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Bibliography

General Relativity cannot be learned by using only one textbook. You need to have at least five of them open at the same time...

Textbooks

- James Hartle, Gravity: An Introduction to Einstein's General Relativity
- Nick E. Mavromatos, General Relativity and Cosmology
- Sean Carroll, Spacetime and Geometry. An Introduction to General Relativity
- Bernard F. Schutz, Geometrical Methods of Mathematical Physics

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General Relativity

- Bernard Schutz, A First Course in General Relativity
- Robert M. Wald, General Relativity
- Charles W. Misner, Kip S. Thorne, John A. Wheeler, Gravitation
- Valeria Ferrari, Leonardo Gualtieri, Paolo Pani, General Relativity and its Applications
- Steven Weinberg, Gravitation and Cosmology
- Norbert Straumann, General Relativity
- Anthony Zee, Einstein Gravity in a Nutshell
- Lewis Ryder, Introduction to General Relativity
- Yvonne Choquet-Bruhat, General Relativity and the Einstein Equation
- Stephen W. Hawking & George F.R. Ellis, The Large Scale Structure of Space-Time
- Achilles Papapetrou, Lectures on General Relativity

Cosmology

- Andrew Liddle, An Introduction to Modern Cosmology
- Steven Weinberg, Cosmology
- Daniel Baumann, Cosmology
- Barbara Ryden, Introduction to Cosmology
- Phillip J.E. Peebles, Principles of Physical Cosmology

Differential Geometry

- Bernard F. Schutz, Geometrical Methods of Mathematical Physics
- Yvonne Choquet-Bruhat, Cecile Dewitt-Morette, Margaret Dillard-Bleick, Analysis, Manifolds and Physics
- Chris J. Isham, Modern Differential Geometry for Physicists
- Mikio Nakahara, Geometry, Topology and Physics
- Charles Nash and Siddhartha Sen, Topology and Geometry for Physicists

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Video Lectures on the General Theory of Relativity

Long and shorter videos on selected topics, problems, and computer exercises.

Virtual lectures by the instructor on selected topics, problems and computer exercises, supplementing the material presented in class. This is an ongoing effort, and you should expect the list to grow in time. Next to the video links, you will find links to transparencies, Mathematica & Maxima notebooks and other relevant material used in the videos.

You can also find the videos in a youtube video list. A separate list with computer exercises using Mathematica, can be found here. The list below contains all the videos in those lists, together with some metadata.

- Unit 1: Differential Geometry
  1. Manifolds: Differential Manifolds, topological spaces, charts, transition functions, atlases. (Slides)
  2. Vectors: Vectors as tangent to curves, tangent space, coordinate basis, component transformations, vector fields, integral curves, Lie bracket, Lie derivative. (Slides)
  3. One forms and tensors: One forms as linear maps on TM, cotagent space, gradient of a function, coordinate bases, tensors, tensor product, contractions, (anti)symmetrization. (Slides)
  4. Differential forms: Differential forms and form fields, wedge product, exterior derivative, interior product, Levi-Civita tensor, duality, Hodge-star operator. (Slides)
  5. Maps: Maps between manifolds, and pullback/pushforward of tensors.
  6. Diffeomorphisms: Diffeomorphic manifolds, pullback/pushforward of tensors, Lie Derivative, components of the Lie derivative in a coordinate basis. (Slides)
  7. Lie Dierivatives: Proof of $\mathcal{L}_V W = [V,W]$, $\mathcal{L}_V f = V(f)$. Computation of $\mathcal{L}_V \omega$. Geometric interpretation of $\mathcal{L}_V W$ and $[V,W]$. (Slides)

Slides

Extras:

- Problem Solving
- Computer Labs (Mathematica, Maxima)
- More Lectures...

Gravity:

- Universal: all forms of matter
  & energy gravitate
- Always attractive: not screened + long range
  (e.g. electrically neutral matter makes EM
  irrelevant @ cosmic scales)

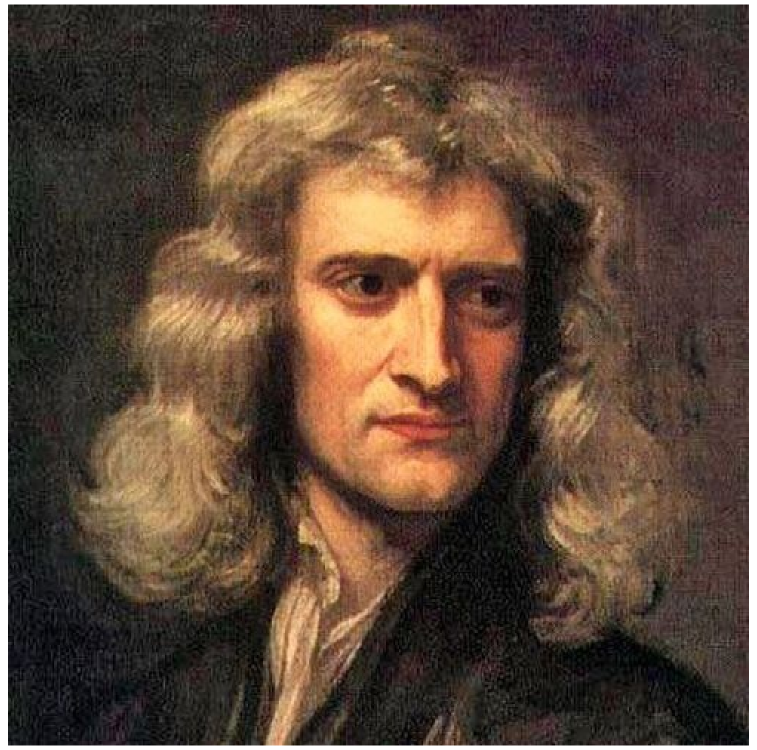
$\Rightarrow$ dominates @ cosmic scales → solar systems, galaxies, ..., cosmology

(despite being ridiculously weak...)

Fun high school physics:

escape velocity!

$$\frac{1}{2} m v_e^2 = \frac{G m M}{R}$$



Newtonian Potential from force of gravity

Fun high school physics:

escape velocity!

$$\left.\begin{array}{l} \frac{1}{2} m v_e^2 = \frac{G m M}{R} \\ \text{set } v_e = c \end{array}\right\} \Rightarrow \quad \frac{2GM}{c^2 R} = 1$$

Fun high school physics:

escape velocity!

$$\left.\begin{array}{l} \frac{1}{2} m v_e^2 = \frac{G m M}{R} \\ \text{set } v_e = c \end{array}\right\} \Rightarrow \underbrace{\frac{2GM}{c^2 R}}_{\text{scale when GR effects become strong!}} = 1$$

Fun high school physics:

escape velocity!

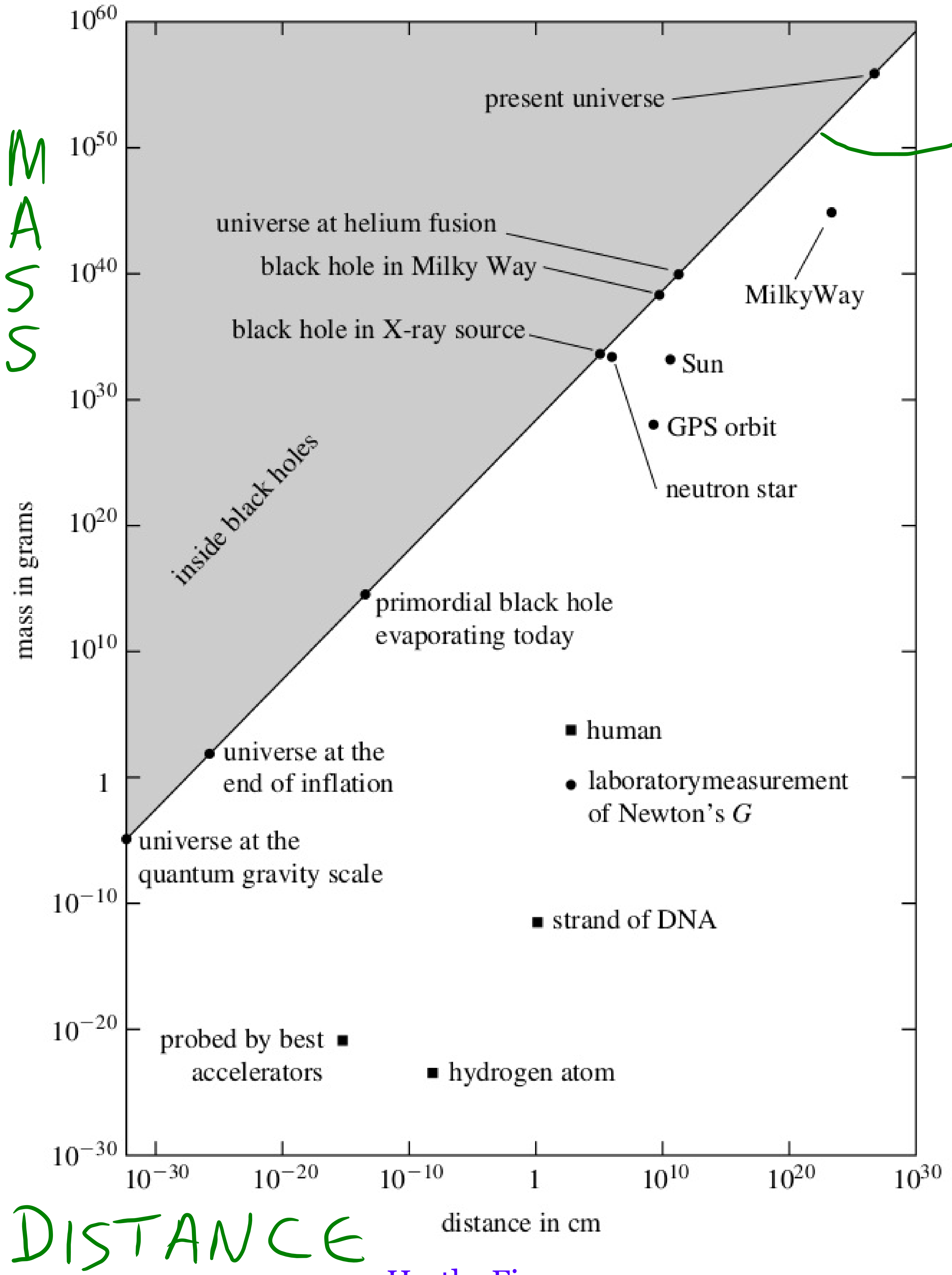
$$\left.\begin{aligned} &\frac{1}{2} m v_e^2 = \frac{GmM}{R} \\ &\text{set } v_e = c \end{aligned}\right\} \Rightarrow \frac{2GM}{c^2 R} = 1$$

Earth $\sim 10^{-9}$

Sun $\sim 10^{-6}$

Neutron Star ~ 0.1

General Relativistic Effects



Hartle, Fig. 1.1

$\frac{2GM}{c^2 R} = 1$ line

- Black Holes
  - stellar
  - supermassive
  - + medium size ...
- Neutron Stars
- G - Waves
- Big bang cosmology

The frontiers

- Quantum Gravity?

$$\ell_{Pl} \equiv (G\hbar/c^3)^{1/2} = 1.62 \times 10^{-33} \text{ cm},$$
$$t_{Pl} \equiv (G\hbar/c^5)^{1/2} = 5.39 \times 10^{-44} \text{ s},$$
$$E_{Pl} \equiv (\hbar c^5/G)^{1/2} = 1.22 \times 10^{19} \text{ GeV},$$
$$\rho_{Pl} \equiv c^5/\hbar G^2 = 5.16 \times 10^{93} \text{ g/cm}^3.$$

$G - \hbar - c$

→ unification ?

→ singularity resolution ?

→ BH information paradox ?

⋮

Newtonian Gravity:

- gravity is a force!

$$\vec{F} = \frac{GmM}{r^2}\hat{r}$$

$$= m\vec{g}$$

$$\vec{g} = -\vec{\nabla}\Phi$$

$$\vec{\nabla}\cdot\vec{g} = -\nabla^2\Phi = -4\pi G\rho(\vec{x})$$

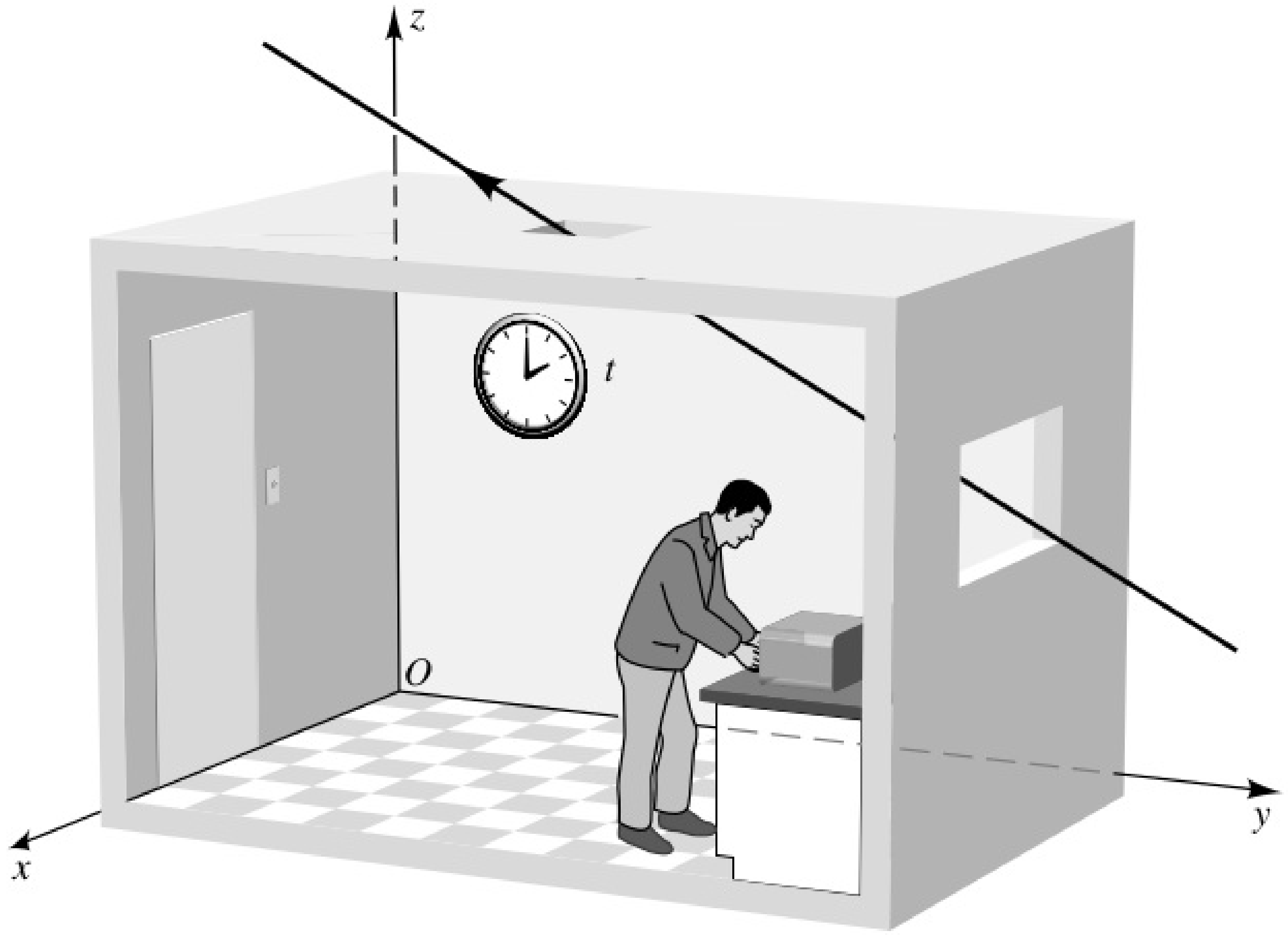
Newtonian Gravity:

- gravity is a force!

Problems:

- Propagates instantaneously
- Singles preferred observer for which Newton's Law applies
- nature does not agree...

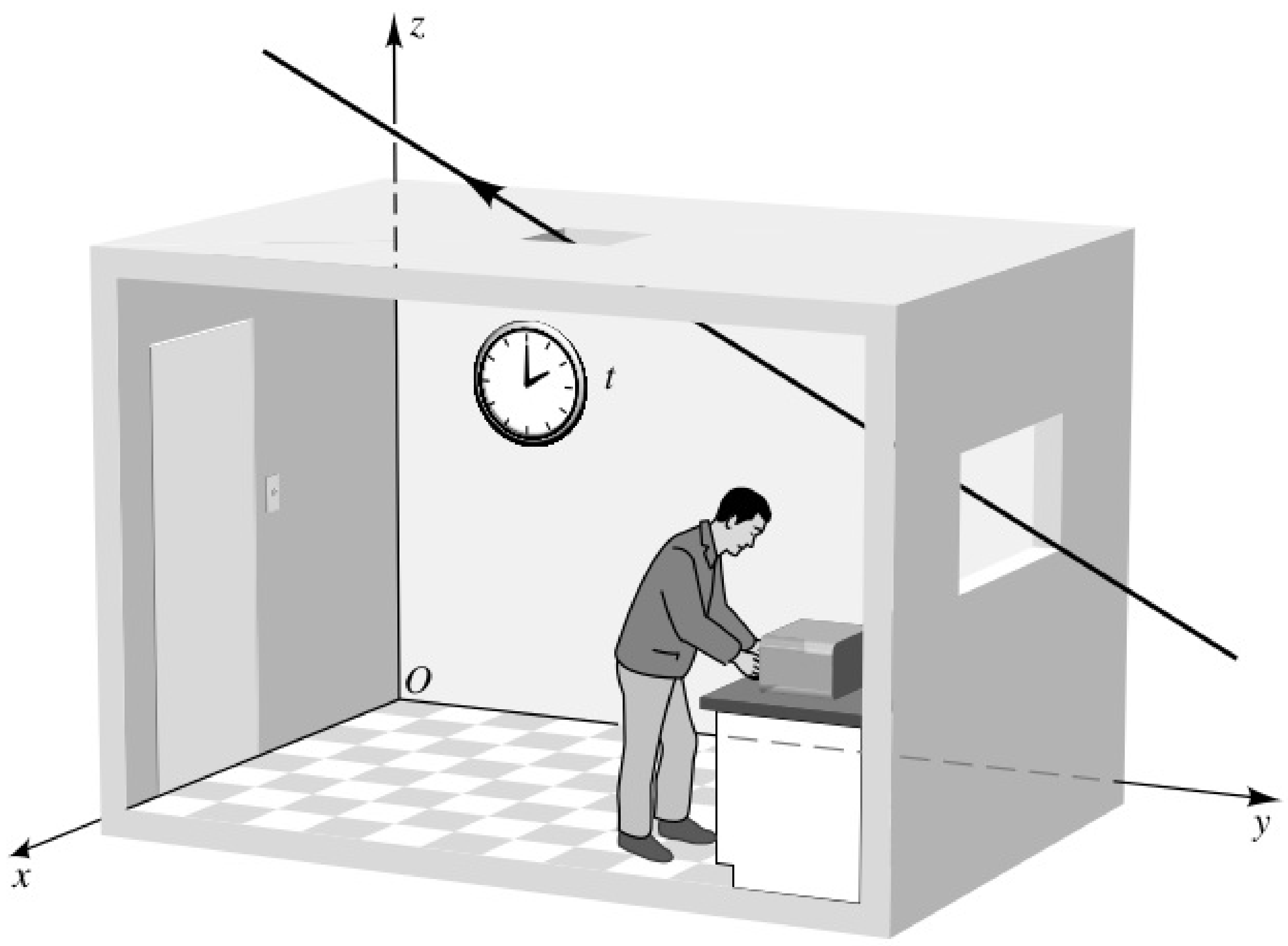
Newtonian Mechanics:



Inertial Observers:

- $\exists$ parameter t s.t. free particles change position @ constant rate $\rightsquigarrow$ universal $\begin{cases} \text{time} \\ \text{simultaneity} \end{cases}$

Newtonian Mechanics:

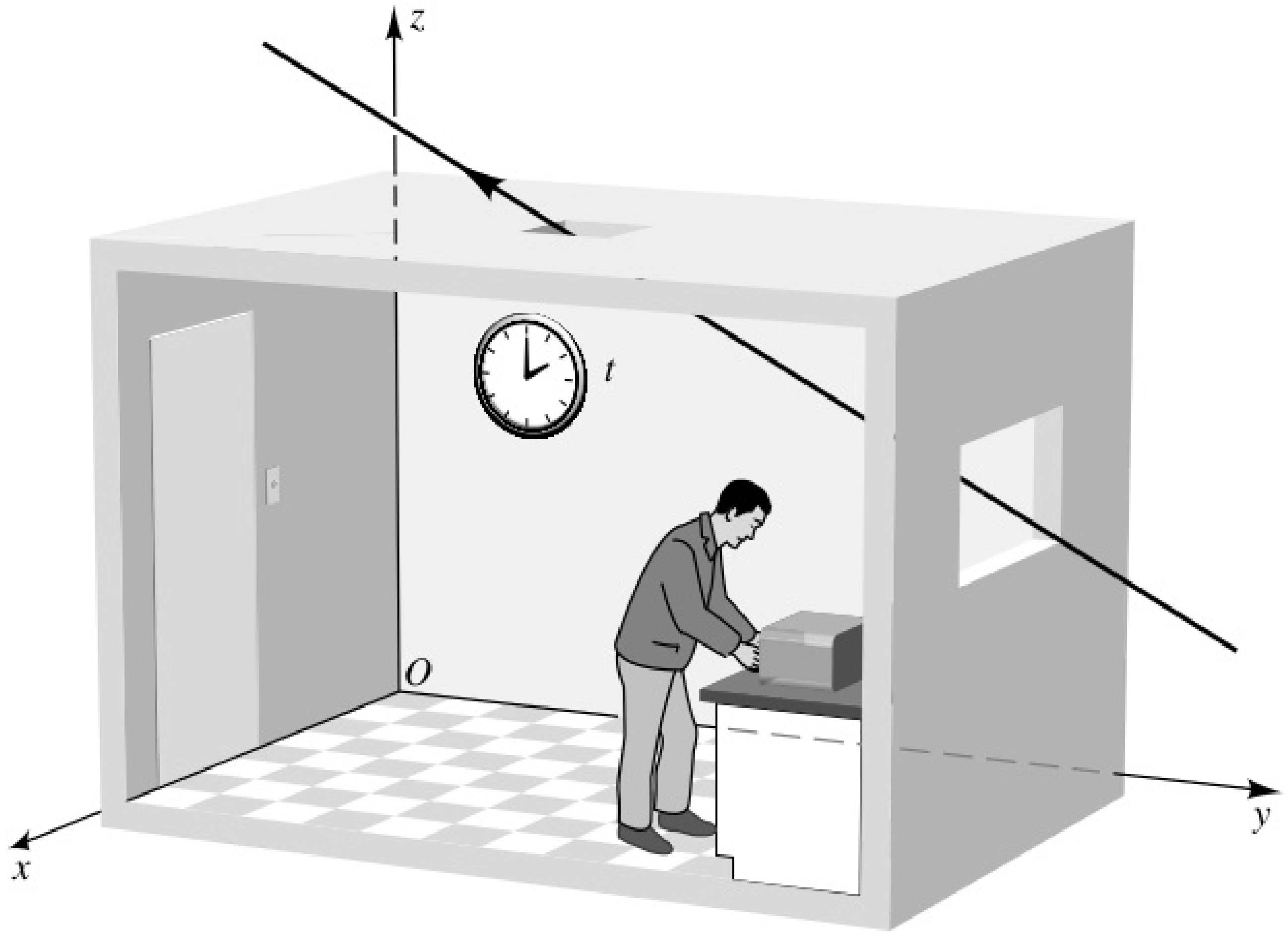


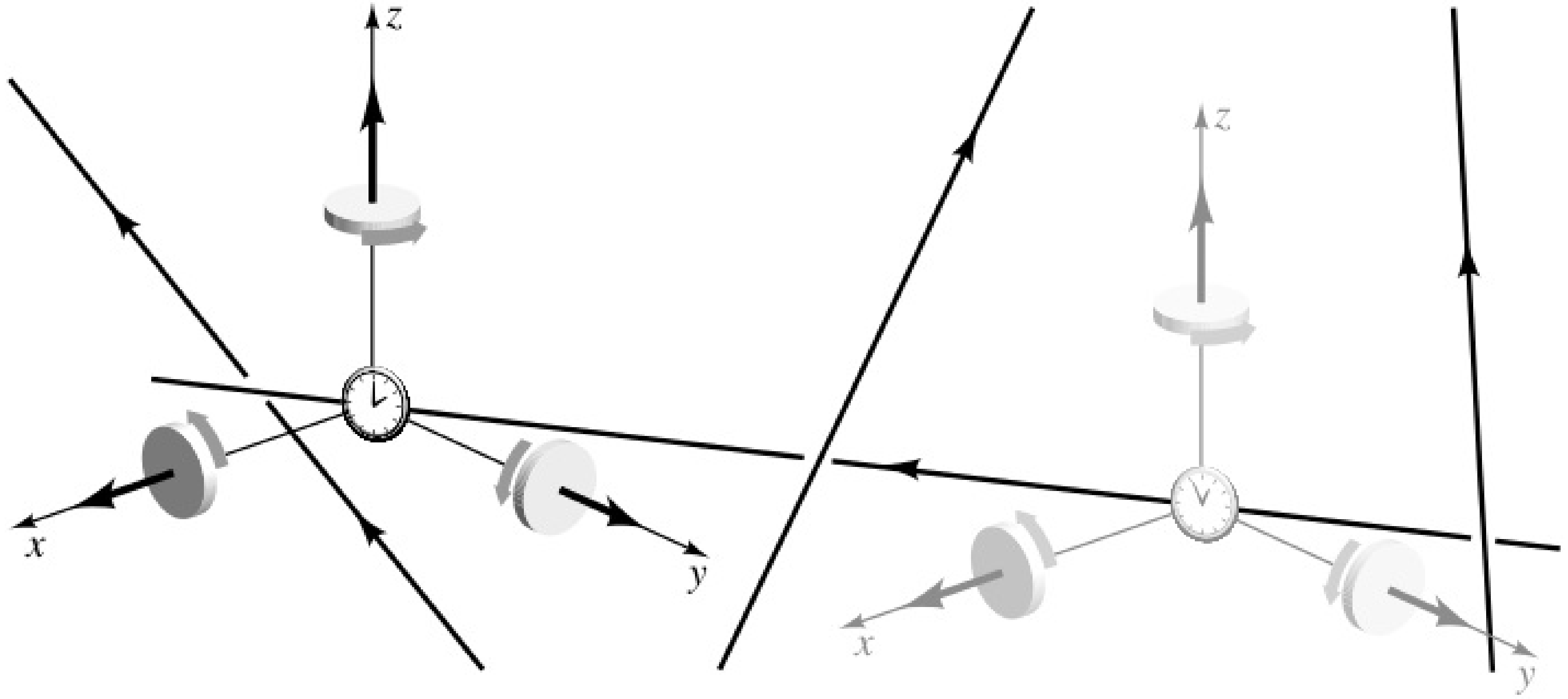
Inertial Observers:

- Move w/ constant relative speed w.r.t. to each other

  (an elite class...)

Newtonian Mechanics:





Free particles:

$$\frac{d^2x}{dt^2}=0 \qquad \frac{d^2y}{dt^2}=0 \qquad \frac{d^2z}{dt^2}=0$$

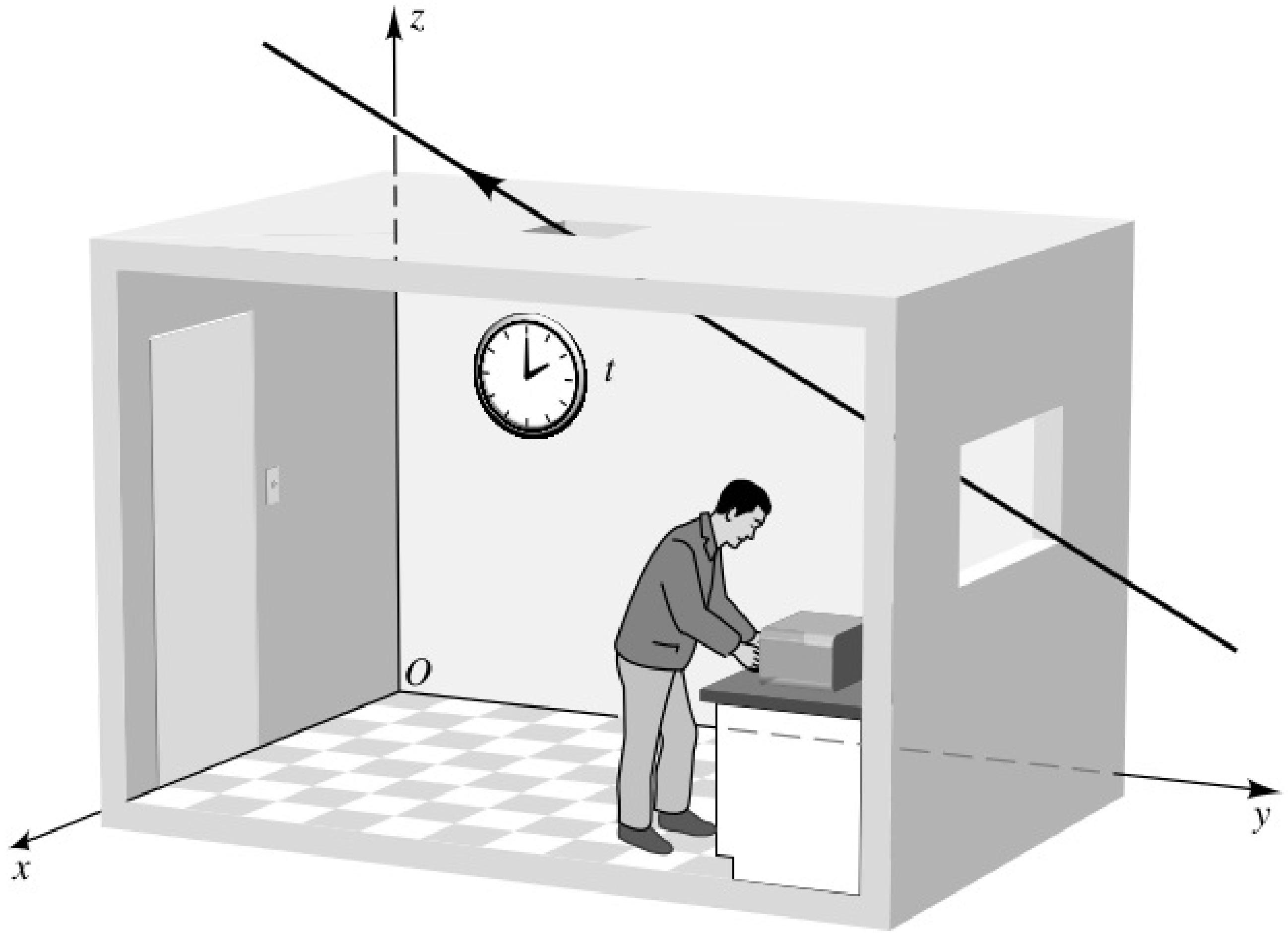
Inertial Observers:

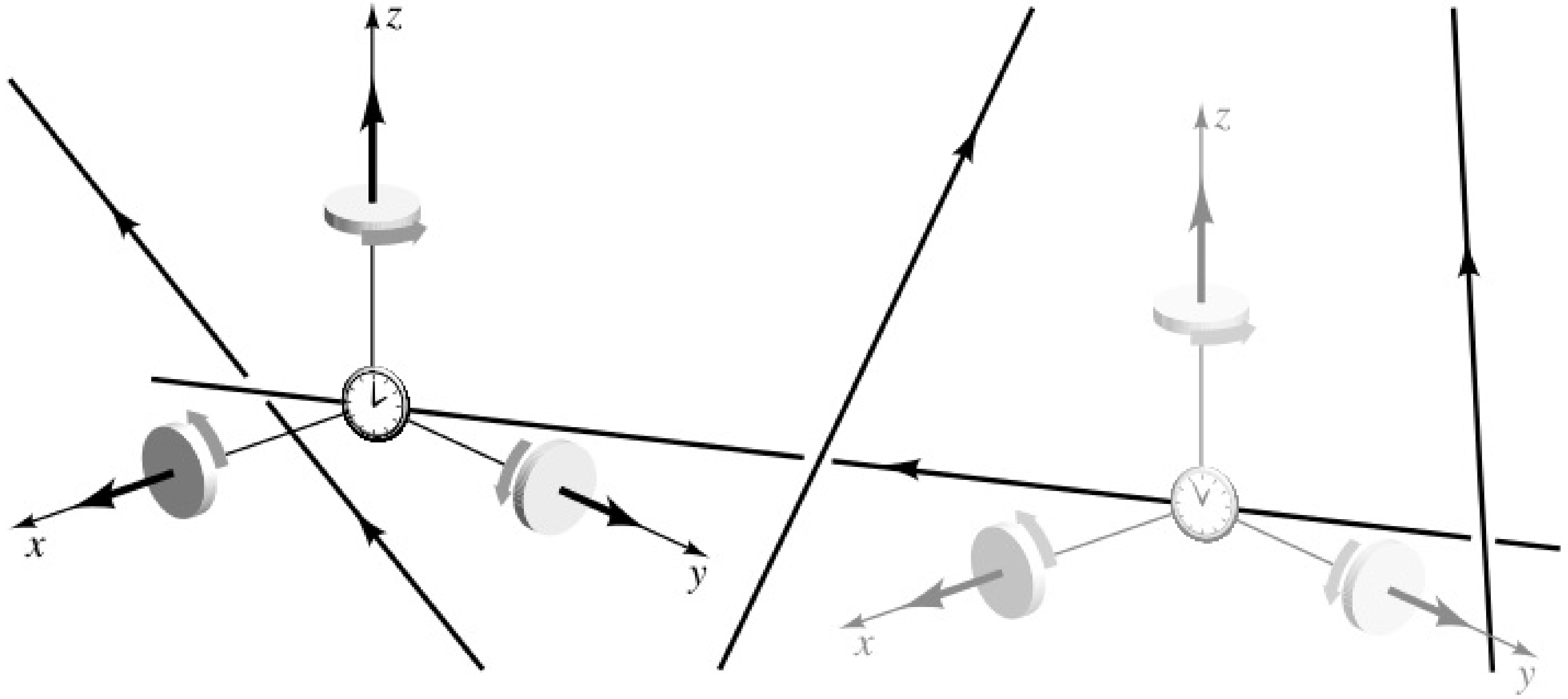
- Construct:

  origin → moving free particle

  axes → 3 orthogonal gyroscopes

Newtonian Mechanics:





Free particles:

$$\frac{d^2x}{dt^2} = 0 \qquad \frac{d^2y}{dt^2} = 0 \qquad \frac{d^2z}{dt^2} = 0$$

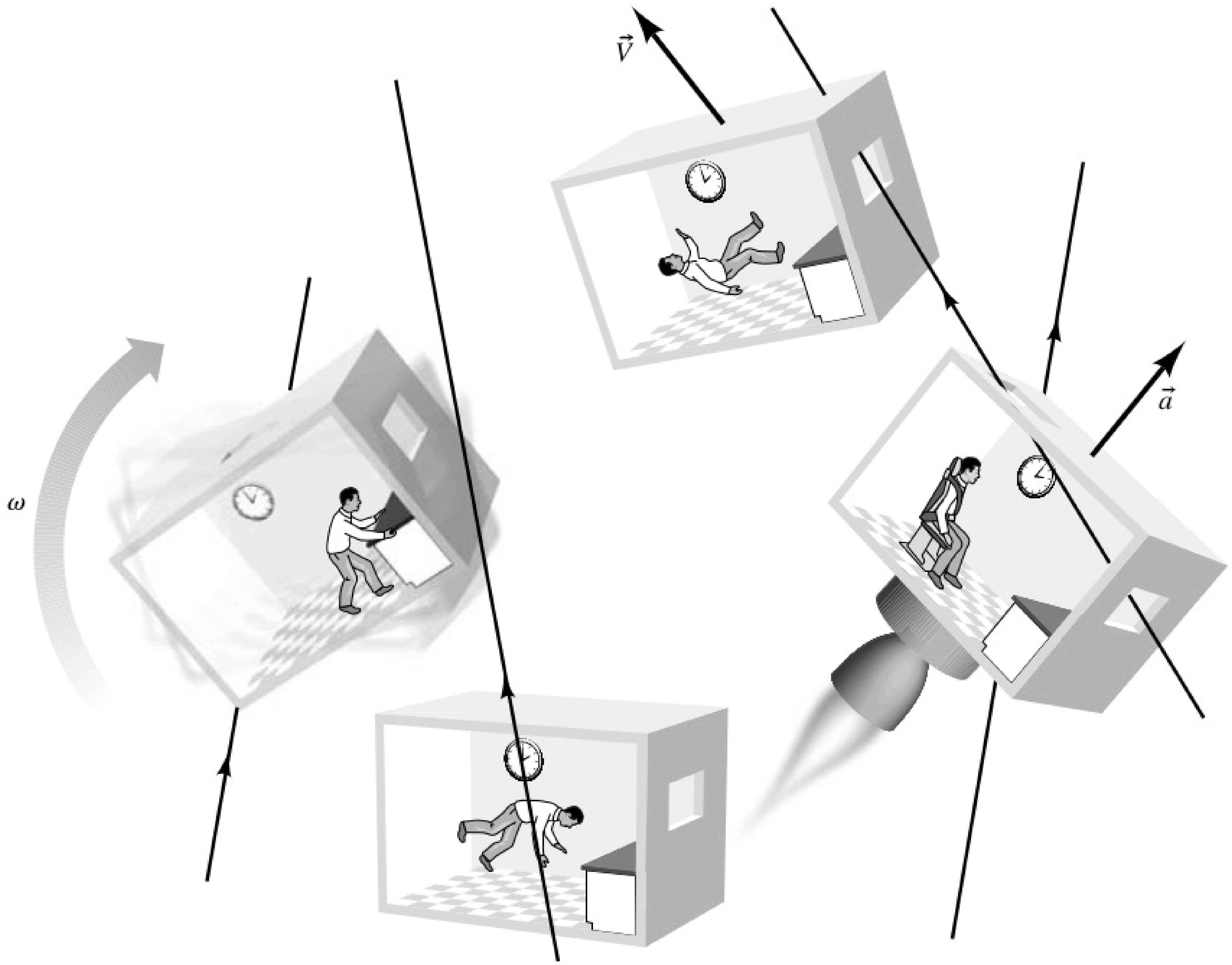
Inertial Observers:

- Construct:

origin → moving free particle

axes → 3 orthogonal gyroscopes

. Special relativity is not special when it comes to I.O.s



Not all observers are inertial

(we are not...)

