physics.ntua.gr/konstant/GR

Konstantinos Anagnostopoulos
Email: konstant@mail.ntua.gr
Internet:physics.ntua.gr/konstant
Office : 104, 1st floor, Physics Bldg.
Tel.: 210 772 1641

Nick Mavromatos
Email: mavroman@mail.ntua.gr
Internet:physics.ntua.gr/members.html#Mavromatos
Office : 218, 2nd floor, Physics Bldg.
Tel.: 210 772 1747

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Bibliography

General Relativity cannot be learned by using only one textbook. You need to have at least five of them open at the same time...

Textbooks

- James Hartle, Gravity: An Introduction to Einstein's General Relativity
- Nick E. Mavromatos, General Relativity and Cosmology
- Sean Carroll, Spacetime and Geometry. An Introduction to General Relativity
- Bernard F. Schutz, Geometrical Methods of Mathematical Physics

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General Relativity

- Bernard Schutz, A First Course in General Relativity
- Robert M. Wald, General Relativity
- Charles W. Misner, Kip S. Thorne, John A. Wheeler, Gravitation
- Valeria Ferrari, Leonardo Gualtieri, Paolo Pani, General Relativity and its Applications
- Steven Weinberg, Gravitation and Cosmology
- Norbert Straumann, General Relativity
- Anthony Zee, Einstein Gravity in a Nutshell
- Lewis Ryder, Introduction to General Relativity
- Yvonne Choquet-Bruhat, General Relativity and the Einstein Equation
- Stephen W. Hawking & George F.R. Ellis, The Large Scale Structure of Space-Time
- Achilles Papapetrou, Lectures on General Relativity

Cosmology

- Andrew Liddle, An Introduction to Modern Cosmology
- Steven Weinberg, Cosmology
- Daniel Baumann, Cosmology
- Barbara Ryden, Introduction to Cosmology
- Phillip J.E. Peebles, Principles of Physical Cosmology

Differential Geometry

- Bernard F. Schutz, Geometrical Methods of Mathematical Physics
- Yvonne Choquet-Bruhat, Cecile Dewitt-Morette, Margaret Dillard-Bleick, Analysis, Manifolds and Physics
- Chris J. Isham, Modern Differential Geometry for Physicists
- Mikio Nakahara, Geometry, Topology and Physics
- Charles Nash and Siddhartha Sen, Topology and Geometry for Physicists

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Video Lectures on the General Theory of Relativity

Long and shorter videos on selected topics, problems, and computer exercises.

Virtual lectures by the instructor on selected topics, problems and computer exercises, supplementing the material presented in class. This is an ongoing effort, and you should expect the list to grow in time. Next to the video links, you will find links to transparencies, Mathematica & Maxima notebooks and other relevant material used in the videos.

You can also find the videos in a youtube video list. A separate list with computer exercises using Mathematica, can be found here. The list below contains all the videos in those lists, together with some metadata.

- Unit 1: Differential Geometry
  1. Manifolds: Differential Manifolds, topological spaces, charts, transition functions, atlases. (Slides) Slides
  2. Vectors: Vectors as tangent to curves, tangent space, coordinate basis, component transformations, vector fields, integral curves, Lie bracket, Lie derivative. (Slides)
  3. One forms and tensors: One forms as linear maps on TM, cotagent space, gradient of a function, coordinate bases, tensors, tensor product, contractions, (anti)symmetrization. (Slides)
  4. Differential forms: Differential forms and form fields, wedge product, exterior derivative, interior product, Levi-Civita tensor, duality, Hodge-star operator. (Slides)
  5. Maps: Maps between manifolds, and pullback/pushforward of tensors.
  6. Diffeomorphisms: Diffeomorphic manifolds, pullback/pushforward of tensors, Lie Derivative, components of the Lie derivative in a coordinate basis. (Slides)
  7. Lie Dierivatives: Proof of $\mathcal{L}_V W = [V, W]$, $\mathcal{L}_V f = V(f)$. Computation of $\mathcal{L}_V \omega$. Geometric interpretation of $\mathcal{L}_V W$ and $[V, W]$. (Slides)

Extras:

- Problem Solving
- Computer Labs (Mathematica, Maxima)
- More Lectures...

Gravity:

- Universal: all forms of matter & energy gravitate

Gravity:

- Universal: all forms of matter & energy gravitate
- Always attractive:

  not screened + long range

e.g. electrically neutral matter makes EM irrelevant @ cosmic scales

Gravity:

- Universal: all forms of matter & energy gravitate
- Always attractive:
  not screened + long range

$\Rightarrow$ Dominant @ cosmic scales

(despite being ridiculously weak ...)

Fun High School Physics: escape velocity

$$\frac{1}{2} m v_e^2 = G \frac{m M}{R}$$

Newtonian Potential

→ Force of Gravity

Fun High School Physics: escape velocity

$$\left.\begin{array}{l} \frac{1}{2} m v_e^2 = G \frac{mM}{R} \\[2em] \text{set } v_e = c \end{array}\right\} \Rightarrow \frac{2GM}{c^2 R} = 1$$

Fun High School Physics: escape velocity

$$\left.\begin{array}{l}\frac{1}{2} m v_e^2 = G\,\frac{m M}{R} \\[2em] \text{set} \quad v_e = c\end{array}\right\} \Rightarrow \underbrace{\frac{2GM}{c^2 R} = 1}$$

scale where GR effects become strong!

Fun High School Physics: escape velocity

$$\left.\begin{array}{l}\frac{1}{2} m v_e^2 = G \frac{mM}{R} \\ \text{set } \quad v_e = c\end{array}\right\} \Rightarrow \underbrace{\frac{2GM}{c^2 R} = 1}$$

scale where GR effects become strong!

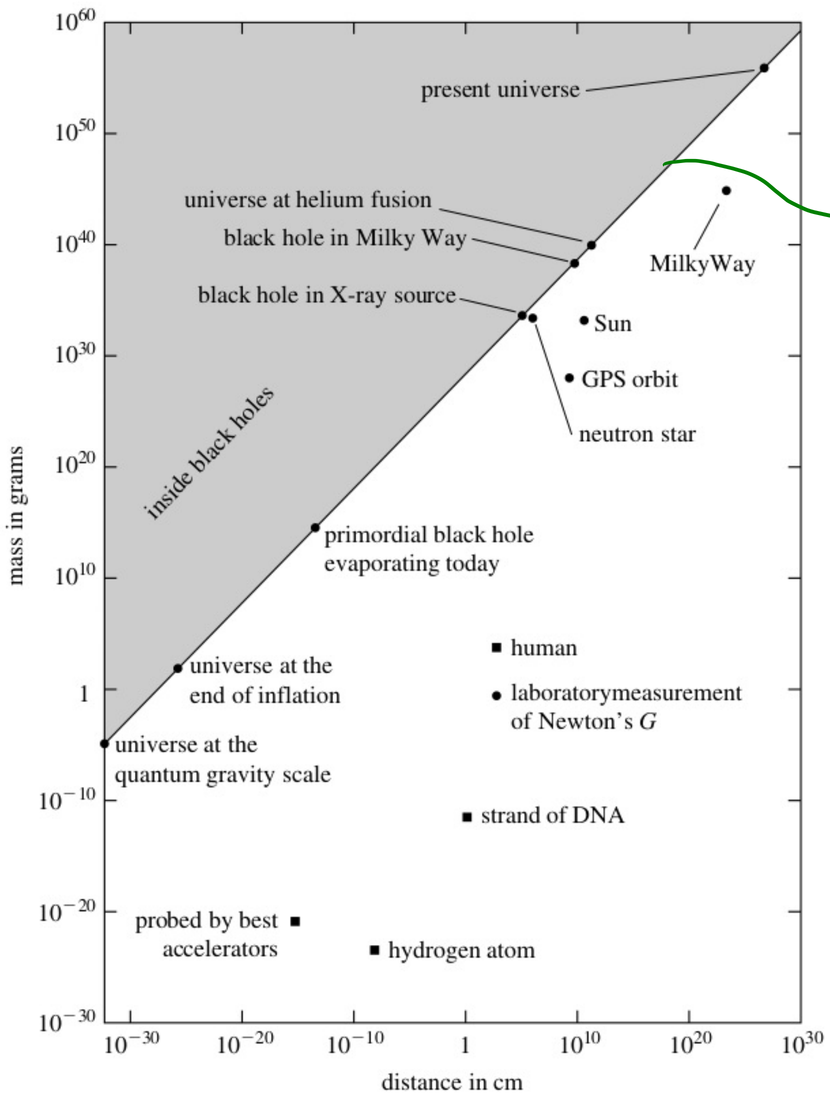
Earth $\sim 10^{-9}$

Sun $\sim 10^{-6}$

Neutron stars $\sim 10^{-1}$!!

