\partial_\mu \phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \partial_\mu (\delta \phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \right] \delta \phi(x) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \right] \right] \delta \phi(x)$$

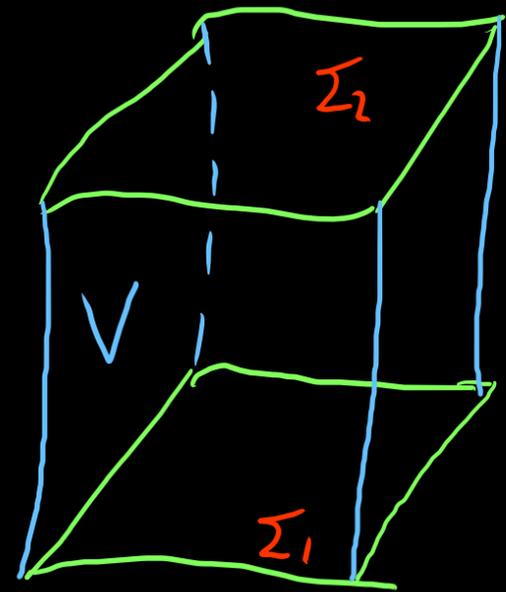


$$\delta \phi|_{\Sigma_1} = \delta \phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi|_{\partial V} = 0$$

Action Principle

$$\delta S = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \right] = 0$$



$$\delta \phi|_{\Sigma_1} = \delta \phi|_{\Sigma_2} = 0$$

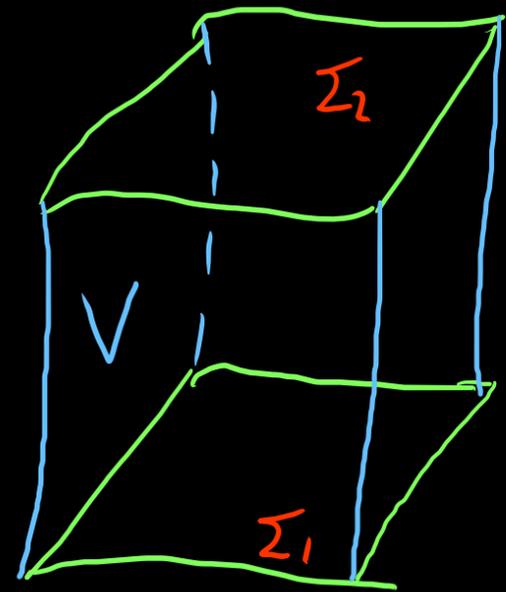
$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi|_{\partial V} = 0$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \right] \right] \delta \phi(x)$$

Action Principle

$$\delta S = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \right] = 0$$

classical eom!



$$\delta \phi|_{\Sigma_1} = \delta \phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi|_{\partial V} = 0$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \right] \right] \delta \phi(x)$$

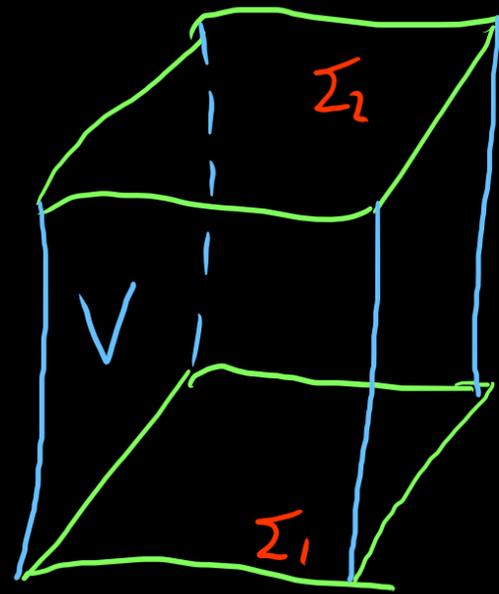
Action Principle

In curved space:

$$\mathcal{L}(\phi, \partial_\mu \phi) \rightarrow \mathcal{L}(\phi, \nabla_\mu \phi)$$

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \rightarrow S = \int d^4x \sqrt{-g} \mathcal{L}(\phi, \nabla_\mu \phi)$$

minimal coupling principle!



$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \delta\phi|_{\partial V} = 0$$

Action Principle

In curved space:

$$\mathcal{L}(\phi, \partial_\mu \phi) \rightarrow \mathcal{L}(\phi, \nabla_\mu \phi)$$

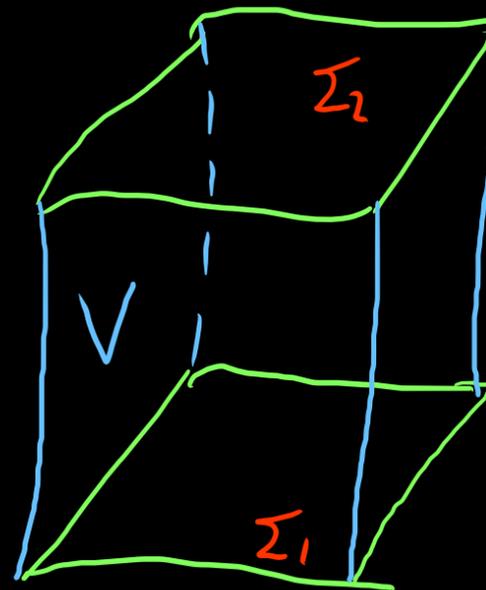
$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \rightarrow S = \int d^4x \sqrt{-g} \mathcal{L}(\phi, \nabla_\mu \phi)$$

Partial integration: use Stokes' theorem!

$$\int_V d^4x \sqrt{-g} \nabla_\mu V^\mu = \int_{\partial V} d^3y \sqrt{\gamma} n_\mu V^\mu$$

induced metric on ∂V by $g_{\mu\nu}$

normal to the 3-surface ∂V



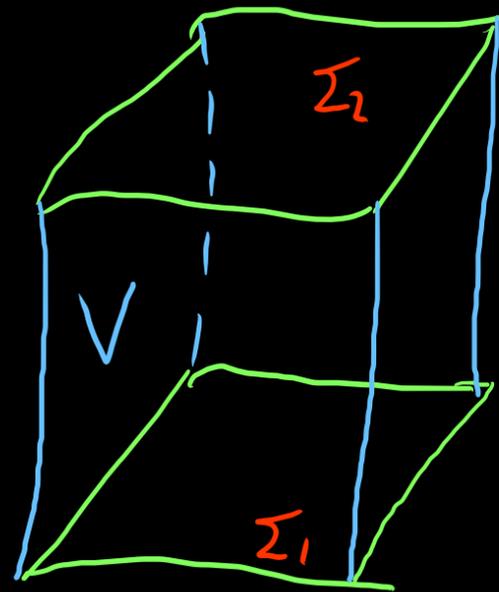
$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \delta\phi|_{\partial V} = 0$$

Action Principle

Then:

$$\int_V d^4x \sqrt{-g} \nabla_\mu (A^\mu B) = \int_V d^4x \sqrt{-g} \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g} A^\mu \nabla_\mu B$$



$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \delta\phi|_{\partial V} = 0$$

... Leibniz rule ...