General Relativity:

* gravity is not a force!

General Relativity:

* gravity is not a force!
* all freely falling observers are inertial

Crucial:
all matter "falls" with the same acceleration!

General Relativity:

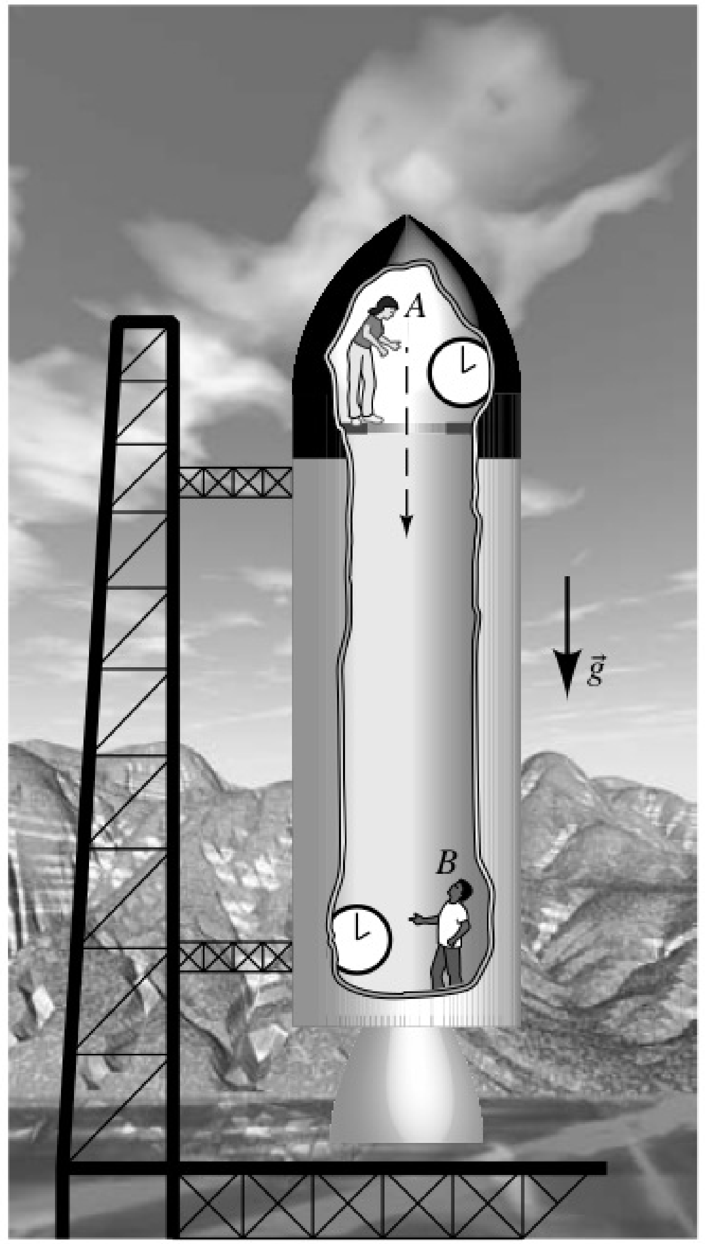
* gravity is not a force!
* all freely falling observers are inertial

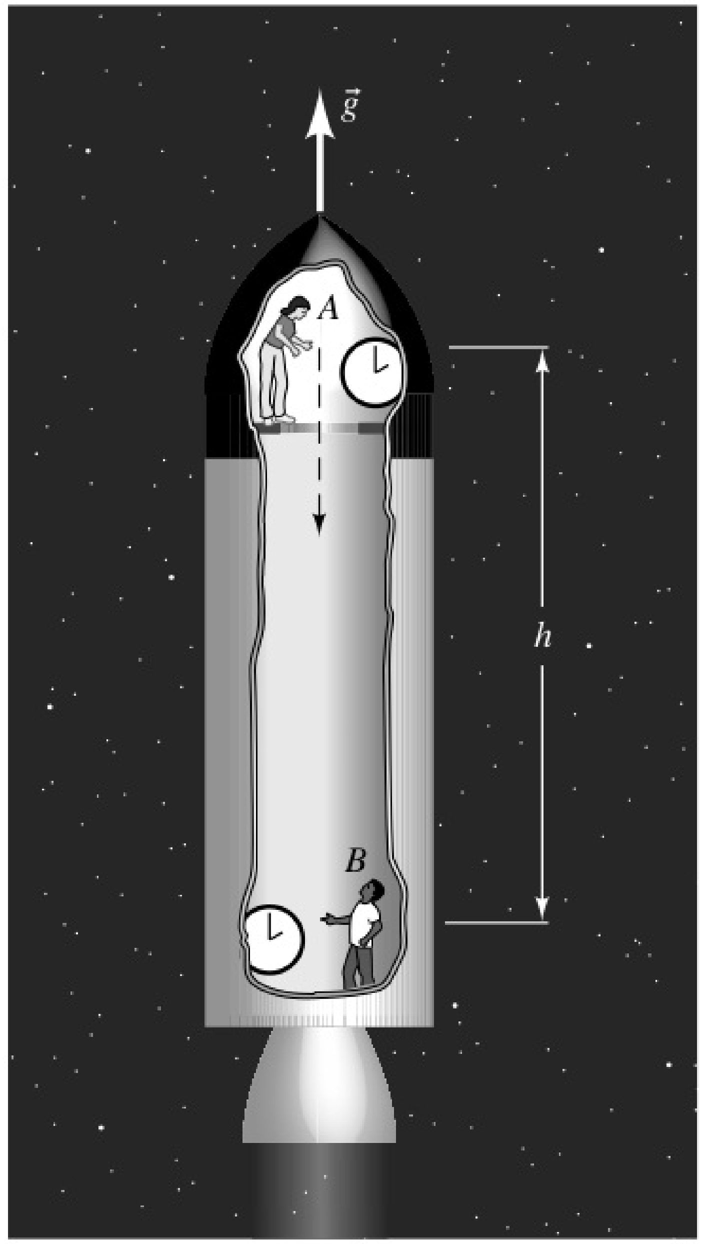
Equivalence Principle

Experiments in a sufficiently small freely falling laboratory, over a sufficiently short time, give results that are indistinguishable from those of the same experiments in an inertial frame in empty space.

General Relativity:

* gravity is not a force!
* all freely falling observers are inertial

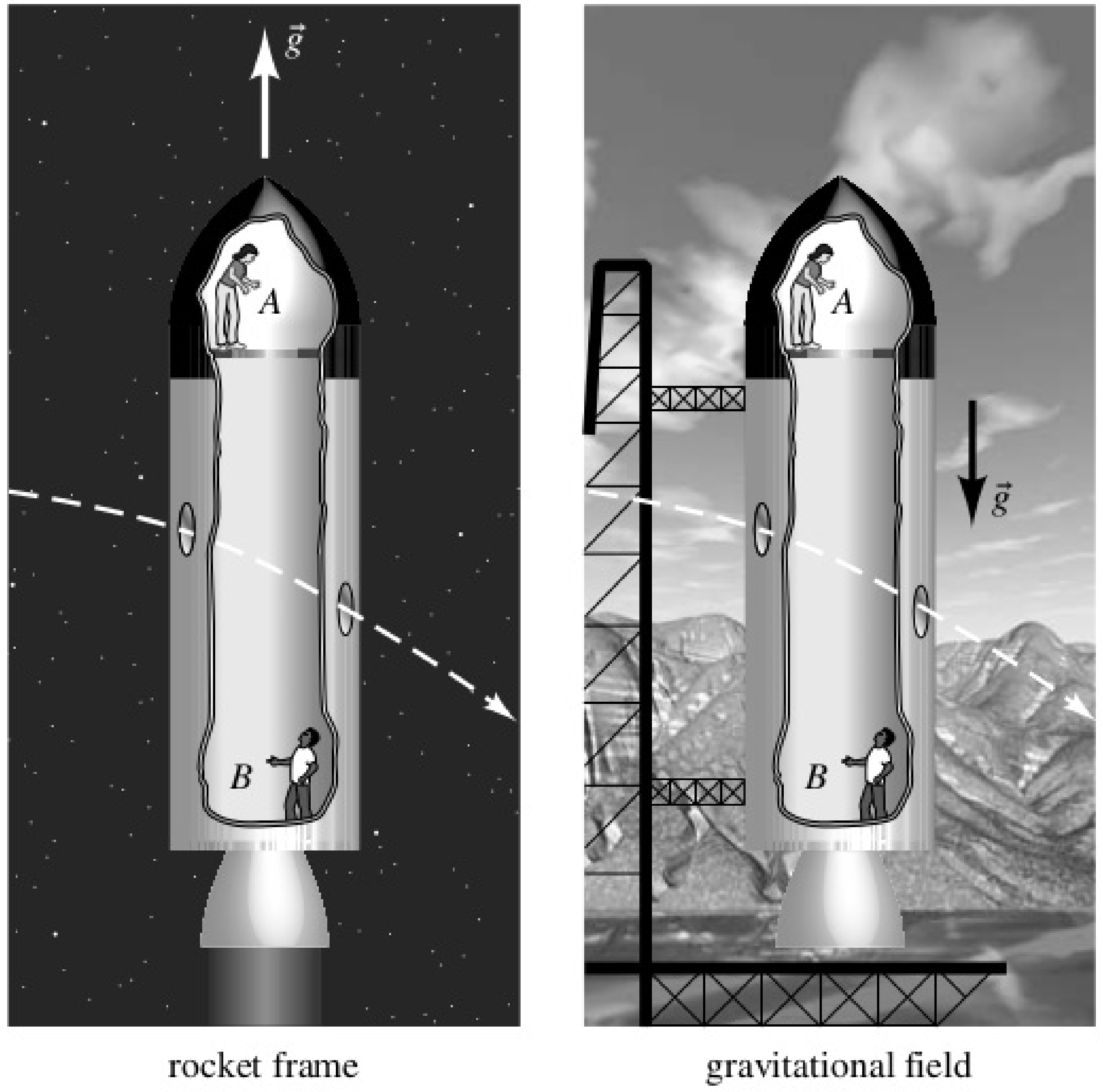




* no experiment can distinguish accelerated motion from being under the influence of gravity

General Relativity:

* gravity is not a force!
* all freely falling observers are inertial



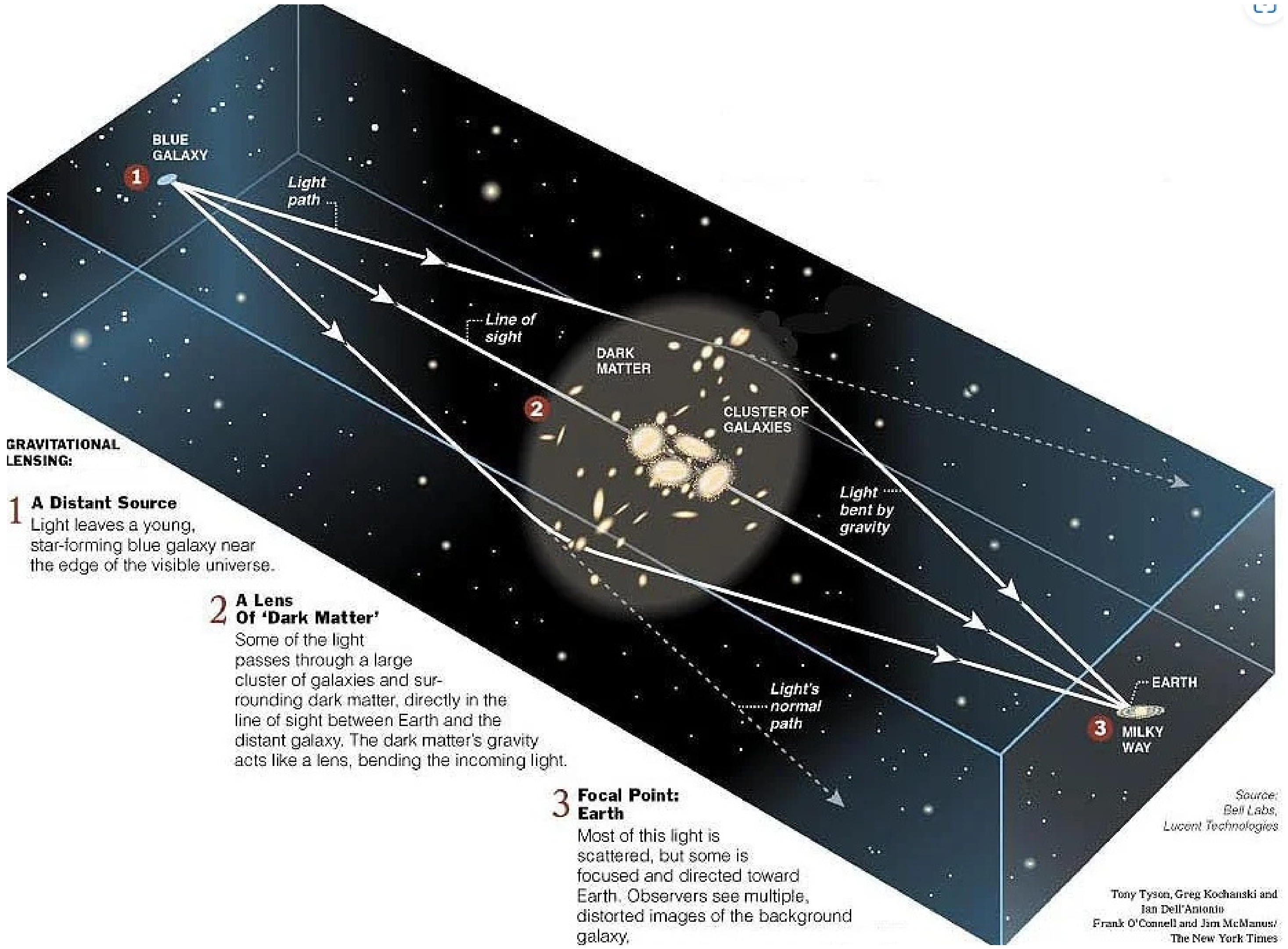
* no experiment can distinguish accelerated motion from being under the influence of gravity

$\rightsquigarrow$ has implications: light appears "falling" in g

General Relativity:

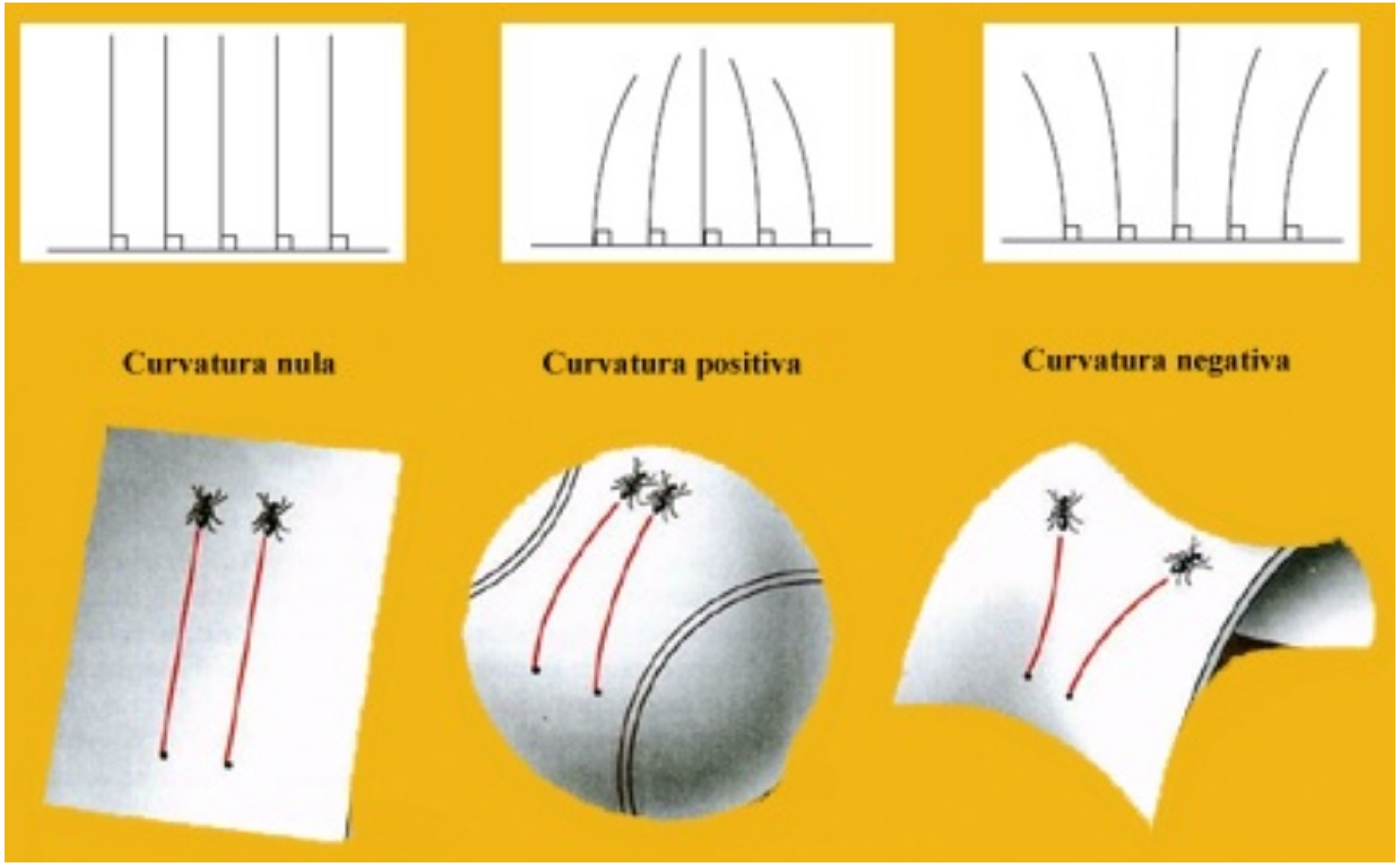
* gravity is geometry

all test particles follow the same trajectories, because they are affected by the shape of spacetime!



General Relativity:

* gravity is curvature



intrinsic notion, no reference to embeddings

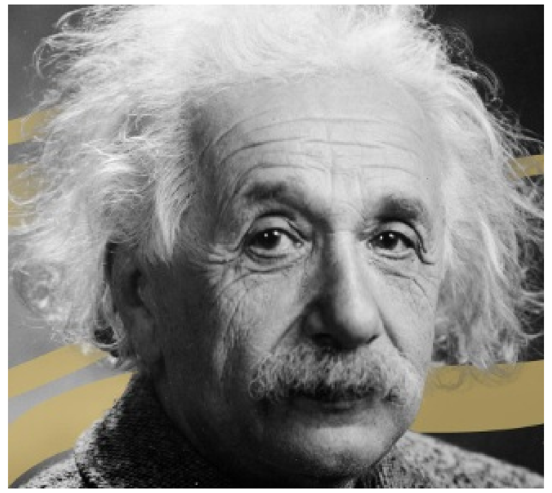
General Relativity:

* gravity is curvature

↳ Depends on dynamics!

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

geometry



matter

General Relativity:

* gravity is curvature

S.R. put grounds of spacetime geometry

$\Rightarrow$ spacetime is a differentiable manifold!

• locally like $\mathbb{R}^4$ (equivalence principle included...)

we can do familiar S.R. physics here

General Relativity:

* gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold!

- locally like $\mathbb{R}^4$. (equivalence principle included ...)

- local observers that communicate!
  (coordinate systems) (coordinate transformations)

General Relativity:

* gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold!

- locally like $\mathbb{R}^4$ (equivalence principle included ...)
- local observers that communicate!
  (coordinate systems) (coordinate transformations)
- can write down dynamical equations
  (do analysis ... on manifolds)

General Relativity:

* gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold!

- locally like $\mathbb{R}^4$ (equivalence principle included ...)
- local observers that communicate!
  (coordinate systems) (coordinate transformations)
- can write down dynamical equations
- physical fields, same for all observers (geometrical objects - tensors)

General Relativity:

* gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold!

+ extra structure: a metric

$\parallel\!\!|$

geometry

measure distances, angles,
compute inner products

General Relativity:

* gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold!

+ extra structure: a metric → singles out time (locally S.R.)

→ causal structure (locally SR)

General Relativity:

* gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold!

+ extra structure: a metric

  $\to$ causal structure

  $\to$ affine connection – parallel transport

  $\hookrightarrow$ Google translate: συγγενική σύνδεση

General Relativity:

* gravity is curvature

⇒ spacetime is a differentiable manifold!

+ extra structure: a metric

→ causal structure

→ affine connection – parallel transport

→ curvature

General Relativity:

* gravity is curvature

⇒ spacetime is a differentiable manifold !

+ extra structure: a metric

background: fixed

dynamical: chosen among many by the dynamics

General Relativity:

* gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold!

+ extra structure: a metric

+ heavy price: physics at global scales is hard.

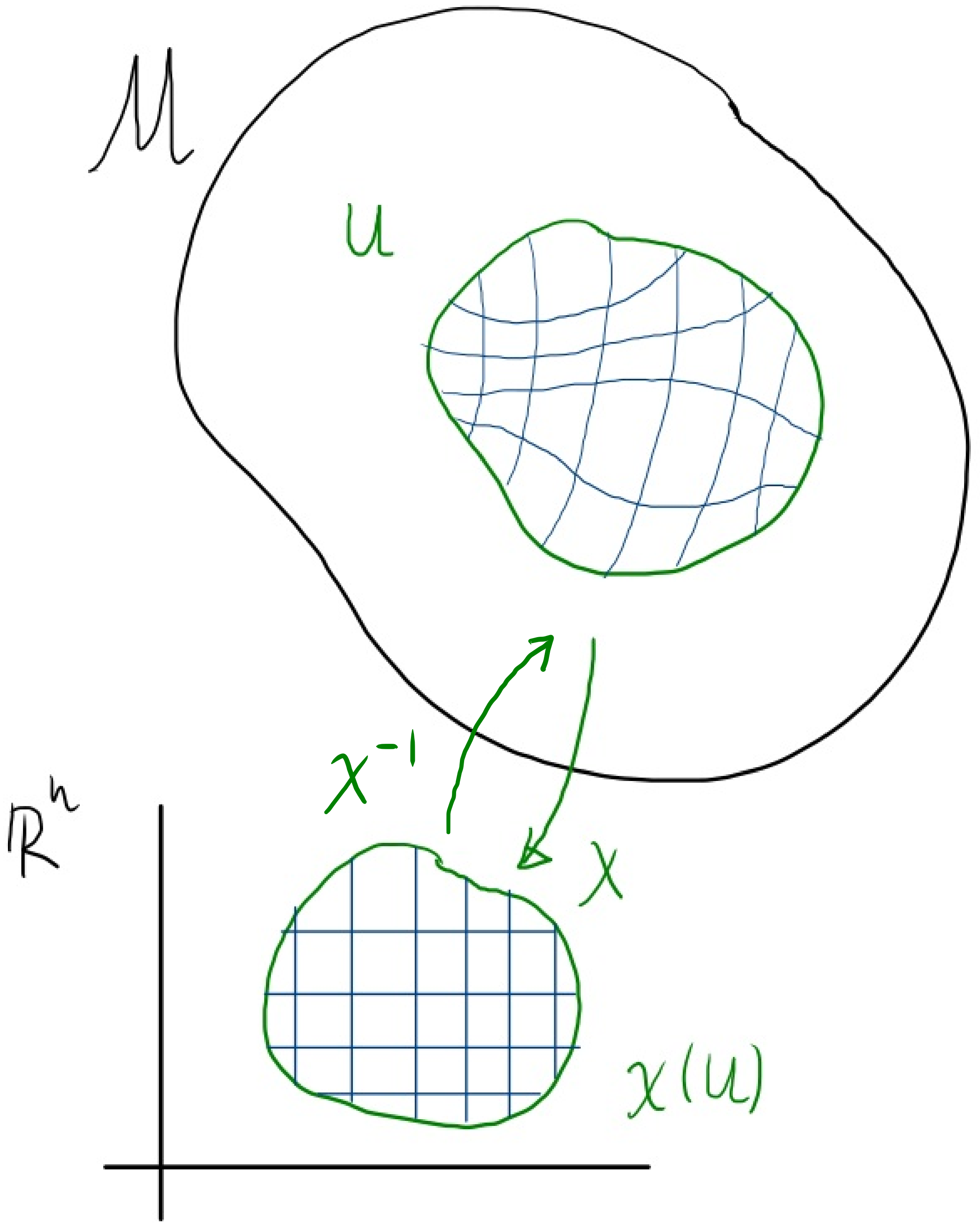
$\rightsquigarrow$ try to bring things to local scales to understand

Differentiable Manifolds

* look locally like $\mathbb{R}^n$

  ! everywhere !

need topology!



Differentiable Manifolds

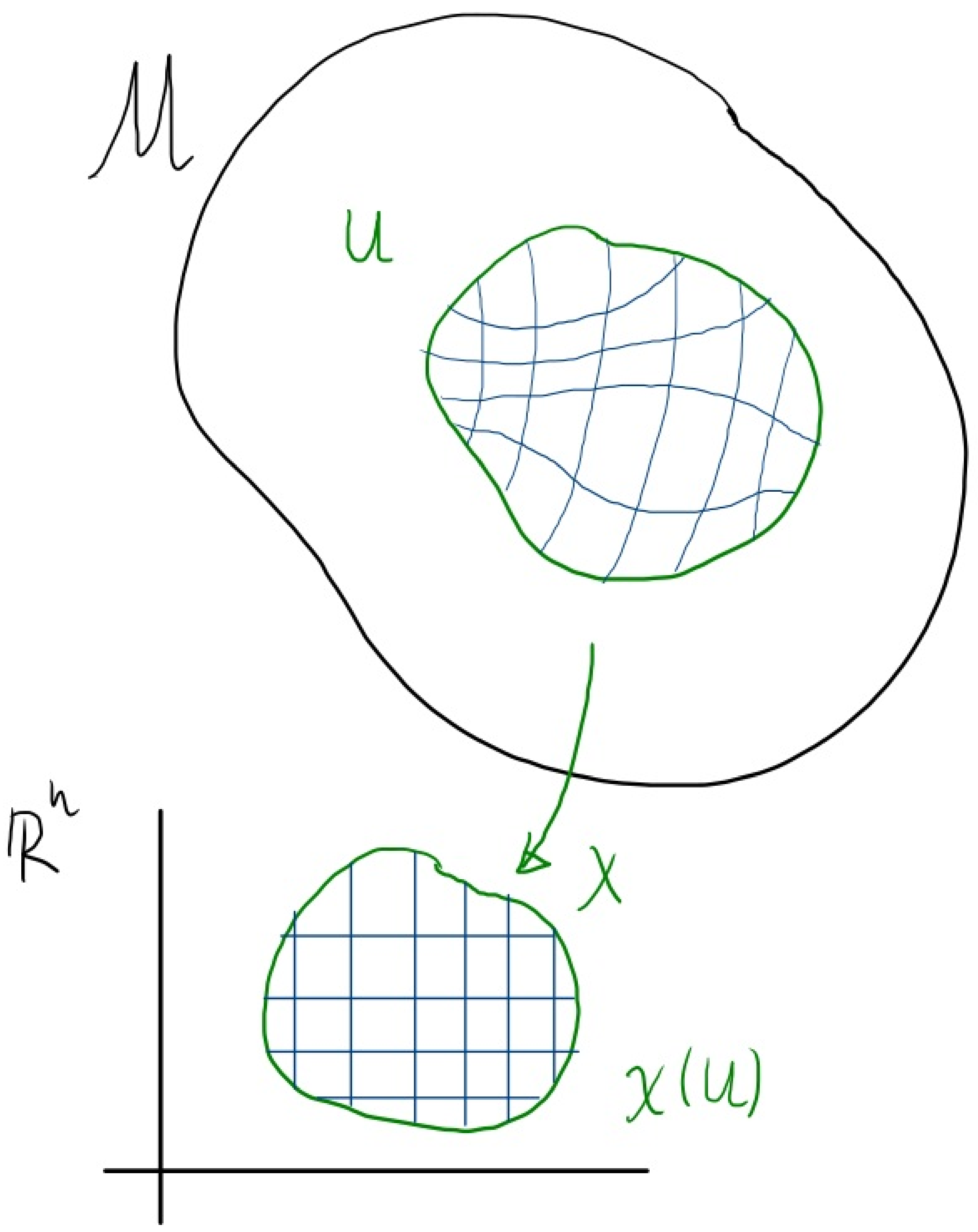
* look locally like $\mathbb{R}^n$

  ! everywhere !