General Relativistic Effects

$$\frac{2GM}{c^2 R} = 1 \quad \text{line}$$

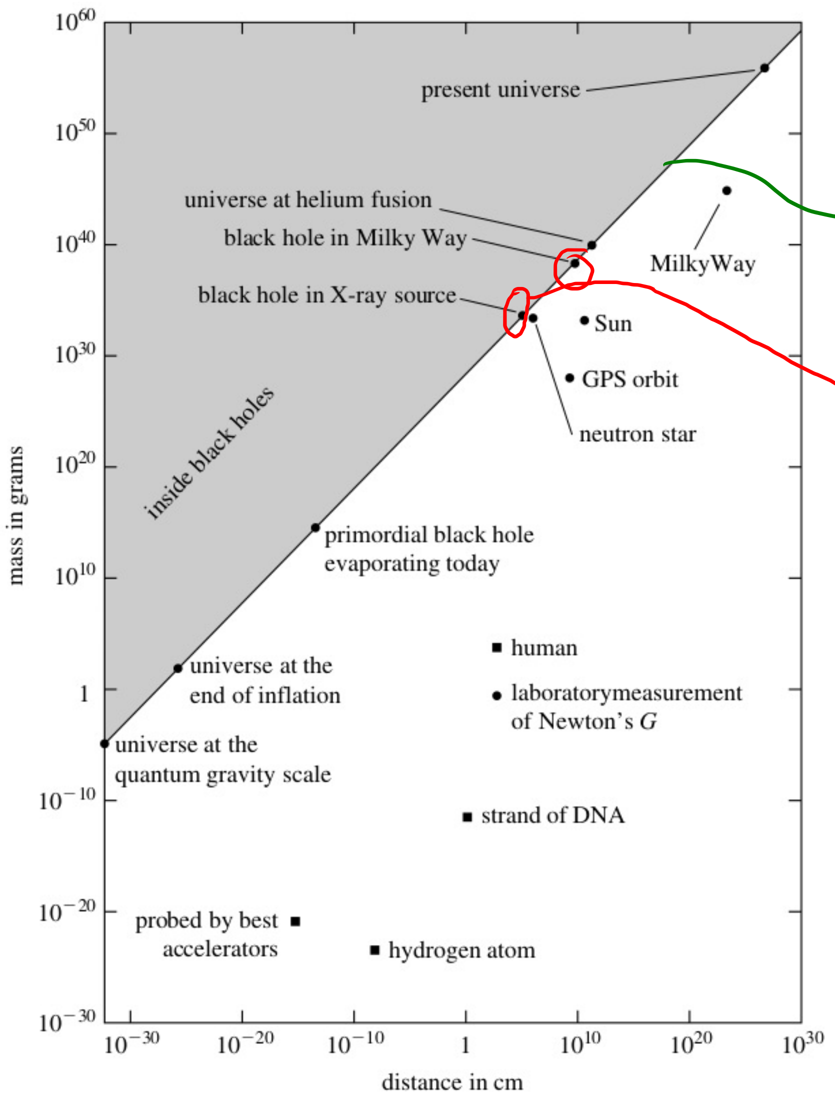


MASS

DISTANCE

Hartle, Fig 1.1

General Relativistic Effects



MASS

DISTANCE

Hartle, Fig 1.1

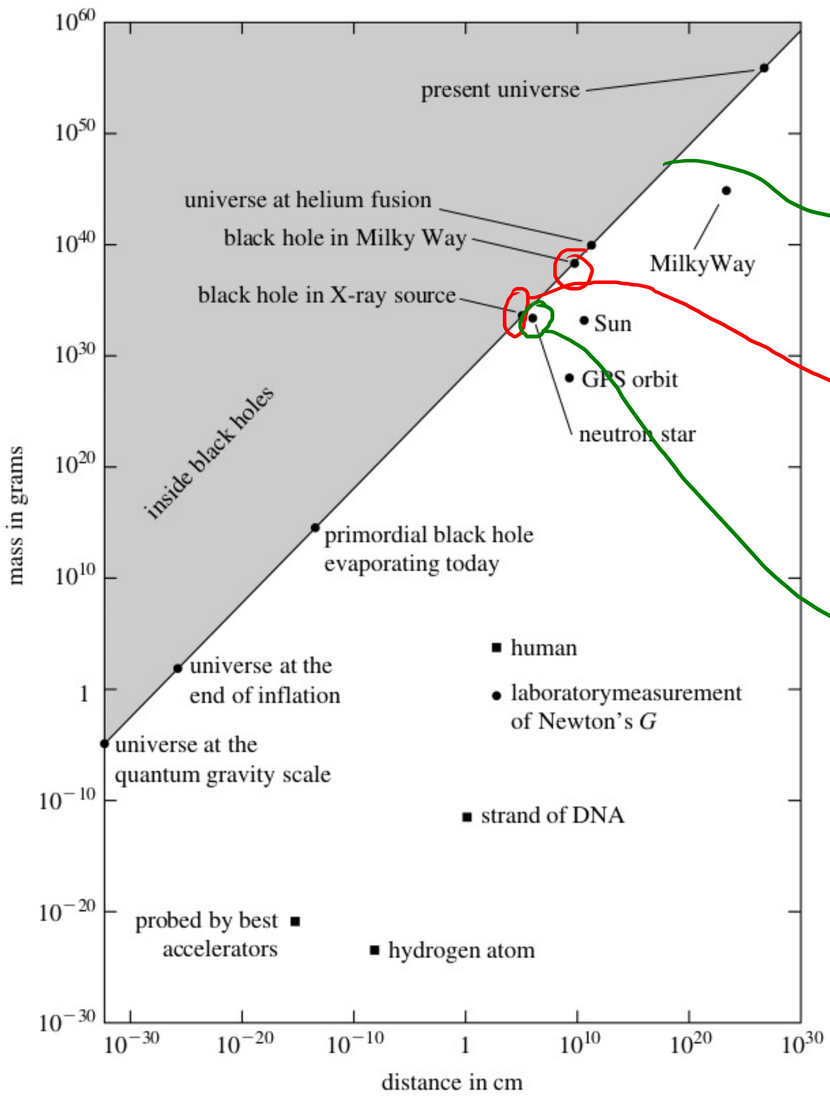
$$\frac{2GM}{c^2 R} = 1 \quad \text{line}$$

Black Holes: stellar

supermassive

General Relativistic Effects

MASS



Hartle, Fig 1.1

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Black Holes: stellar

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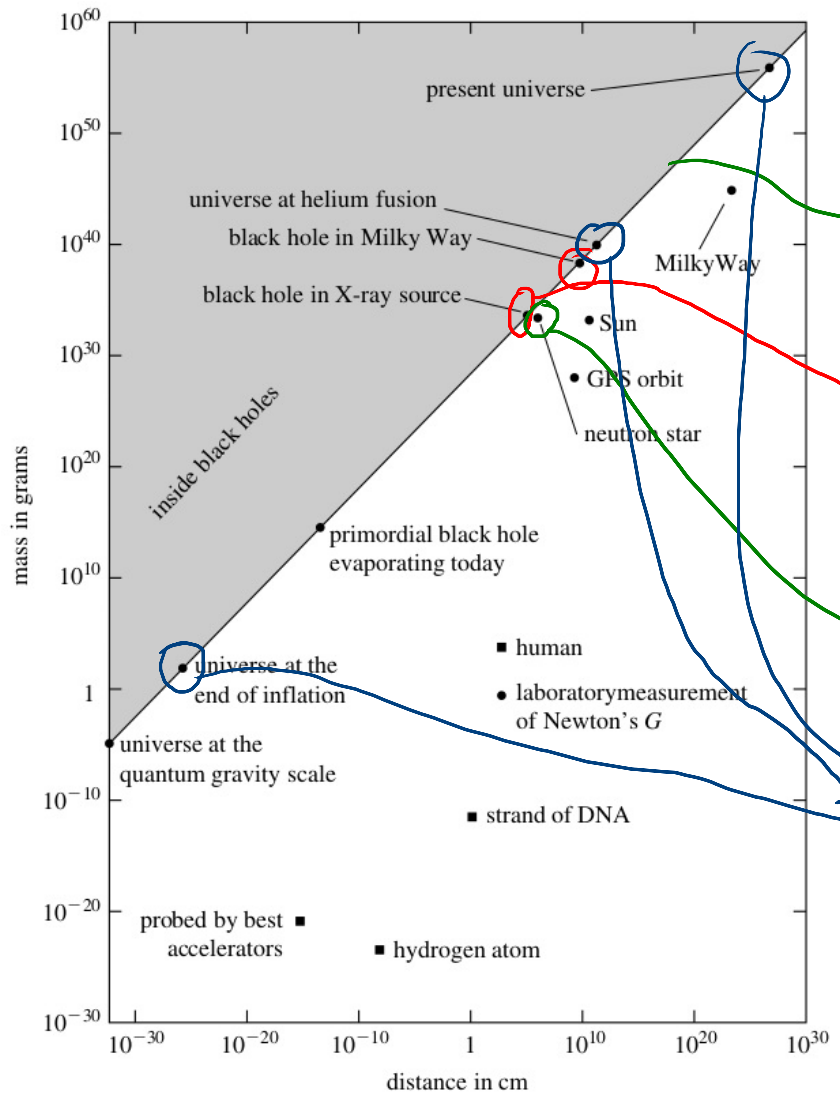
General Relativistic Effects

MASS



Hartle, Fig 1.1

DISTANCE

$$\frac{2GM}{c^2 R} = 1 \quad \text{line}$$

Black Holes: stellar

supermassive

Neutron Stars

Cosmology

Planck Scales:

$$\ell_{\rm Pl} \equiv (G\hbar/c^3)^{1/2} = 1.62 \times 10^{-33}\ \text{cm},$$

$$t_{\rm Pl} \equiv (G\hbar/c^5)^{1/2} = 5.39 \times 10^{-44}\ \text{s},$$

$$E_{\rm Pl} \equiv (\hbar c^5/G)^{1/2} = 1.22 \times 10^{19}\ \text{GeV},$$

$$\rho_{\rm Pl} \equiv c^5/\hbar G^2 = 5.16 \times 10^{93}\ \text{g/cm}^3.$$

$G - \hbar - c$

fundamental constants

FRONTIERS!