Partial integration: use Stokes' theorem!

$$\int_V d^4x \sqrt{-g} \nabla_\mu V^\mu = \int_{\partial V} d^3y \sqrt{\gamma} n_\mu V^\mu$$

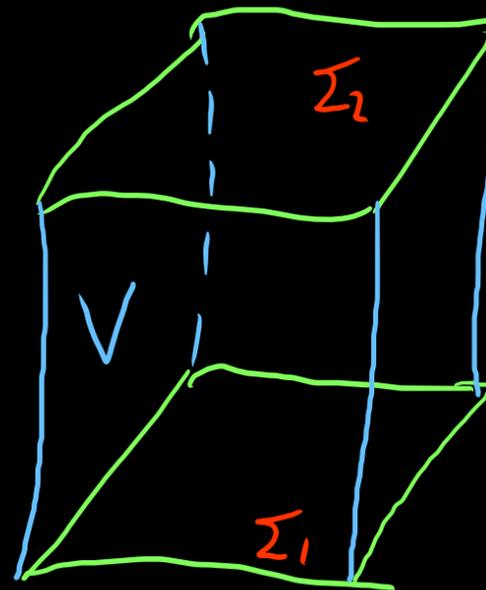
induced metric on ∂V by $g_{\mu\nu}$

normal to the 3-surface ∂V

Action Principle

Then:

$$\int_V d^4x \sqrt{-g} \nabla_\mu (A^\mu B) = \int_V d^4x \sqrt{-g} \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g} A^\mu \nabla_\mu B$$



↓ Stokes

$$\int_{\partial V} d^3y \sqrt{\gamma} n_\mu A^\mu B = \int_V d^4x \sqrt{-g} \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g} A^\mu \nabla_\mu B$$

$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

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Partial integration: use Stokes' theorem!

$$\int_V d^4x \sqrt{-g} \nabla_\mu V^\mu = \int_{\partial V} d^3y \sqrt{\gamma} n_\mu V^\mu$$

induced metric on ∂V by $g_{\mu\nu}$

normal to the 3-surface ∂V

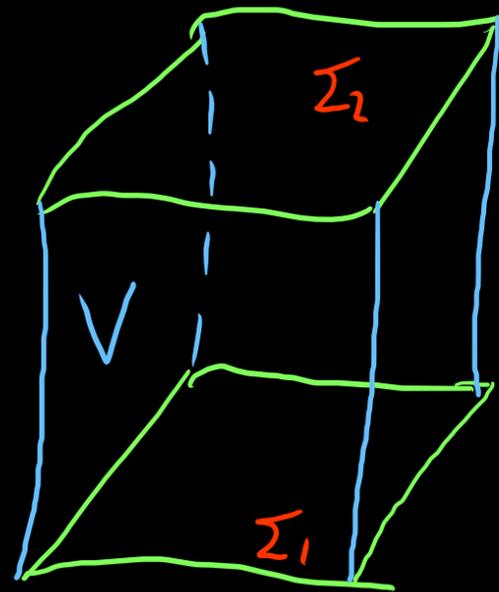
Action Principle

Then:

$$\int_V d^4x \sqrt{-g} \nabla_\mu (A^\mu B) = \int_V d^4x \sqrt{-g} \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g} A^\mu \nabla_\mu B$$

$$\int_{\partial V} d^3y \sqrt{\gamma} n_\mu A^\mu B = \int_V d^4x \sqrt{-g} \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g} A^\mu \nabla_\mu B$$

$$\Rightarrow \int_V d^4x \sqrt{-g} A^\mu \nabla_\mu B = - \int_V d^4x \sqrt{-g} \nabla_\mu A^\mu B + \int_{\partial V} d^3y \sqrt{\gamma} n_\mu A^\mu B$$



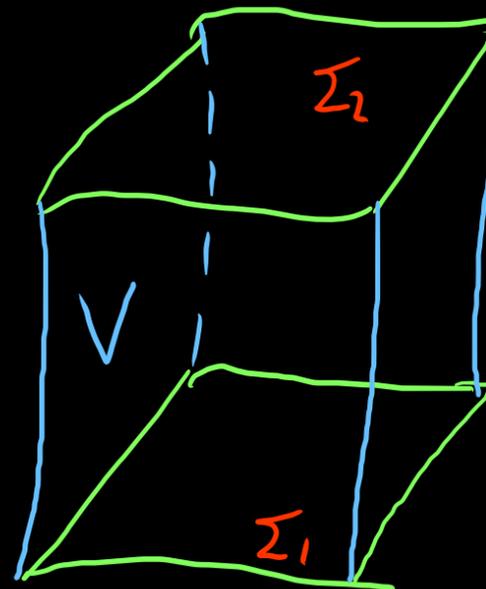
$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi|_{\partial V} = 0$$

Action Principle

Then:

$$\int_V d^4x \sqrt{-g} \nabla_\mu (A^\mu B) = \int_V d^4x \sqrt{-g} \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g} A^\mu \nabla_\mu B$$



$$\int_{\partial V} d^3y \sqrt{\gamma} n_\mu A^\mu B = \int_V d^4x \sqrt{-g} \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g} A^\mu \nabla_\mu B$$

$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \delta\phi|_{\partial V} = 0$$

$$\Rightarrow \int_V d^4x \sqrt{-g} A^\mu \nabla_\mu B = - \int_V d^4x \sqrt{-g} \nabla_\mu A^\mu B + \int_{\partial V} d^3y \sqrt{\gamma} n_\mu A^\mu B$$

"passed over"
ignoring $\sqrt{-g}$
(magic of ∇_μ !) \rightsquigarrow Stokes...

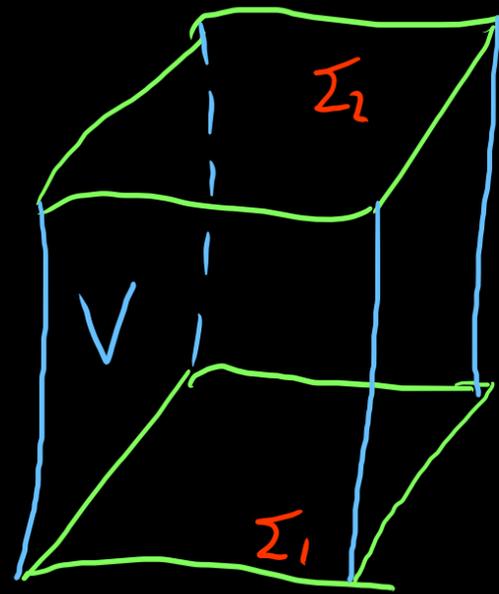
Boundary term:
set to 0 or cancel otherwise
(careful...)