Perfect! Promising to work with equivalence principle!

n: defines dimensionality

defines coordinate system

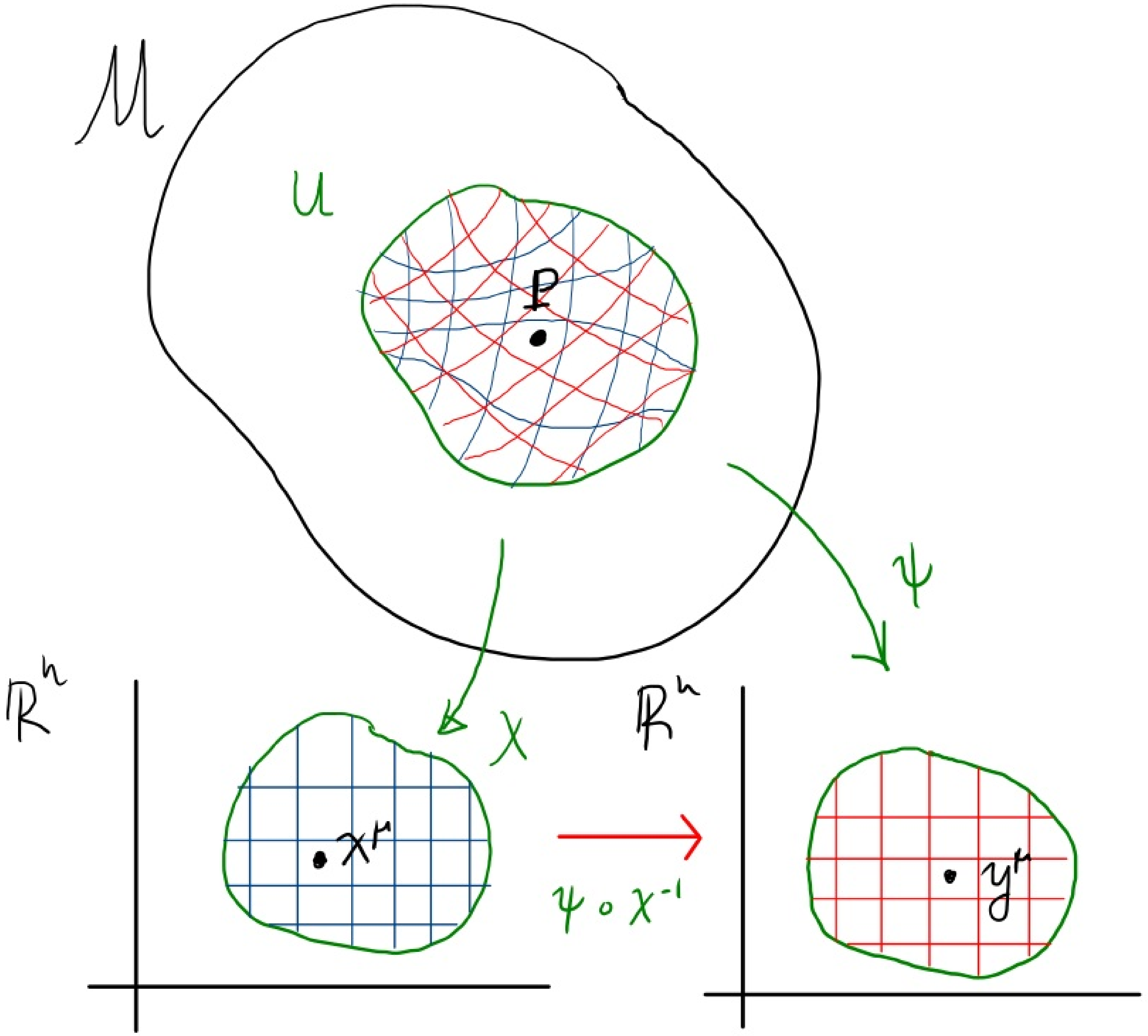


Differentiable Manifolds

* look locally like $\mathbb{R}^n$

* coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$



Differentiable Manifolds

* look locally like $\mathbb{R}^n$
* coordinate transformations

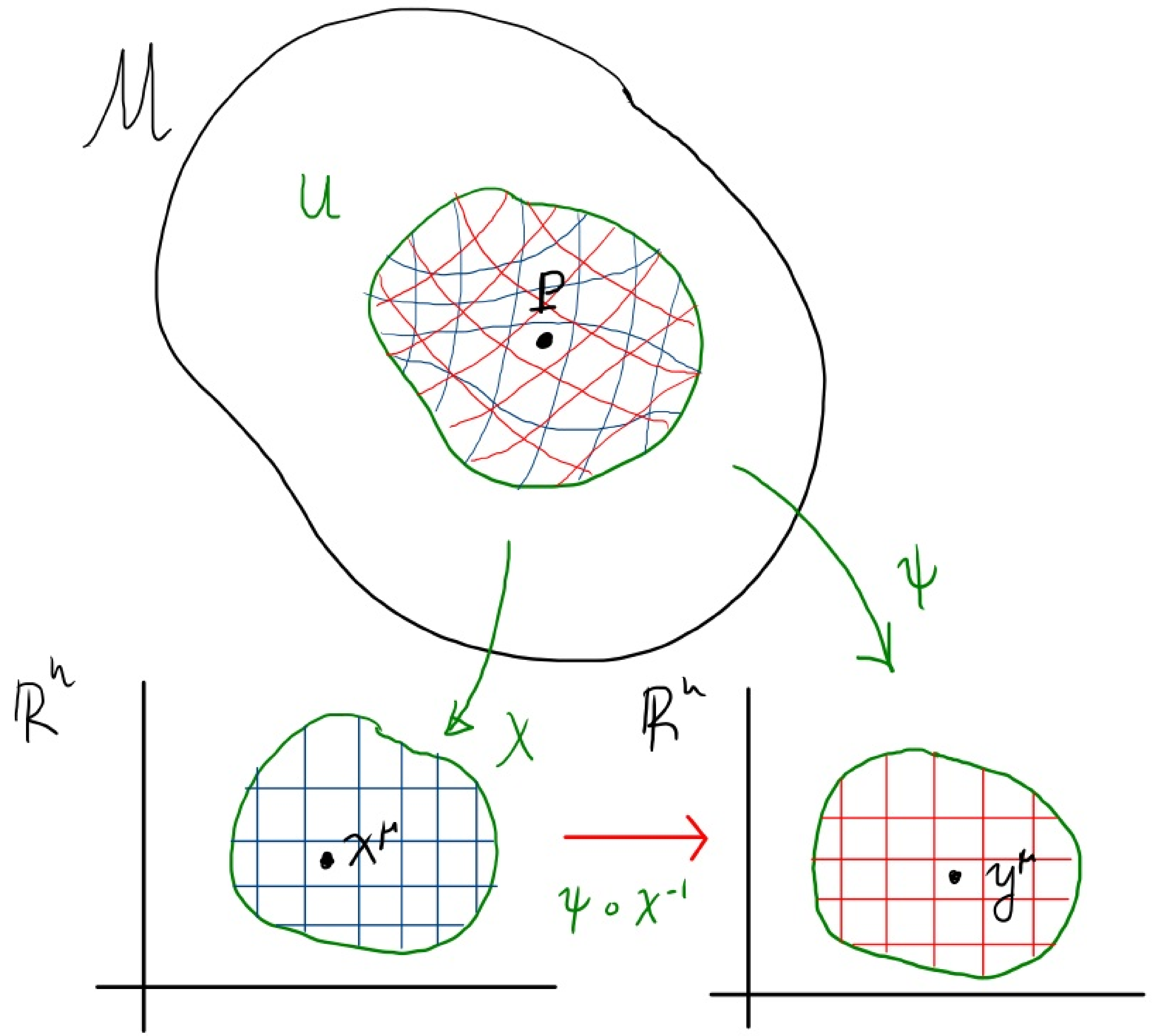
$$y^\mu = y^\mu(x^\nu)$$

"observer" ψ ↵ ↳ "observer" χ

!Perfect! • Communication between observers

• observers agree on geometry ("coordinate invariance")

"absolute" → ↑ "relative"



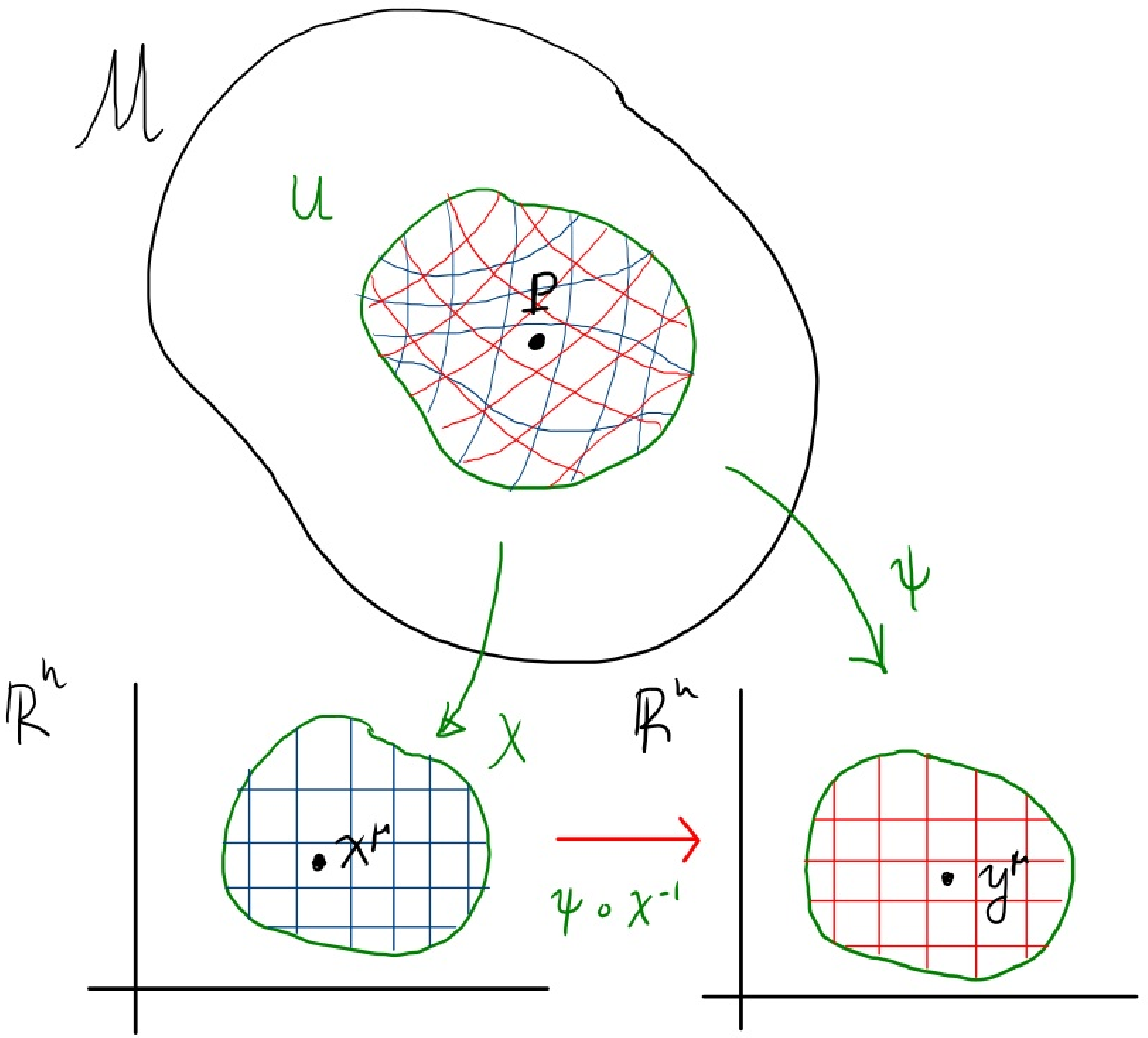
Differentiable Manifolds

* look locally like $\mathbb{R}^n$
* coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$

* differential structure

$$\frac{\partial y^\mu}{\partial x^\nu}$$



Differentiable Manifolds

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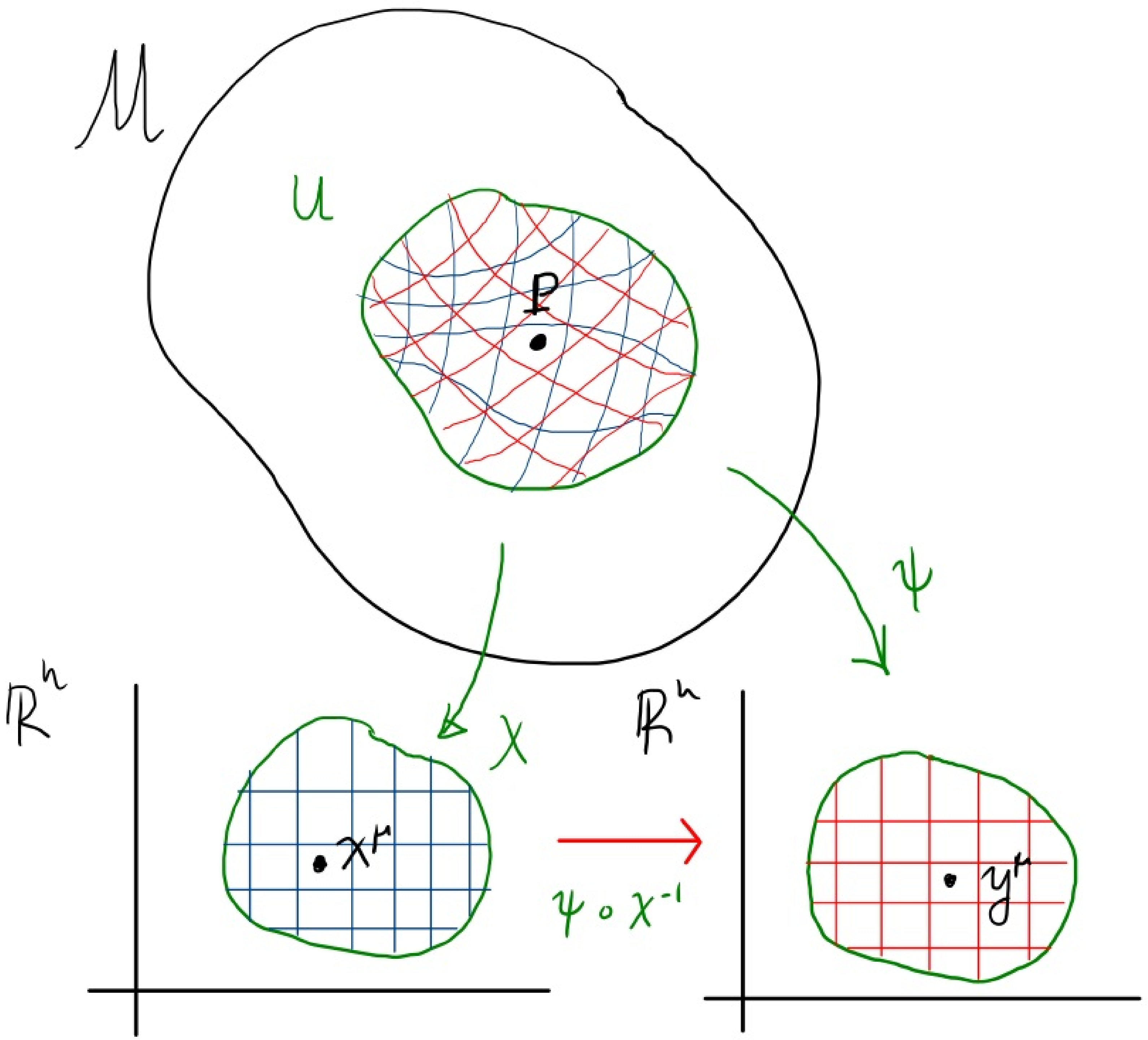


! Perfect ! Derivatives, integrals - all geometric !

Differentiable Manifolds

* look locally like $\mathbb{R}^n$

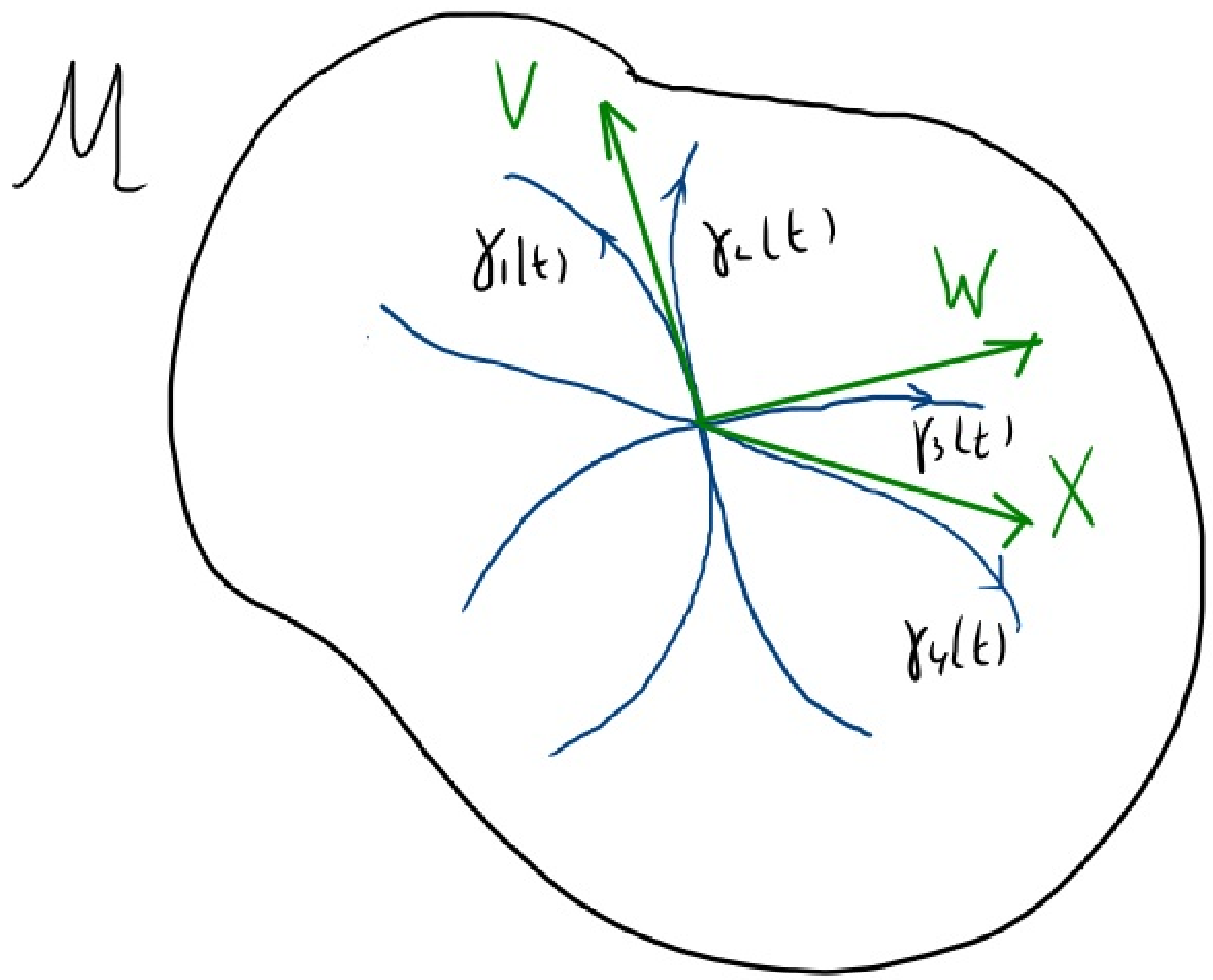
* coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$

* differential structure

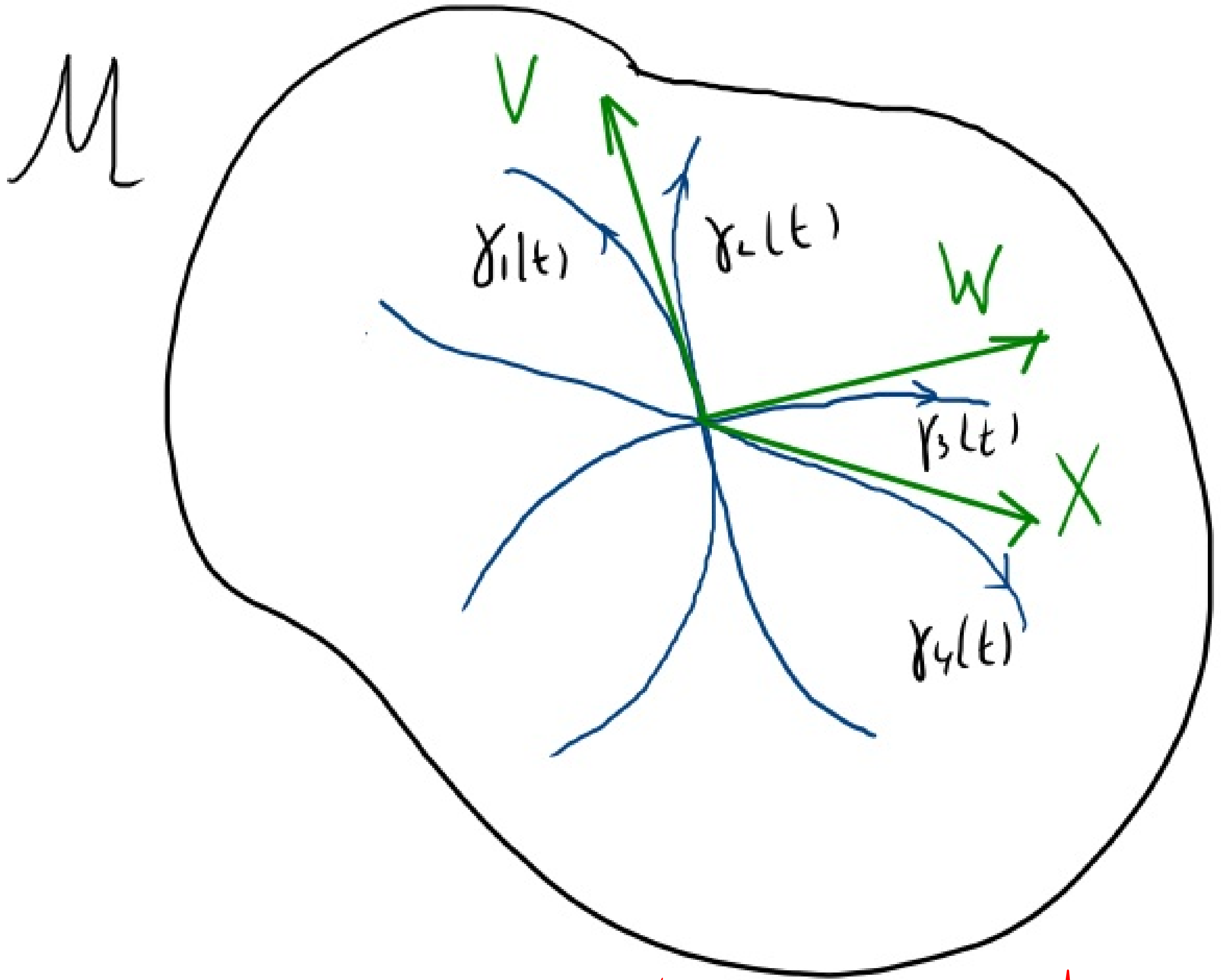
$$\frac{\partial y^\mu}{\partial x^\nu}$$

* fields: curves $\rightarrow$ vectors $\rightarrow$ tensor fields



Differentiable Manifolds

* look locally like $\mathbb{R}^n$



* coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$

* differential structure

$$\frac{\partial y^\mu}{\partial x^\nu}$$

geometric objects
"absolute"

* fields : curves $\rightarrow$ vectors $\rightarrow$ tensor fields

!Perfect theoretial physicist's playground!

Differentiable Manifolds

The map:

Topological spaces, locally $\mathbb{R}^n$

Differentiable Manifolds

The map:

Topological spaces, locally $\mathbb{R}^n$

$\downarrow$

Local coordinate systems

Differentiable Manifolds

The map:

Topological spaces, locally $\mathbb{R}^n$

$\downarrow$

Local coordinate systems

$\downarrow$

Differentiable coordinate xfms

Differentiable Manifolds

The map:

Topological spaces, locally $\mathbb{R}^n$

$\downarrow$

Local coordinate systems

$\downarrow$

Differentiable coordinate xfms

$\downarrow$

Differential Structure

Differentiable Manifolds

Differential Structure

$$\downarrow$$

Fields

Differentiable Manifolds

Differential Structure

$\downarrow$

Fields

$\swarrow$

Vector

(rate of change
of fns on curves)

Differentiable Manifolds

Differential Structure

↓

Fields

↙

vector → 1-form

(rate of change of fns on curves)

Differentiable Manifolds

Differential Structure

↓

Fields

↙

vector $\longrightarrow$ 1-form

(rate of change of fns on curves)

↓

tensor

Differentiable Manifolds

Differential Structure

$\downarrow$

Fields

$\swarrow$

vector $\longrightarrow$ 1-form $\longrightarrow$ n-form

(rate of change of fns on curves)

vector $\downarrow$ Lie derivatives

1-form $\downarrow$ tensor

n-form $\swarrow$ exterior derivative

n-form $\searrow$ integration

Differentiable Manifolds

Differential Structure

↓

Fields

vector → 1-form → n-form

(rate of change of fns on curves)

vector → Lie derivatives

1-form → tensor

n-form → exterior derivative, integration

! all with diff manifold structure only !

Differentiable Manifolds

Additional Structure:

* Affine connection
  - covariant derivative
  - parallel transport
  - geodesics
  - curvature

Differentiable Manifolds

Additional Structure:

* Affine connection
  - covariant derivative
  - parallel transport
  - geodesics
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compare objects @ differet points (the "absolute" way...)

straightest curves

initially parallel geodesics deviate

Differentiable Manifolds

Additional Structure:

* Affine connection
  - covariant derivative
  - parallel transport
  - geodesics
  - curvature

} compare objects @ differet points (the "absolute" way...)

straightest curves

initially parallel geodesics deviate

⇝ Careful: No geometry yet!

⇝ Gauge theories have it!

Differentiable Manifolds

Additional Structure:

* Affine connection
  - covariant derivative
  - parallel transport
  - geodesics
  - curvature
* Metric
  - distances, angles, inner product
  - causal structure (for GR metrics)
  - chooses unique "physical" affine connection + curvature

Differentiable Manifolds

Additional Structure:

* Affine connection
  - covariant derivative
  - parallel transport
  - geodesics
  - curvature

now we can do GR,
have "spacetime"

* Metric
  - distances, angles, inner product
  - causal structure (for GR metrics)
  - chooses unique "physical" affine connection + curvature

Differentiable Manifolds

Additional Structure:

* Affine connection
  - covariant derivative
  - parallel transport
  - geodesics
  - curvature

infinite possibilities
Dynamics choose!
now we can do GR, have "spacetime"

* Metric
  - distances, angles, inner product
  - causal structure (for GR metrics)
  - chooses unique "physical" affine connection + curvature

Differentiable Manifolds

Additional Structure:

* Affine connection
  - covariant derivative
  - parallel transport
  - geodesics
  - curvature

change of metric $\overset{\text{may}}{\Longrightarrow}$ change of curvature

infinite possibilities

Dynamics choose!

now we can do GR, have "spacetime"

* Metric
  - distances, angles, inner product
  - causal structure (for GR metrics)
  - chooses unique "physical" affine connection + curvature

Examples of Manifolds

- $\mathbb{R}^n$: $\mathbb{R}$ (line), $\mathbb{R}^2$ (plane), $\mathbb{R}^3$ (space), ...
- S^n : S^0 (2 points), S^1 (circle), S^2 (sphere), ...
- T^n : T^2 (torus), ...
- Lie groups: rotations, Lorentz xfms
- $M = M_1 \times M_2$ $\quad P = (P_1, P_2)$ $\quad P_1 \in M_1, P_2 \in M_2$

Examples of Manifolds

- $\mathbb{R}^n$: $\mathbb{R}$ (line), $\mathbb{R}^2$ (plane), $\mathbb{R}^3$ (space), ...
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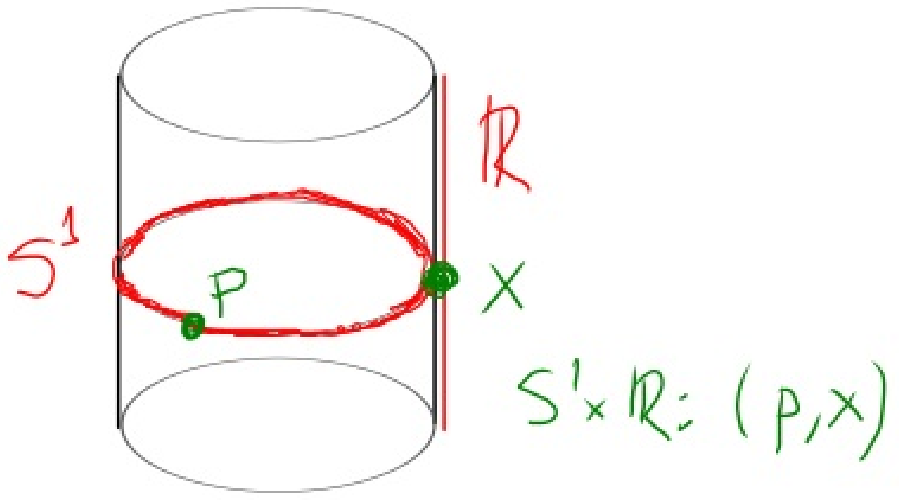
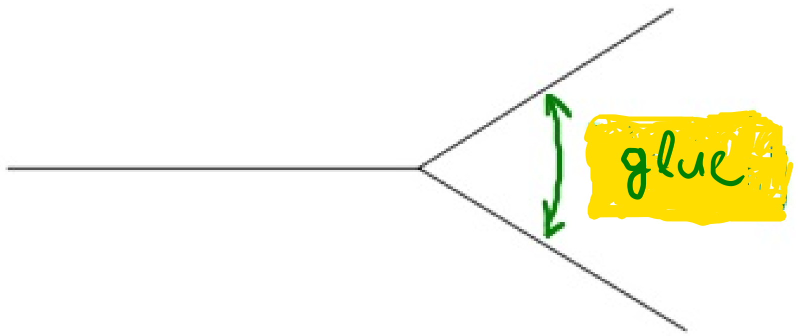


$S^1 \times \mathbb{R}$ (cylinder)

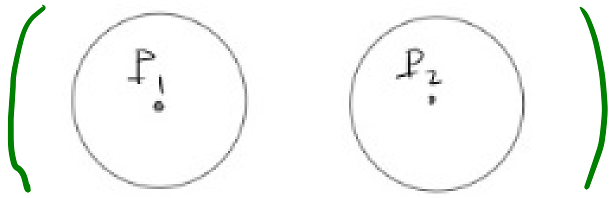
$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

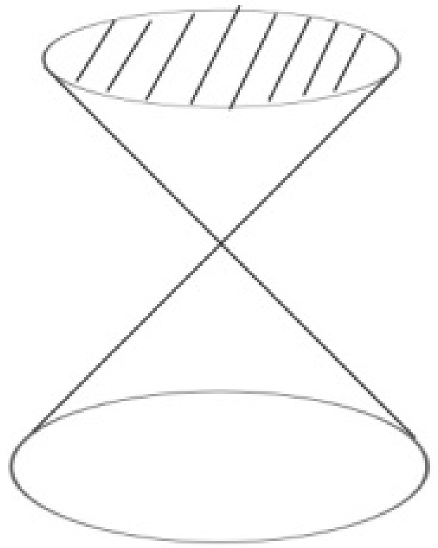
$T^2 = S^1 \times S^1$

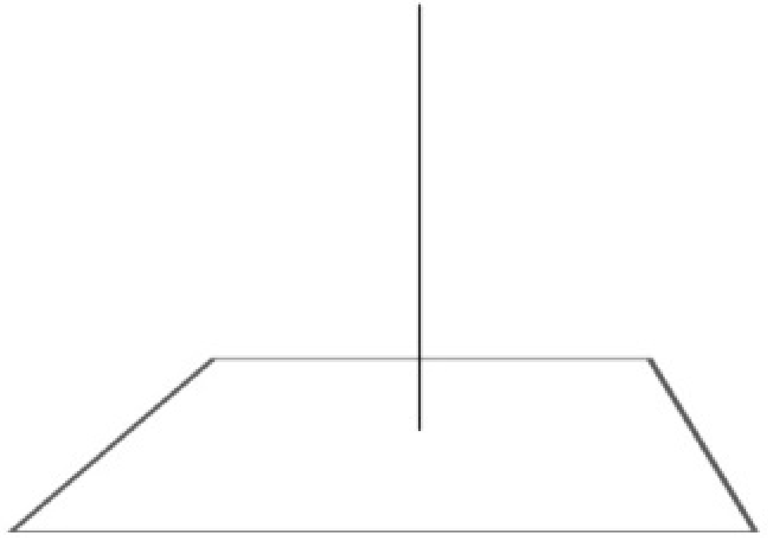
Examples of non-Manifolds



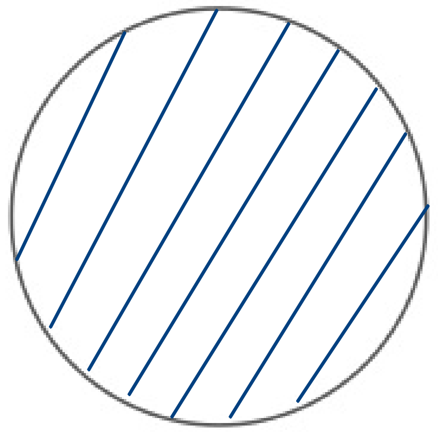
non Hausdorff





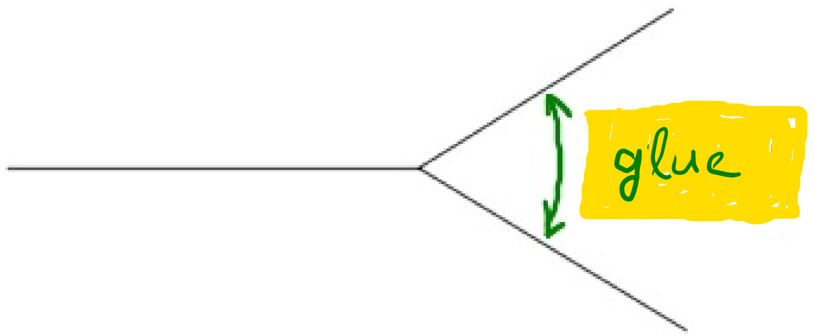


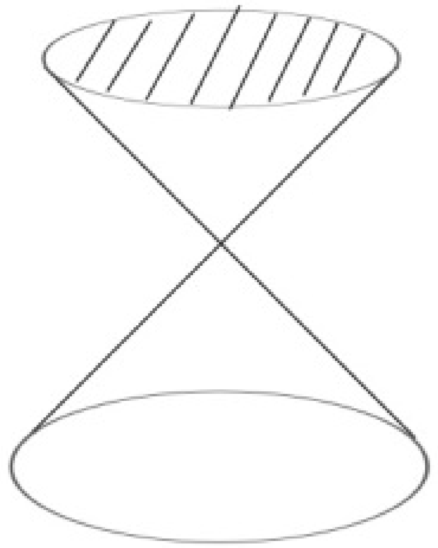
not locally everywhere $\cong \mathbb{R}^n$

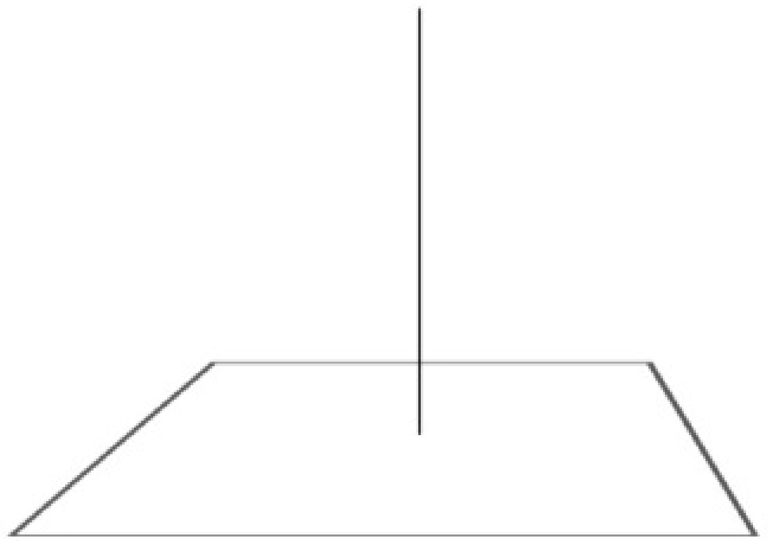


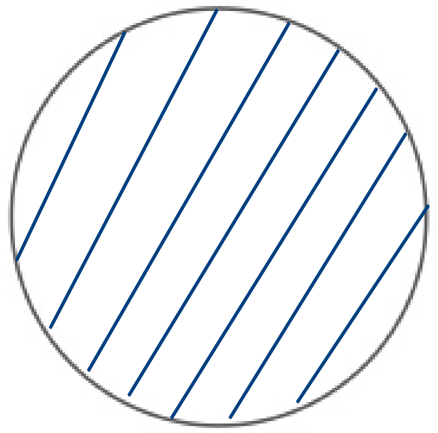
Manifolds with boundary

Examples of non-Manifolds

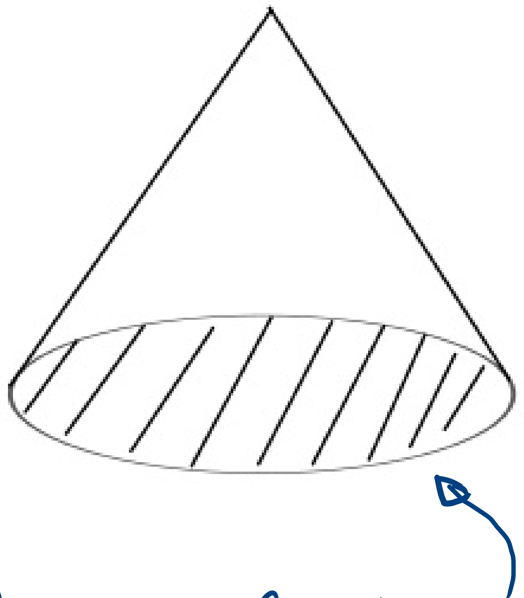








Quiz: is it a manifold?