Planck Scales:

$$\ell_{\mathrm{Pl}} \equiv (G\hbar/c^3)^{1/2} = 1.62 \times 10^{-33}\ \mathrm{cm},$$
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$G - \hbar - c$

fundamental constants

? Quantum Gravity ?

? Unification ?

? Singularity Resolution ?

? BH Information Paradox ?

? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ?

- Newtonian Gravity:

→ gravity is a force!

$$\vec{F} = \frac{G m M}{r^2} \hat{r} = m \vec{g}$$

! ! ! ! Universal for ! all kinds of matter! !

- Newtonian Gravity:

→ gravity is a force!

$$\vec{F} = \frac{G m M}{r^2} \hat{r} = m \vec{g}$$

$$\vec{g} = - \vec{\nabla} \Phi(\vec{x})$$

$$\vec{\nabla} \cdot \vec{g} = - \nabla^2 \Phi(\vec{x}) = - 4\pi G \rho(\vec{x})$$

- Newtonian Gravity:

→ gravity is a force!

$$\vec{F} = \frac{G m M}{r^2} \hat{r} = m \vec{g}$$

gravity

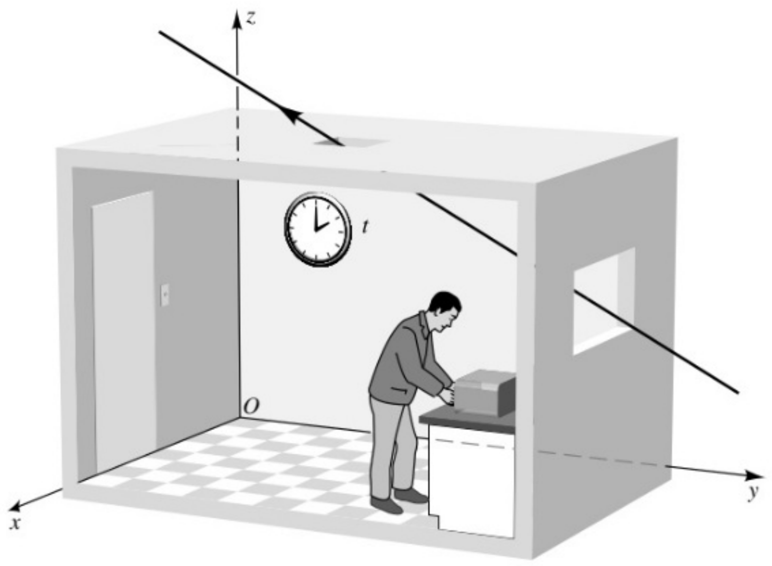
matter

$$\nabla^2 \Phi(\vec{x}) = 4\pi G \rho(\vec{x})$$

Problems

- Propagates instantaneously
- Singles preferred observer for which Newton's law applies
- Nature does not agree...

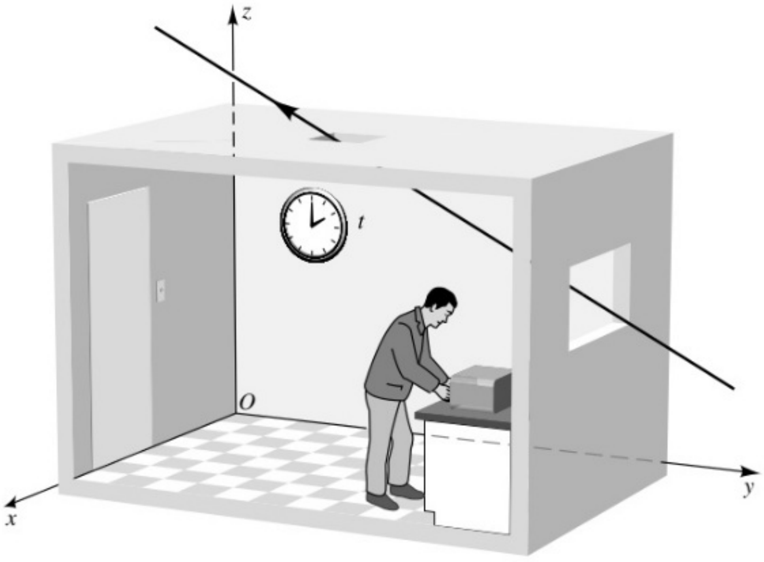
Newtonian Mechanics:



Inertial Observers:

$\exists$ parameter t s.t. free particles change position @ constant rate

Newtonian Mechanics:



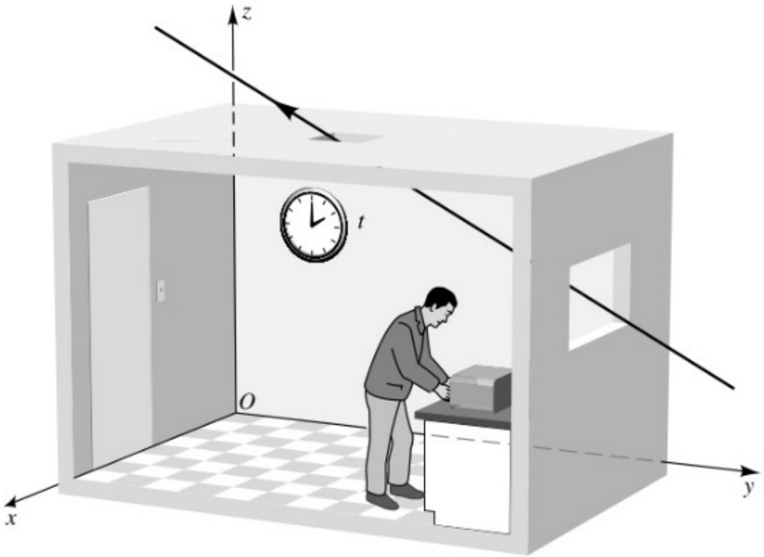
special class of observers

Universal time
and
simultaneity

Inertial Observers:

$\exists$ parameter t s.t. free particles change position @ constant rate

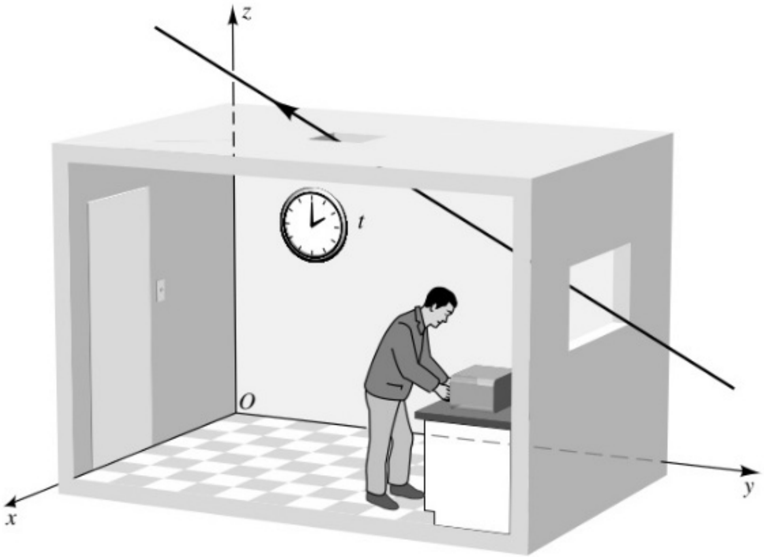
Newtonian Mechanics:



Inertial Observers:

Move @ constant relative speed

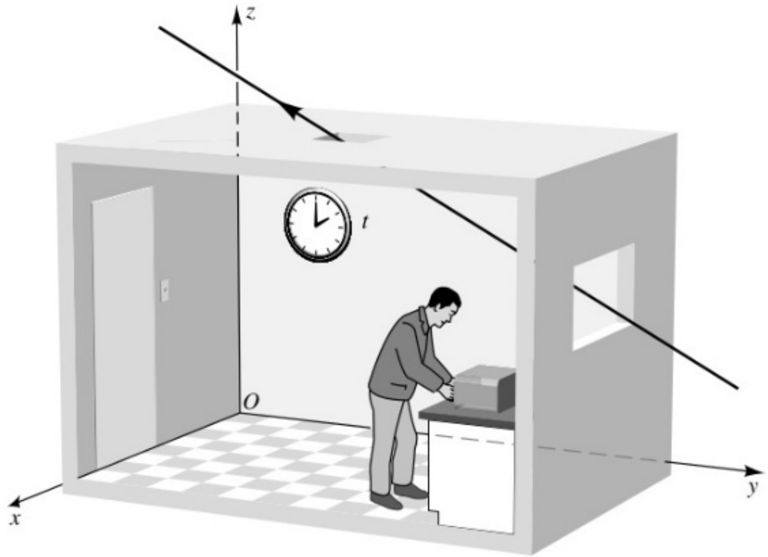
Newtonian Mechanics:

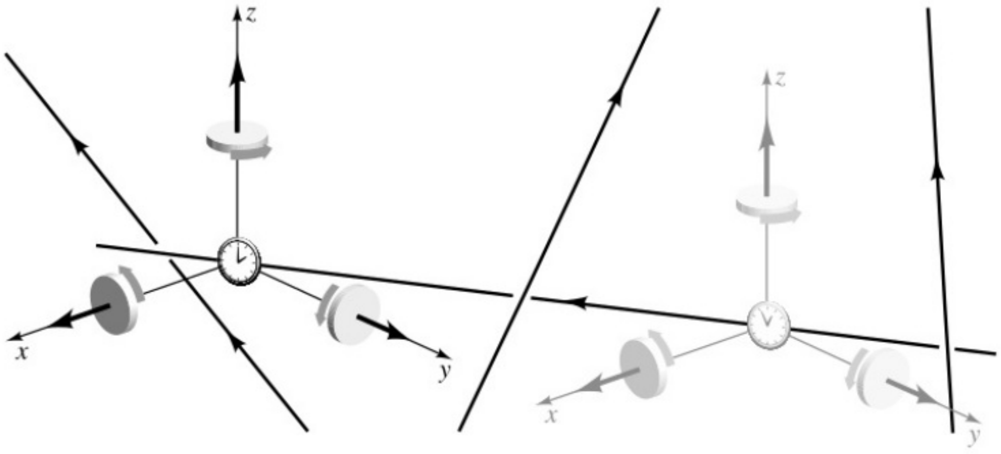


Construct:

1. pick axes' origin a free particle

Newtonian Mechanics:

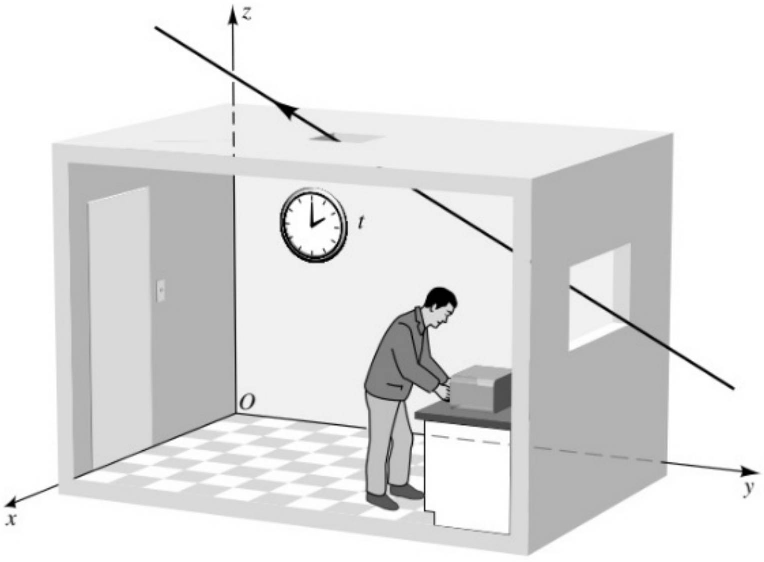




Construct:

1. pick axes' origin a free particle
2. choose axes using 3 orthogonal gyroscopes

Newtonian Mechanics:

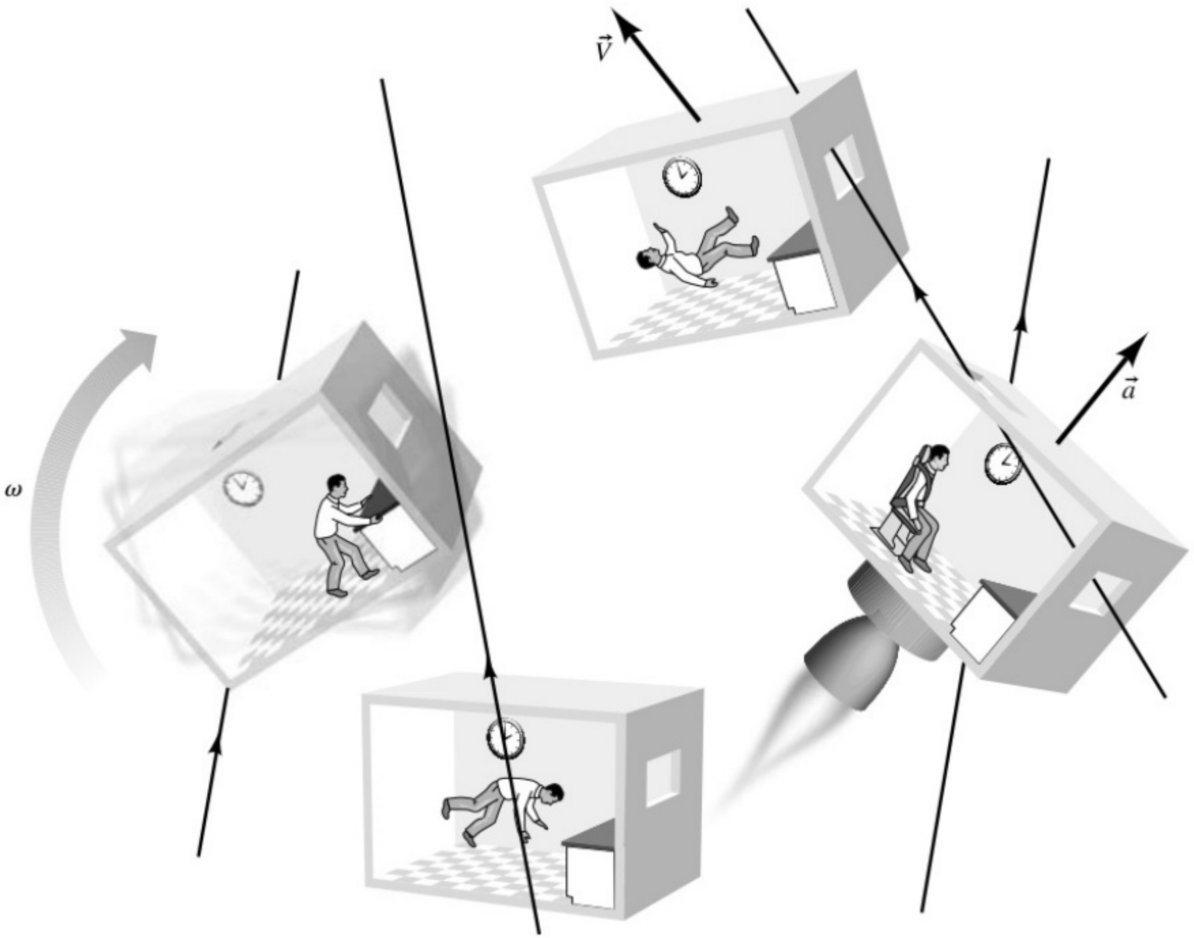


Free particles:

$$\frac{d^2x}{dt^2} = 0 \qquad \frac{d^2y}{dt^2} = 0 \qquad \frac{d^2z}{dt^2} = 0$$

Construct:

1. Pick axes' origin a free particle
2. choose axes using 3 orthogonal gyroscopes



Not all observers are inertial
(we are not!)

General Relativity:

- gravity is NOT a force!

General Relativity:

- gravity is NOT a force!
- all freely falling observers are inertial

General Relativity:

- gravity is NOT a force!
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Crucial:

all matter "falls" w/ same acceleration!

General Relativity:

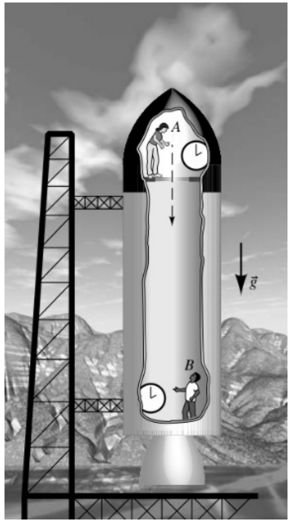
- gravity is NOT a force!
- all freely falling observers are inertial

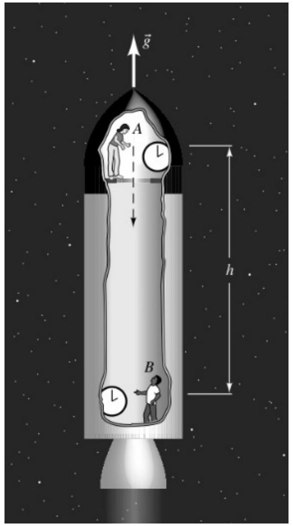
Equivalence Principle

Experiments in a sufficiently small freely falling laboratory, over a sufficiently short time, give results that are indistinguishable from those of the same experiments in an inertial frame in empty space.