Action Principle

Example: scalar field

$$S = \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right)$$

minimal coupling, no e.g. $R\phi^2$ term



$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

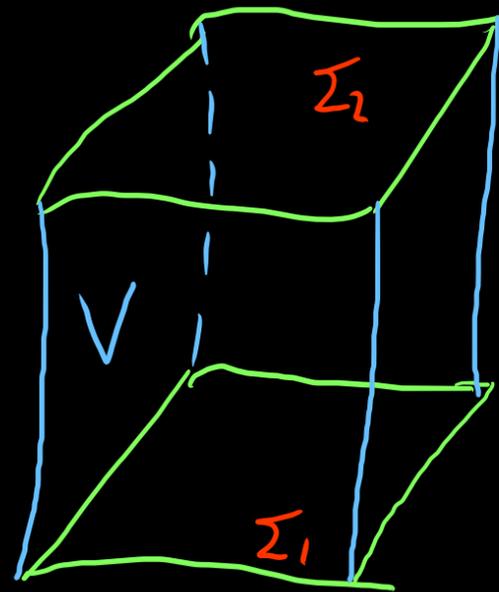
$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \delta\phi|_{\partial V} = 0$$

Action Principle

Example: scalar field

$$S = \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right)$$

$$S[\phi + \delta\phi] = \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu (\phi + \delta\phi) \nabla^\mu (\phi + \delta\phi) - V(\phi + \delta\phi) \right)$$



$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \delta\phi|_{\mathcal{V}} = 0$$

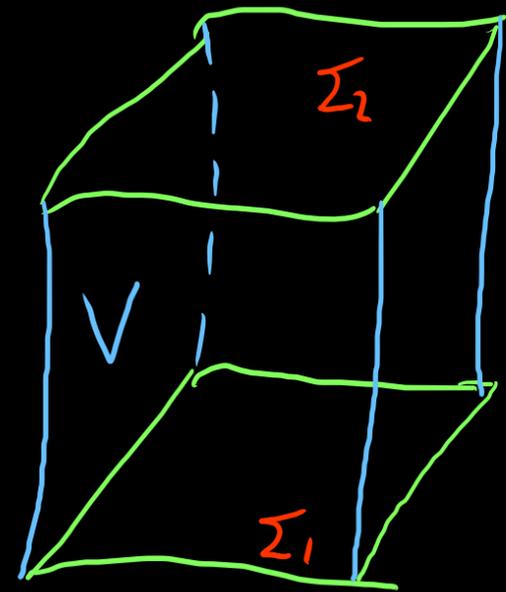
Action Principle

Example: scalar field

$$S = \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right)$$

$$S[\phi + \delta\phi] = \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu (\phi + \delta\phi) \nabla^\mu (\phi + \delta\phi) - V(\phi + \delta\phi) \right)$$

$$= \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right) + \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \delta\phi \nabla^\mu \phi - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \delta\phi - \frac{dV}{d\phi} \delta\phi \right) + \dots$$



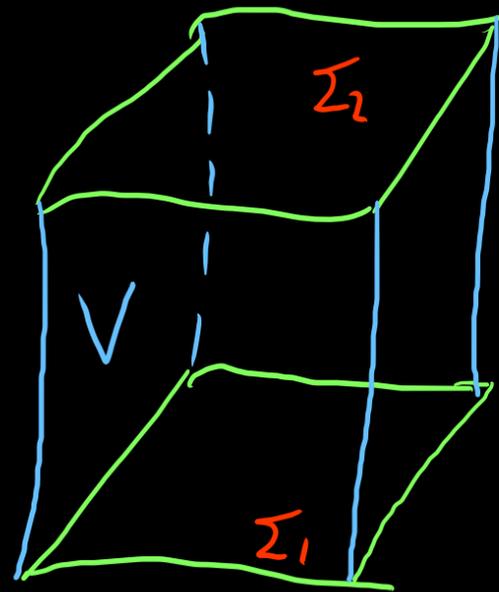
$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

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Action Principle

Example: scalar field

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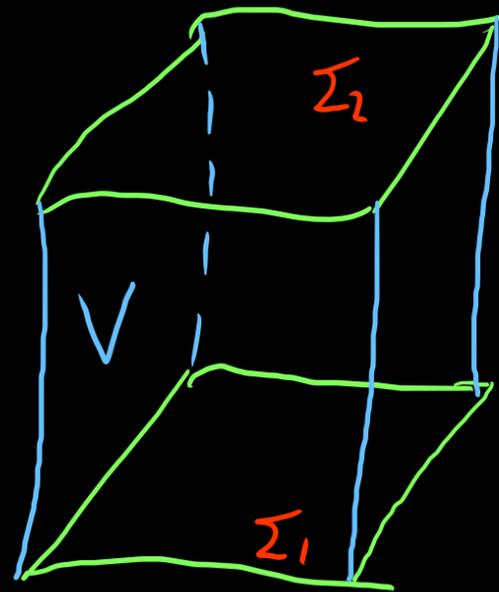
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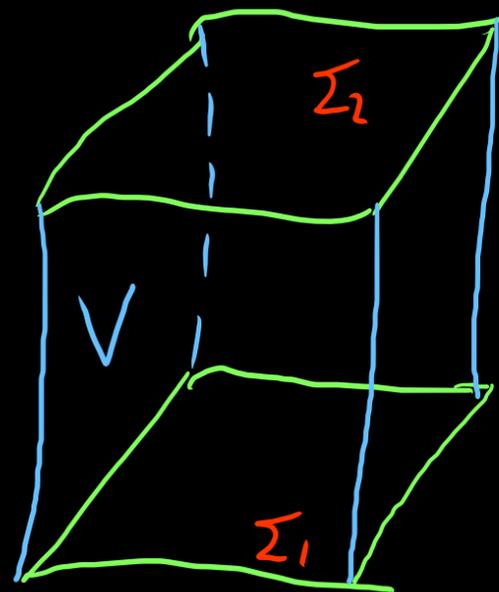
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Take:

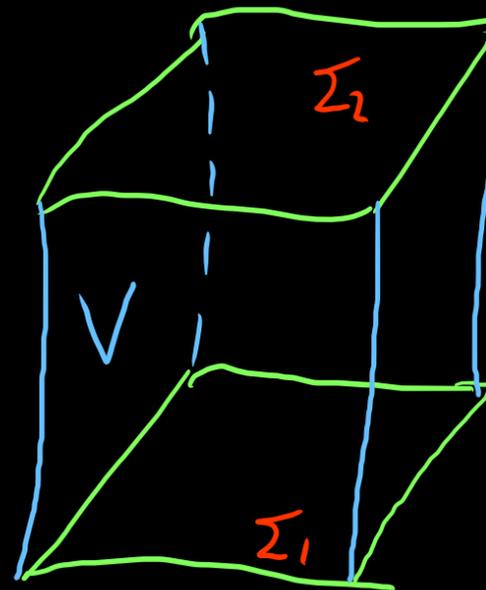
$$\int_{\partial V} \sqrt{\gamma} n^\mu \nabla_\mu \phi \delta\phi = 0$$

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ϕ has compact support
($\phi=0$ at ∞)

or

$\phi, \delta\phi \rightarrow 0$ at ∞
fast enough



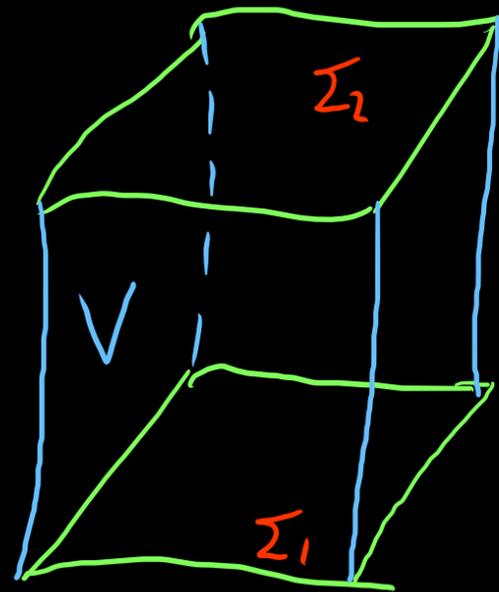
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$$\delta S = 0 \quad \forall \delta \phi \Rightarrow \nabla^2 \phi - \frac{dV}{d\phi} = 0$$

$$\text{or} \quad \square \phi - \frac{dV}{d\phi} = 0$$



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Gravity:

- Ingredients:
1. $g_{\mu\nu}$ (degrees of freedom - not all independent)
 2. scalar Lagrangian (diffeomorphism invariance)
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- e. enjoy!
 $\delta S = 0$

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$$\delta(S^{EH} + S^\Lambda + S^M) = 0 \Rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

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not inverses to
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almost like
lowering
indices

Variation of $\sqrt{-g}$

Temporary notation

g : the matrix $g_{\mu\nu}$

$\det g$: its determinant

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$$\text{tr } g = \sum_r g_r$$

\hookrightarrow • diagonalize $g \rightarrow \Lambda g_D \Lambda^{-1}$

$$\bullet \text{tr } g = \text{tr } \Lambda g_D \Lambda^{-1} = \text{tr } \Lambda^{-1} \Lambda g_D = \text{tr } g_D$$

$$= \sum g_r$$

$$g_D = \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & g_3 & \\ & & & g_4 \end{pmatrix}$$

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as promised!

Variation of S^{EH}

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$$16\pi G \delta S^{EH} = \int \delta \sqrt{-g} g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}$$

Variation of S^{EH}

$$16\pi G S^{EH} = \int \sqrt{-g} R = \int \sqrt{-g} g^{\mu\nu} R_{\mu\nu} \Rightarrow$$

$$\begin{aligned} 16\pi G \delta S^{EH} &= \int \delta \sqrt{-g} g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \\ &= \int \sqrt{-g} \left(-\frac{1}{2} g_{\rho\sigma} \delta g^{\rho\sigma}\right) R + \int \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \end{aligned}$$

Variation of S^{EH}

$$16\pi G S^{EH} = \int \sqrt{-g} R = \int \sqrt{-g} g^{\mu\nu} R_{\mu\nu} \Rightarrow$$

$$\begin{aligned} 16\pi G \delta S^{EH} &= \int \delta \sqrt{-g} g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \\ &= \int \sqrt{-g} \left(-\frac{1}{2} g_{\rho\sigma} \delta g^{\rho\sigma} \right) R + \int \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \\ &= \int \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} + \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \end{aligned}$$

Variation of S^{EH}

$$16\pi G S^{EH} = \int \sqrt{-g} R = \int \sqrt{-g} g^{\mu\nu} R_{\mu\nu} \Rightarrow$$

$$\begin{aligned} 16\pi G \delta S^{EH} &= \int \delta \sqrt{-g} g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \\ &= \int \sqrt{-g} \left(-\frac{1}{2} g_{\rho\sigma} \delta g^{\rho\sigma}\right) R + \int \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \\ &= \int \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} + \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \\ &= \int \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} + \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \end{aligned}$$

\hookrightarrow almost there... $\underbrace{\hspace{10em}}$ \hookrightarrow get rid of this!

Compute $\delta R_{\mu\nu}$:

$$R^{\rho}_{\mu\sigma\nu} = \partial_{\sigma}\Gamma^{\rho}_{\nu\mu} - \partial_{\nu}\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}$$

Compute $\delta R_{\mu\nu}$:

$$R^{\rho}_{\mu\sigma\nu} = \partial_{\sigma}\Gamma^{\rho}_{\nu\mu} - \partial_{\nu}\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}$$

$$R^{\rho}_{\mu\rho\nu} = \partial_{\rho}\Gamma^{\rho}_{\nu\mu} - \partial_{\nu}\Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu}$$

Compute $\delta R_{\mu\nu}$:

$$R^{\rho}_{\mu\sigma\nu} = \partial_{\sigma}\Gamma^{\rho}_{\nu\mu} - \partial_{\nu}\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}$$

$$R_{\mu\nu} = \partial_{\rho}\Gamma^{\rho}_{\nu\mu} - \partial_{\nu}\Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu}$$

For $\delta R_{\mu\nu}$ we need $\delta\Gamma^{\rho}_{\nu\mu}$

Compute $\delta \Gamma^{\mu}_{\nu\rho}$

Consider $\Gamma^{\mu}_{\nu\rho} = g_{\mu\lambda} \Gamma^{\lambda}_{\nu\rho} = \frac{1}{2} \left(\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\nu\mu} - \partial_{\mu} g_{\nu\rho} \right)$