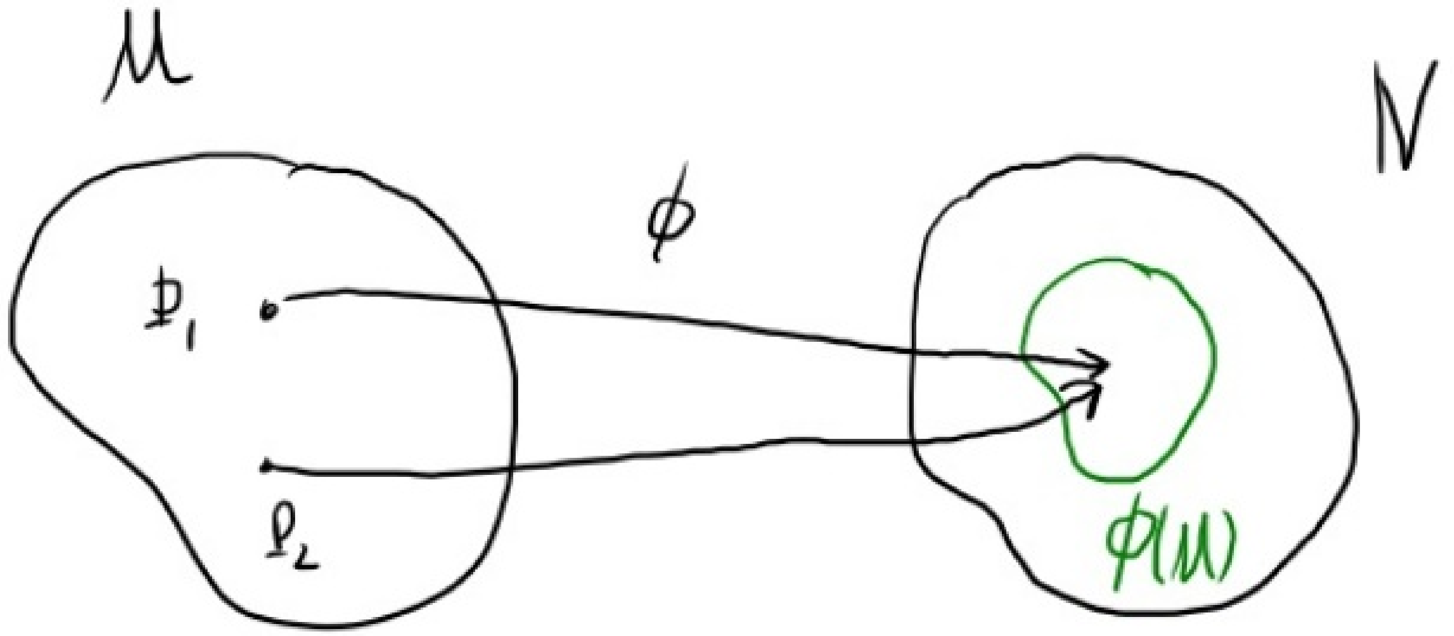


(extends infinitely)

(tip included)

ended on 230220

Maps



- $\mathcal{M}$: domain
- $\mathcal{N}$: codomain
- $\phi(\mathcal{M})$: range or image of $\mathcal{M}$

– onto: $\phi(\mathcal{M}) = \mathcal{N}$ (surjective)

– 1-1 : $p_1 \neq p_2 \Rightarrow \phi(p_1) \neq \phi(p_2)$ (injective)

– invertible: 1-1 and onto (bijective)

$\phi^{-1} : \mathcal{N} \to \mathcal{M}$ a map

Topology


topology
[top-ol-uh-jee]
noun
1. The study of adding tasty toppings.
CHOOSE FROM 7 BREAKFAST SANDWICHES
buy
any Coke product
PRO-GO PROTEIN PACK
16g PROTEIN

Topology

* $\mathcal{U}$ a topological space:
covered by open sets $\{U_\alpha\}$ s.t
$U_\alpha \cap U_\beta$ and $\bigcup_\alpha U_\alpha$ are open sets
↳ possibly infinite #

$\emptyset$ and $\mathcal{U}$ are open

Topology

* $\mathcal{M}$ a topological space:
  covered by open sets $\{U_\alpha\}$ s.t.
  $U_\alpha \cap U_\beta$ and $\bigcup_\alpha U_\alpha$ are open sets
* $\{U_\alpha\}$ define a topology on $\mathcal{M}$

there are many and inequivalent:
(some also trivial, like $\{\phi, \mathcal{M}\}$)

Topology

* $\mathcal{M}$ a topological space:
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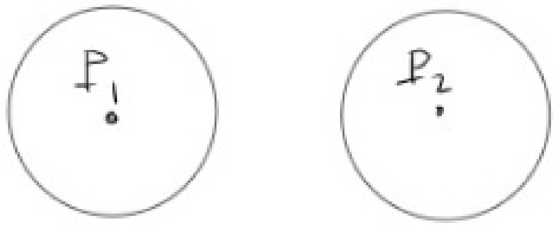
* $\mathbb{R}^n$ is assumed with the topology
  generated by open balls

$$B_{x_0}(r) \subset \mathbb{R}^n$$

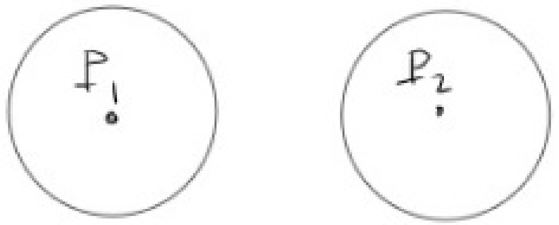
$$B_{x_0}(r) = \{x \mid x \in \mathbb{R}^n,\ |x - x_0| < r\}$$

Topology

* M is Hausdorff: 

$\forall\ p_1 \neq p_2$, $\exists$ open U_1, U_2 s.t. $U_1 \cap U_2 = \emptyset$

Topology

* M is Hausdorff: 

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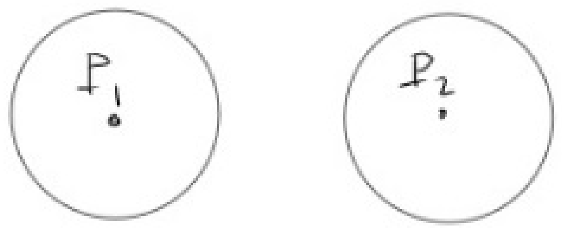
* $\mathcal{M}$ is Hausdorff: 

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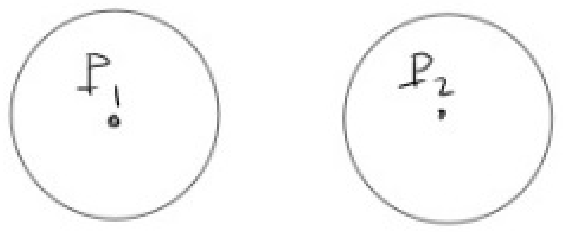
* $\mathcal{M}$ is Hausdorff: 

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Topology

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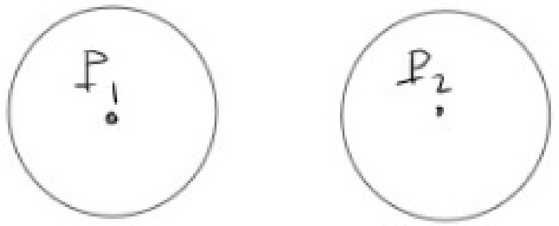

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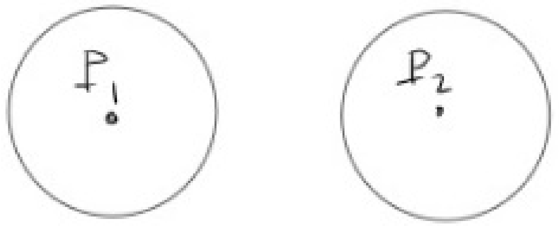
* $\mathcal{M}$ is Hausdorff: 

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* ∂W is the boundary of W if $\partial W = \overline{W} \setminus W^\circ$

Topology

* W is compact if $\forall \{U_\alpha\}$ open covering of W, $\exists$ finite $\{U_{\alpha'}\} \subset \{U_\alpha\}$, also open covering of W

(theorem)

- In $\mathbb{R}^n$ with usual topology it is easy:

$W \subseteq \mathbb{R}^n$ compact $\Leftrightarrow$ W is closed + bounded

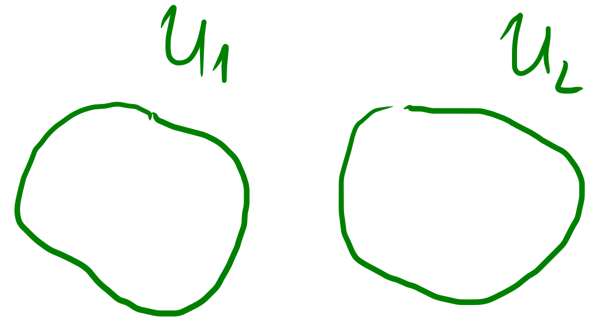
e.g. $\overline{B}_{x_0}(r) = \{x \mid |x - x_0| \leq r\}$, $\underbrace{S^n, T^n}_{\text{because closed + bounded in } \mathbb{R}^{n+1}}$

Topology

* W is compact if $\forall \{U_\alpha\}$ open covering of W, $\exists$ finite $\{U_{\alpha'}\} \subset \{U_\alpha\}$, also open covering of W

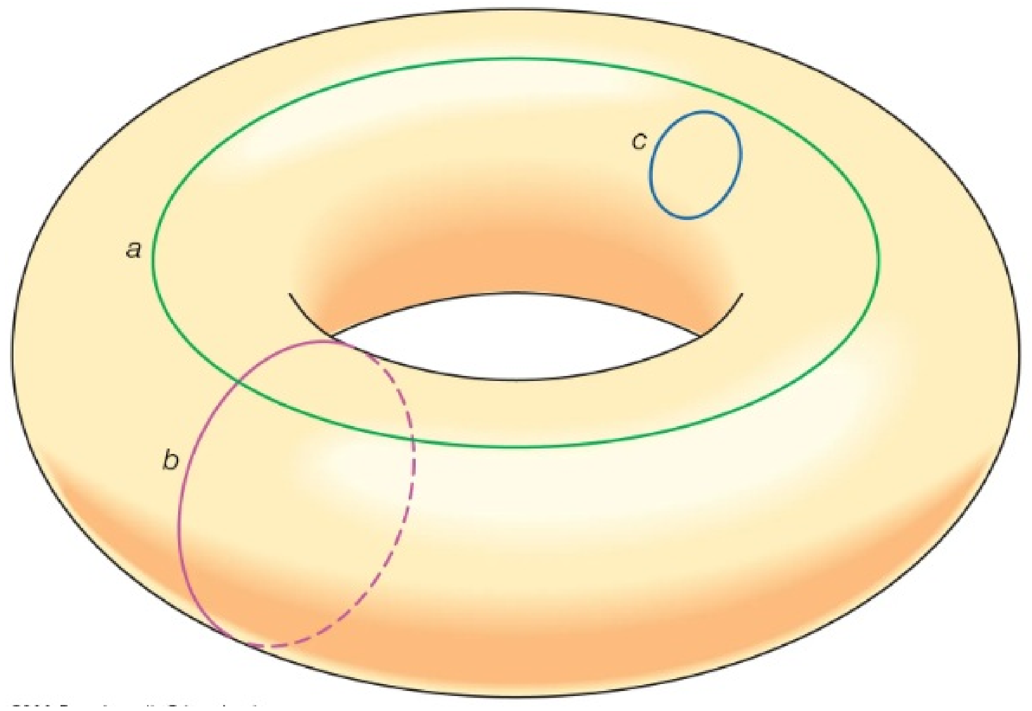
* W connected if $\nexists$ U_1, U_2 s.t. $U_1 \cup U_2 = W$ and $U_1 \cap U_2 = \emptyset$

  " disconnected if not connected



$W = U_1 \cup U_2$
disconnected

Topology



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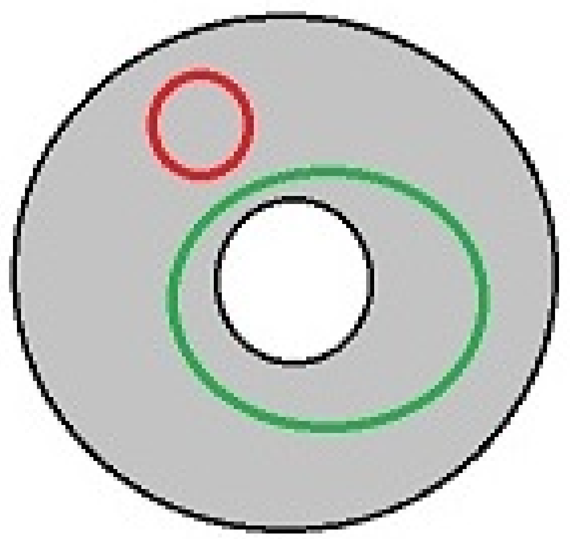
* W connected if $\nexists\, U_1, U_2$ s.t. $U_1 \cup U_2 = W$ and $U_1 \cap U_2 = \emptyset$

  " disconnected if not connected
* W simply connected if all loops contractible to a point

  $\mathbb{R}^2, S^2$: simply connected

  $\mathbb{R}^2 \setminus \{0\}, T^2, S^1$: not simply connected



Topology

* Continuity:

$\phi : M \to N$ continuous iff

$\forall$ open $V \subseteq N \Rightarrow \underbrace{\phi^{-1}(V)}_{\text{inverse image}} \subseteq M$ is open

ϕ^{-1} may not be a map

Topology

* Continuity:

$\phi : M \to N$ continuous iff
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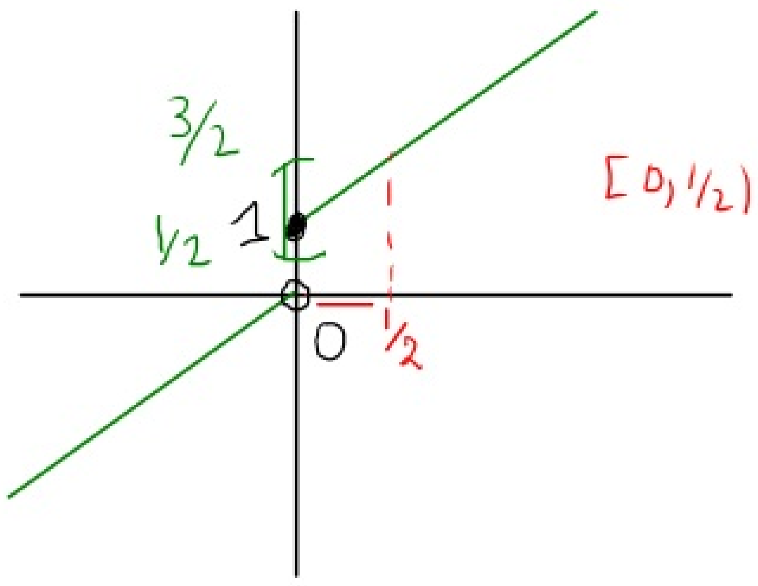
e.g. $f(x) = \begin{cases} x & x < 0 \\ x+1 & x \geq 0 \end{cases}$



$[0, 1/2) \to [1, 3/2) \subset (1/2, 3/2)$

$f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right) = \left[0, \frac{1}{2}\right)$

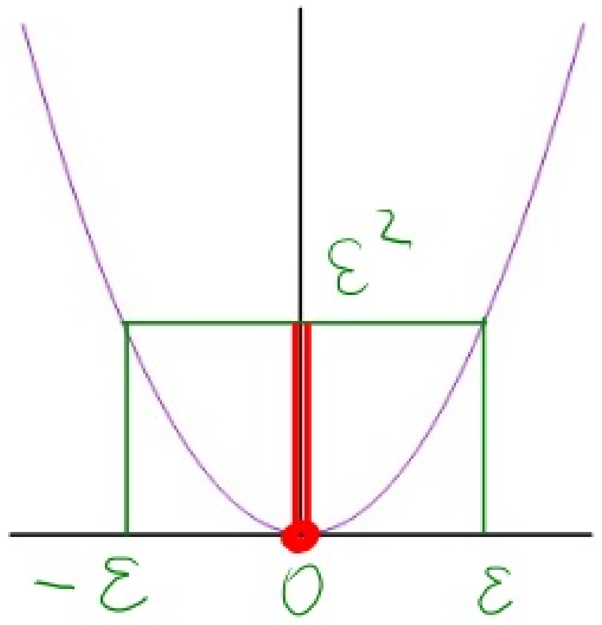
not open

Topology

* Continuity:

$\phi: \mathcal{M} \to \mathcal{N}$ continuous iff

$\forall$ open $V \subseteq \mathcal{N} \Rightarrow \phi^{-1}(V) \subseteq \mathcal{M}$ is open



Continuous ϕ does not necessarily map open to open

$$f(x) = x^2$$

$f\big((-\epsilon, \epsilon)\big) = [0, \epsilon^2)$ not open

Topology

* Continuity:

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* Homeomorphism:

$$\phi \begin{cases} \text{continuous} \\ \text{invertible} \end{cases}$$

ϕ^{-1} continuous

Topology

* M, N are homeomorphic if $\exists$ homeomorphism $\phi: M \to N$

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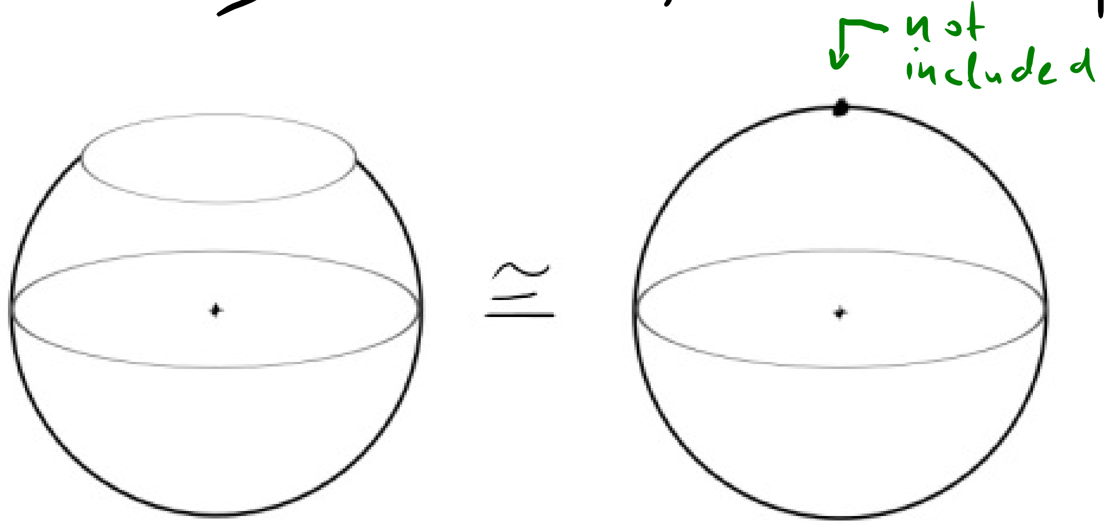

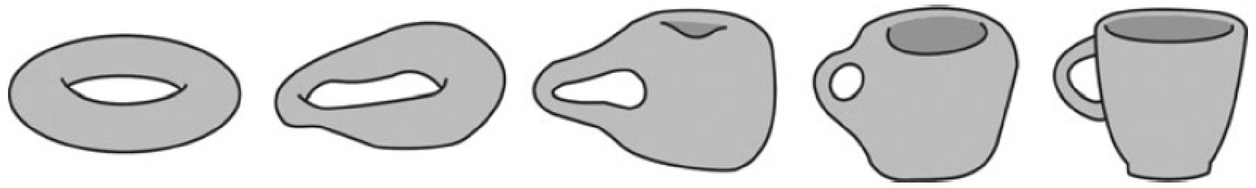

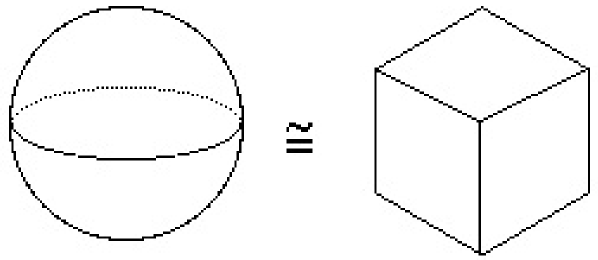

punctures, cuts, ... prohibited!

Differentiable Manifolds

* M is a diff. manifold of $\dim M = n$
  w/ maximal atlas if:
  - M is a Hausdorff topological space

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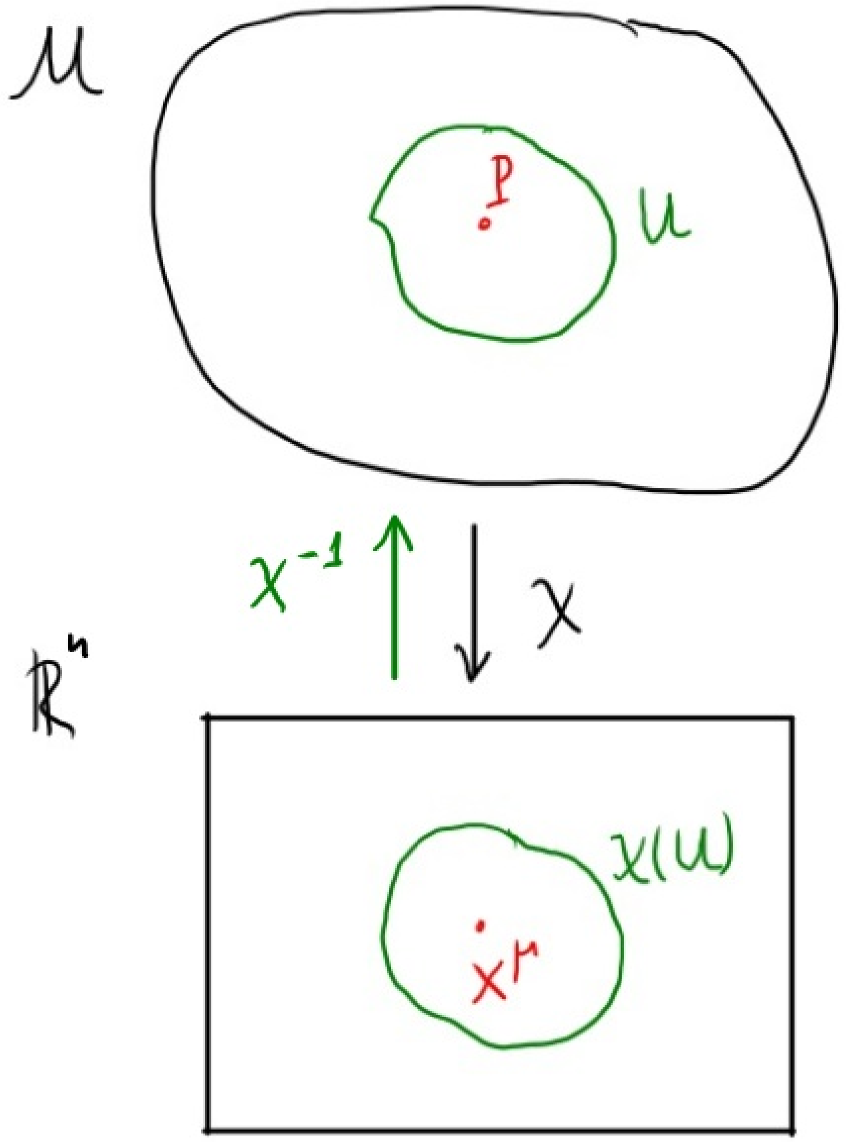
    - $\forall P \in M$ is in a chart (U, x) s.t.
      - U open neighborhood of P
      - $x : U \to x(U) \subseteq \mathbb{R}^n$ a homeo

        $P \to x^\mu(P) \equiv x(P)$



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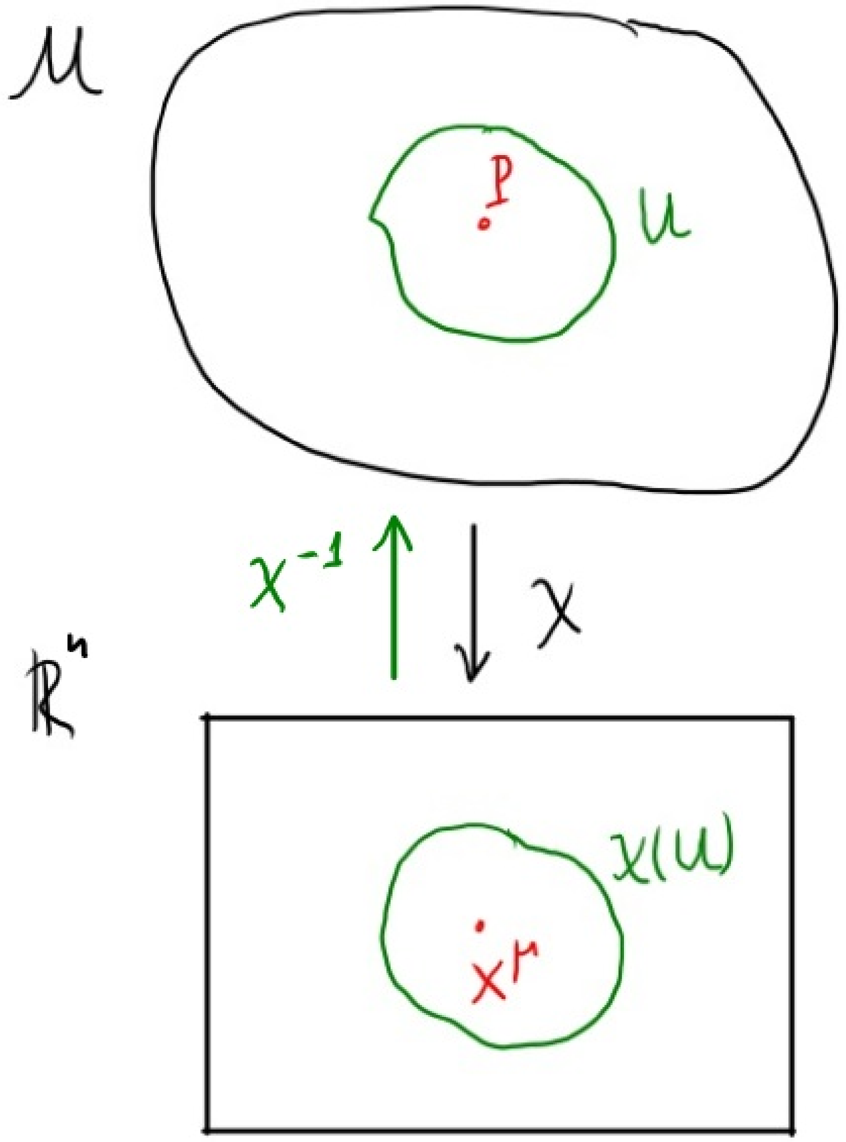
    $\forall P \in \mathcal{M}$ is in a chart $(\mathcal{U}, \chi)$ s.t.

    - $\mathcal{U}$ open neighborhood of P
    - $\chi : \mathcal{U} \to \chi(\mathcal{U}) \subseteq \mathbb{R}^n$ a homeo

      $$P \to x^\mu(P) \equiv \chi(P)$$

+ $x^\mu(P)$ coordinates of P
+ $\mathcal{U} \cong \chi(\mathcal{U}) \subseteq \mathbb{R}$ topologically equivalent



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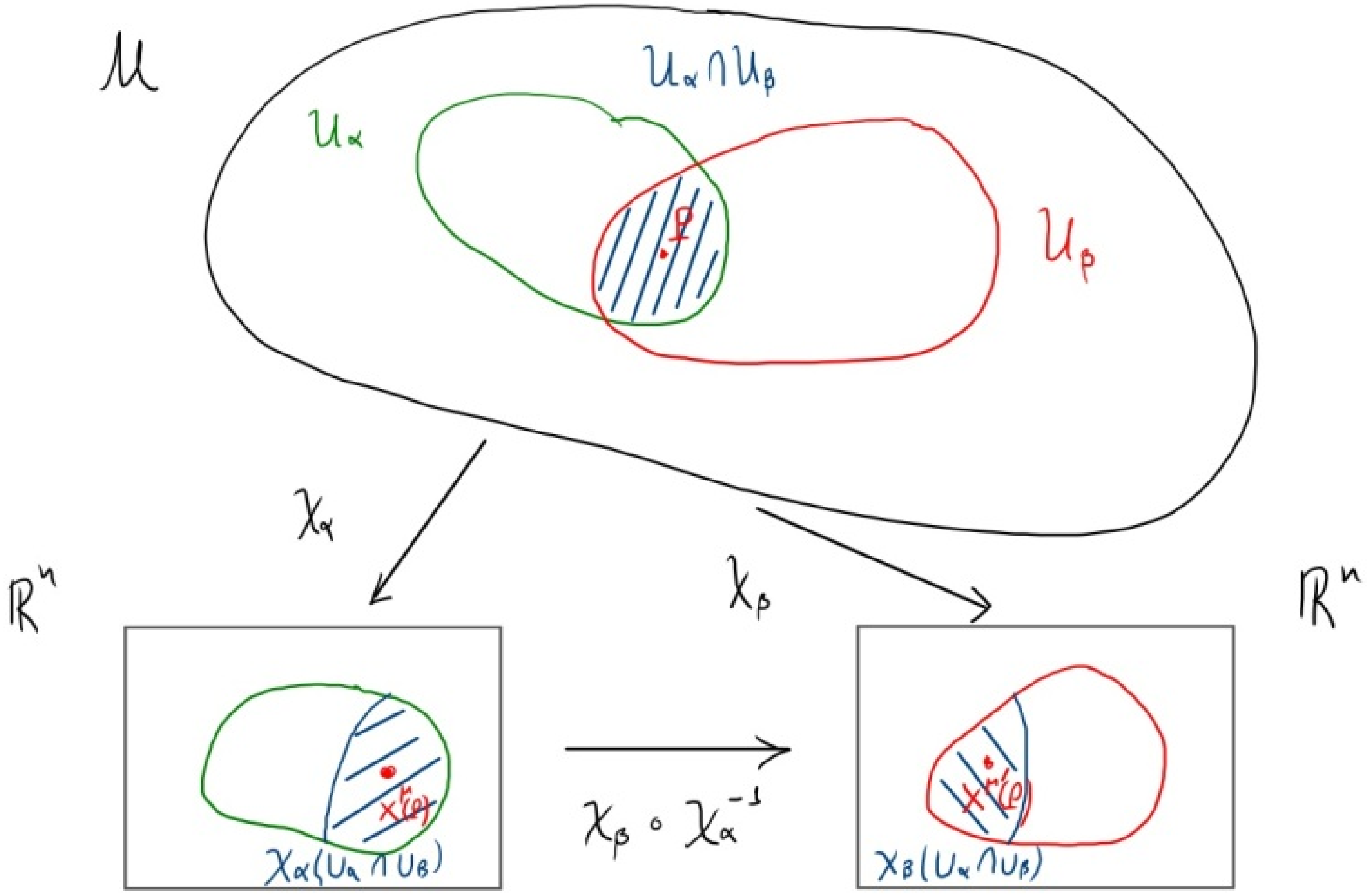
    $\forall\, P \in \mathcal{M}$ is in a chart $(\mathcal{U}, x)$
  - Coordinate xfms are differentiable

$\chi_\beta \circ \chi_\alpha^{-1}$: transition function

$\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$



$$\chi_\beta \circ \chi_a^{-1} : \chi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \longrightarrow \chi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

$$x^\mu \longrightarrow x^{\mu'}(x^\nu) \qquad \text{differentiable}$$

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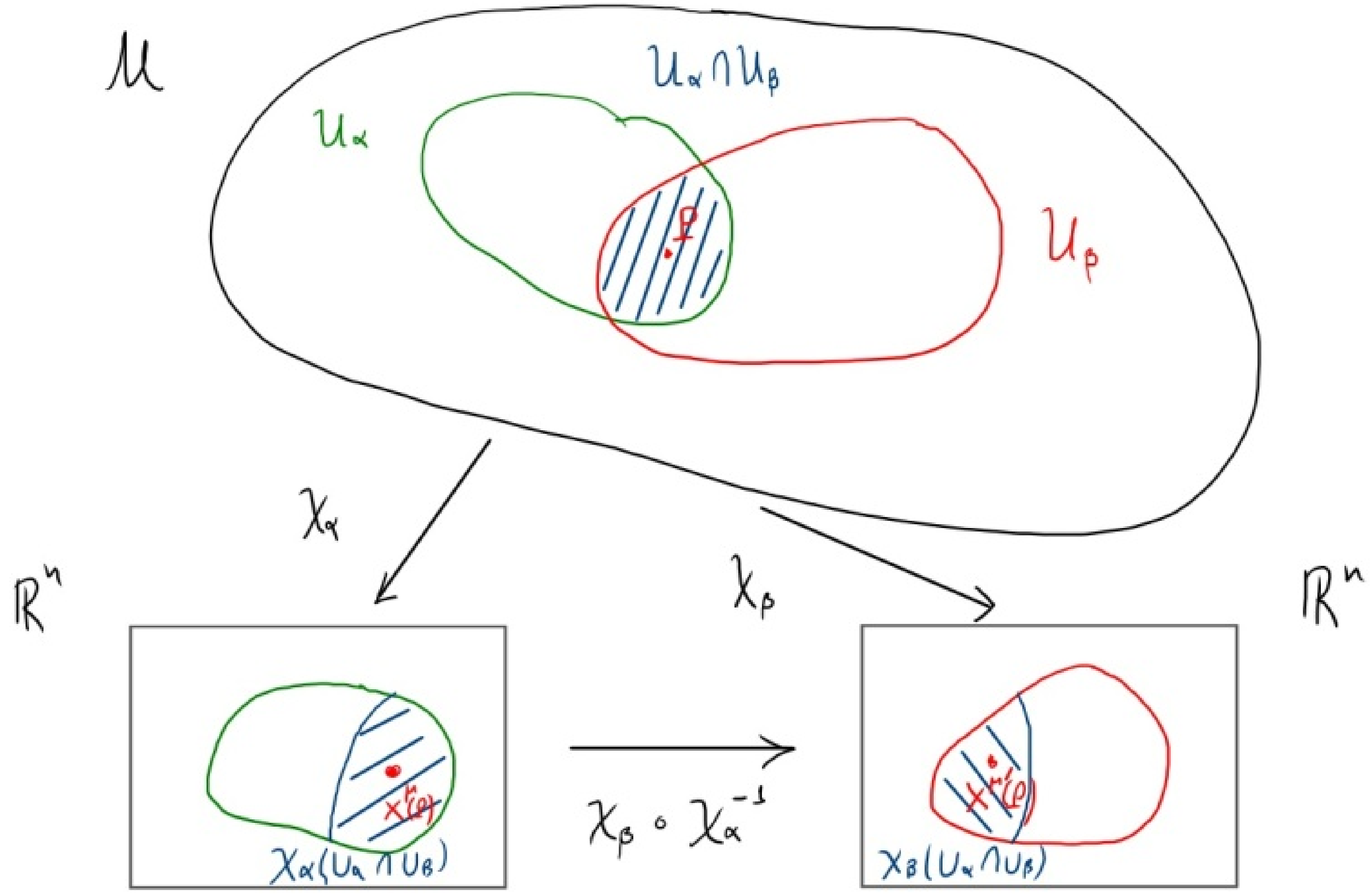

$$\chi_\beta \circ \chi_\alpha^{-1} : \quad x^{\mu'} = x^{\mu'}(x^\nu) \equiv \chi_\beta \circ \chi_\alpha^{-1}(x^\nu)$$