General Relativity:

- gravity is NOT a force!
- all freely falling observers are inertial

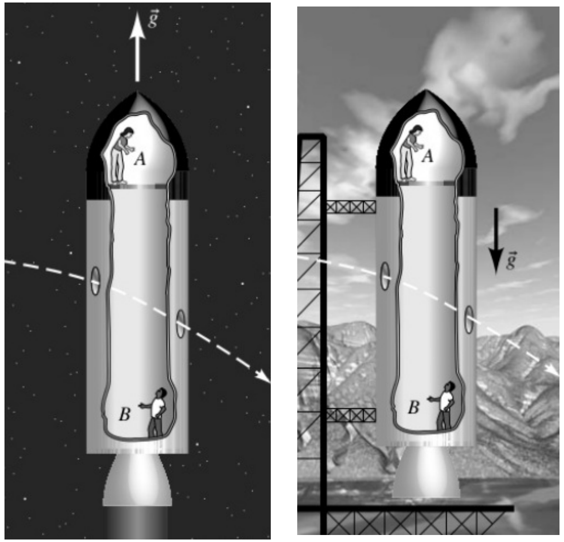




No experiment can distinguish accelerated motion and gravitational falling

General Relativity:

- gravity is NOT a force!
- all freely falling observers are inertial



$\Rightarrow$ light appears to be "falling" in g

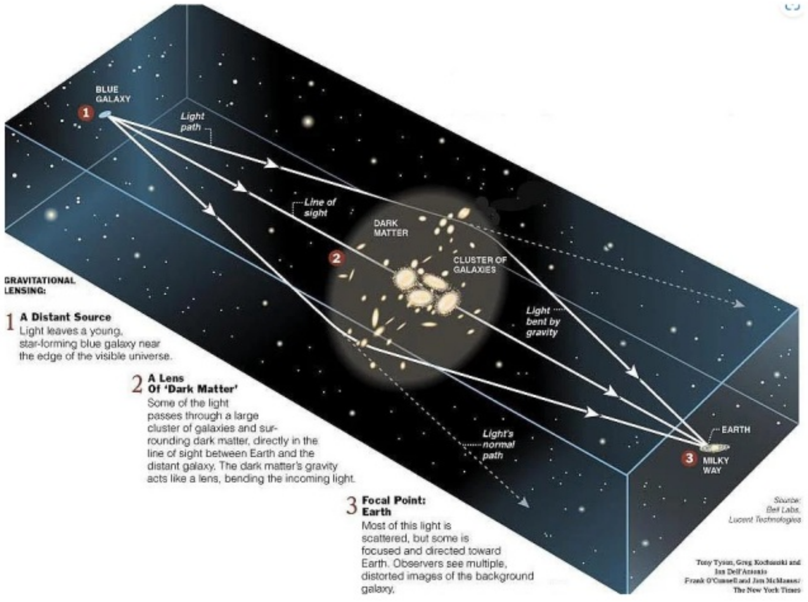
General Relativity:

- gravity is geometry

General Relativity:

- gravity is geometry

all test particles in free fall follow the same trajectories because they are affected by the shape of spacetime!

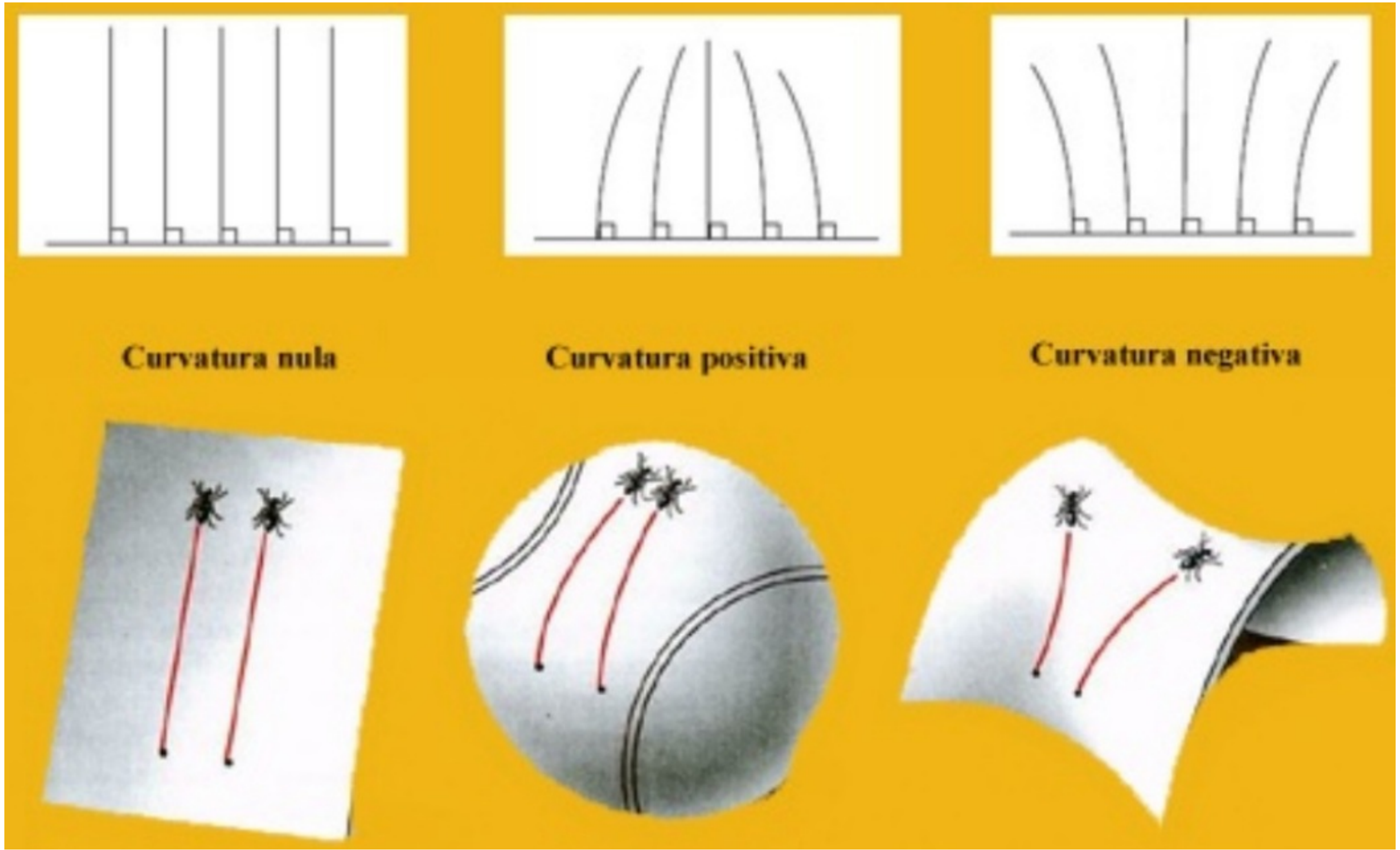


General Relativity:

- gravity is curvature

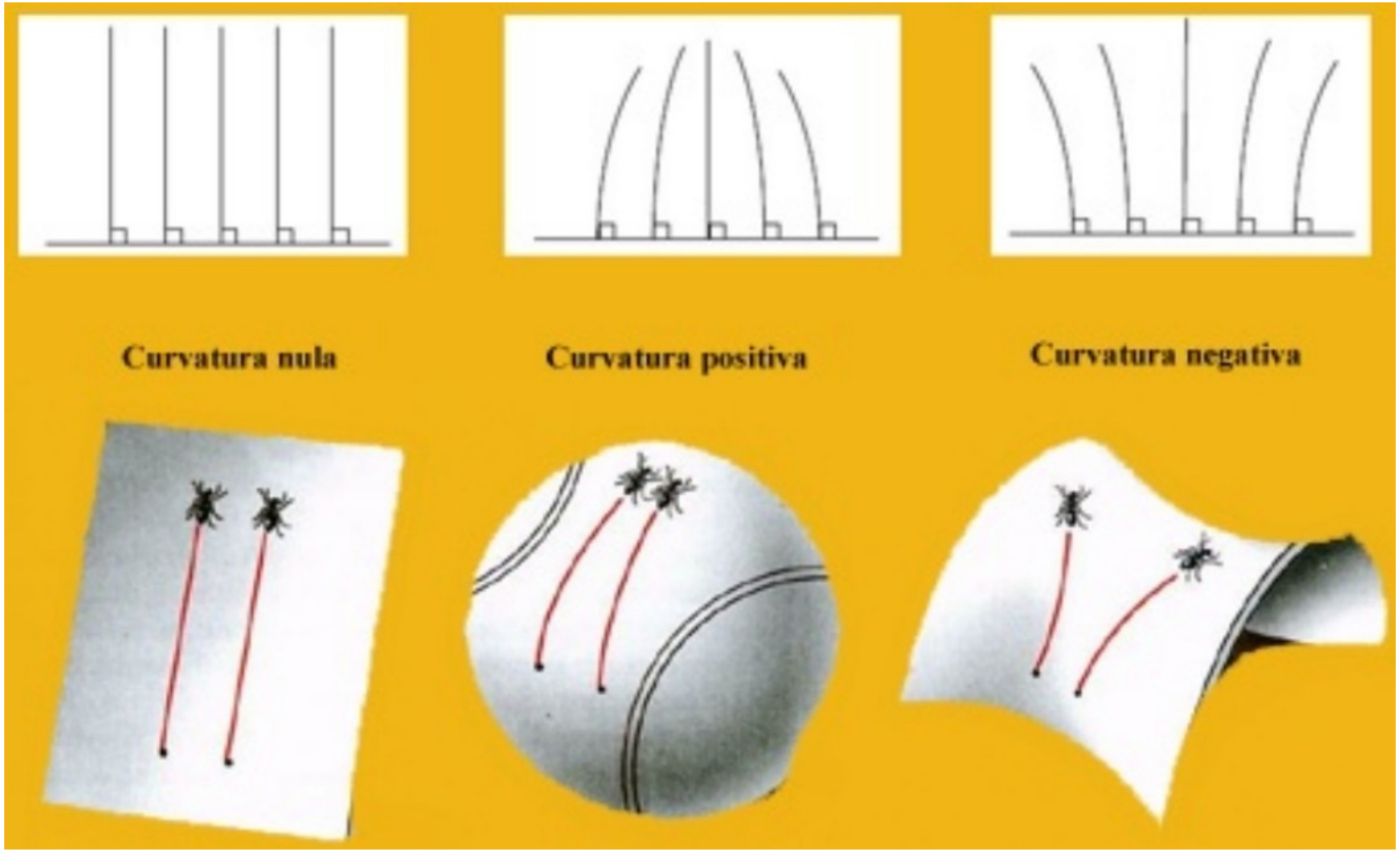
General Relativity:

- gravity is curvature, causes relative accelerations!



General Relativity:

- gravity is curvature, causes relative accelerations!



intrinsic notion: no reference to embeddings

General Relativity:

- gravity is curvature determined by dynamics!

General Relativity:

- gravity is curvature determined by dynamics!

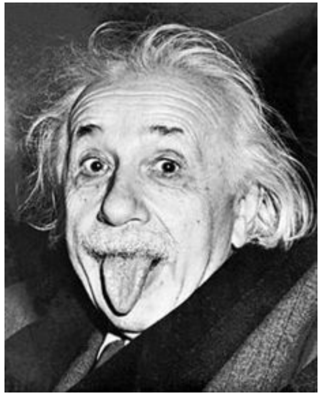
$$G_{\mu\nu} = 8\pi\, T_{\mu\nu}$$

General Relativity:

- gravity is curvature determined by dynamics!

$$G_{\mu\nu} = 8\pi\, T_{\mu\nu}$$

geometry



matter

General Relativity:

- gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold

General Relativity:

- gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold

- locally like $\mathbb{R}^4$

General Relativity:

- gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold

- locally like $\mathbb{R}^4$
- locally nearly flat + Minkowski

(equivalence principle)

General Relativity:

- gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold

- locally like $\mathbb{R}^4$
- locally nearly flat + Minkowski
- observers (coordinate systems) communicate (coordinate transformations)

General Relativity:

- gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold

- physical fields, same for all observers (tensor fields)

- observers (coordinate systems) communicate (coordinate transformations)

General Relativity:

- gravity is curvature

$\Rightarrow$ spacetime is a differentiable manifold

- physical fields, same for all observers (tensor fields)
- dynamical equations (analysis on manifolds)
- observers (coordinate systems) communicate (coordinate transformations)

General Relativity:

- gravity is curvature

Curvature comes from extra structure

crucial: a dynamical quantity,
not a background

General Relativity:

- gravity is curvature

Curvature comes from extra structure

The metric $g_{\mu\nu}$

General Relativity:

- gravity is curvature

Curvature comes from extra structure

The metric $g_{\mu\nu}$

- locally special relativity

General Relativity:

- gravity is curvature

Curvature comes from extra structure

The metric $g_{\mu\nu}$

- locally special relativity
- causal structure

General Relativity:

- gravity is curvature

Curvature comes from extra structure

The metric $g_{\mu\nu}$

- locally special relativity
- causal structure
- parallel transport, geodesics

General Relativity:

- gravity is curvature

Curvature comes from extra structure

The metric $g_{\mu\nu}$

- locally special relativity
- causal structure
- parallel transport, geodesics
- curvature

General Relativity:

- gravity is curvature

Differentiable manifold

(analysis)

Metric

(geometry)

General Relativity:

- gravity is curvature

Differentiable manifold   Background

(analysis)

Metric   Dynamical

(geometry)

General Relativity:

- gravity is curvature

Differentiable manifold

(analysis)

Background

fixed

Metric

(geometry)

Dynamical

chosen among many
by the dynamics

General Relativity:

- gravity is curvature

Physics at global scales is hard!

General Relativity:

- gravity is curvature

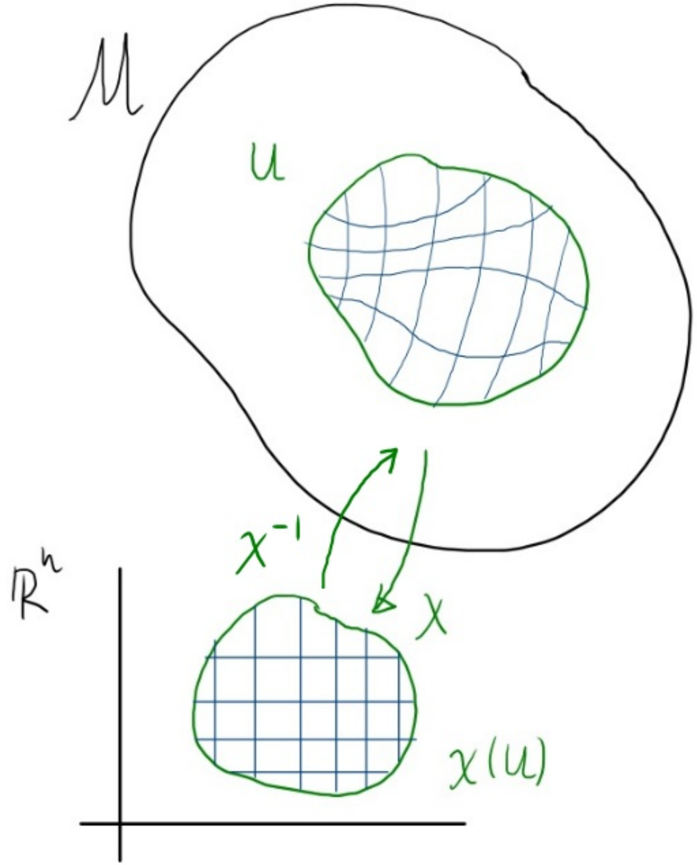
Physics at global scales is hard!

fortunately:

Physics at local scales is easy

Differentiable Manifords

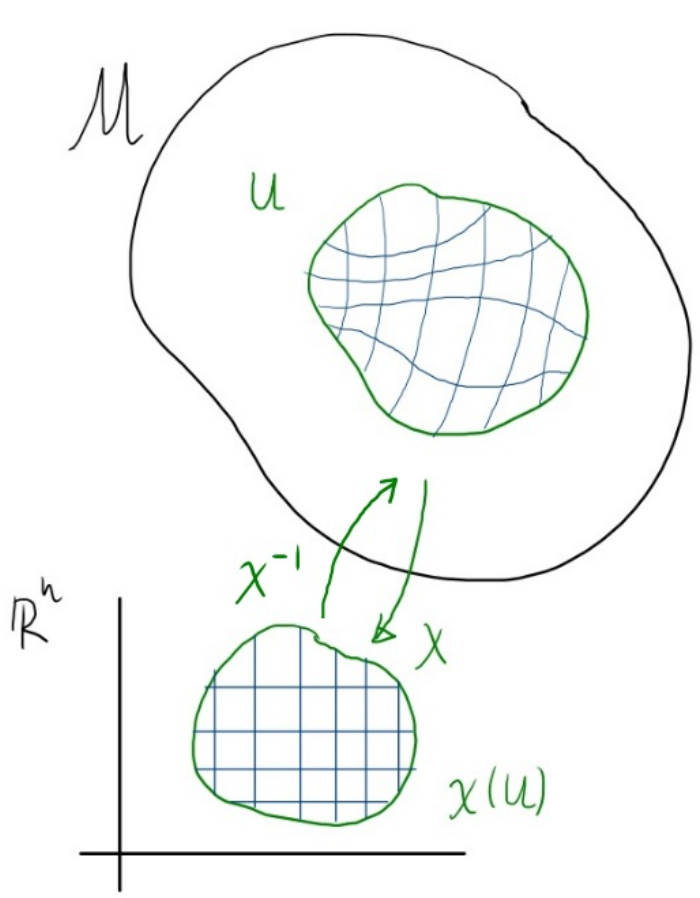
- look locally like $\mathbb{R}^n$



Differentiable Manifords

- look locally like $\mathbb{R}^n$

dimension

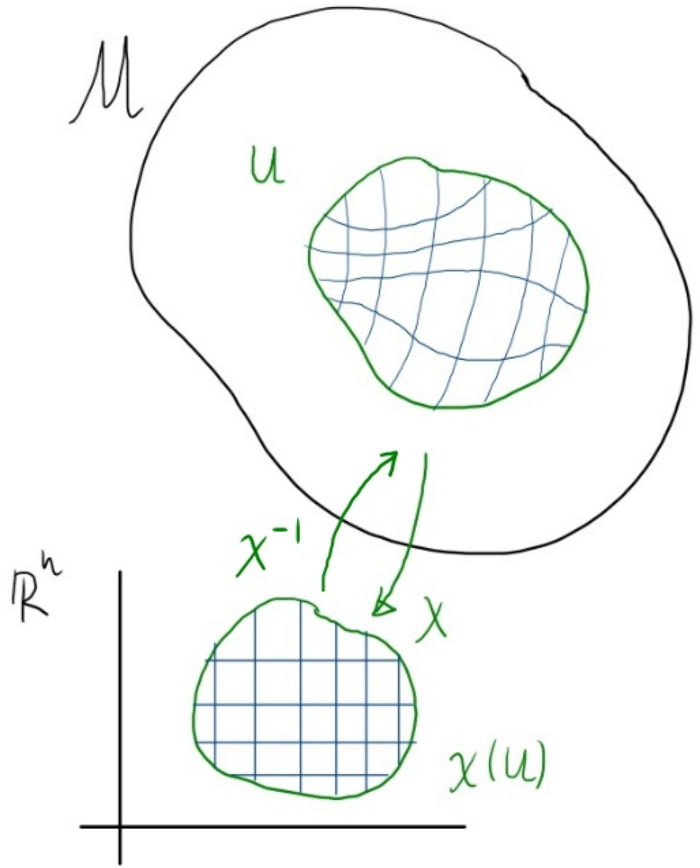


Differentiable Manifords

- look locally like $\mathbb{R}^n$

need topology

dimension



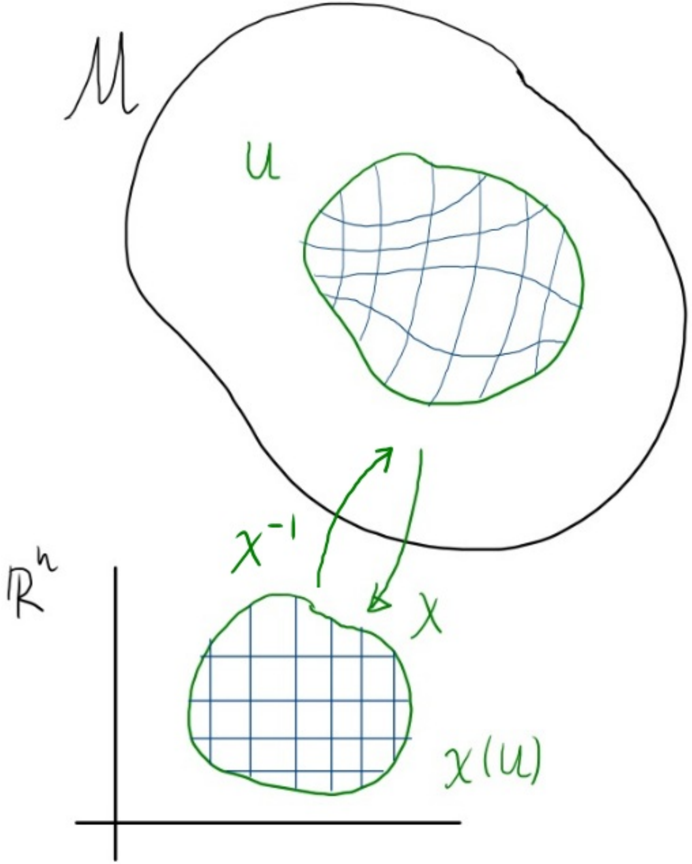
Differentiable Manifords

- look locally like $\mathbb{R}^n$

everywhere!

dimension

need topology



Differentiable Manifords

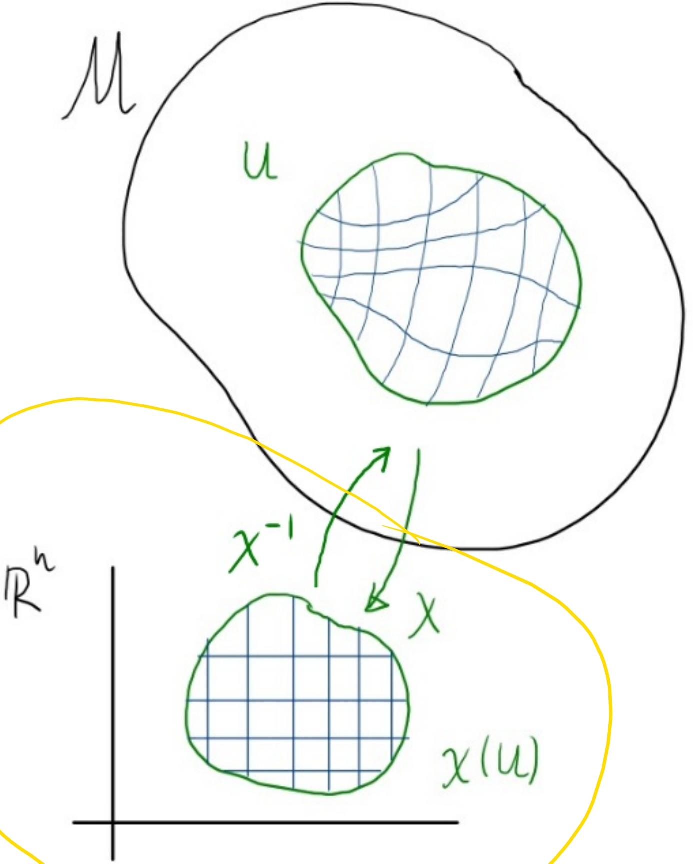
- look locally like $\mathbb{R}^n$

everywhere!