Compute $\delta \Gamma^{\mu}_{\nu\rho}$

Consider $\Gamma^{\mu}_{\nu\rho} = g_{\mu\lambda} \Gamma^{\lambda}_{\nu\rho} = \frac{1}{2} (\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\nu\mu} - \partial_{\mu} g_{\nu\rho})$

$$\Rightarrow \delta \Gamma^{\mu}_{\nu\rho} = \frac{1}{2} (\partial_{\nu} \delta g_{\rho\mu} + \partial_{\rho} \delta g_{\nu\mu} - \partial_{\mu} \delta g_{\nu\rho})$$

Compute $\delta \Gamma^{\mu}_{\nu\rho}$

Consider $\Gamma_{\mu\nu\rho} = g_{\mu\lambda} \Gamma^{\lambda}_{\nu\rho} = \frac{1}{2} (\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\nu\mu} - \partial_{\mu} g_{\nu\rho})$

$$\Rightarrow \delta \Gamma_{\mu\nu\rho} = \frac{1}{2} (\partial_{\nu} \delta g_{\rho\mu} + \partial_{\rho} \delta g_{\nu\mu} - \partial_{\mu} \delta g_{\nu\rho})$$

$$\Gamma^{\lambda}_{\nu\rho} = g^{\lambda\mu} \Gamma_{\mu\nu\rho} \Rightarrow \delta \Gamma^{\lambda}_{\nu\rho} = \delta g^{\lambda\mu} \Gamma_{\mu\nu\rho} + g^{\lambda\mu} \delta \Gamma_{\mu\nu\rho}$$

Compute $\delta \Gamma^{\mu}_{\nu\rho}$

Consider $\Gamma^{\lambda}_{\mu\nu\rho} = g_{\mu\lambda} \Gamma^{\lambda}_{\nu\rho} = \frac{1}{2} (\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\nu\mu} - \partial_{\mu} g_{\nu\rho})$

$$\Rightarrow \delta \Gamma^{\lambda}_{\mu\nu\rho} = \frac{1}{2} (\partial_{\nu} \delta g_{\rho\mu} + \partial_{\rho} \delta g_{\nu\mu} - \partial_{\mu} \delta g_{\nu\rho})$$

$$\Gamma^{\lambda}_{\nu\rho} = g^{\lambda\mu} \Gamma_{\mu\nu\rho} \Rightarrow \delta \Gamma^{\lambda}_{\nu\rho} = \delta g^{\lambda\mu} \Gamma_{\mu\nu\rho} + g^{\lambda\mu} \delta \Gamma_{\mu\nu\rho}$$

$$= -g^{\lambda\alpha} g^{\mu\beta} \delta g_{\alpha\beta} \Gamma_{\mu\nu\rho} + g^{\lambda\mu} \delta \Gamma_{\mu\nu\rho}$$

Compute $\delta \Gamma^{\mu}_{\nu\rho}$

Consider $\Gamma_{\mu\nu\rho} = g_{\mu\lambda} \Gamma^{\lambda}_{\nu\rho} = \frac{1}{2} (\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\nu\mu} - \partial_{\mu} g_{\nu\rho})$

$\Rightarrow \delta \Gamma_{\mu\nu\rho} = \frac{1}{2} (\partial_{\nu} \delta g_{\rho\mu} + \partial_{\rho} \delta g_{\nu\mu} - \partial_{\mu} \delta g_{\nu\rho})$

$\Gamma^{\lambda}_{\nu\rho} = g^{\lambda\mu} \Gamma_{\mu\nu\rho} \Rightarrow \delta \Gamma^{\lambda}_{\nu\rho} = \delta g^{\lambda\mu} \Gamma_{\mu\nu\rho} + g^{\lambda\mu} \delta \Gamma_{\mu\nu\rho}$

$= -g^{\lambda\alpha} g^{\mu\beta} \delta g_{\alpha\beta} \Gamma_{\mu\nu\rho} + g^{\lambda\mu} \delta \Gamma_{\mu\nu\rho}$

$= -g^{\lambda\alpha} \delta g_{\alpha\beta} \Gamma^{\beta}_{\nu\rho} + g^{\lambda\mu} \frac{1}{2} (\partial_{\nu} \delta g_{\rho\mu} + \partial_{\rho} \delta g_{\nu\mu} - \partial_{\mu} \delta g_{\nu\rho})$

Compute $\delta \Gamma^{\mu}_{\nu\rho}$

Consider $\Gamma_{\mu\nu\rho} = g_{\mu\lambda} \Gamma^{\lambda}_{\nu\rho} = \frac{1}{2} (\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\nu\mu} - \partial_{\mu} g_{\nu\rho})$

$$\Rightarrow \delta \Gamma_{\mu\nu\rho} = \frac{1}{2} (\partial_{\nu} \delta g_{\rho\mu} + \partial_{\rho} \delta g_{\nu\mu} - \partial_{\mu} \delta g_{\nu\rho})$$

$$\Gamma^{\lambda}_{\nu\rho} = g^{\lambda\mu} \Gamma_{\mu\nu\rho} \Rightarrow \delta \Gamma^{\lambda}_{\nu\rho} = \delta g^{\lambda\mu} \Gamma_{\mu\nu\rho} + g^{\lambda\mu} \delta \Gamma_{\mu\nu\rho}$$

$$= -g^{\lambda\alpha} g^{\mu\beta} \delta g_{\alpha\beta} \Gamma_{\mu\nu\rho} + g^{\lambda\mu} \delta \Gamma_{\mu\nu\rho}$$

$$= -g^{\lambda\alpha} \delta g_{\alpha\beta} \Gamma^{\beta}_{\nu\rho} + g^{\lambda\mu} \frac{1}{2} (\partial_{\nu} \delta g_{\rho\mu} + \partial_{\rho} \delta g_{\nu\mu} - \partial_{\mu} \delta g_{\nu\rho})$$

$$= \frac{1}{2} g^{\lambda\mu} [\partial_{\nu} \delta g_{\rho\mu} + \partial_{\rho} \delta g_{\nu\mu} - \partial_{\mu} \delta g_{\nu\rho} - 2 \Gamma^{\sigma}_{\nu\rho} \delta g_{\mu\sigma}]$$

$$\begin{aligned}
&= \frac{1}{2} g^{\lambda\mu} \left[\left(\partial_\nu \delta g_{\rho\mu} - \underbrace{\Gamma^\sigma_{\nu\rho} \delta g_{\sigma\mu}}_{\text{green}} - \underbrace{\Gamma^\sigma_{\nu\mu} \delta g_{\rho\sigma}}_{\text{red}} \right) \right. \\
&\quad + \left(\partial_\rho \delta g_{\nu\mu} - \underbrace{\Gamma^\sigma_{\rho\nu} \delta g_{\sigma\mu}}_{\text{green}} - \underbrace{\Gamma^\sigma_{\rho\mu} \delta g_{\nu\sigma}}_{\text{yellow}} \right) \\
&\quad \left. - \left(\partial_\mu \delta g_{\nu\rho} - \underbrace{\Gamma^\sigma_{\mu\nu} \delta g_{\sigma\rho}}_{\text{red}} - \underbrace{\Gamma^\sigma_{\mu\rho} \delta g_{\nu\sigma}}_{\text{yellow}} \right) \right]
\end{aligned}$$

