(1) Consider the curve $x^2 + t^2 = R^2$ $y = z = 0$.

Compute its spacetime length and compare it to its Euclidean length

(2) Consider the curve:

$$x = R\cos(\omega s)$$
$$y = R\sin(\omega s)$$
$$z = 0$$
$$t = v \cdot s$$

(α) Write down the condition that R, ω, v must satisfy so that the curve is timelike

(β) Compute its length from $s = 0$ as a function of s

(γ) Write the parametric equations in terms of its proper time τ

(3) Consider the coordinate systems

(t, ρ, φ, z) $\quad x = \rho\cos\varphi$, $\quad y = \rho\sin\varphi$

(t, r, θ, ϕ) $\quad x = r\sin\theta\cos\varphi$, $\quad y = r\sin\theta\sin\varphi$, $\quad z = r\cos\theta$

If $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$ in the (t,x,y,z) coordinate system,

compute $\eta_{\mu'\nu'} = \dfrac{\partial x^{\mu}}{\partial x^{\mu'}} \dfrac{\partial x^{\nu}}{\partial x^{\nu'}} \eta_{\mu\nu}$ in the (t,ρ,φ,z) and (t,r,θ,ϕ) coordinate systems

(4) Express the components of the tensor $S_{\mu\nu\rho}$ in terms of the components of $S^{\mu\nu\rho}$, by lowering the indices by $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$

(5) Let $\epsilon_{\mu\nu\rho\lambda}$ be the completely antisymmetric Levi-Civita symbol with $\epsilon_{0123} = +1$

Compute the components of the tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}$ in terms of the components of $\vec{\mathcal{E}}$ and $\vec{B}$.

(6) Compute $F_{\mu\nu}F^{\mu\nu}$, $\tilde{F}_{\mu\nu}F^{\mu\nu}$, $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ in terms of the components of $\vec{\mathcal{E}}$ and $\vec{B}$

(7) Write out the contraction $S_{\mu\nu}U^{\mu}W^{\nu}$ in terms of the components of $S^{\mu\nu}$, U^{μ}, W^{ν} in Cartesian coordinates and $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$

Write the same expression if $(\eta_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$

(8) Consider the inertial observers

$$A: (t, x, y, z) \qquad B: (t', x', y', z') \qquad C: (t'', x'', y'', z'')$$

so that at $t = t' = t'' = 0$ their xyz axes coincide. Oberver B moves with rapidity β_1 in the $+x$ direction with respect to A, and C moves with rapidity β_2 in the $-y$ direction with respect to B ($v_1 = \tanh\beta_1$, $v_2 = \tanh\beta_2$). Observer C observes a massive particle moving with velocity $\vec{V} = (V^1, V^2, V^3)$ and an electrostatic field $\vec{\mathcal{E}} = \mathcal{E}\,\hat{z}$. Compute the velocity of the particle, and $\vec{\mathcal{E}}, \vec{B}$ as measured by observer A.

Suggestion: (α) compute the composite Lorentz transformation

(β) compute U^μ (4-velocity), $F^{\mu\nu}$ (EM-tensor) in the C-frame

(c) use (α) to transform to the A-frame the components of U^μ, $F^{\mu\nu}$

(d) compute $\vec{V}$, $\vec{\mathcal{E}}$, $\vec{B}$ from U^μ, $F^{\mu\nu}$