Differentiability: $\dfrac{\partial x^{\mu'}}{\partial x^\mu}$ continuous $\rightarrow C^1$

$\dfrac{\partial^p x^{\mu'}}{\partial x^{\mu_1} \cdots \partial x^{\mu_p}}$ " $\rightarrow C^p$

Analytic $\rightarrow C^\infty$

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  - $\{(U_\alpha, x_\alpha)\}$, $\{U_\alpha\}$ open covering of M is an atlas of M

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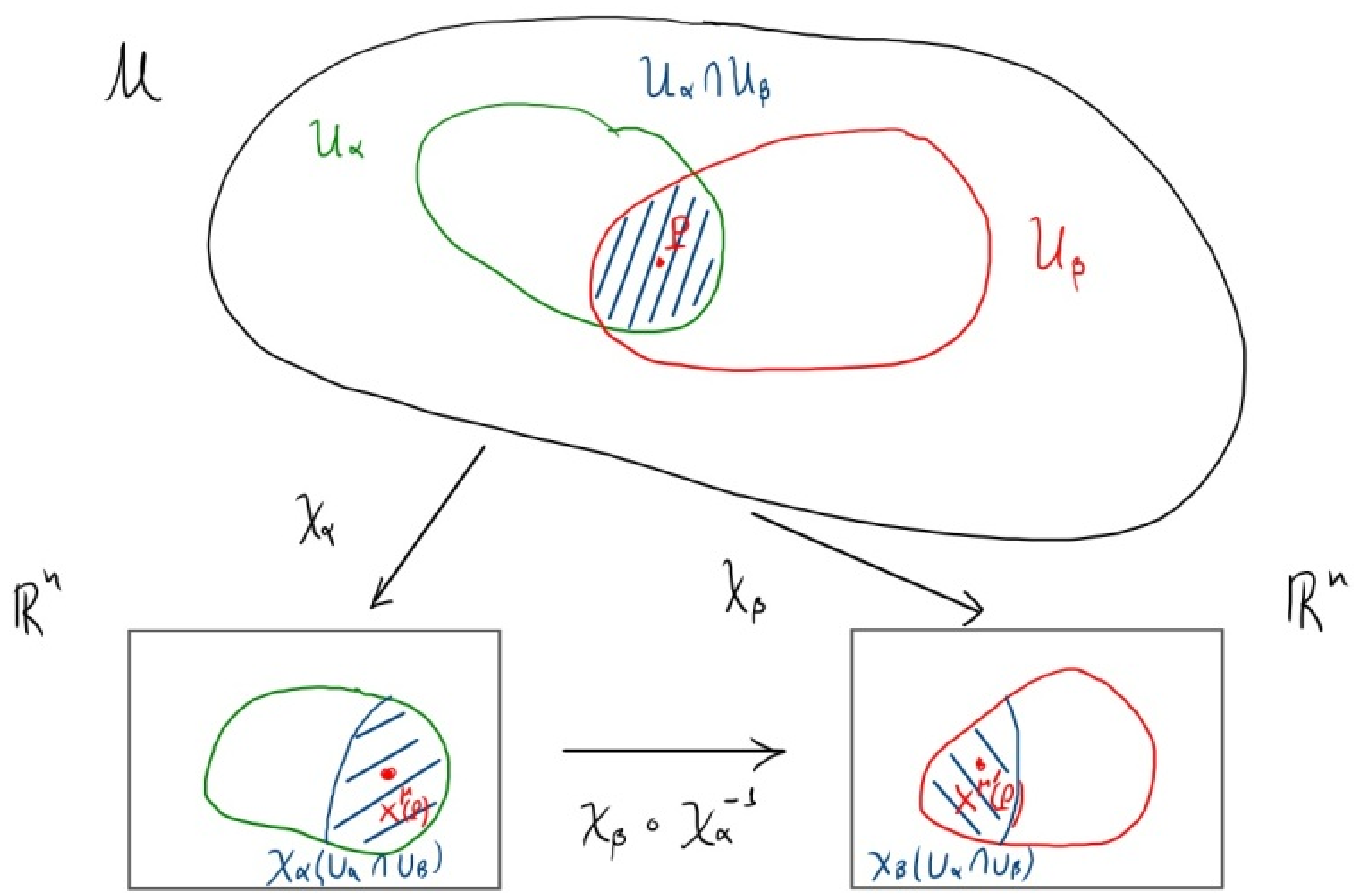

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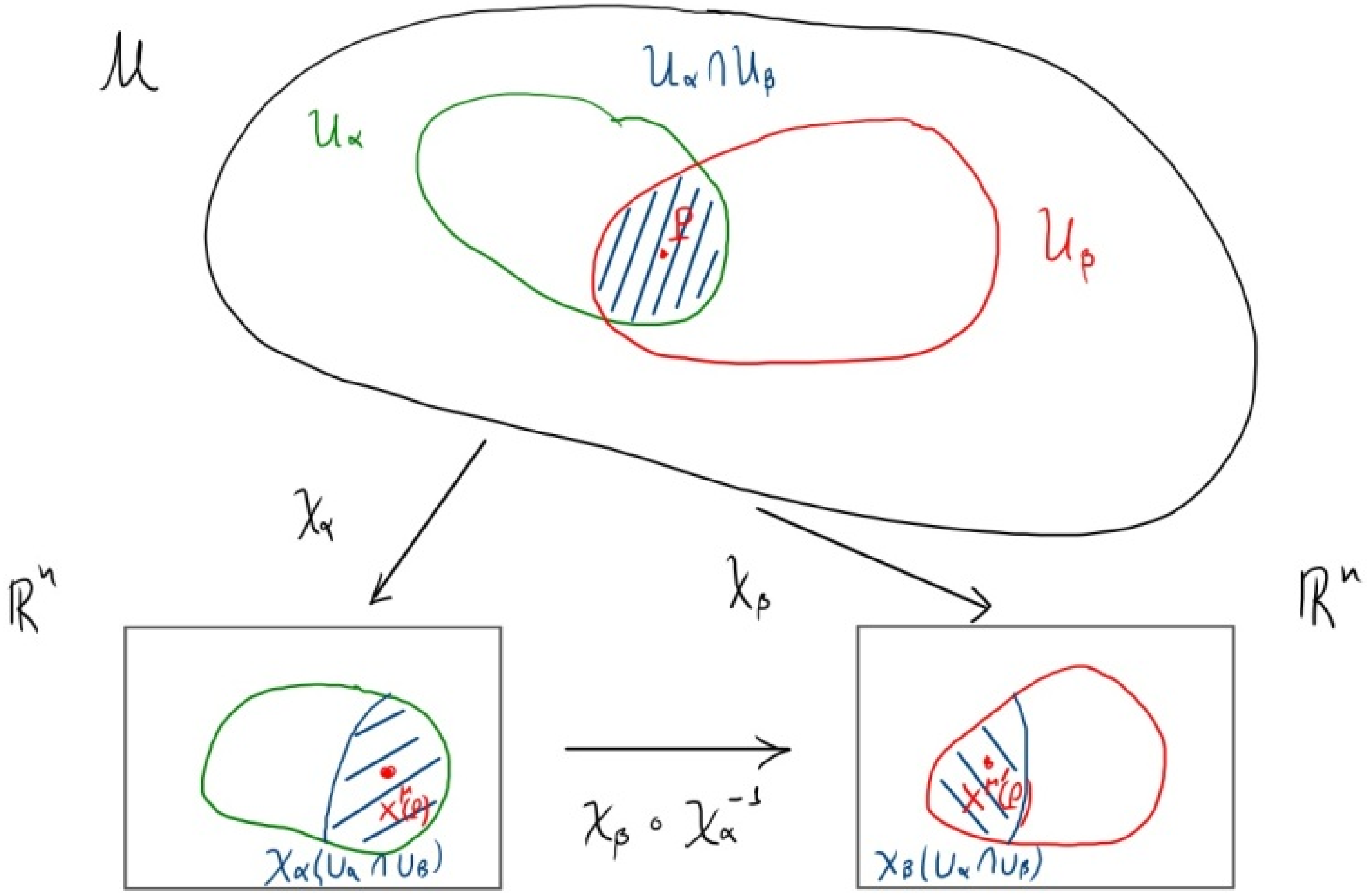
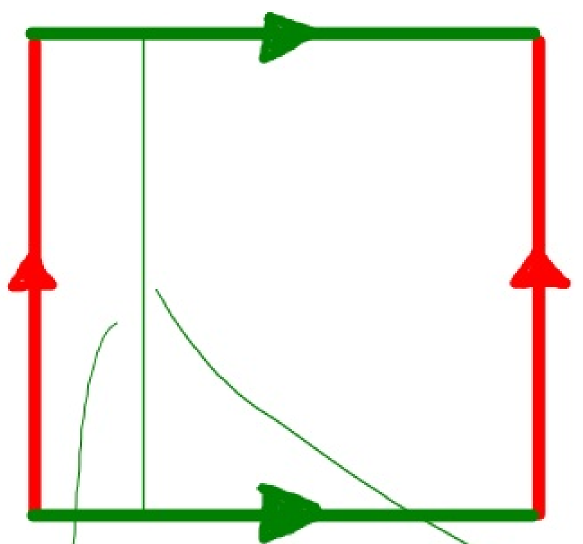
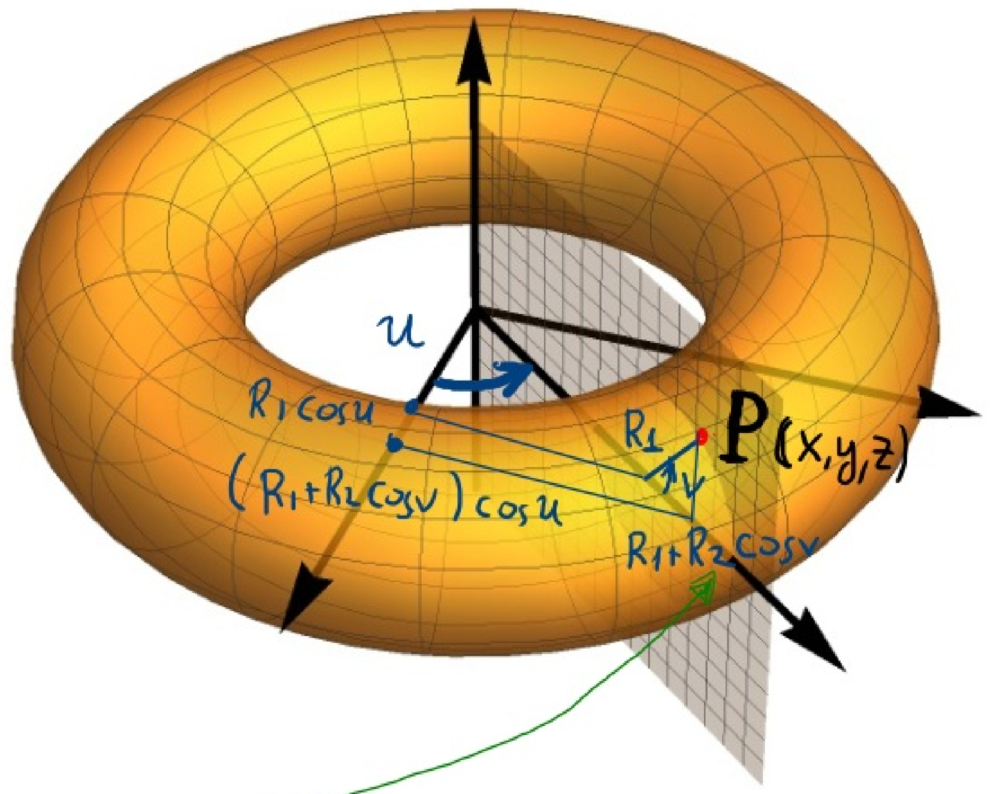


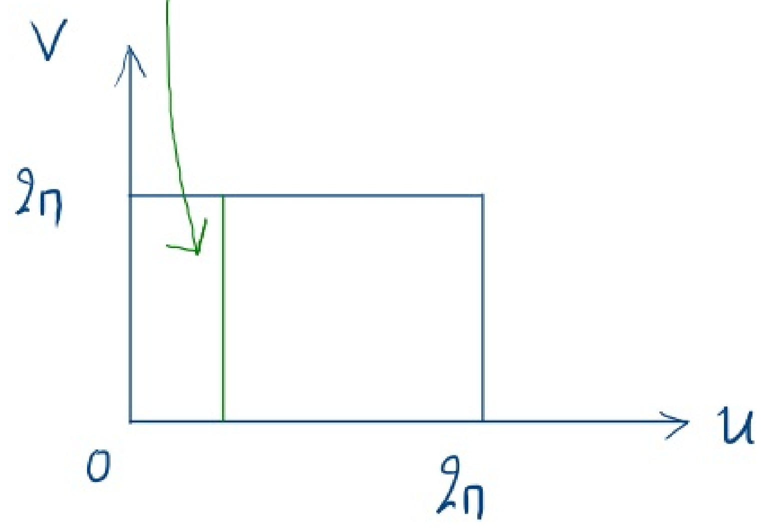
- Maximal atlas: contains all compatible charts

* Many times we define manifolds using embeddings

the manifold







the embedding

$x = (R_1+R_2\cos v)\cos u$

$y = (R_1+R_2\cos v)\sin u$

$z = R_2 \sin v$

a coordinate system

* Many times we define manifolds using embeddings

- manifolds need not embeddings to exist
  can be embedded in many ways

spacetime is not embedded anywhere

(in GR, beware of string theorists!)

* Many times we define manifolds using embeddings

- manifolds need not embeddings to exist
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- embeddings can be useful: Any n-dim manifold embeddable in $\mathbb{R}^{2n}$ (Whitney's embedding theorem)

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  - manifolds need not embeddings to exist
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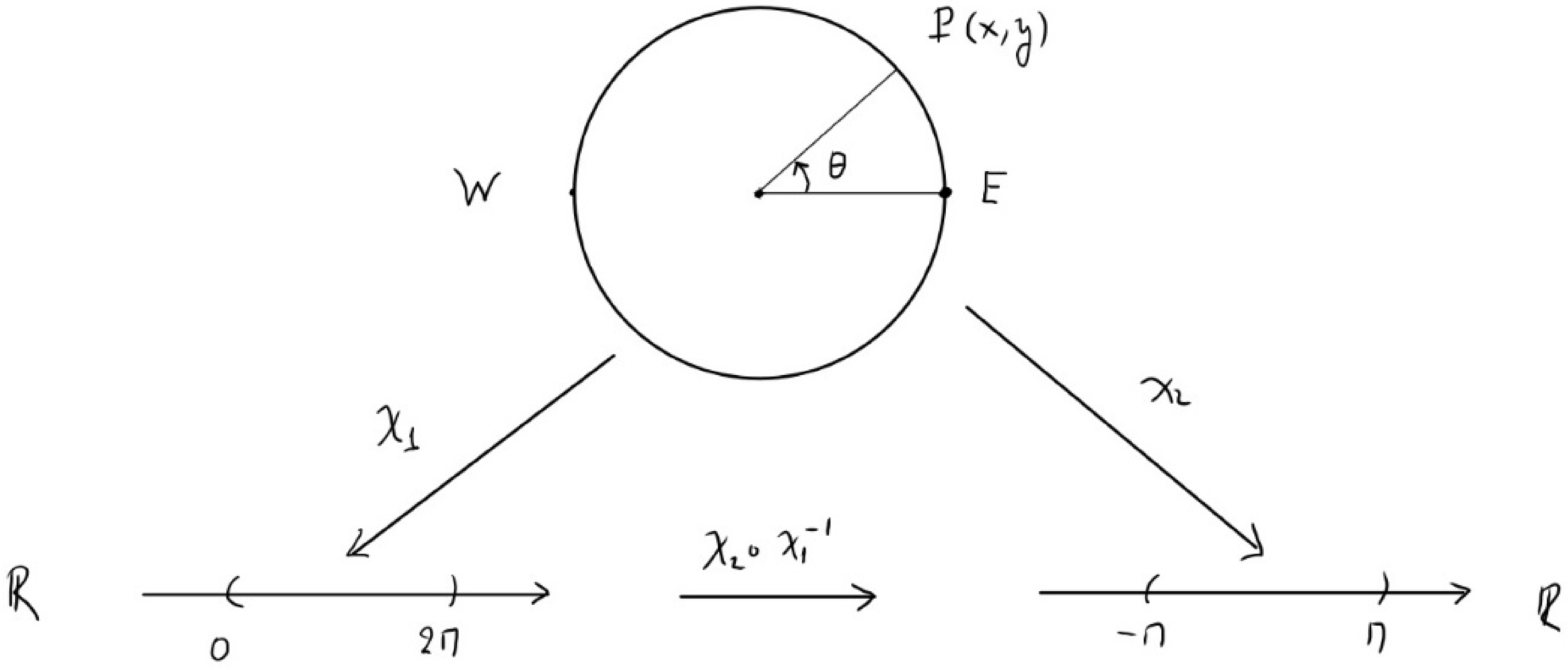
* Manifolds may need more than one chart to be covered

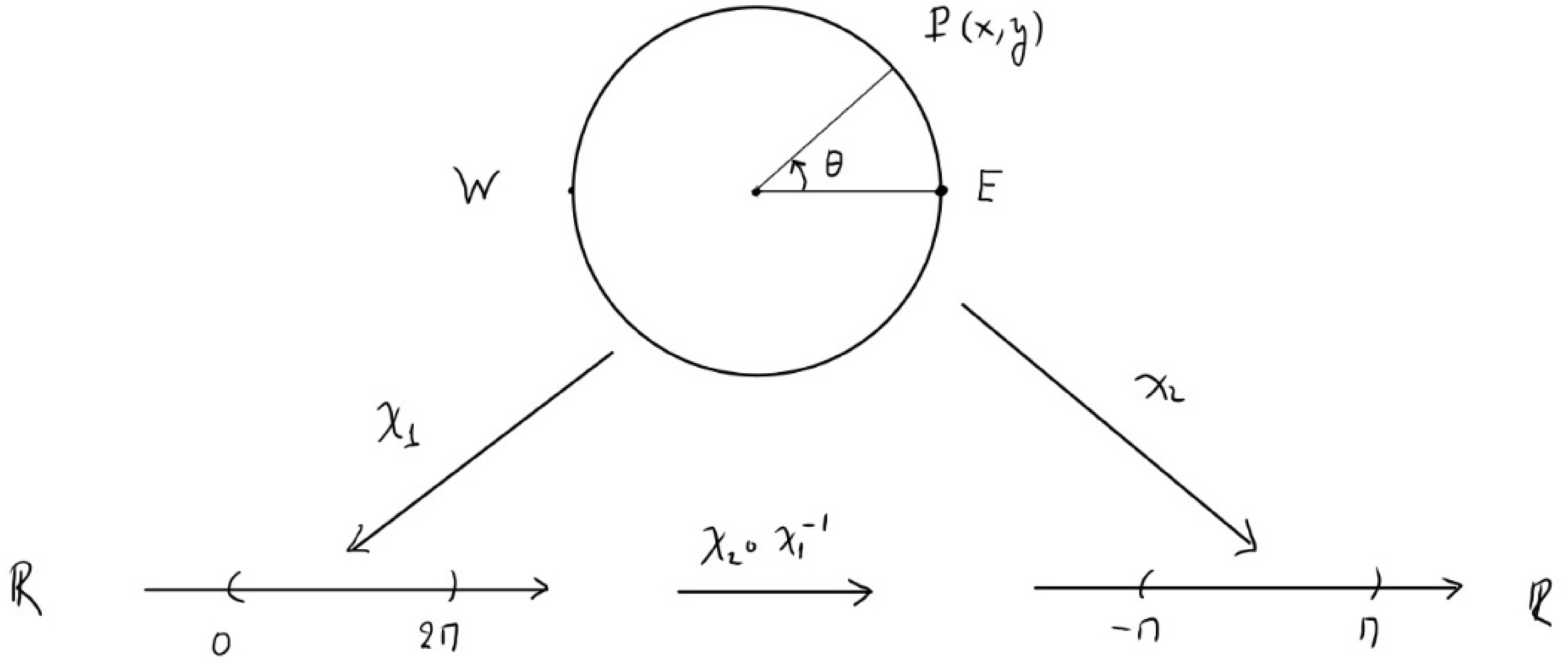
Examples: S^1



(U^1, χ_1): $U_1 = S^1 \setminus \{E\}$, $\chi_1(\underline{P}) = \theta \quad 0 < \theta < 2\pi$

$\chi_1 : (x, y) \mapsto \theta$

Examples: S^1



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(U^2, χ_2): $U_2 = S^1 \setminus \{W\}$, $\chi_2(P) = \theta$ $\quad -\pi < \theta < \pi$

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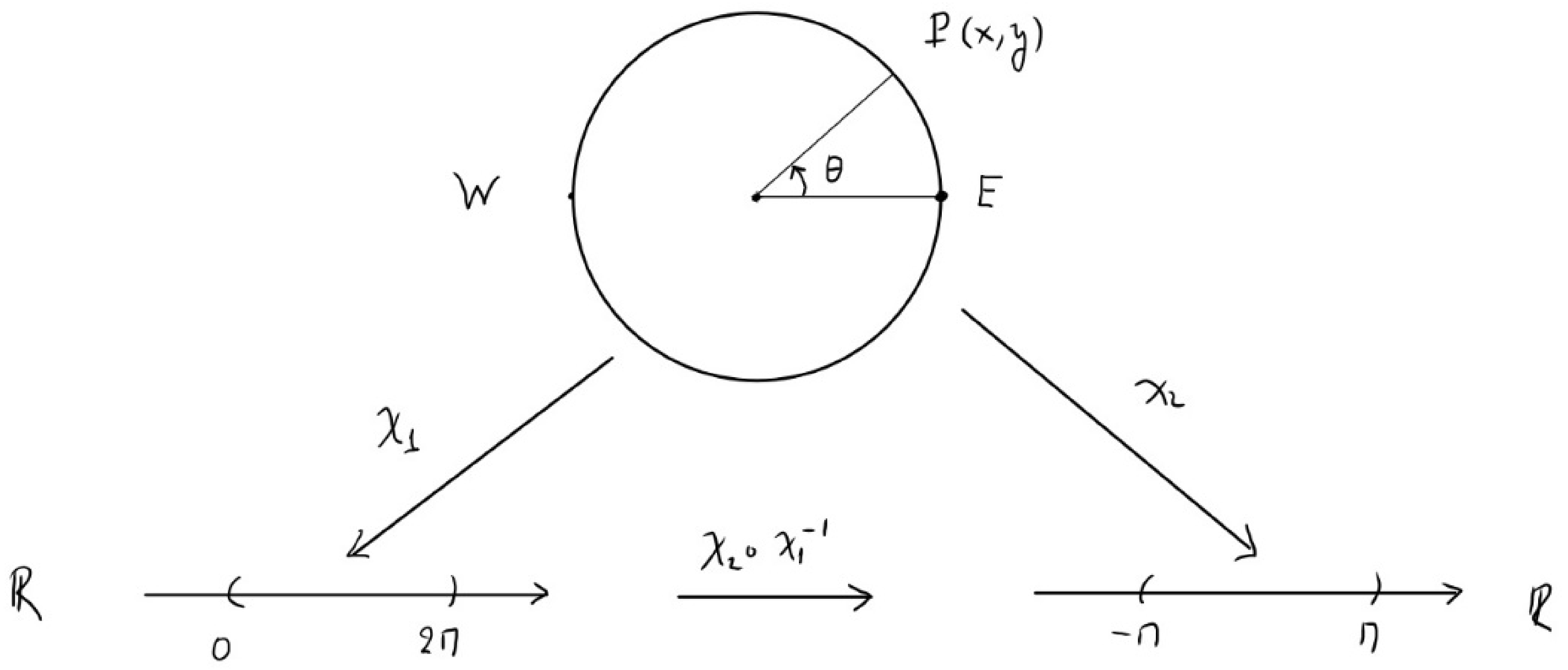
Examples: S^1



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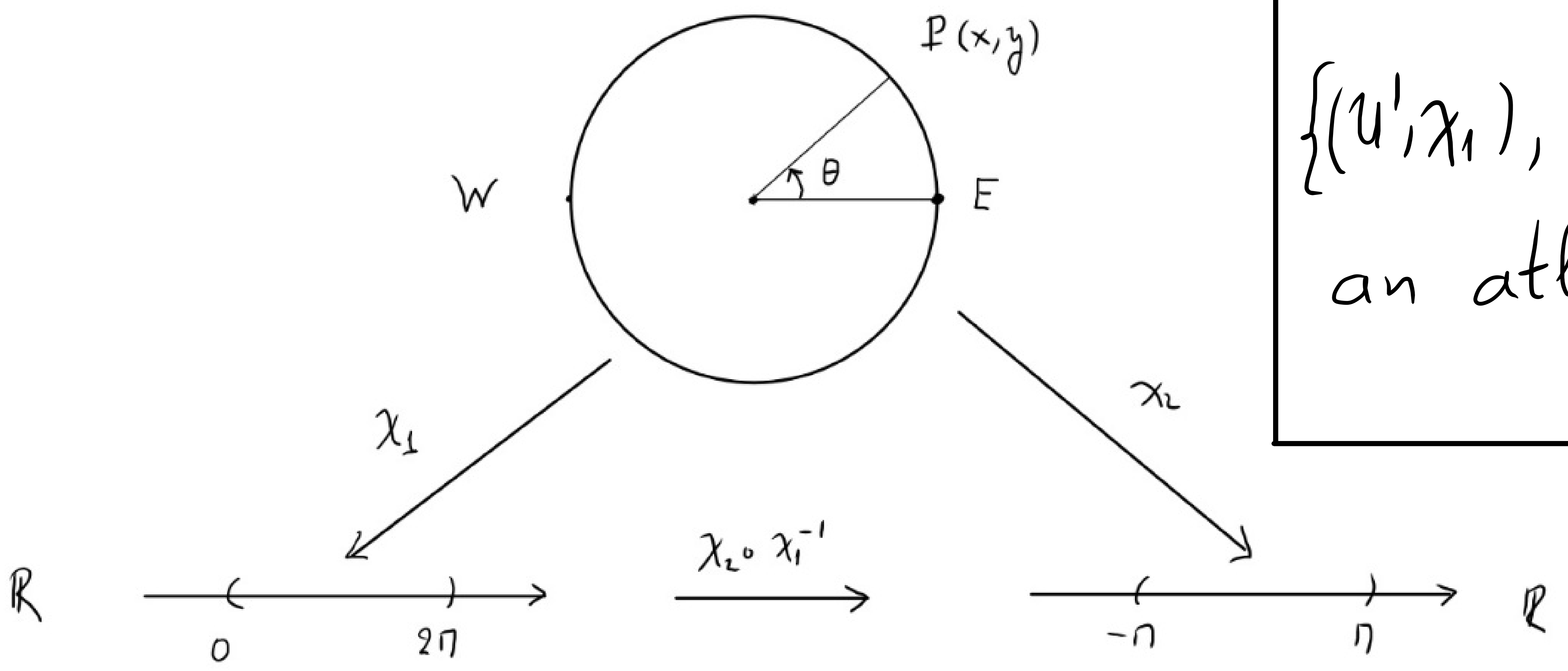
$$\chi_2 \circ \chi_1^{-1}(\theta) = \begin{cases} \theta & 0 < \theta < \pi \\ \theta - 2\pi & \pi < \theta < 2\pi \end{cases}$$

differentiable

Examples: S^1



$S^1 = U^1 \cup U^2$

$\{(U^1, \chi_1), (U^2, \chi_2)\}$

an atlas of S^1

(U^1, χ_1): $U_1 = S^1 \setminus \{E\}$, $\chi_1(P) = \theta$ $\quad 0 < \theta < 2\pi$

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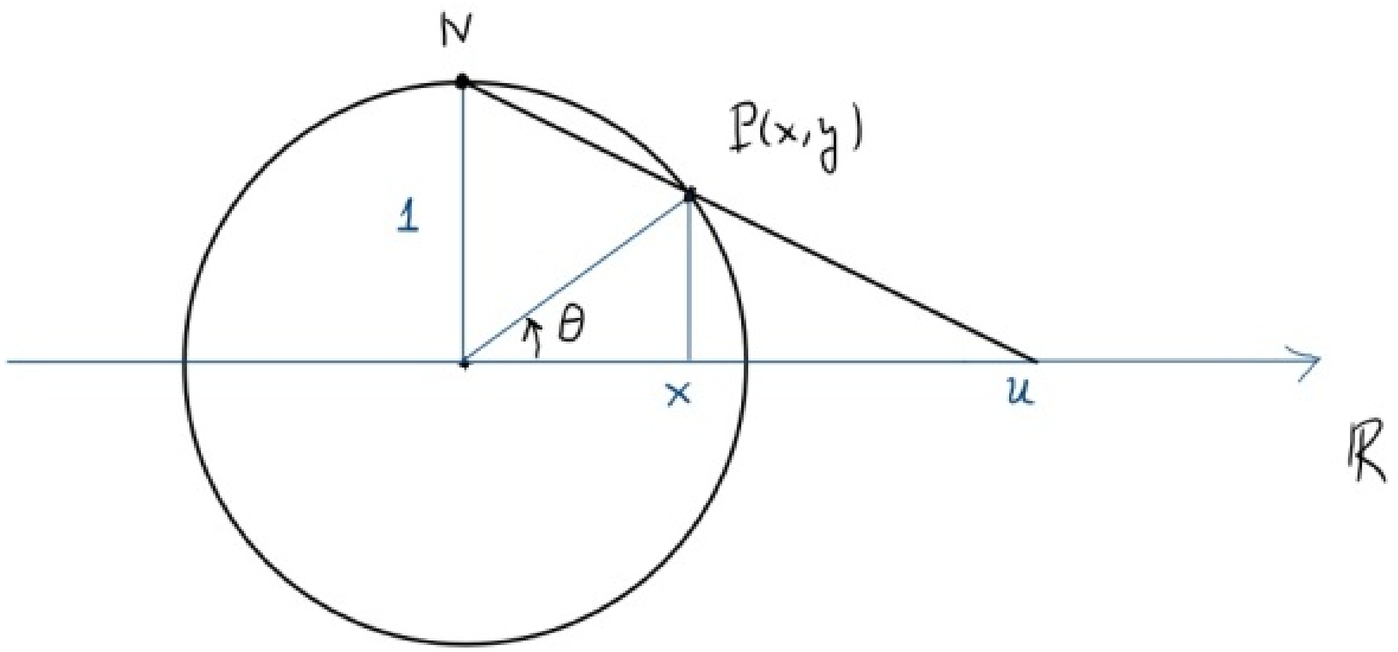
differentiable

Examples: S^1 another atlas

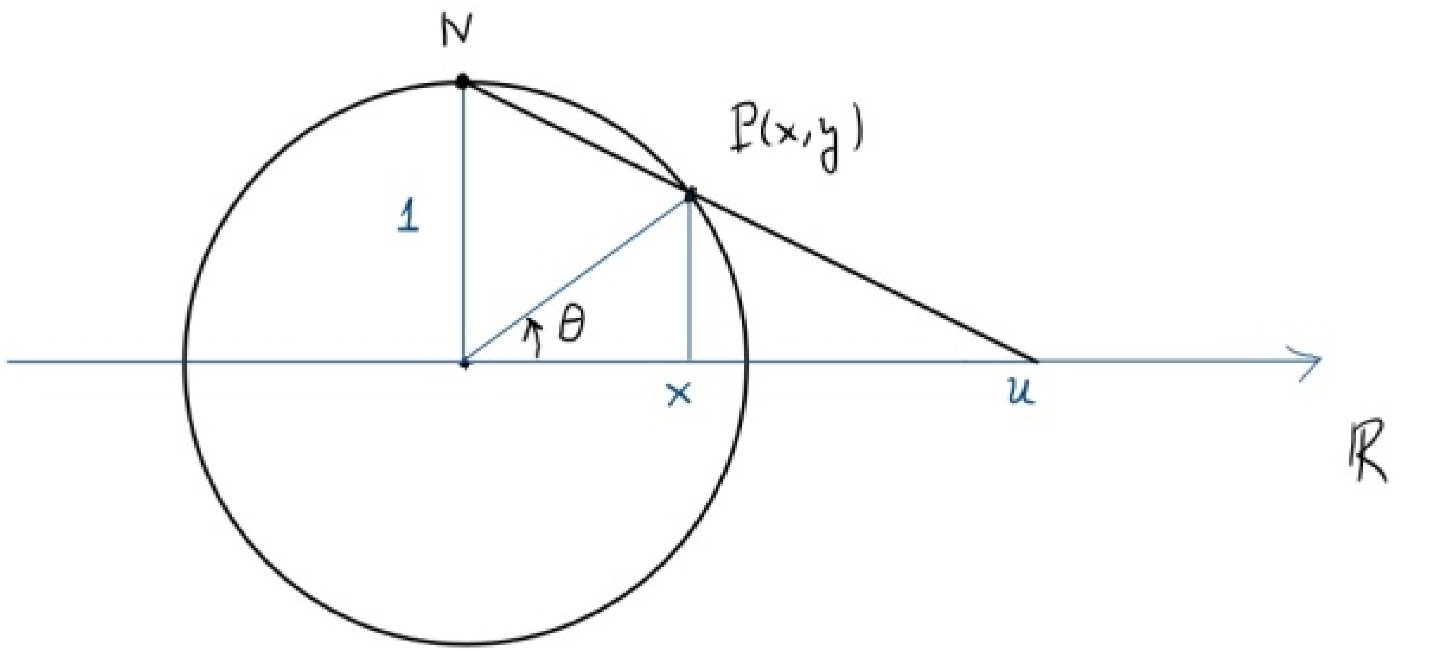


(U^3, χ_3): $U^3 = S^1 \setminus \{N\}$, $\chi_3 : (x,y) \mapsto u = \dfrac{x}{1-y}$ $\quad -\infty < u < +\infty$

From triangle similarity:

$$\frac{u}{1} = \frac{u-x}{y} \Rightarrow u = \frac{x}{1-y}$$

Examples: S^1 another atlas



(U^3, χ_3): $U^3 = S^1 \setminus \{N\}$, $\chi_3 : (x,y) \mapsto u = \dfrac{x}{1-y}$ $\quad -\infty < u < +\infty$