$$\delta \Gamma^\lambda_{\nu\rho} = \frac{1}{2} g^{\lambda\mu} \left[\partial_\nu \delta g_{\rho\mu} + \partial_\rho \delta g_{\nu\mu} - \partial_\mu \delta g_{\nu\rho} - 2 \Gamma^\sigma_{\nu\rho} \delta g_{\mu\sigma} \right]$$

$$\begin{aligned}
&= \frac{1}{2} g^{\lambda\mu} \left[\left(\partial_\nu \delta g_{\rho\mu} - \underbrace{\Gamma^\sigma_{\nu\rho} \delta g_{\sigma\mu}}_{\text{green}} - \underbrace{\Gamma^\sigma_{\nu\mu} \delta g_{\rho\sigma}}_{\text{red}} \right) \right. \\
&\quad + \left(\partial_\rho \delta g_{\nu\mu} - \underbrace{\Gamma^\sigma_{\rho\nu} \delta g_{\sigma\mu}}_{\text{green}} - \underbrace{\Gamma^\sigma_{\rho\mu} \delta g_{\nu\sigma}}_{\text{yellow}} \right) \\
&\quad \left. - \left(\partial_\mu \delta g_{\nu\rho} - \underbrace{\Gamma^\sigma_{\mu\nu} \delta g_{\sigma\rho}}_{\text{red}} - \underbrace{\Gamma^\sigma_{\mu\rho} \delta g_{\nu\sigma}}_{\text{yellow}} \right) \right]
\end{aligned}$$

$$= \frac{1}{2} g^{\lambda\mu} \left[\nabla_\nu \delta g_{\rho\mu} + \nabla_\rho \delta g_{\nu\mu} - \nabla_\mu \delta g_{\nu\rho} \right]$$

$$\delta \Gamma^\lambda_{\nu\rho} = \frac{1}{2} g^{\lambda\mu} \left[\partial_\nu \delta g_{\rho\mu} + \partial_\rho \delta g_{\nu\mu} - \partial_\mu \delta g_{\nu\rho} - 2 \Gamma^\sigma_{\nu\rho} \delta g_{\mu\sigma} \right]$$

$$\begin{aligned}
&= \frac{1}{2} g^{\lambda\mu} \left[\left(\partial_\nu \delta g_{\rho\mu} - \underbrace{\Gamma^\sigma_{\nu\rho} \delta g_{\sigma\mu}}_{\text{green}} - \underbrace{\Gamma^\sigma_{\nu\mu} \delta g_{\rho\sigma}}_{\text{red}} \right) \right. \\
&\quad + \left(\partial_\rho \delta g_{\nu\mu} - \underbrace{\Gamma^\sigma_{\rho\nu} \delta g_{\sigma\mu}}_{\text{green}} - \underbrace{\Gamma^\sigma_{\rho\mu} \delta g_{\nu\sigma}}_{\text{yellow}} \right) \\
&\quad \left. - \left(\partial_\mu \delta g_{\nu\rho} - \underbrace{\Gamma^\sigma_{\mu\nu} \delta g_{\sigma\rho}}_{\text{red}} - \underbrace{\Gamma^\sigma_{\mu\rho} \delta g_{\nu\sigma}}_{\text{yellow}} \right) \right]
\end{aligned}$$

$$= \frac{1}{2} g^{\lambda\mu} \left[\nabla_\nu \delta g_{\rho\mu} + \nabla_\rho \delta g_{\nu\mu} - \nabla_\mu \delta g_{\nu\rho} \right]$$

$\Rightarrow \delta \Gamma^\lambda_{\nu\rho}$ is a tensor!

$$\begin{aligned}
&= \frac{1}{2} g^{\lambda\mu} \left[\left(\partial_\nu \delta g_{\rho\mu} - \underbrace{\Gamma_{\nu\rho}^\sigma \delta g_{\sigma\mu}}_{\text{green}} - \underbrace{\Gamma_{\nu\mu}^\sigma \delta g_{\rho\sigma}}_{\text{red}} \right) \right. \\
&\quad + \left(\partial_\rho \delta g_{\nu\mu} - \underbrace{\Gamma_{\rho\nu}^\sigma \delta g_{\sigma\mu}}_{\text{green}} - \underbrace{\Gamma_{\rho\mu}^\sigma \delta g_{\nu\sigma}}_{\text{yellow}} \right) \\
&\quad \left. - \left(\partial_\mu \delta g_{\nu\rho} - \underbrace{\Gamma_{\mu\nu}^\sigma \delta g_{\sigma\rho}}_{\text{red}} - \underbrace{\Gamma_{\mu\rho}^\sigma \delta g_{\nu\sigma}}_{\text{yellow}} \right) \right]
\end{aligned}$$

$$= \frac{1}{2} g^{\lambda\mu} \left[\nabla_\nu \delta g_{\rho\mu} + \nabla_\rho \delta g_{\nu\mu} - \nabla_\mu \delta g_{\nu\rho} \right]$$

$\Rightarrow \delta \Gamma^\lambda_{\nu\rho}$ is a tensor!

$$\Rightarrow \nabla_\mu \delta \Gamma^\lambda_{\nu\rho} = \partial_\mu \delta \Gamma^\lambda_{\nu\rho} + \Gamma_{\mu\sigma}^\lambda \delta \Gamma^\sigma_{\nu\rho} - \Gamma_{\mu\nu}^\sigma \delta \Gamma^\lambda_{\sigma\rho} - \Gamma_{\mu\rho}^\sigma \delta \Gamma^\lambda_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma_{\nu\mu}^\rho - \partial_\nu \Gamma_{\rho\mu}^\rho + \Gamma_{\rho\lambda}^\rho \Gamma_{\nu\mu}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\rho\mu}^\lambda$$

$$\Rightarrow \nabla_\mu \delta \Gamma_{\nu\rho}^\lambda = \partial_\mu \delta \Gamma_{\nu\rho}^\lambda + \Gamma_{\mu\sigma}^\lambda \delta \Gamma_{\nu\rho}^\sigma - \Gamma_{\mu\nu}^\sigma \delta \Gamma_{\sigma\rho}^\lambda - \Gamma_{\mu\rho}^\sigma \delta \Gamma_{\nu\sigma}^\lambda$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}$$

$$\begin{aligned} \delta R_{\mu\nu} &= \partial_\rho \delta \Gamma^\rho_{\nu\mu} - \partial_\nu \delta \Gamma^\rho_{\rho\mu} \\ &+ \delta \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} + \Gamma^\rho_{\rho\lambda} \delta \Gamma^\lambda_{\nu\mu} \\ &- \delta \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu} - \Gamma^\rho_{\nu\lambda} \delta \Gamma^\lambda_{\rho\mu} \end{aligned}$$

$$\Rightarrow \nabla_\mu \delta \Gamma^\lambda_{\nu\rho} = \partial_\mu \delta \Gamma^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} \delta \Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\mu\nu} \delta \Gamma^\lambda_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} \delta \Gamma^\lambda_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}$$

$$\begin{aligned} \delta R_{\mu\nu} &= \partial_\rho \delta \Gamma^\rho_{\nu\mu} - \partial_\nu \delta \Gamma^\rho_{\rho\mu} \\ &+ \delta \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} + \Gamma^\rho_{\rho\lambda} \delta \Gamma^\lambda_{\nu\mu} \\ &- \delta \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu} - \Gamma^\rho_{\nu\lambda} \delta \Gamma^\lambda_{\rho\mu} \\ &- \Gamma^\lambda_{\rho\nu} \delta \Gamma^\rho_{\lambda\mu} + \Gamma^\lambda_{\nu\rho} \delta \Gamma^\rho_{\lambda\mu} \end{aligned}$$

add + subtract the same term

$$\Rightarrow \nabla_\mu \delta \Gamma^\lambda_{\nu\rho} = \partial_\mu \delta \Gamma^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} \delta \Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\mu\nu} \delta \Gamma^\lambda_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} \delta \Gamma^\lambda_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}$$

$$\begin{aligned} \delta R_{\mu\nu} = & \underbrace{\partial_\rho \delta \Gamma^\rho_{\nu\mu}} - \partial_\nu \delta \Gamma^\rho_{\rho\mu} \\ & + \delta \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} + \underbrace{\Gamma^\rho_{\rho\lambda} \delta \Gamma^\lambda_{\nu\mu}} \\ & - \underbrace{\delta \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}} - \Gamma^\rho_{\nu\lambda} \delta \Gamma^\lambda_{\rho\mu} \\ & - \underbrace{\Gamma^\lambda_{\rho\nu} \delta \Gamma^\rho_{\lambda\mu}} + \Gamma^\lambda_{\nu\rho} \delta \Gamma^\rho_{\lambda\mu} \end{aligned}$$

$$\Rightarrow \nabla_\mu \delta \Gamma^\lambda_{\nu\rho} = \partial_\mu \delta \Gamma^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} \delta \Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\mu\nu} \delta \Gamma^\lambda_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} \delta \Gamma^\lambda_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}$$

$$\begin{aligned} \delta R_{\mu\nu} &= \underbrace{\partial_\rho \delta \Gamma^\rho_{\nu\mu}} - \partial_\nu \delta \Gamma^\rho_{\rho\mu} \\ &+ \delta \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} + \underbrace{\Gamma^\rho_{\rho\lambda} \delta \Gamma^\lambda_{\nu\mu}} \\ &- \underbrace{\delta \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}} - \Gamma^\rho_{\nu\lambda} \delta \Gamma^\lambda_{\rho\mu} \\ &- \underbrace{\Gamma^\lambda_{\rho\nu} \delta \Gamma^\rho_{\lambda\mu}} + \Gamma^\lambda_{\nu\rho} \delta \Gamma^\rho_{\lambda\mu} \\ &= \nabla_\rho \delta \Gamma^\rho_{\nu\mu} \end{aligned}$$

$$\Rightarrow \nabla_\mu \delta \Gamma^\lambda_{\nu\rho} = \partial_\mu \delta \Gamma^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} \delta \Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\mu\nu} \delta \Gamma^\lambda_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} \delta \Gamma^\lambda_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}$$

$$\begin{aligned} \delta R_{\mu\nu} &= \underbrace{\partial_\rho \delta \Gamma^\rho_{\nu\mu}}_{\text{orange}} - \underbrace{\partial_\nu \delta \Gamma^\rho_{\rho\mu}}_{\text{green}} \\ &+ \underbrace{\delta \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu}}_{\text{green}} + \underbrace{\Gamma^\rho_{\rho\lambda} \delta \Gamma^\lambda_{\nu\mu}}_{\text{orange}} \\ &- \underbrace{\delta \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}}_{\text{orange}} - \underbrace{\Gamma^\rho_{\nu\lambda} \delta \Gamma^\lambda_{\rho\mu}}_{\text{green}} \\ &- \underbrace{\Gamma^\lambda_{\rho\nu} \delta \Gamma^\rho_{\lambda\mu}}_{\text{orange}} + \underbrace{\Gamma^\lambda_{\nu\rho} \delta \Gamma^\rho_{\lambda\mu}}_{\text{green}} \\ &= \nabla_\rho \delta \Gamma^\rho_{\nu\mu} - \nabla_\nu \delta \Gamma^\rho_{\rho\mu} \end{aligned}$$

$$\Rightarrow \nabla_\mu \delta \Gamma^\lambda_{\nu\rho} = \partial_\mu \delta \Gamma^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} \delta \Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\mu\nu} \delta \Gamma^\lambda_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} \delta \Gamma^\lambda_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda} \Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\rho\mu}$$

$$\begin{aligned} \delta R_{\mu\nu} &= \underbrace{\partial_\rho \delta \Gamma^{\rho}_{\nu\mu}}_{\text{orange}} - \underbrace{\partial_\nu \delta \Gamma^{\rho}_{\rho\mu}}_{\text{green}} \\ &+ \underbrace{\delta \Gamma^{\rho}_{\rho\lambda} \Gamma^{\lambda}_{\nu\mu}}_{\text{green}} + \underbrace{\Gamma^{\rho}_{\rho\lambda} \delta \Gamma^{\lambda}_{\nu\mu}}_{\text{orange}} \\ &- \underbrace{\delta \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\rho\mu}}_{\text{orange}} - \underbrace{\Gamma^{\rho}_{\nu\lambda} \delta \Gamma^{\lambda}_{\rho\mu}}_{\text{green}} \\ &- \underbrace{\Gamma^{\lambda}_{\rho\nu} \delta \Gamma^{\rho}_{\lambda\mu}}_{\text{orange}} + \underbrace{\Gamma^{\lambda}_{\nu\rho} \delta \Gamma^{\rho}_{\lambda\mu}}_{\text{green}} \\ &= \nabla_\rho \delta \Gamma^{\rho}_{\nu\mu} - \nabla_\nu \delta \Gamma^{\rho}_{\rho\mu} \end{aligned}$$