$x = \cos\theta,\ y = \sin\theta \Rightarrow \qquad u = \dfrac{\cos\theta}{1-\sin\theta} \Rightarrow$

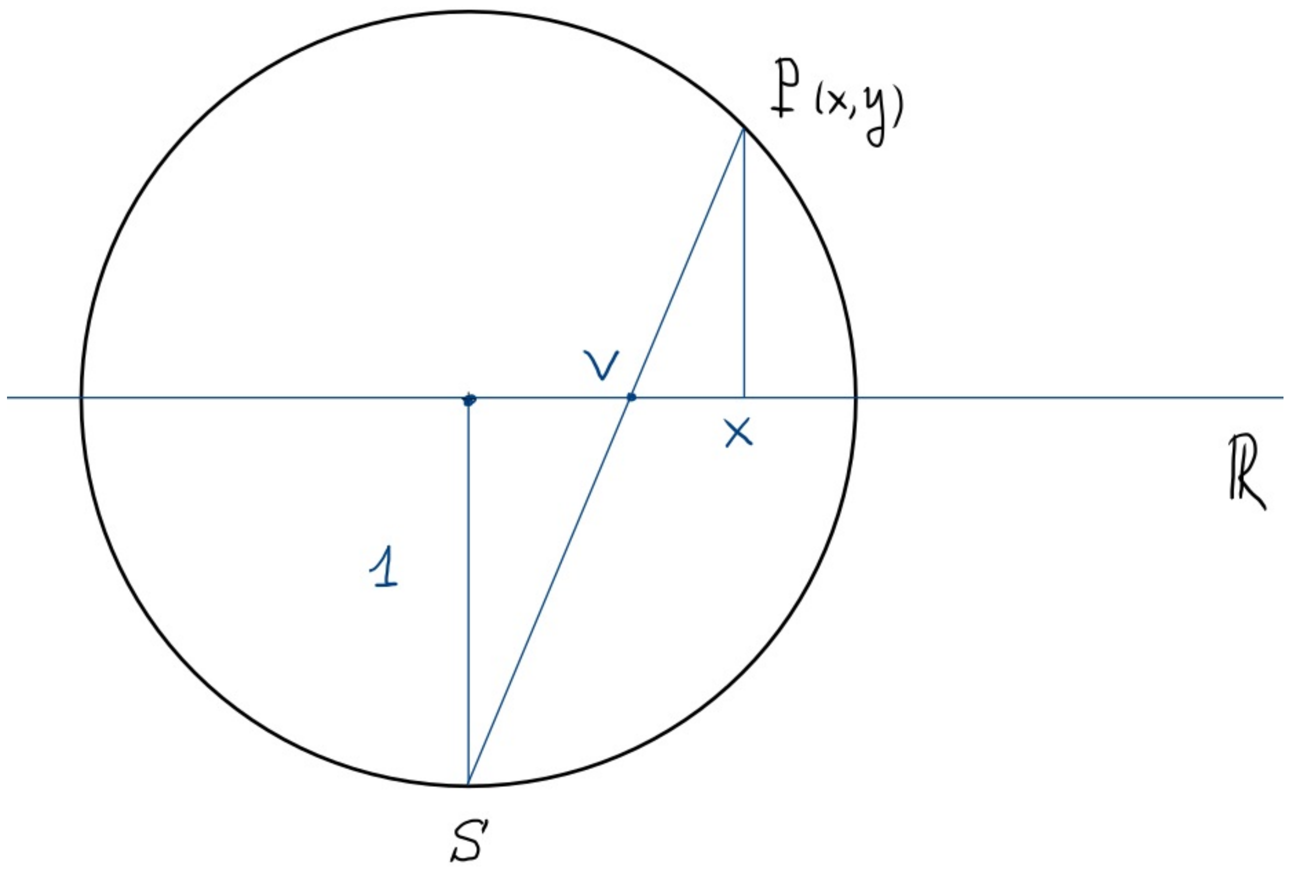
$u = \chi_3 \circ \chi_1^{-1}(\theta) = \dfrac{\cos\theta}{1-\sin\theta}$ is differentiable

Examples: S^1 another atlas

(U^1, χ_1): $U_1 = S^1 \setminus \{E\}$ $\quad \chi_1(x,y) = \theta$, $\quad 0 < \theta < 2\pi$

(U^2, χ_2): $U_2 = S^1 \setminus \{W\}$ $\quad \chi_2(x,y) = \theta$, $\quad -\pi < \theta < \pi$

(U^3, χ_3): $U_3 = S^1 \setminus \{N\}$ $\quad \chi_3(x,y) = \dfrac{x}{1-y}$



(U^4, χ_4): $U^4 = S^1 \setminus \{S\}$

$$\chi_4 : (x,y) \mapsto v = \frac{x}{1+y} \qquad -\infty < v < +\infty$$

$$v = \frac{\cos\theta}{1+\sin\theta} = \chi_4 \circ \chi_1^{-1}$$

differentiable

Examples: S^1 another atlas

(U^1, χ_1): $U_1 = S^1 \setminus \{E\}$ $\quad \chi_1(x,y) = \theta$, $0 < \theta < 2\pi$ $\qquad (U^2, \chi_2)$: $U_2 = S^1 \setminus \{W\}$ $\quad \chi_2(x,y) = \theta$, $-\pi < \theta < \pi$

(U^3, χ_3): $U_3 = S^1 \setminus \{N\}$ $\quad \chi_3(x,y) = \dfrac{x}{1-y}$ $\qquad (U^4, \chi_4)$: $U_4 = S^1 \setminus \{S\}$ $\quad \chi_4(x,y) = \dfrac{x}{1+y}$

Transition maps:

$$u = \chi_3 \circ \chi_1^{-1}(\theta) = \frac{\cos\theta}{1 - \sin\theta}$$

$$v = \chi_4 \circ \chi_1^{-1}(\theta) = \frac{\cos\theta}{1 + \sin\theta}$$

$$v = \chi_4 \circ \chi_3^{-1}(u) = \frac{1}{u}$$

Examples: S^1 another atlas

(U^1, x_1): $U_1 = S^1 \setminus \{E\}$ $\quad x_1(x,y) = \theta$, $0 < \theta < 2\pi$ $\qquad (U^2, x_2)$: $U_2 = S^1 \setminus \{W\}$ $\quad x_2(x,y) = \theta$, $-\pi < \theta < \pi$

(U^3, x_3): $U_3 = S^1 \setminus \{N\}$ $\quad x_3(x,y) = \dfrac{x}{1-y}$ $\qquad (U^4, x_4)$: $U_4 = S^1 \setminus \{S\}$ $\quad x_4(x,y) = \dfrac{x}{1+y}$

Transition maps:

$$u = x_3 \circ x_1^{-1}(\theta) = \frac{\cos\theta}{1-\sin\theta}$$

$$v = x_4 \circ x_1^{-1}(\theta) = \frac{\cos\theta}{1+\sin\theta}$$

$$u = x_4 \circ x_3^{-1}(u) = \frac{1}{u}$$

$$\left.\begin{array}{l} u \cdot v = \dfrac{x}{1-y} \cdot \dfrac{x}{1+y} = \dfrac{x^2}{1-y^2} \\ x^2 + y^2 = 1 \end{array}\right\} \Rightarrow$$

$$u \cdot v = \frac{x^2}{1-(1-x^2)} = \frac{x^2}{x^2} = 1$$

Examples: S^1 another atlas

(U^1, x_1): $U_1 = S^1 \setminus \{E\}$ $\quad x_1(x,y) = \theta$, $0 < \theta < 2\pi$

(U^2, x_2): $U_2 = S^1 \setminus \{W\}$ $\quad x_2(x,y) = \theta$, $-\pi < \theta < \pi$

(U^3, x_3): $U_3 = S^1 \setminus \{N\}$ $\quad x_3(x,y) = \frac{x}{1-y}$

(U^4, x_4): $U_4 = S^1 \setminus \{S\}$ $\quad x_4(x,y) = \frac{x}{1+y}$

Transition maps:

$$u = x_3 \circ x_1^{-1}(\theta) = \frac{\cos\theta}{1-\sin\theta}$$

$$v = x_4 \circ x_1^{-1}(\theta) = \frac{\cos\theta}{1+\sin\theta}$$

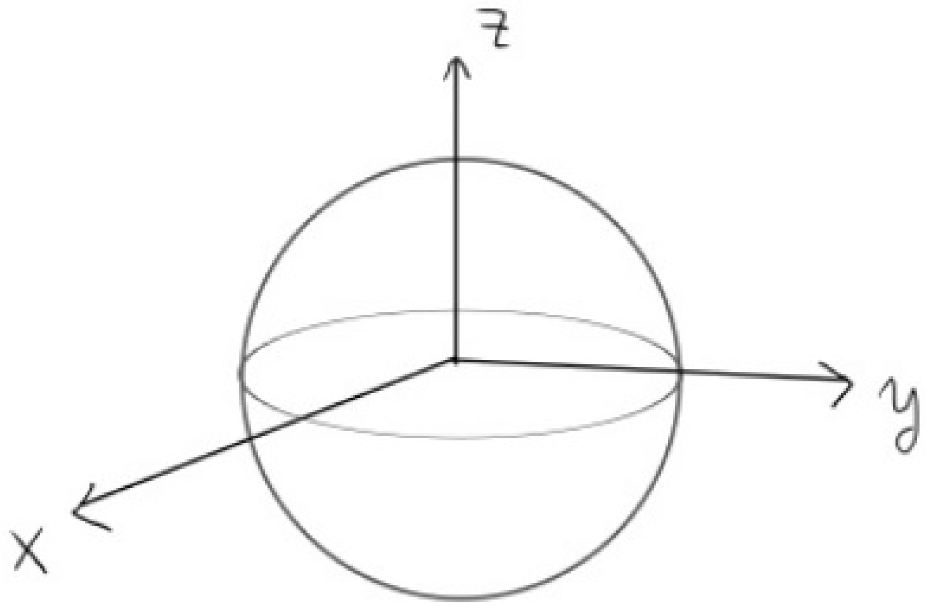
$$u = x_4 \circ x_3^{-1}(u) = \frac{1}{u}$$

- All differentiable

- $\left.\begin{array}{l} \mathcal{A}_1 = \{(U^1, x_1), (U^2, x_2)\} \\ \mathcal{A}_2 = \{(U^3, x_3), (U^4, x_4)\} \end{array}\right\}$ atlases

- $\nexists$ atlas with only one chart ! $\quad (\mathbb{R} \not\cong S^1)$

S^2: The sphere

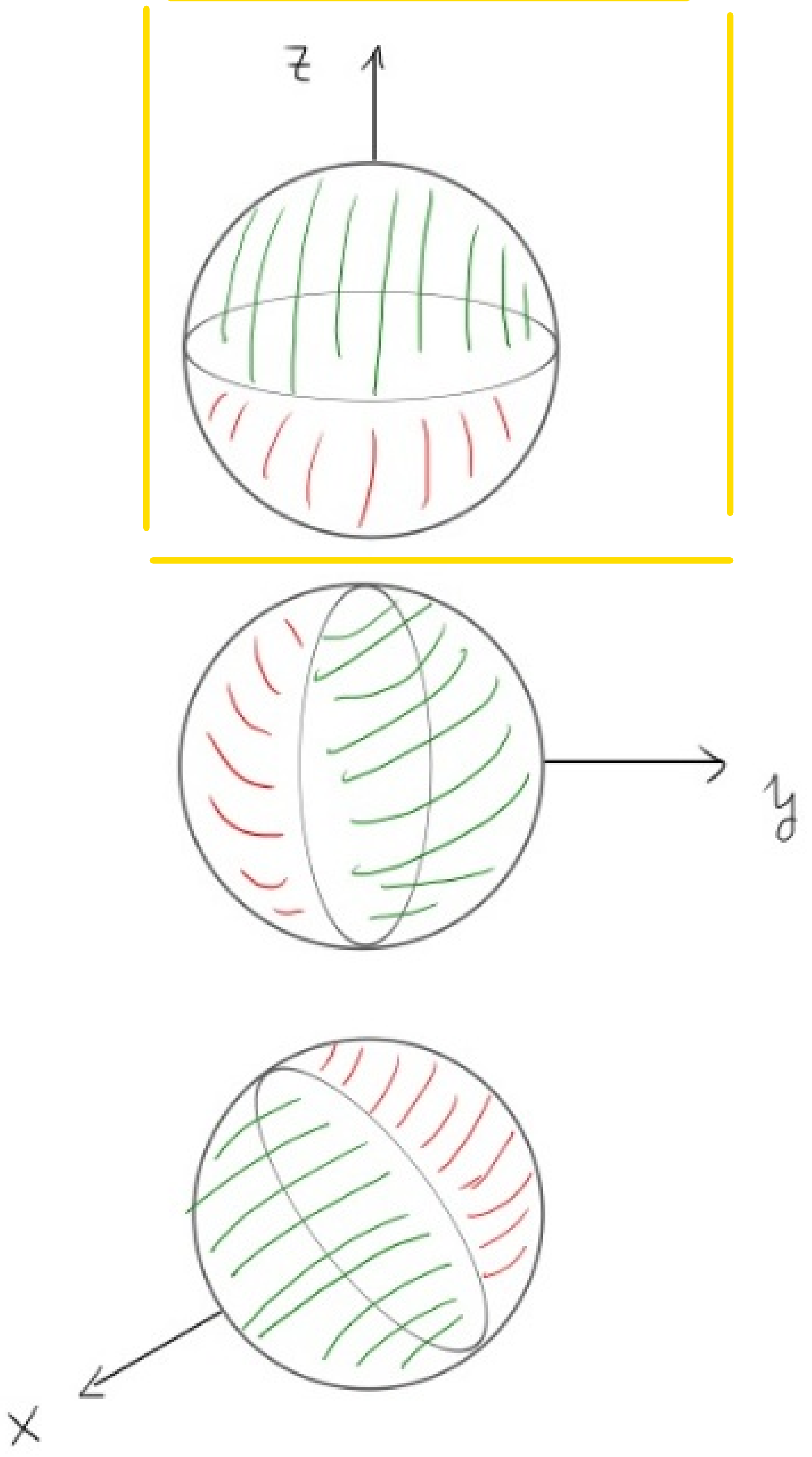


Define S^2 by (x,y,z), s.t.

$$x^2+y^2+z^2=1$$

S^2: The sphere

Define 6 charts - hemispheres:



$$U_{z_+} = \{(x,y,z) \in S^2 \mid z > 0\}$$

$$\chi_{z+} : (x, y, +\sqrt{1-x^2-y^2}) \mapsto (x, y)$$

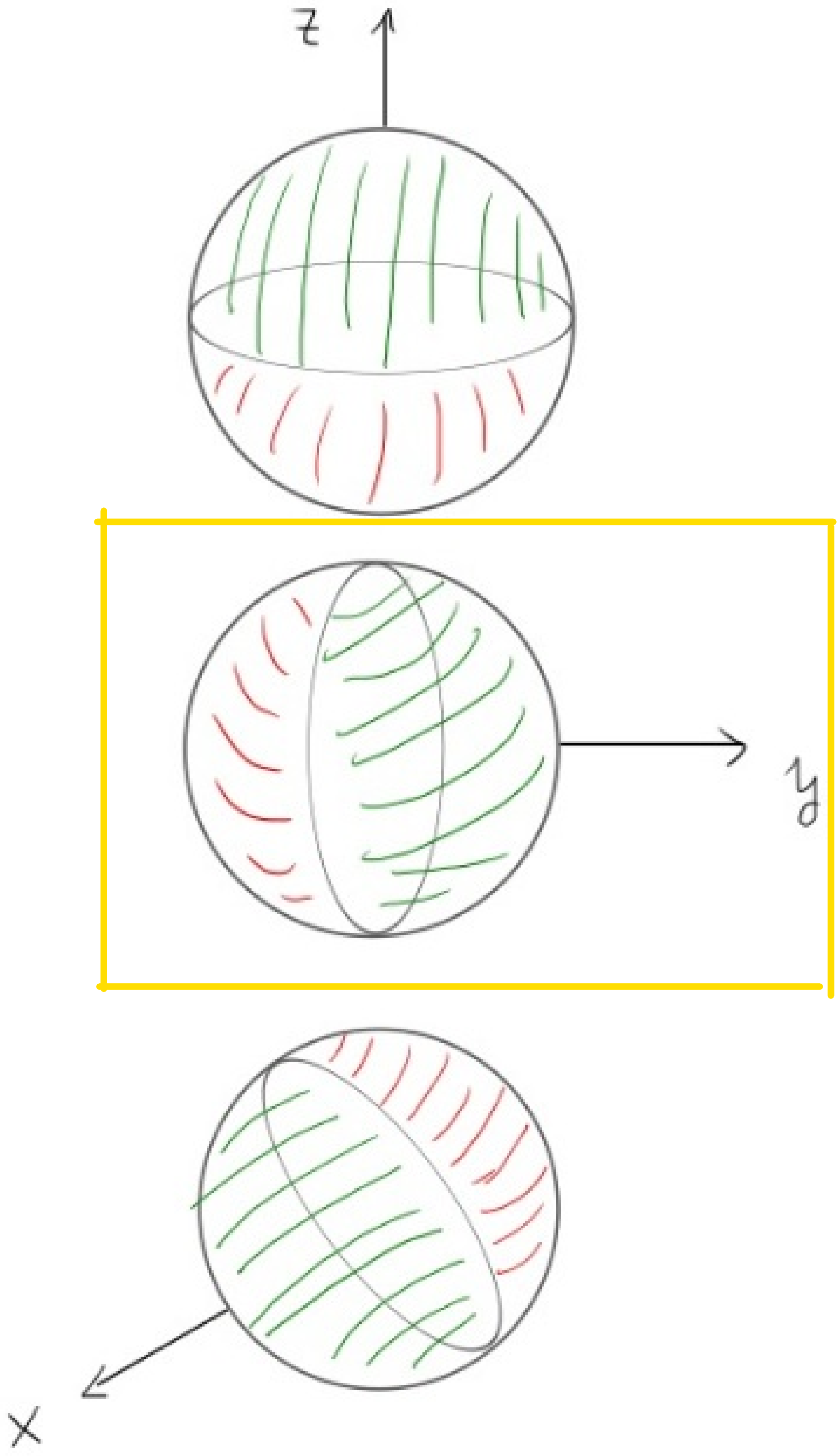
$$U_{z_-} = \{(x,y,z) \in S^2 \mid z < 0\}$$

$$\chi_{z-} : (x, y, -\sqrt{1-x^2-y^2}) \mapsto (x, y)$$

projected on same point on (x,y)-plane

S^2: The sphere

Define 6 charts - hemispheres:



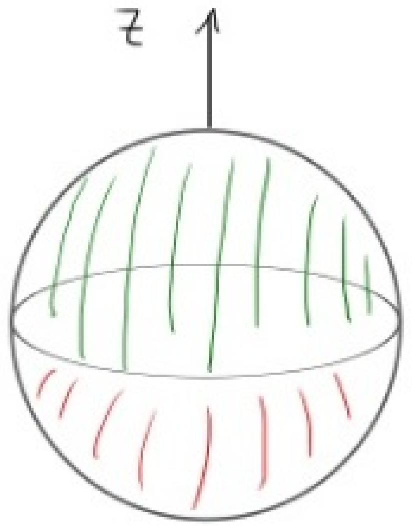
$$\chi_{z+}: (x, y, +\sqrt{1-x^2-y^2}) \mapsto (x, y)$$

$$\chi_{z-}: (x, y, -\sqrt{1-x^2-y^2}) \mapsto (x, y)$$

$$\left.\begin{array}{l} \chi_{y+}: (x, +\sqrt{1-x^2-z^2}) \mapsto (x, z) \\[2em] \chi_{y-}: (x, -\sqrt{1-x^2-z^2}) \mapsto (x, z) \end{array}\right\} \text{projection on } (x,z) \text{ plane}$$

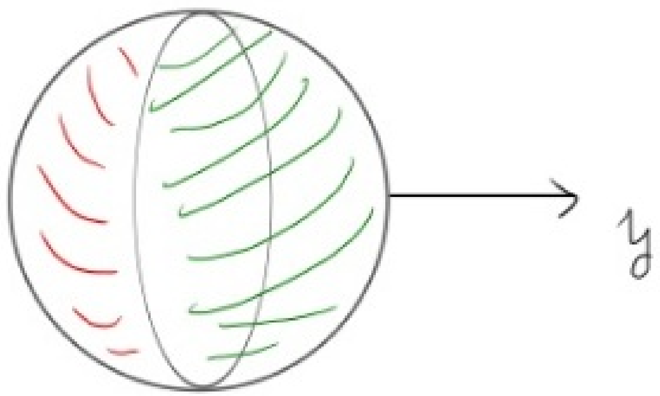
S^2: The sphere

Define 6 charts - hemispheres:



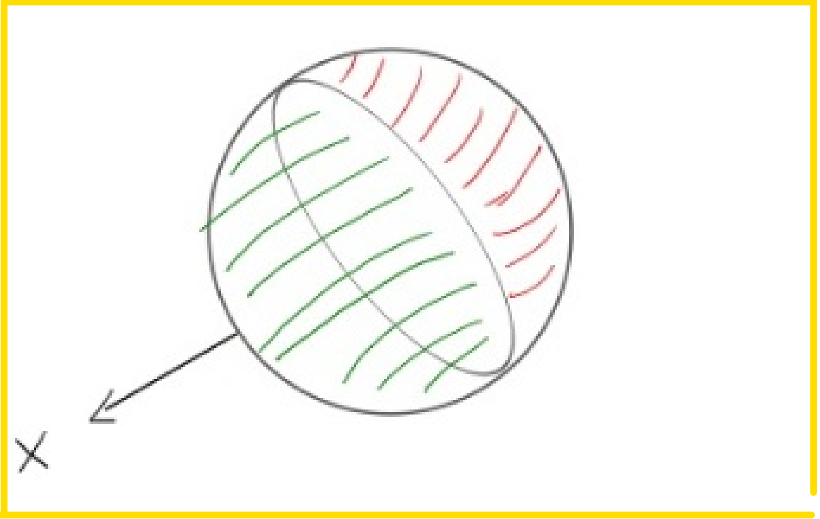
$$\chi_{z+}: (x, y, +\sqrt{1-x^2-y^2}) \mapsto (x,y)$$

$$\chi_{z-}: (x, y, -\sqrt{1-x^2-y^2}) \mapsto (x,y)$$



$$\chi_{y+}: (x, +\sqrt{1-x^2-z^2}) \mapsto (x,z)$$

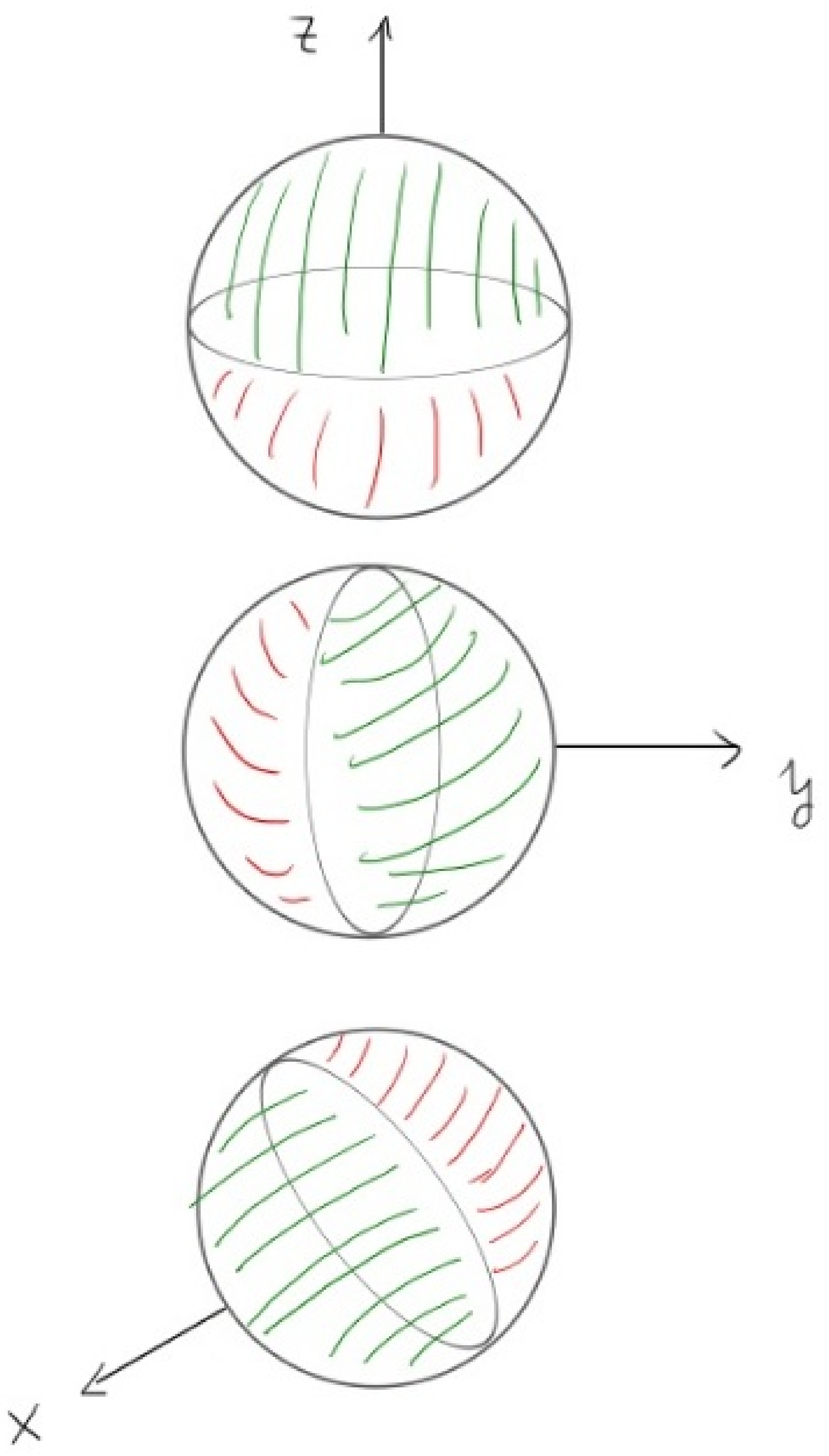
$$\chi_{y-}: (x, -\sqrt{1-x^2-z^2}) \mapsto (x,z)$$



$$\left.\begin{aligned} \chi_{x+} &: (+\sqrt{1-y^2-z^2}, y, z) \mapsto (y,z) \\ \chi_{x-} &: (-\sqrt{1-y^2-z^2}, y, z) \mapsto (y,z) \end{aligned}\right\} \text{projection on } (y,z)\text{-plane}$$

S^2: The sphere

Define 6 charts - hemispheres:



$\chi_{z+}: (x, y, +\sqrt{1-x^2-y^2}) \mapsto (x,y)$

$\chi_{z-}: (x, y, -\sqrt{1-x^2-y^2}) \mapsto (x,y)$

$\chi_{y+}: (x, +\sqrt{1-x^2-z^2}) \mapsto (x,z)$

$\chi_{y-}: (x, -\sqrt{1-x^2-z^2}) \mapsto (x,z)$

$\chi_{x+}: (+\sqrt{1-y^2-z^2}, y, z) \mapsto (y,z)$

$\chi_{x-}: (-\sqrt{1-y^2-z^2}, y, z) \mapsto (y,z)$

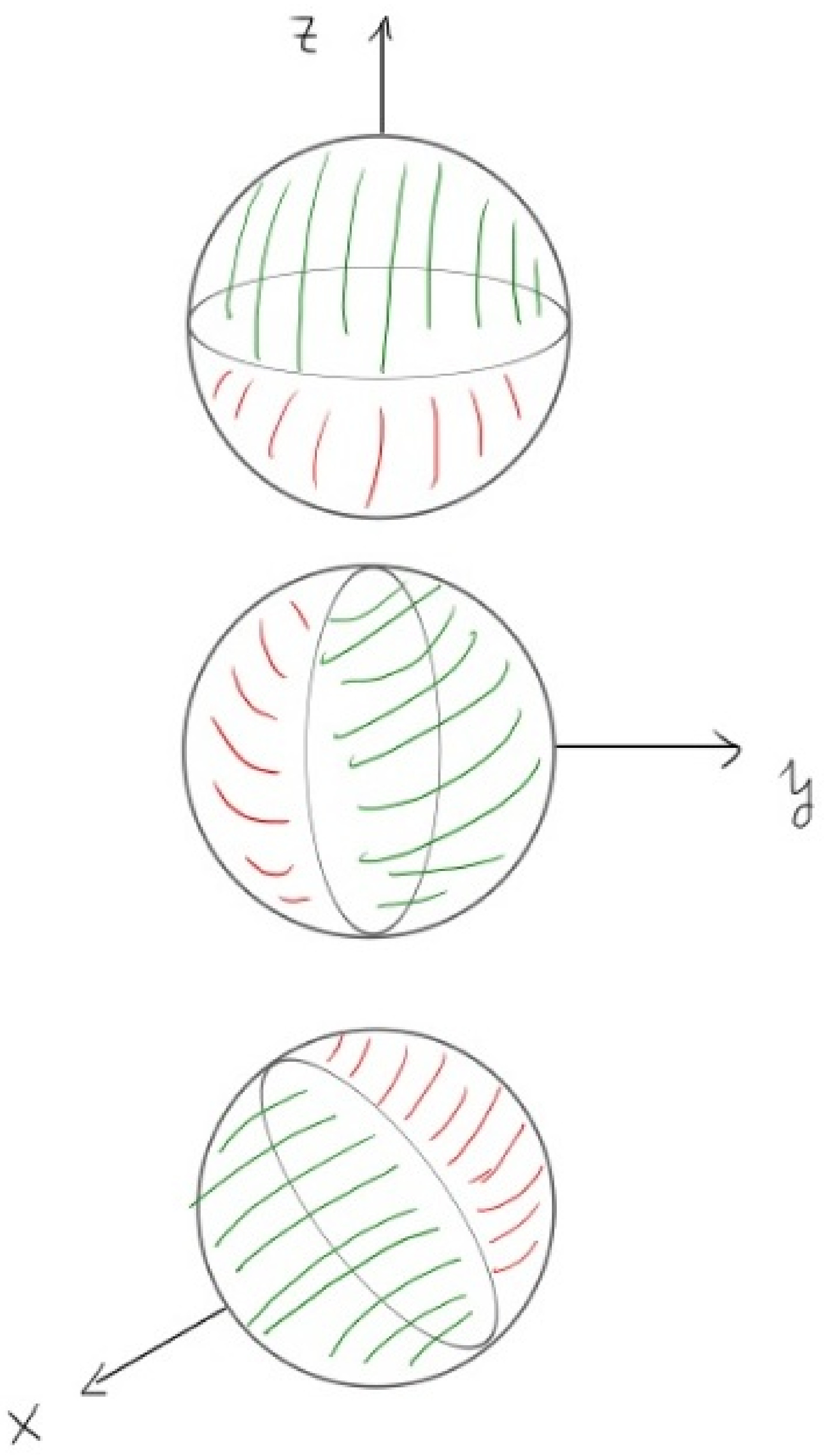
A transition map:

$\chi_{y-} \circ \chi_{x+}^{-1}: (y,z) \mapsto (x,z)$

with $\begin{cases} x = \sqrt{1-y^2-z^2} \\ z = z \end{cases}$

S^2: The sphere

Define 6 charts - hemispheres:



$$\chi_{z+}: (x, y, +\sqrt{1-x^2-y^2}) \mapsto (x, y)$$

$$\chi_{z-}: (x, y, -\sqrt{1-x^2-y^2}) \mapsto (x, y)$$

$$\chi_{y+}: (x, +\sqrt{1-x^2-z^2}) \mapsto (x, z)$$

$$\chi_{y-}: (x, -\sqrt{1-x^2-z^2}) \mapsto (x, z)$$

$$\chi_{x+}: (+\sqrt{1-y^2-z^2}, y, z) \mapsto (y, z)$$

$$\chi_{x-}: (-\sqrt{1-y^2-z^2}, y, z) \mapsto (y, z)$$

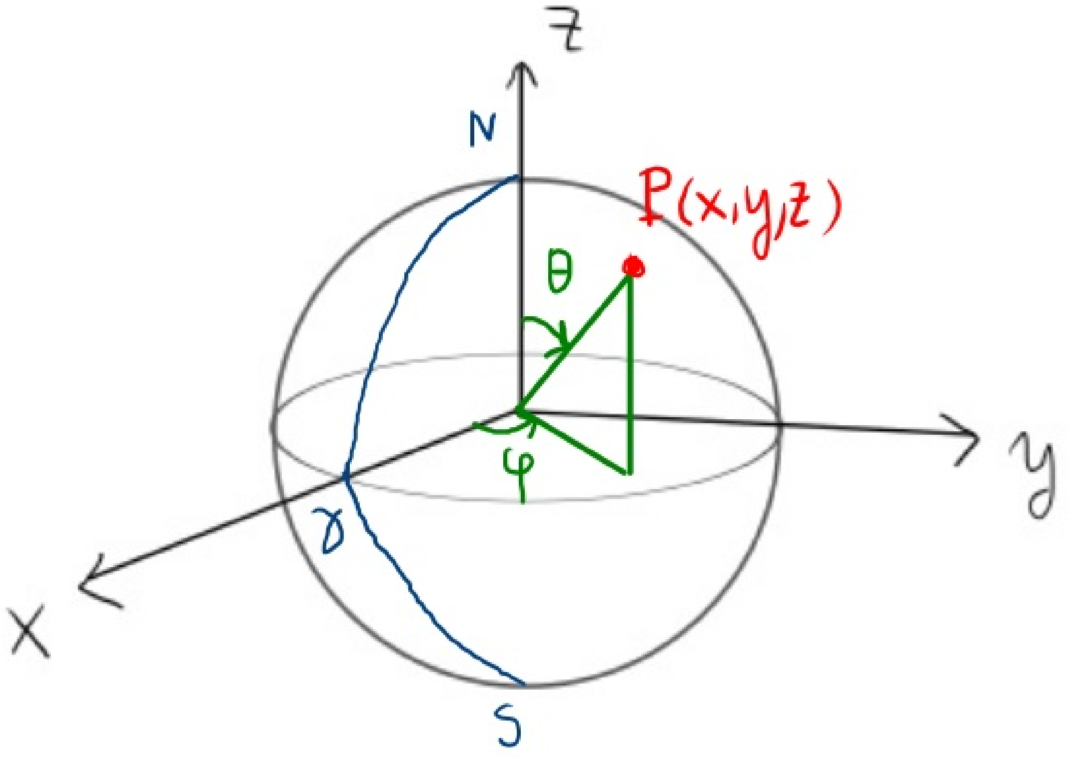
A transition map:

$$\chi_{y-} \circ \chi_{x+}^{-1}: (y, z) \mapsto (x, z)$$

with $\begin{cases} x = \sqrt{1-y^2-z^2} \\ z = z \end{cases}$

- Differentiable
- Need all 6 to cover S^2

S^2: The sphere



Another chart:

$(\mathcal{U}_\theta, \chi_\theta)$: $\mathcal{U}_\theta = S^2 \setminus \{\gamma\}$

$\chi_\theta : (x, y, z) \mapsto (\theta, \varphi)$ $\quad 0<\theta<\pi$

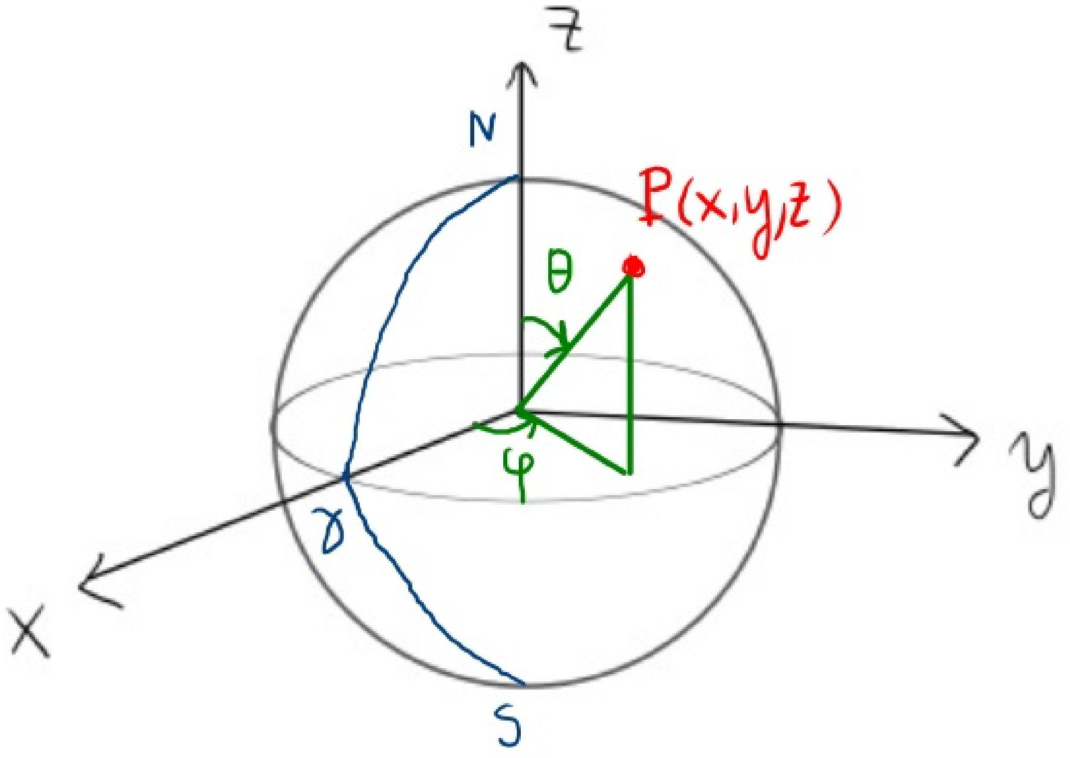
$0<\varphi<2\pi$

$x = \sin\theta \cos\varphi$

$y = \sin\theta \sin\varphi$

$z = \cos\theta$

S^2: The sphere



Another chart:

$$(U_\theta, \chi_\theta): \quad U_\theta = S^2 \setminus \{\gamma\}$$

$$\chi_\theta : (x,y,z) \mapsto (\theta, \varphi) \qquad \begin{array}{l} 0<\theta<\pi \\ 0<\varphi<2\pi \end{array}$$

$$x = \sin\theta\cos\varphi$$
$$y = \sin\theta\sin\varphi$$
$$z = \cos\theta$$

Transition Functions:

$$\chi_{y_+} \circ \chi_\theta^{-1} : (\theta, \varphi) \mapsto (x, z)$$

$$x = \sin\theta\cos\varphi \qquad 0<\theta<\pi$$
$$z = \cos\theta \qquad 0<\varphi<\pi$$

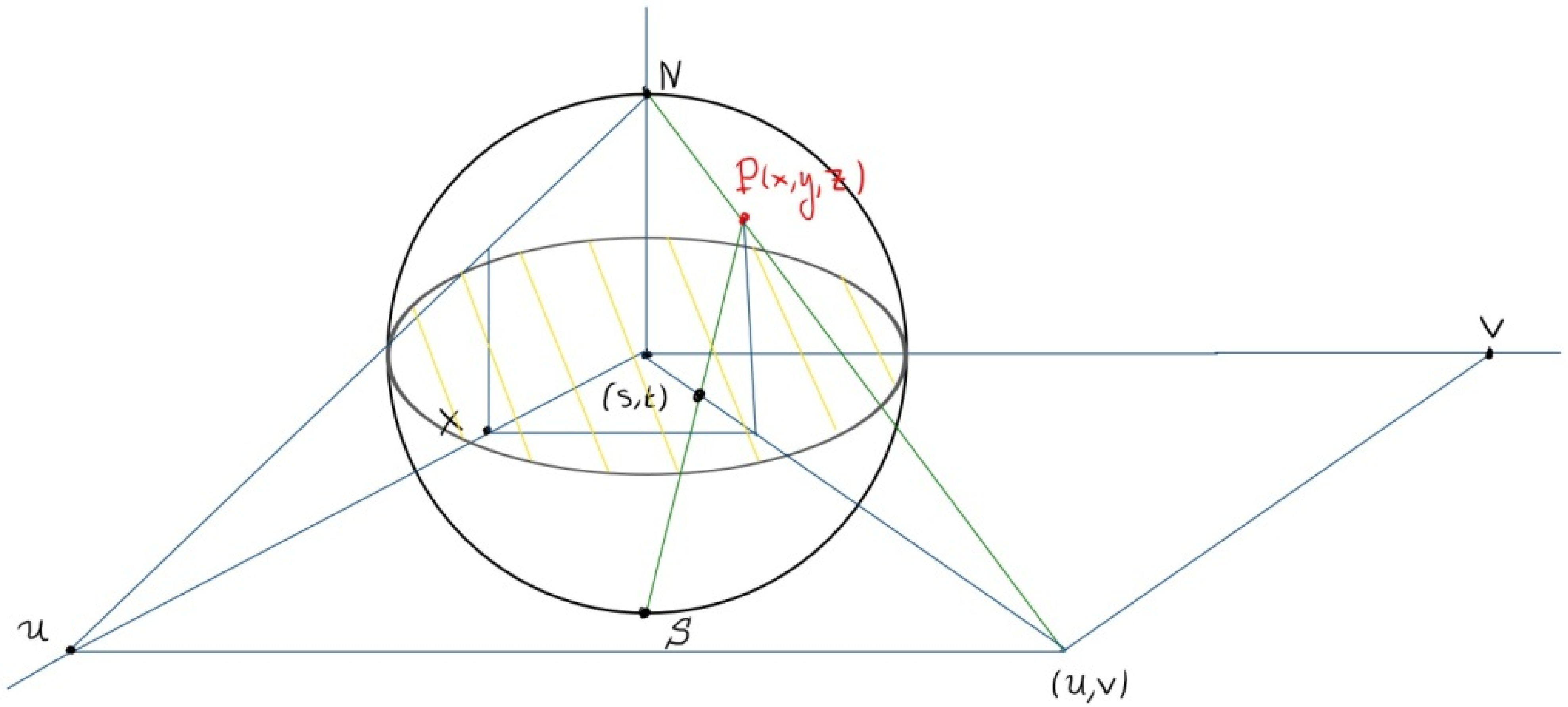
$$\chi_\theta \circ \chi_{y_+}^{-1} : (x,z) \mapsto (\theta, \varphi)$$

$$0 < \theta = \tan^{-1}\sqrt{\frac{1}{z^2} - 1} < \pi \qquad \left(\tan\theta = \frac{\sqrt{x^2+y^2}}{z}\right)$$

$$0 < \varphi = \tan^{-1}\frac{\sqrt{1-x^2-z^2}}{x} < \pi \qquad \left(\tan\varphi = \frac{y}{x}\right)$$

$$-1 < x < 1$$
$$-1 < z < 1$$

S^2: The sphere

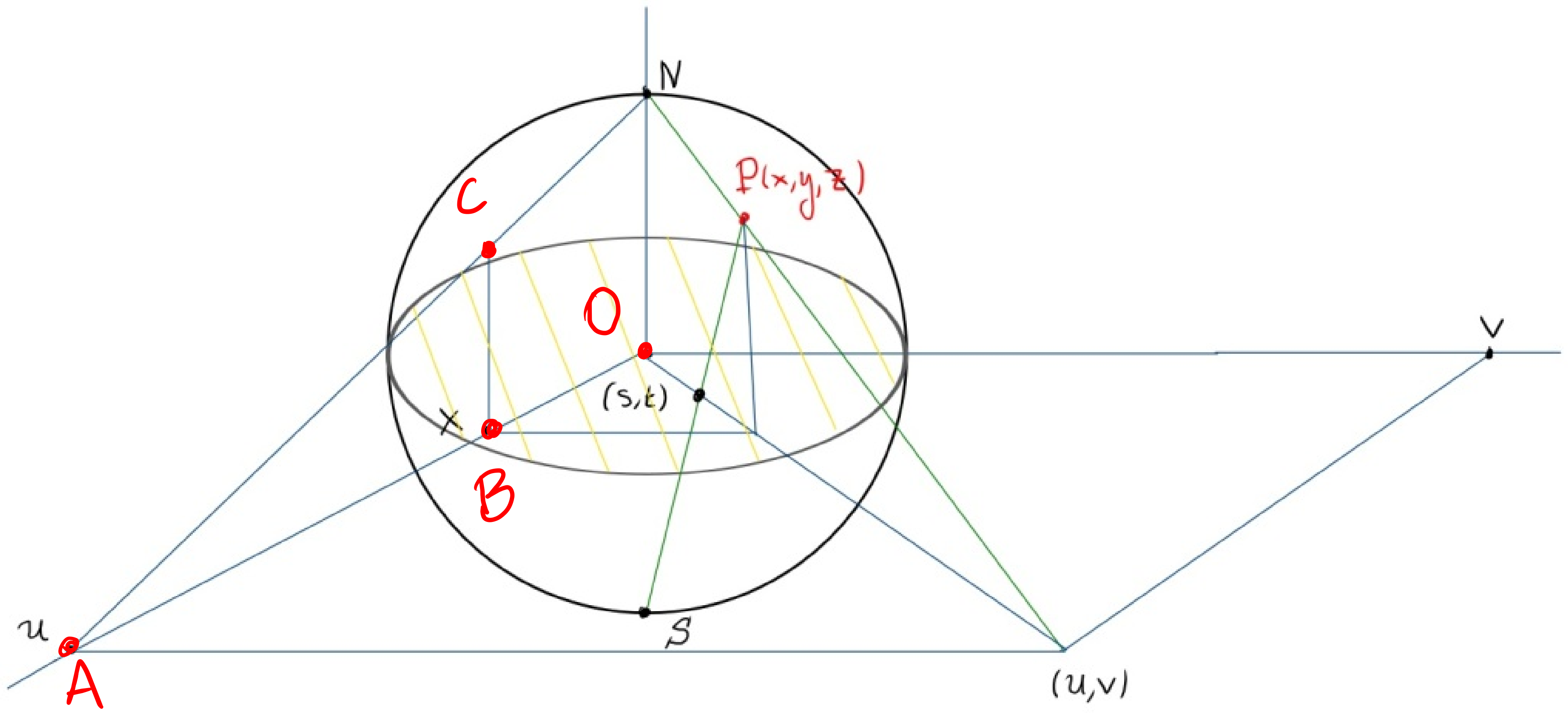


$$(U_N, \chi_N) : \quad U_N = S^2 \setminus \{N\}$$

$$\chi_N : (x,y,z) \mapsto (u,v) \qquad \begin{matrix} -\infty < u < +\infty \\ -\infty < v < +\infty \end{matrix}$$

$$u = \frac{x}{1-z} \qquad v = \frac{y}{1-z}$$

S^2: The sphere



$$\frac{OA}{AB} = \frac{ON}{CB} \Rightarrow \frac{u}{u-x} = \frac{1}{z} \Rightarrow \frac{u}{u-(u-x)} = \frac{1}{1-z} \Rightarrow$$

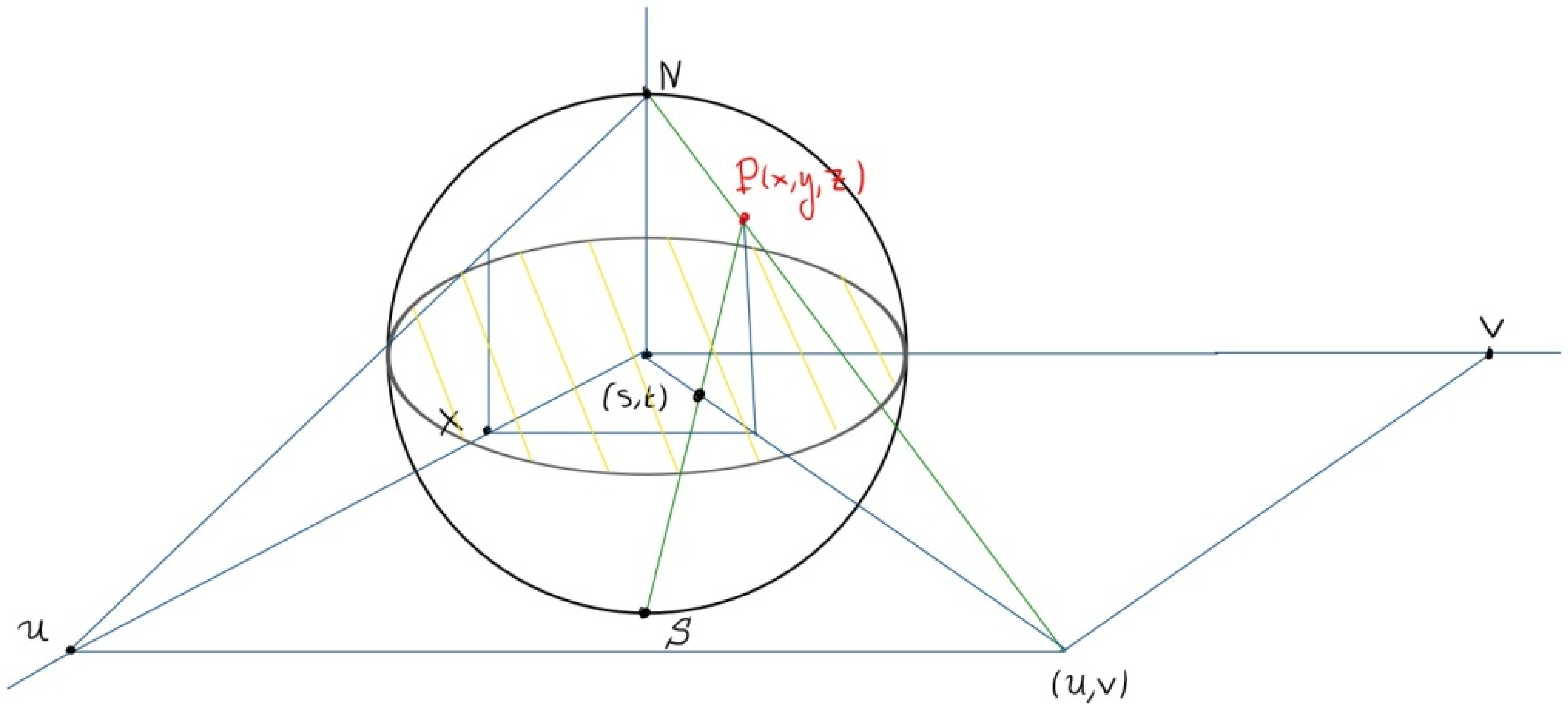
$$\frac{u}{x} = \frac{1}{1-z} \Rightarrow u = \frac{x}{1-z}$$

$$(U_N, X_N): \quad U_N = S^2 \setminus \{N\}$$

$$X_N : (x,y,z) \mapsto (u,v) \qquad \begin{array}{l} -\infty < u < +\infty \\ -\infty < v < +\infty \end{array}$$

$$u = \frac{x}{1-z} \qquad v = \frac{y}{1-z}$$

S^2: The sphere



(U_N, χ_N): $U_N = S^2 \setminus \{N\}$

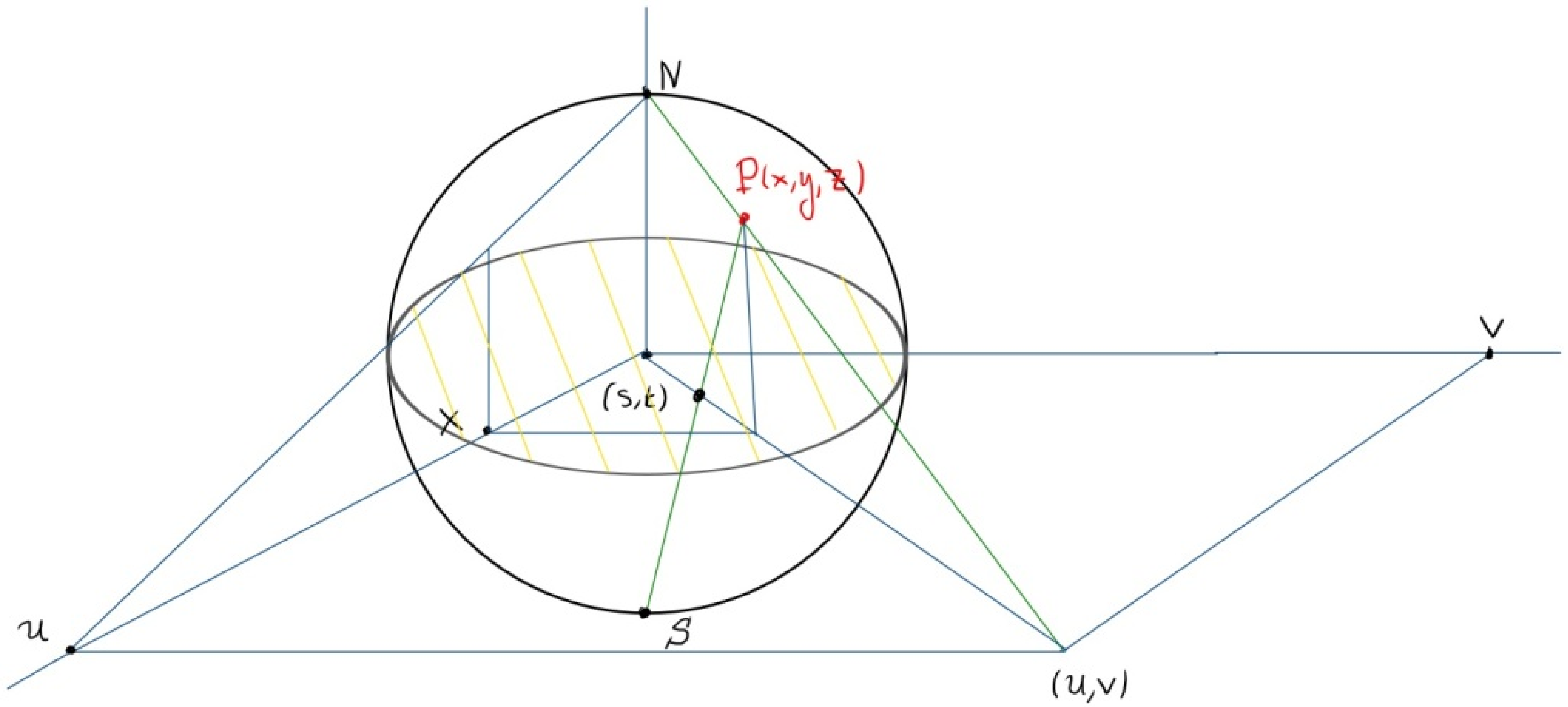
$\chi_N : (x,y,z) \mapsto (u,v)$

$$u = \frac{x}{1-z} \qquad v = \frac{y}{1-z}$$

$\chi_N \circ \chi_\theta^{-1} : (\theta, \varphi) \mapsto (u,v)$

$$u = \frac{\sin\theta\cos\varphi}{1-\cos\theta} \qquad V = \frac{\sin\theta\sin\varphi}{1-\cos\theta}$$

S^2: The sphere



(U_N, χ_N): $U_N = S^2 \setminus \{N\}$

$\chi_N : (x,y,z) \mapsto (u,v)$

$$u = \frac{x}{1-z} \qquad v = \frac{y}{1-z}$$

(U_S, χ_S): $U_S = S^2 \setminus \{S\}$

$\chi_S : (x,y,z) \mapsto (s,t)$

$$s = \frac{x}{1+z} \qquad t = \frac{y}{1+z}$$

$\rightsquigarrow$ Exercise: prove!

S^2: The sphere

$$\chi_S \circ \chi_\Theta^{-1} : (\theta, \varphi) \mapsto (s, t) \qquad s = \frac{\sin\theta\cos\varphi}{1+\cos\theta}$$

$$t = \frac{\sin\theta \sin\varphi}{1+\cos\theta}$$

$$\chi_S \circ \chi_N^{-1} : (u, v) \mapsto (s, t) \qquad s = \frac{u}{u^2+v^2} \quad (1)$$

$$t = \frac{v}{u^2+v^2} \quad (2)$$

$$\left.\begin{aligned} us + vt &= \frac{x^2}{1-z^2} + \frac{y^2}{1-z^2} = \frac{x^2+y^2}{x^2+y^2} = 1 \\ ut &= vs = \frac{xy}{1-z^2} \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} us + tv &= 1 \\ vs - ut &= 0 \end{aligned}\right\} \Rightarrow (1)\ (2)$$

S^2: The sphere

$$\chi_S \circ \chi_\theta^{-1} : (\theta, \varphi) \mapsto (s,t) \qquad s = \frac{\sin\theta\cos\varphi}{1+\cos\theta} \qquad t = \frac{\sin\theta\sin\varphi}{1+\cos\theta}$$

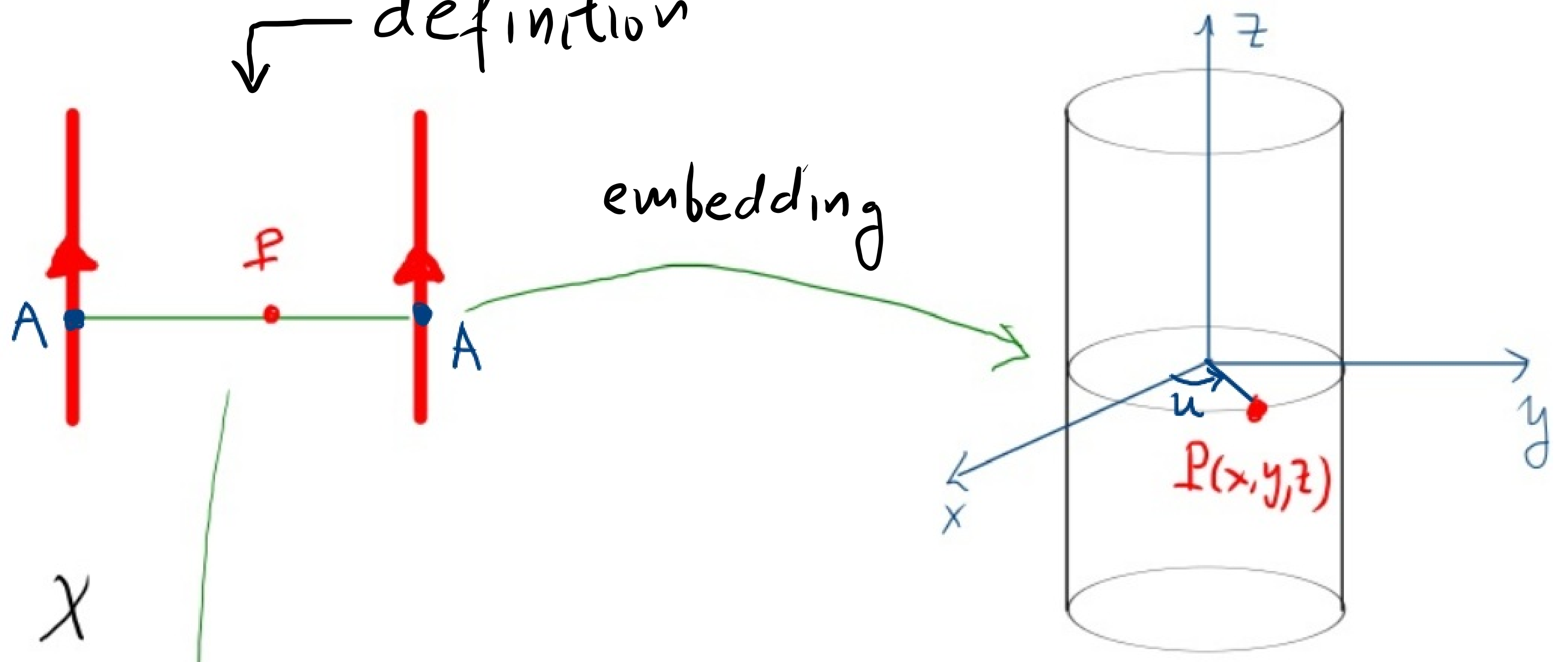
$$\chi_S \circ \chi_N^{-1} : (u,v) \mapsto (s,t) \qquad s = \frac{u}{u^2+v^2} \qquad t = \frac{v}{u^2+v^2}$$

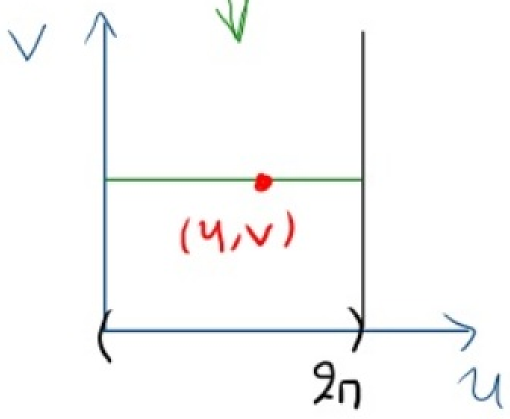
* all transition maps differentiable
* $\{(U_N, \chi_N), (U_S, \chi_S)\}$ an atlas of S^2

  minimal: can't find w/ 1 chart ($S^2 \not\cong \mathbb{R}^2$)

$S^1 \times \mathbb{R}$: The Cylinder

definition





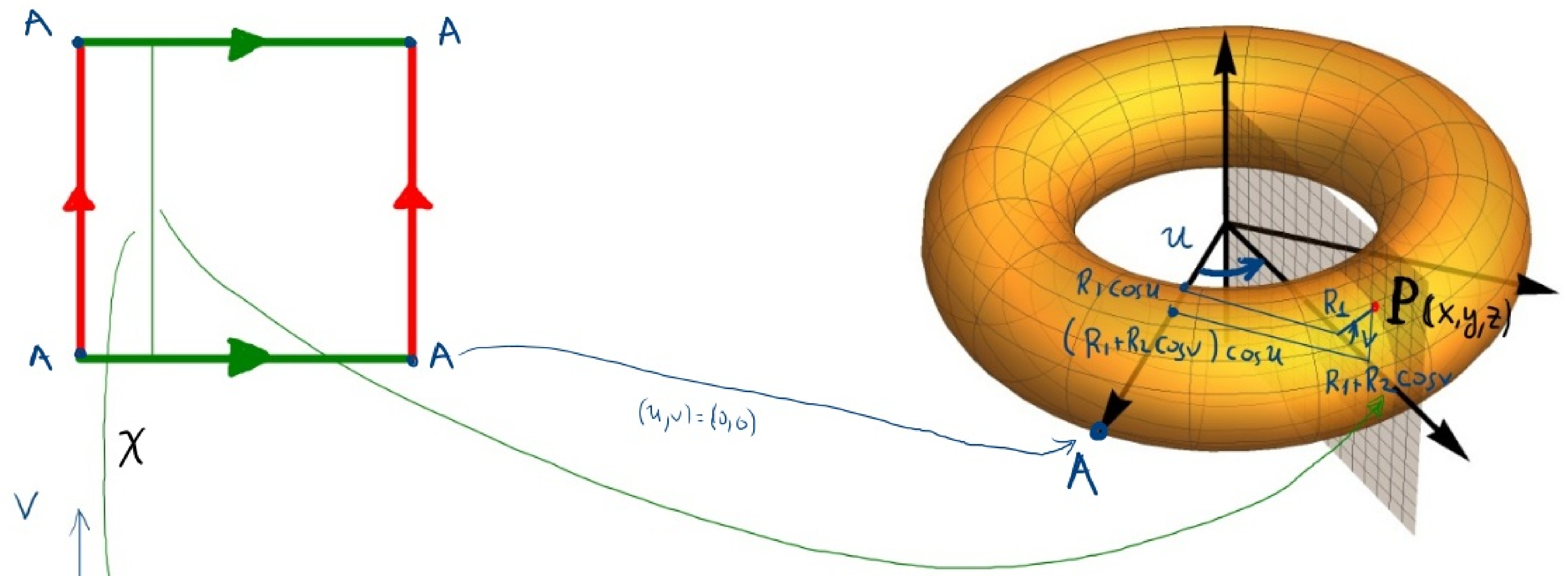
$$x = R\cos u$$
$$y = R\sin u$$
$$z = v$$

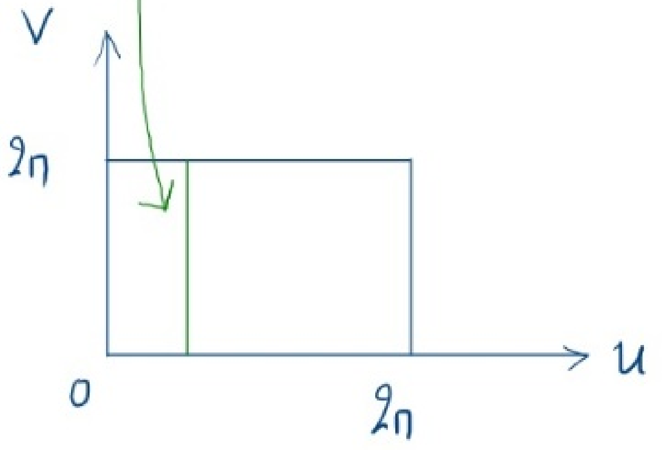
(U, X) : $U = S^1 \times \mathbb{R} \setminus \{x = R \text{ line}\}$

$X : (x, y, z) \mapsto (u, v)$ $\quad 0 < u < 2\pi$

$\quad -\infty < v < +\infty$

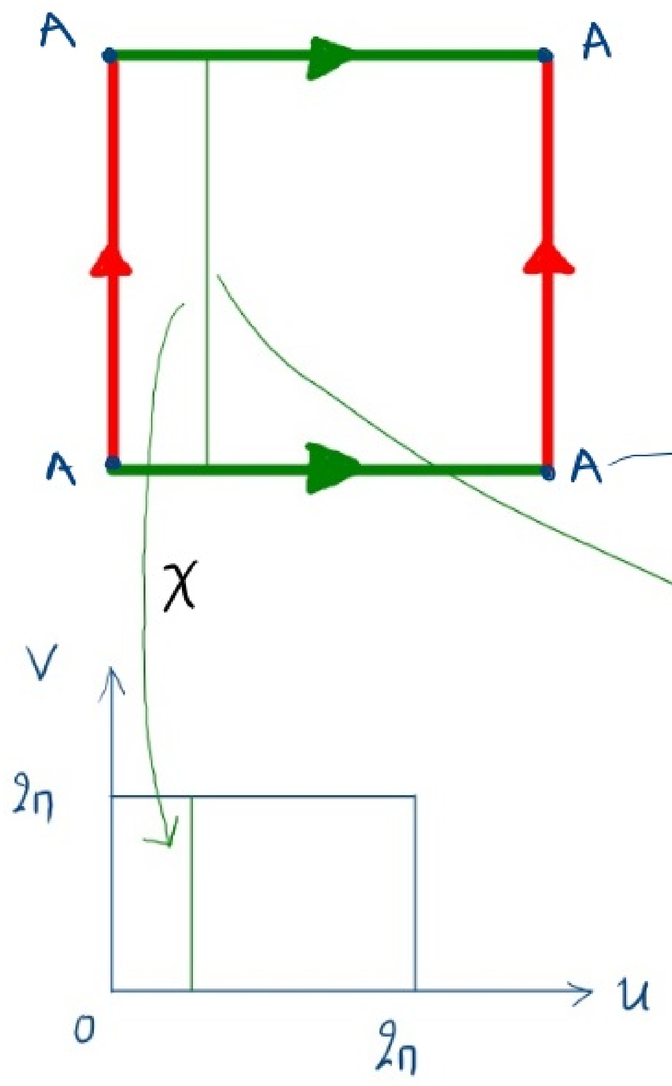
$T^2 = S^1 \times S^1$: The Torus

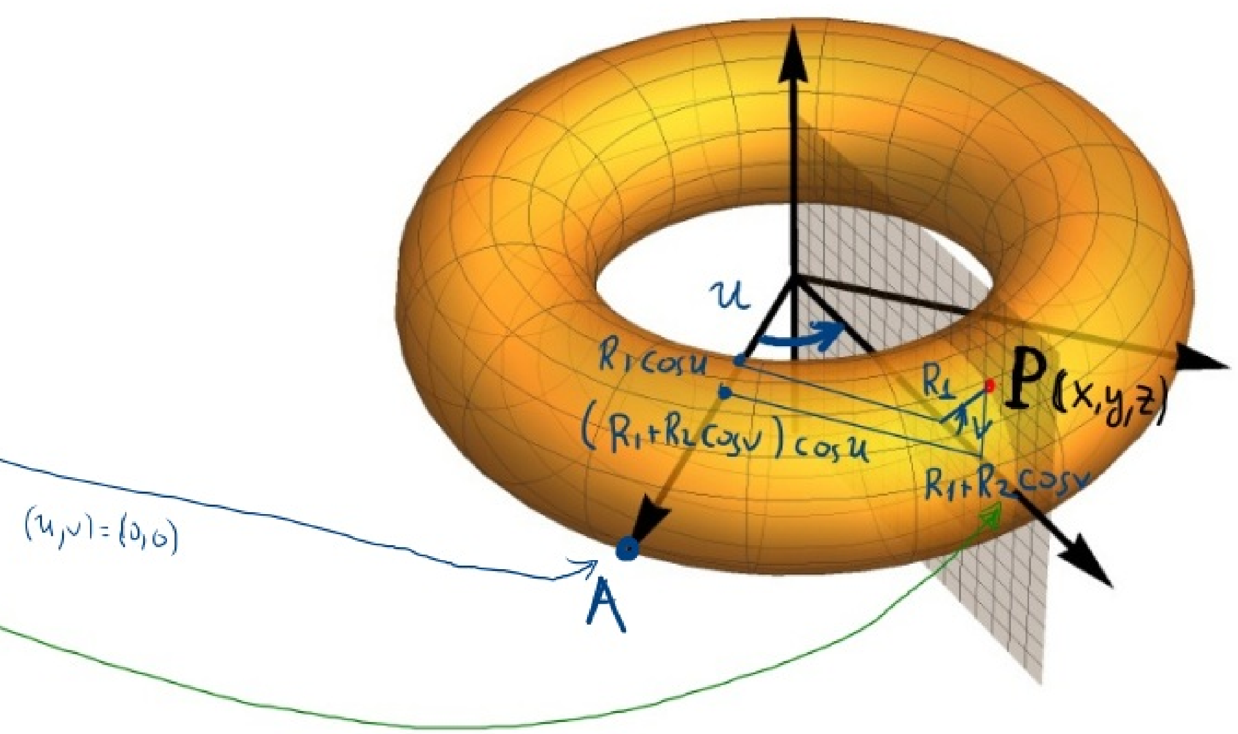




$$x = (R_1 + R_2 \cos v) \cos u$$
$$y = (R_1 + R_2 \cos v) \sin u$$
$$z = R_2 \sin v$$