$$\Rightarrow g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} \nabla_\rho \delta \Gamma^{\rho}_{\nu\mu} - g^{\mu\nu} \nabla_\nu \delta \Gamma^{\rho}_{\rho\mu}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}$$

$$\begin{aligned} \delta R_{\mu\nu} &= \underbrace{\partial_\rho \delta \Gamma^\rho_{\nu\mu}}_{\text{orange}} - \underbrace{\partial_\nu \delta \Gamma^\rho_{\rho\mu}}_{\text{green}} \\ &+ \underbrace{\delta \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu}}_{\text{green}} + \underbrace{\Gamma^\rho_{\rho\lambda} \delta \Gamma^\lambda_{\nu\mu}}_{\text{orange}} \\ &- \underbrace{\delta \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}}_{\text{orange}} - \underbrace{\Gamma^\rho_{\nu\lambda} \delta \Gamma^\lambda_{\rho\mu}}_{\text{green}} \\ &- \underbrace{\Gamma^\lambda_{\rho\nu} \delta \Gamma^\rho_{\lambda\mu}}_{\text{orange}} + \underbrace{\Gamma^\lambda_{\nu\rho} \delta \Gamma^\rho_{\lambda\mu}}_{\text{green}} \\ &= \nabla_\rho \delta \Gamma^\rho_{\nu\mu} - \nabla_\nu \delta \Gamma^\rho_{\rho\mu} \end{aligned}$$

$$\begin{aligned} \Rightarrow g^{\mu\nu} \delta R_{\mu\nu} &= g^{\mu\nu} \nabla_\rho \delta \Gamma^\rho_{\nu\mu} - g^{\mu\nu} \nabla_\nu \delta \Gamma^\rho_{\rho\mu} \\ &= \nabla_\rho [g^{\mu\nu} \delta \Gamma^\rho_{\nu\mu}] - \nabla_\nu [g^{\mu\nu} \delta \Gamma^\rho_{\rho\mu}] \end{aligned}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda} \Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\rho\mu}$$

$$\begin{aligned} \delta R_{\mu\nu} &= \underbrace{\partial_\rho \delta \Gamma^{\rho}_{\nu\mu}}_{\text{orange}} - \underbrace{\partial_\nu \delta \Gamma^{\rho}_{\rho\mu}}_{\text{green}} \\ &+ \underbrace{\delta \Gamma^{\rho}_{\rho\lambda} \Gamma^{\lambda}_{\nu\mu}}_{\text{green}} + \underbrace{\Gamma^{\rho}_{\rho\lambda} \delta \Gamma^{\lambda}_{\nu\mu}}_{\text{orange}} \\ &- \underbrace{\delta \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\rho\mu}}_{\text{orange}} - \underbrace{\Gamma^{\rho}_{\nu\lambda} \delta \Gamma^{\lambda}_{\rho\mu}}_{\text{green}} \\ &- \underbrace{\Gamma^{\lambda}_{\rho\nu} \delta \Gamma^{\rho}_{\lambda\mu}}_{\text{orange}} + \underbrace{\Gamma^{\lambda}_{\nu\rho} \delta \Gamma^{\rho}_{\lambda\mu}}_{\text{green}} \\ &= \nabla_\rho \delta \Gamma^{\rho}_{\nu\mu} - \nabla_\nu \delta \Gamma^{\rho}_{\rho\mu} \end{aligned}$$

$$\begin{aligned} \Rightarrow g^{\mu\nu} \delta R_{\mu\nu} &= g^{\mu\nu} \nabla_\rho \delta \Gamma^{\rho}_{\nu\mu} - g^{\mu\nu} \nabla_\nu \delta \Gamma^{\rho}_{\rho\mu} \\ &= \nabla_\rho [g^{\mu\nu} \delta \Gamma^{\rho}_{\nu\mu}] - \nabla_\nu [g^{\mu\nu} \delta \Gamma^{\rho}_{\rho\mu}] \\ &= \nabla_\rho [g^{\mu\nu} \delta \Gamma^{\rho}_{\nu\mu} - g^{\mu\rho} \delta \Gamma^{\nu}_{\nu\mu}] \end{aligned}$$

almost there...

$$\int_V \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \int_V \sqrt{-g} \nabla_\rho [g^{\mu\nu} \delta \Gamma^\rho_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu_{\nu\mu}]$$

$$= \int_{\partial V} \sqrt{\delta} n_\rho [g^{\mu\nu} \delta \Gamma^\rho_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu_{\nu\mu}]$$

$$\Rightarrow g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} \nabla_\rho \delta \Gamma^\rho_{\nu\mu} - g^{\mu\nu} \nabla_\nu \delta \Gamma^\rho_{\rho\mu}$$

$$= \nabla_\rho [g^{\mu\nu} \delta \Gamma^\rho_{\nu\mu}] - \nabla_\nu [g^{\mu\nu} \delta \Gamma^\rho_{\rho\mu}]$$

$$= \nabla_\rho [g^{\mu\nu} \delta \Gamma^\rho_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu_{\nu\mu}]$$

almost there...

$$\int_V \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \int_V \sqrt{-g} \nabla_\rho [g^{\mu\nu} \delta \Gamma^\rho_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu_{\nu\mu}]$$

$$= \int_{\partial V} \sqrt{\gamma} n_\rho [g^{\mu\nu} \delta \Gamma^\rho_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu_{\nu\mu}]$$

oops! ... not zero!

(can't have $\delta g|_{\partial V} = 0$ and $\delta \Gamma|_{\partial V} = 0$ at the same time)

$$\Rightarrow g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} \nabla_\rho \delta \Gamma^\rho_{\nu\mu} - g^{\mu\nu} \nabla_\nu \delta \Gamma^\rho_{\rho\mu}$$

$$= \nabla_\rho [g^{\mu\nu} \delta \Gamma^\rho_{\nu\mu}] - \nabla_\nu [g^{\mu\nu} \delta \Gamma^\rho_{\rho\mu}]$$

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Can get rid of this term by adding a boundary term:

$$S^k_\alpha \frac{1}{2} \int_{\partial V} \sqrt{\gamma} K, \quad K = \gamma_{ij} k^{ij} \quad k^{ij} \text{ the extrinsic curvature!}$$

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Then $S = S^{\text{EH}} + S^\Lambda + S^M + S^k$,

and δS^k cancels the $\int_{\partial V} \sqrt{\gamma} n_\rho [g^{\mu\nu} \delta \Gamma^\rho_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu_{\nu\mu}]$ term

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