$T^2 = S^1 \times S^1$: The Torus



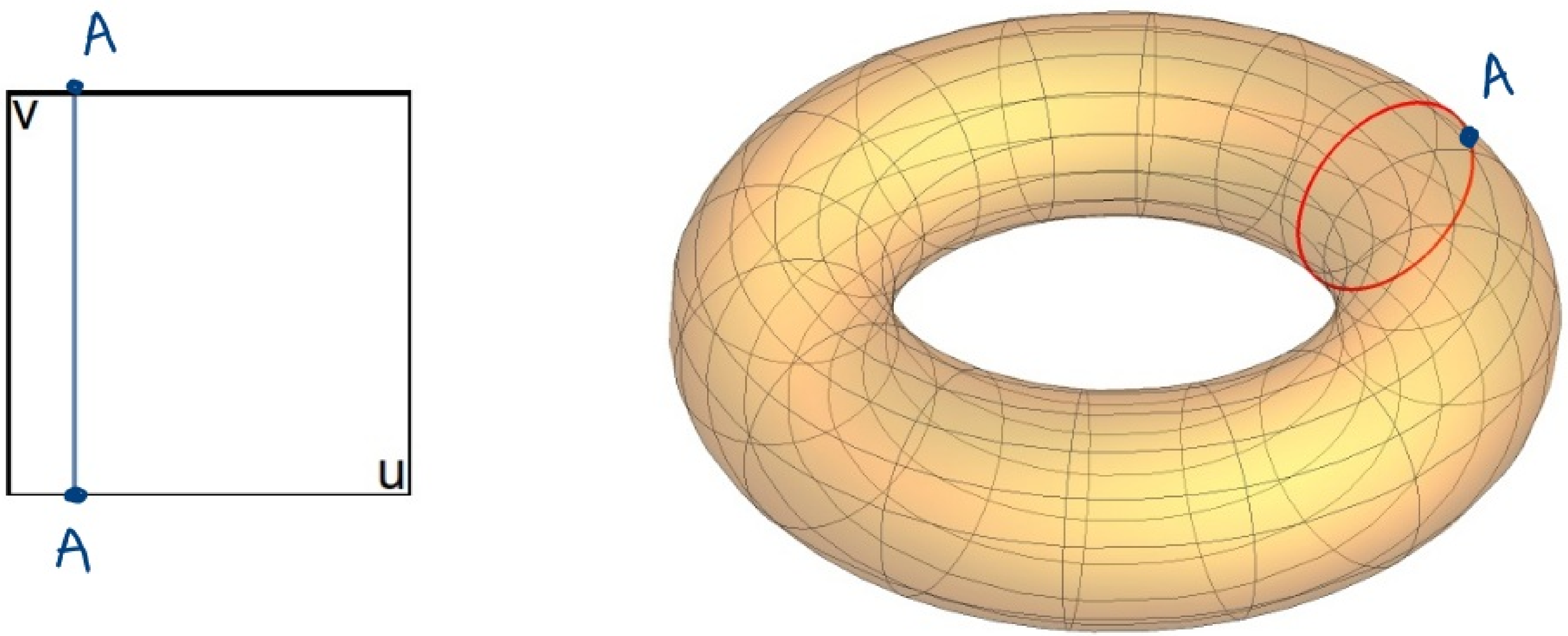


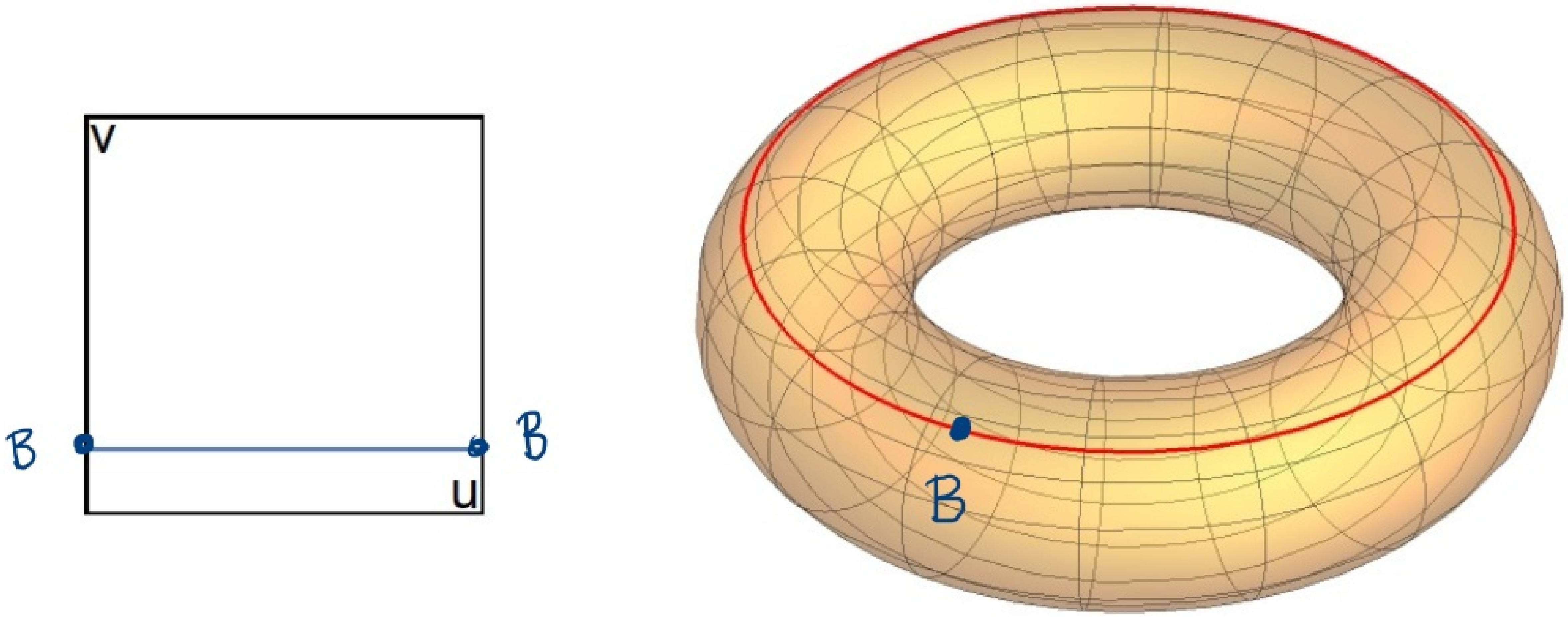
$$x = (R_1 + R_2 \cos v)\cos u$$
$$y = (R_1 + R_2 \cos v)\sin u$$
$$z = R_2 \sin v$$

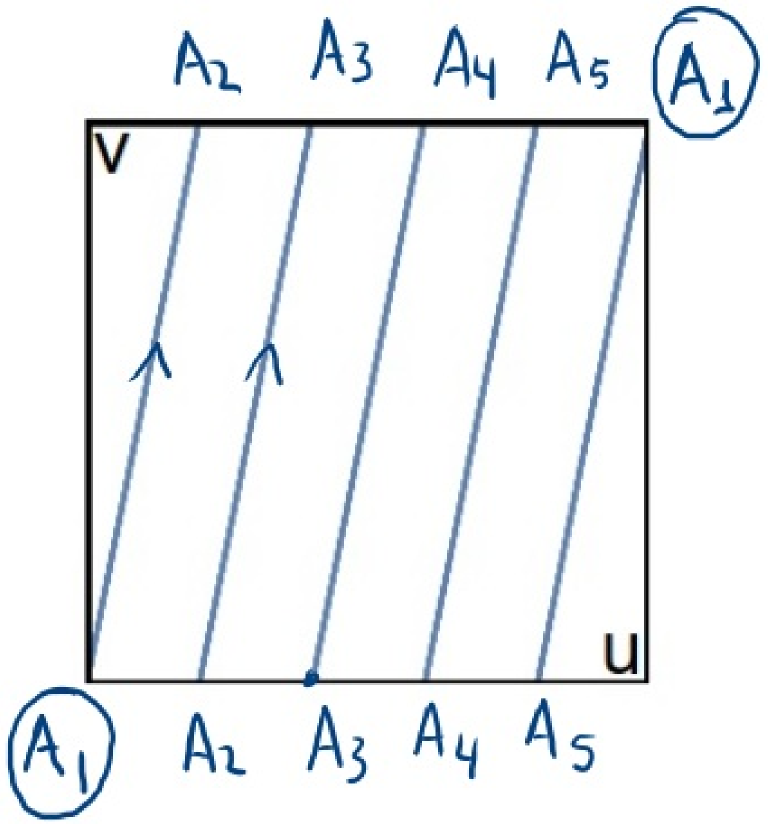
$(\mathcal{U}, \chi)$: $\mathcal{U} = T^2 \setminus (C_1 \cup C_2)$

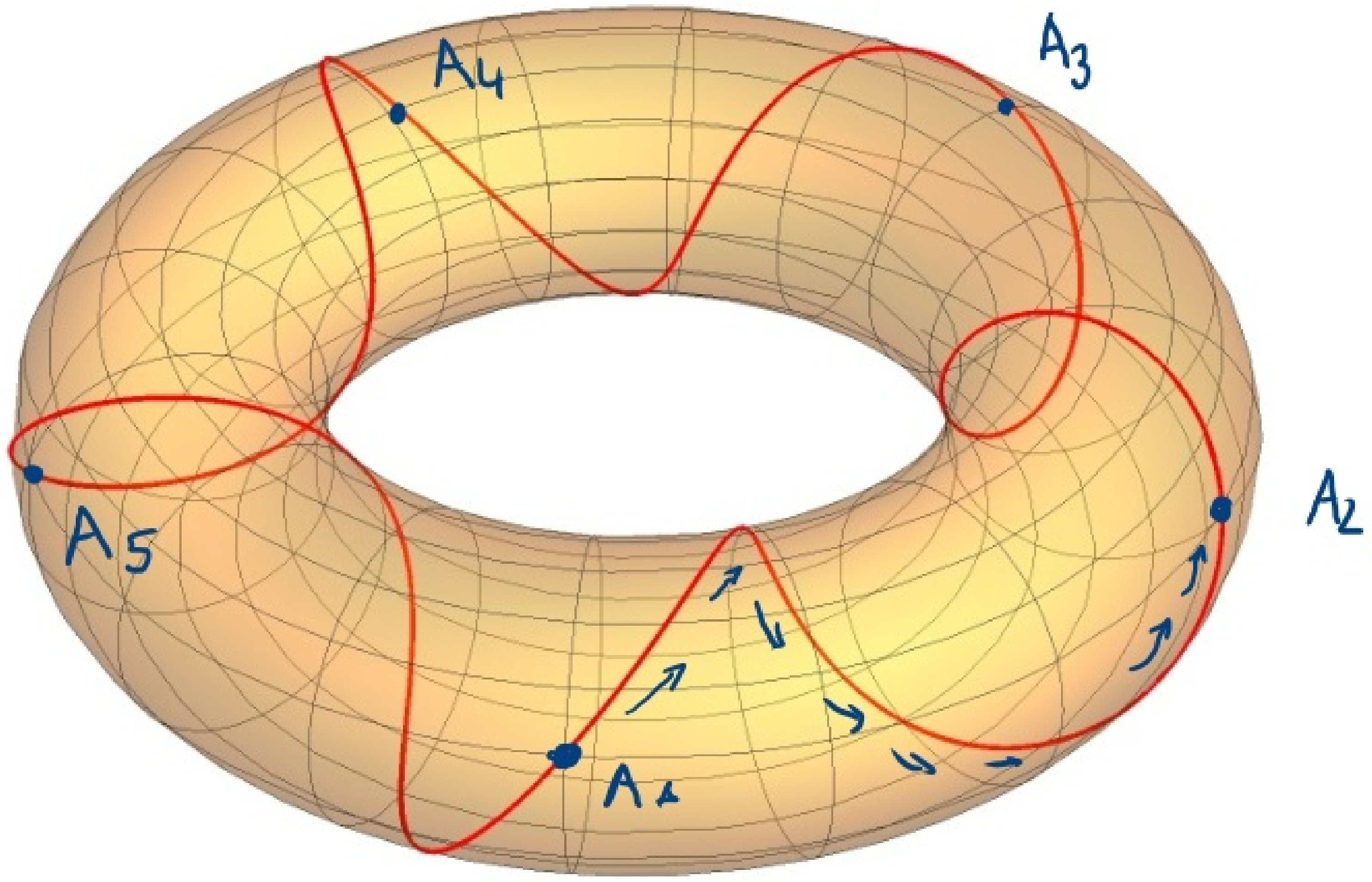
$C_1 = \{\text{the "red" circle}\}$ $\chi : (x, y, z) \mapsto (u, v)$

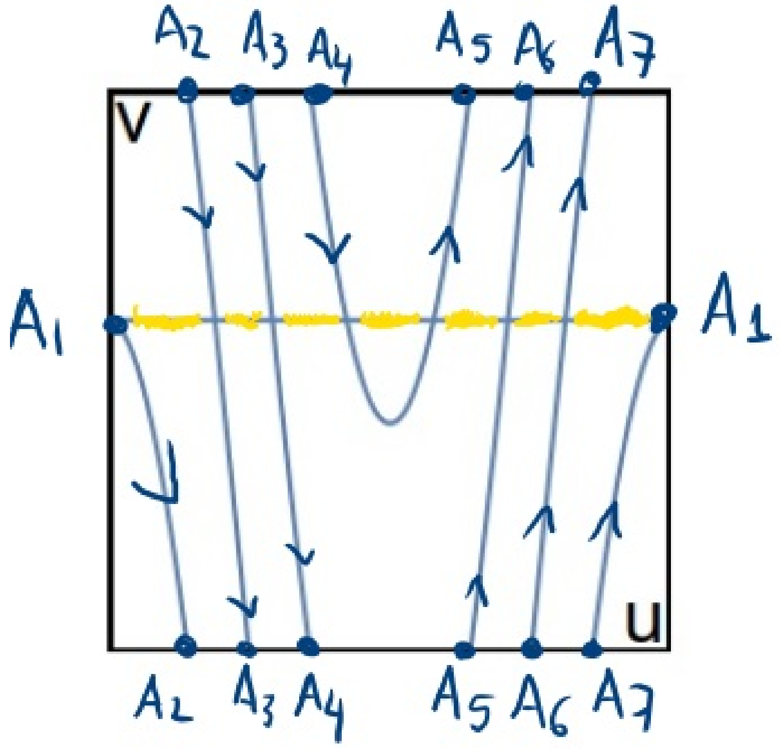
$C_2 = \{\text{the "green" circle}\}$ $0 < u < 2\pi$ $0 < v < 2\pi$

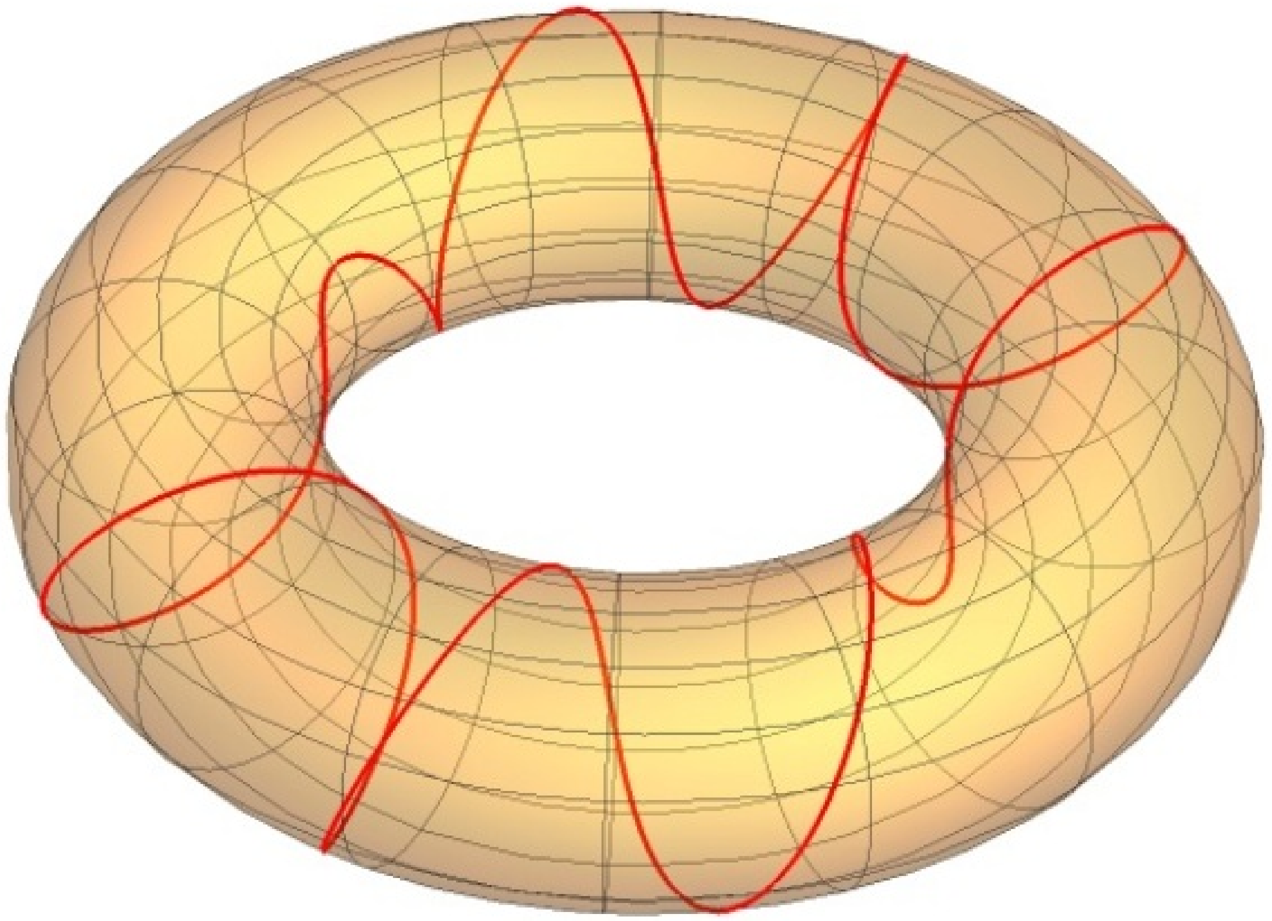

A
v
u
A
A


v
B
B
u
B

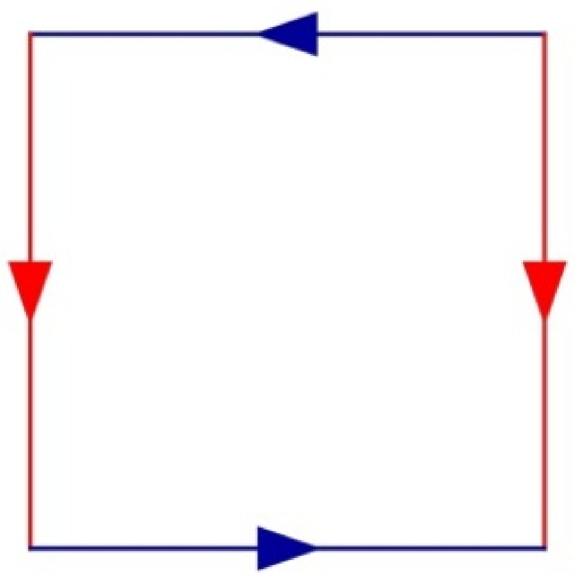

A2 A3 A4 A5 A1
v
u
A1 A2 A3 A4 A5


A4
A3
A5
A2
A1


A2 A3 A4 A5 A6 A7
v
A1
A1
u
A2 A3 A4 A5 A6 A7

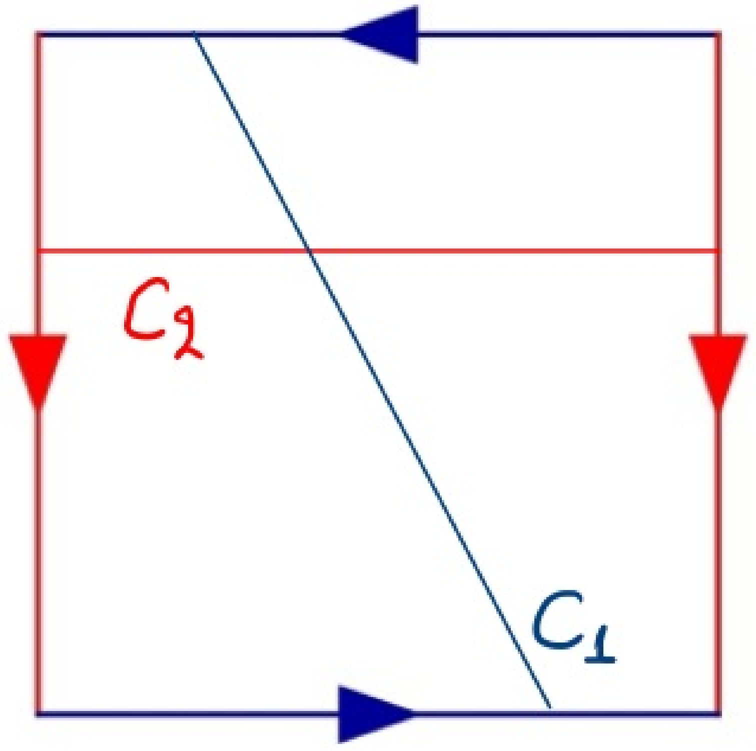


Klein Bottle

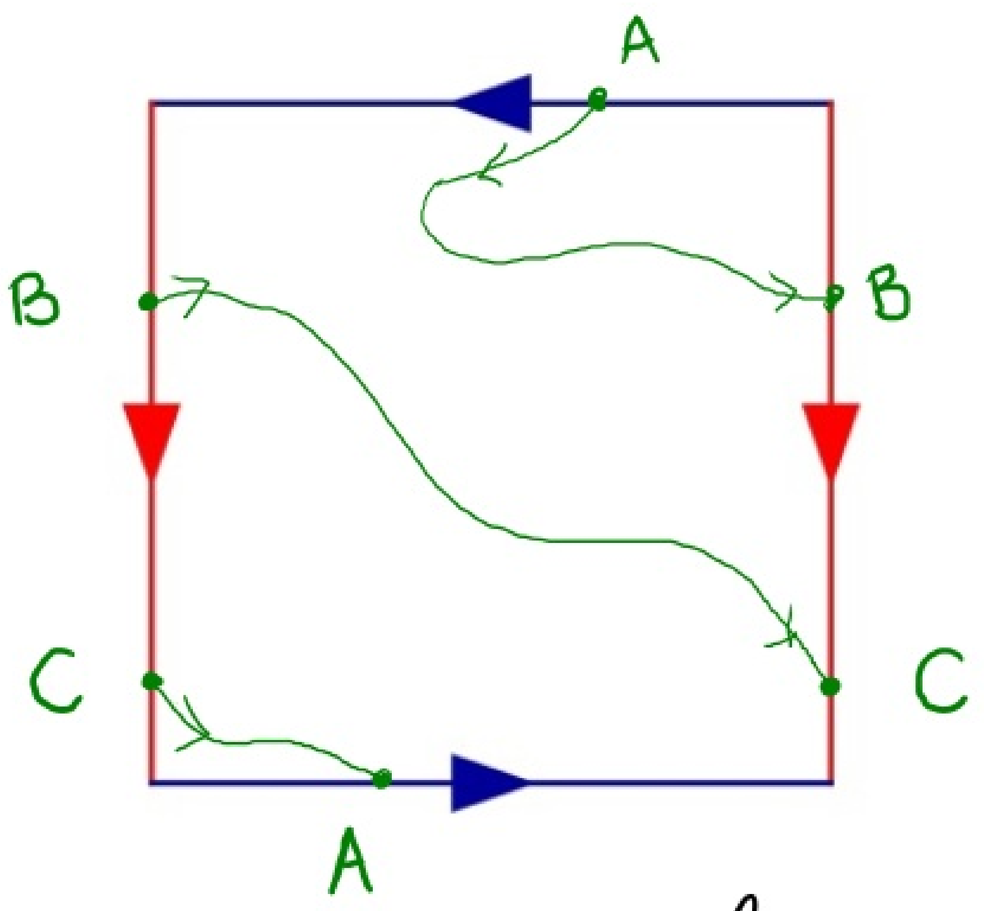


- Non - orientable
- Not embeddable in $\mathbb{R}^3$
  (Ok in $\mathbb{R}^4$ - Whitney's embedding theorem)

Klein Bottle

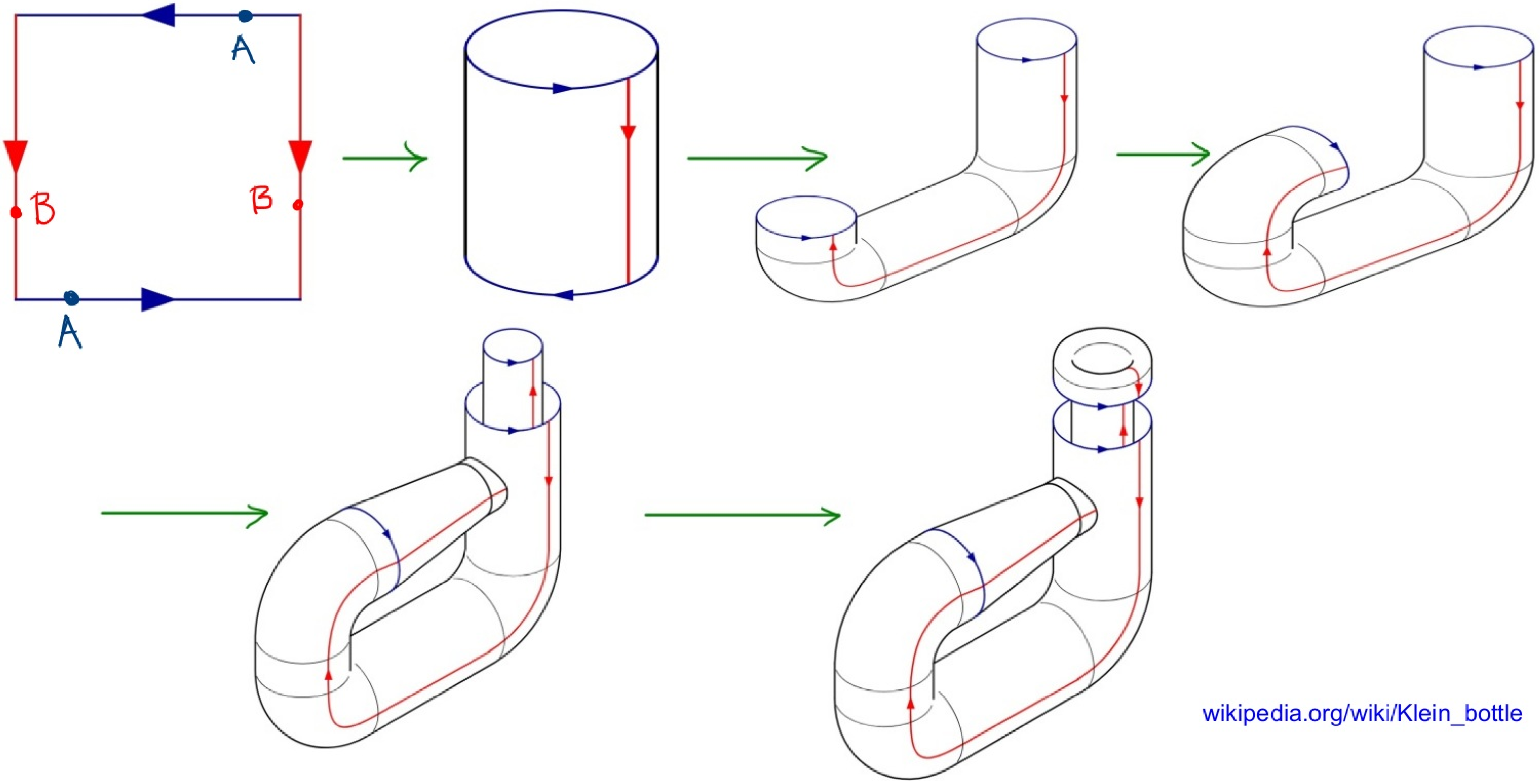


C_1 and C_2 are circles



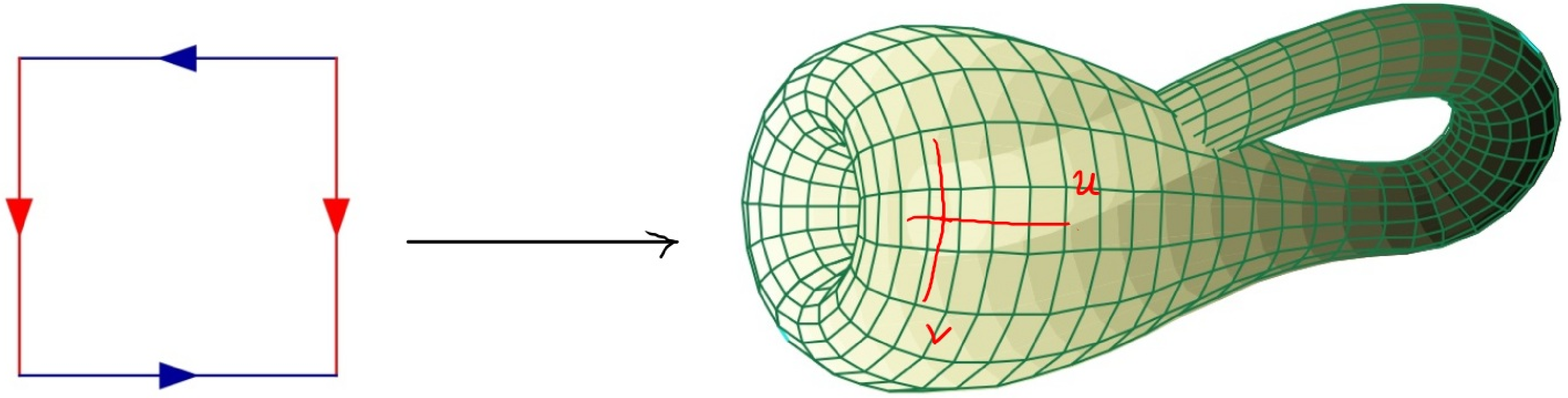
- a closed loop
- notice the direction of velocities at A, C

Klein Bottle



wikipedia.org/wiki/Klein_bottle

Klein Bottle



$$x(u,v) = -\frac{2}{15}\cos u\,(3\cos v - 30\sin u + 90\cos^4 u\sin u - \\ 60\cos^6 u\sin u + 5\cos u\cos v\sin u)$$

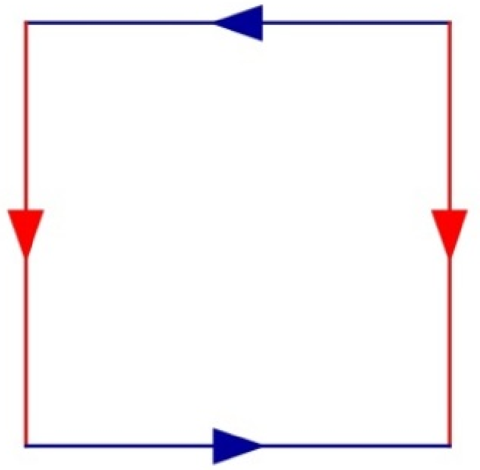
$$y(u,v) = -\frac{1}{15}\sin u\,(3\cos v - 3\cos^2 u\cos v - 48\cos^4 u\cos v + 48\cos^6 u\cos v - \\ 60\sin u + 5\cos u\cos v\sin u - 5\cos^3 u\cos v\sin u - \\ 80\cos^5 u\cos v\sin u + 80\cos^7 u\cos v\sin u)$$

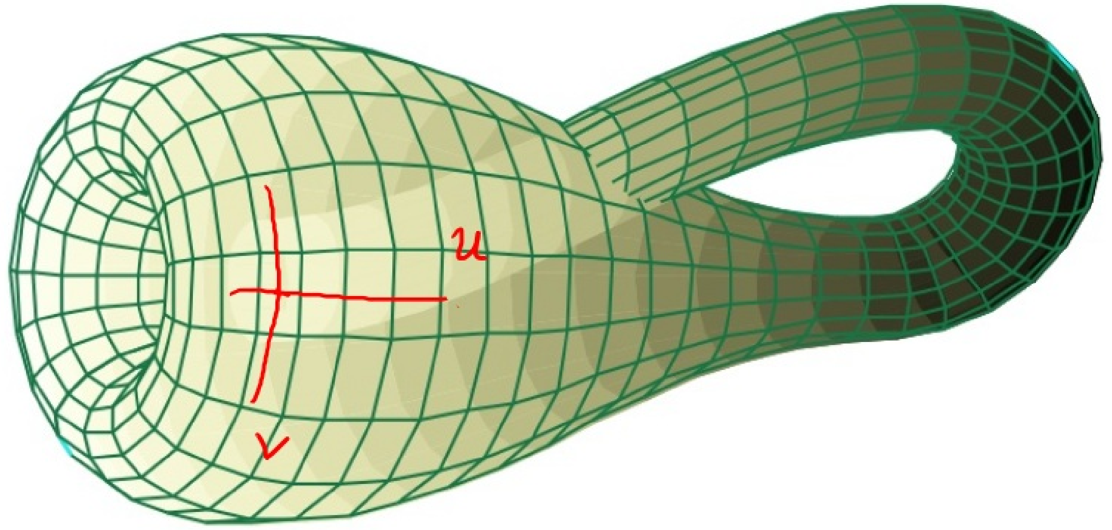
$$z(u,v) = \frac{2}{15}(3 + 5\cos u\sin u)\sin v$$

$0 < u < \pi \qquad 0 < v < 2\pi$

wikipedia.org/wiki/Klein_bottle

Klein Bottle





Embedding in $\mathbb{R}^4$

4-D non-intersecting [edit]

A non-intersecting 4-D parametrization can be modeled after that of the flat torus:

$$x = R\left(\cos\frac{\theta}{2}\cos v - \sin\frac{\theta}{2}\sin 2v\right)$$
$$y = R\left(\sin\frac{\theta}{2}\cos v + \cos\frac{\theta}{2}\sin 2v\right)$$
$$z = P\cos\theta\,(1 + \epsilon\sin v)$$
$$w = P\sin\theta\,(1 + \epsilon\sin v)$$

wikipedia.org/wiki/Klein_bottle