need topology

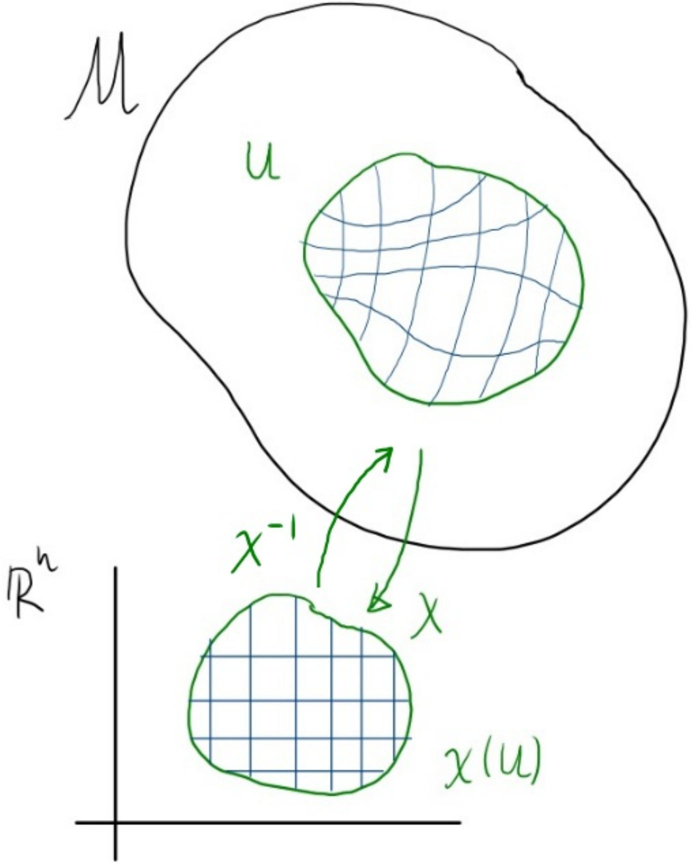
dimension

defines coordinate system



Differentiable Manifords

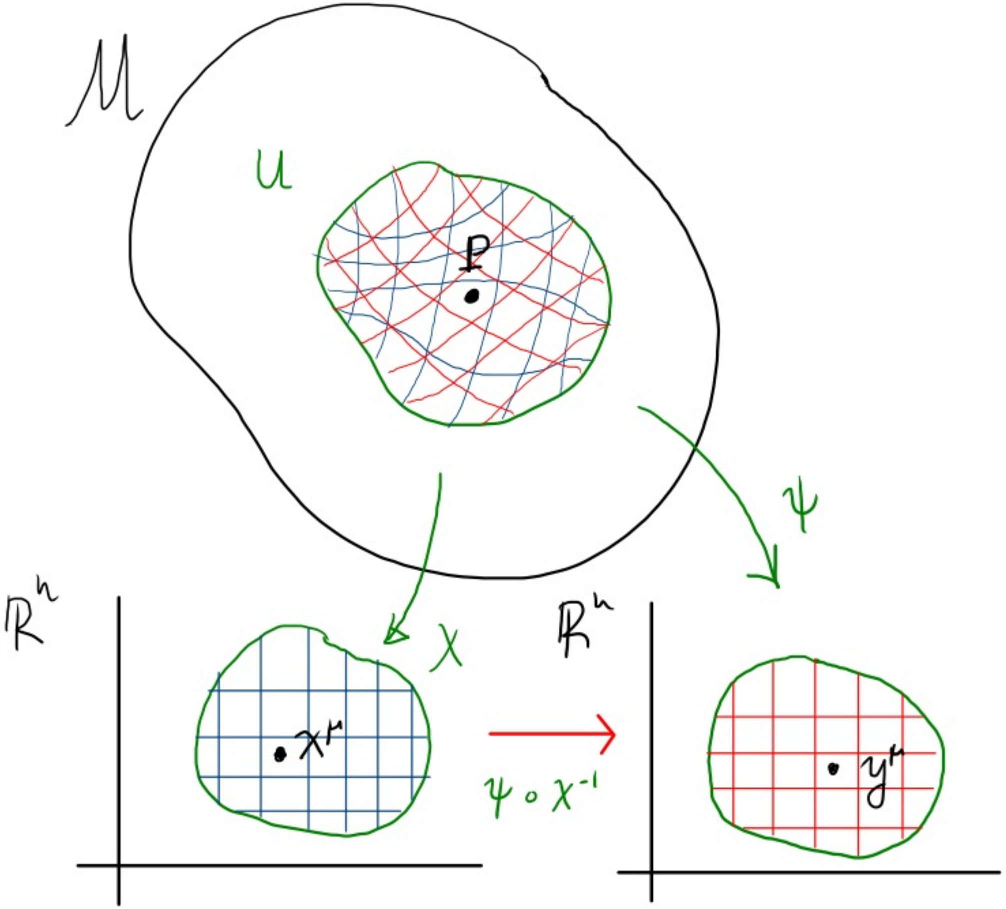
- $\underbrace{\text{look locally like } \mathbb{R}^n}_{\text{required by equivalence principle!}}$



Differentiable Manifords

- look locally like $\mathbb{R}^n$
- coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$



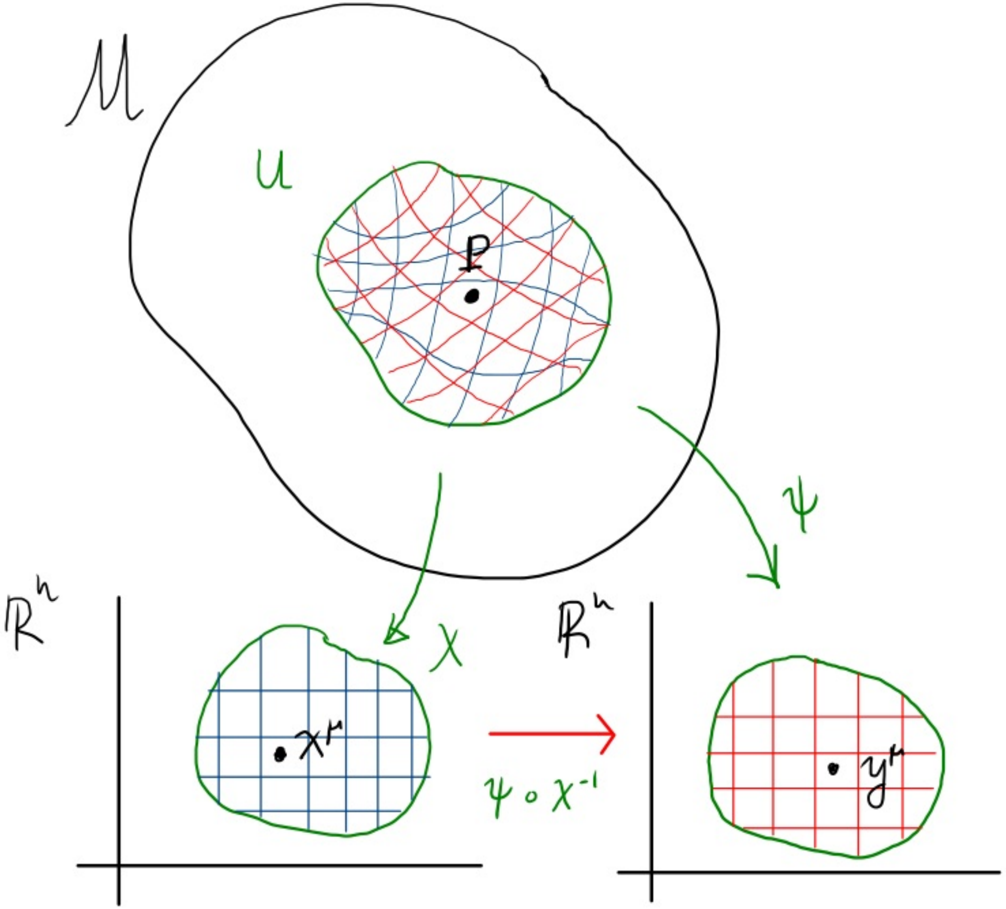
Differentiable Manifords

- look locally like $\mathbb{R}^n$
- coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$

observer ψ

observer χ



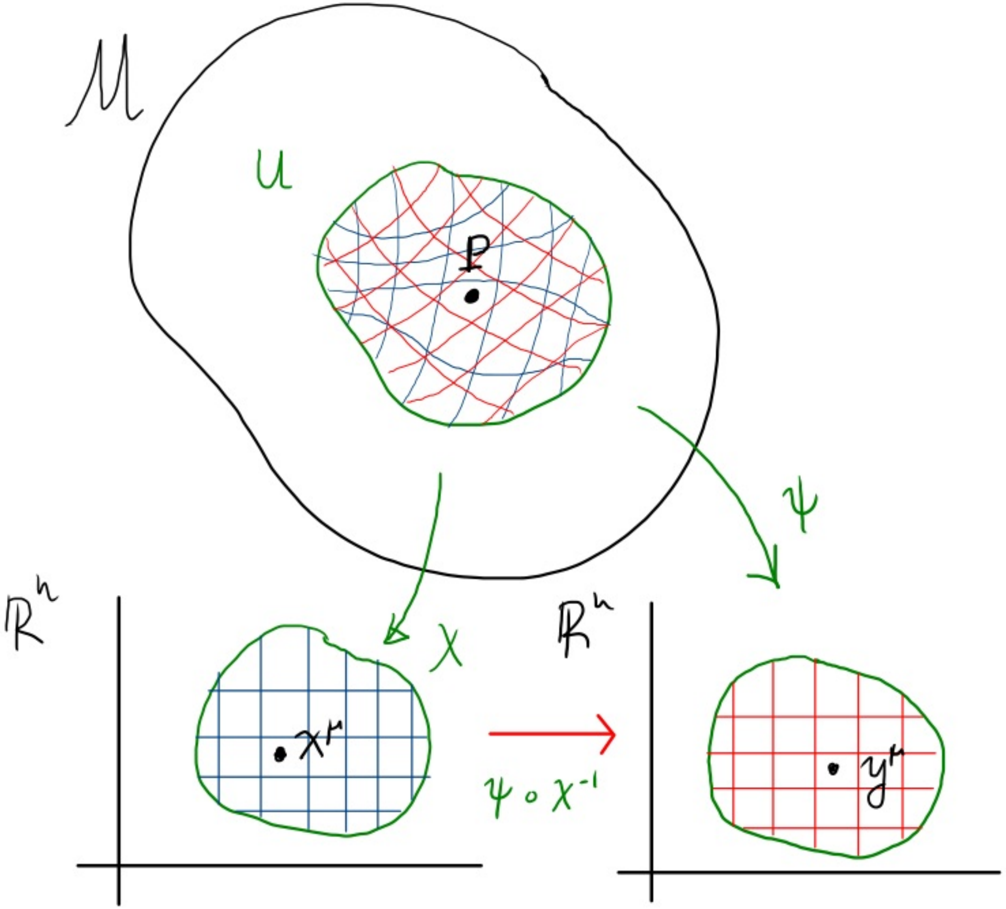
Differentiable Manifords

- look locally like $\mathbb{R}^n$
- coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$

observer ψ $\longleftrightarrow$ observer χ

communication



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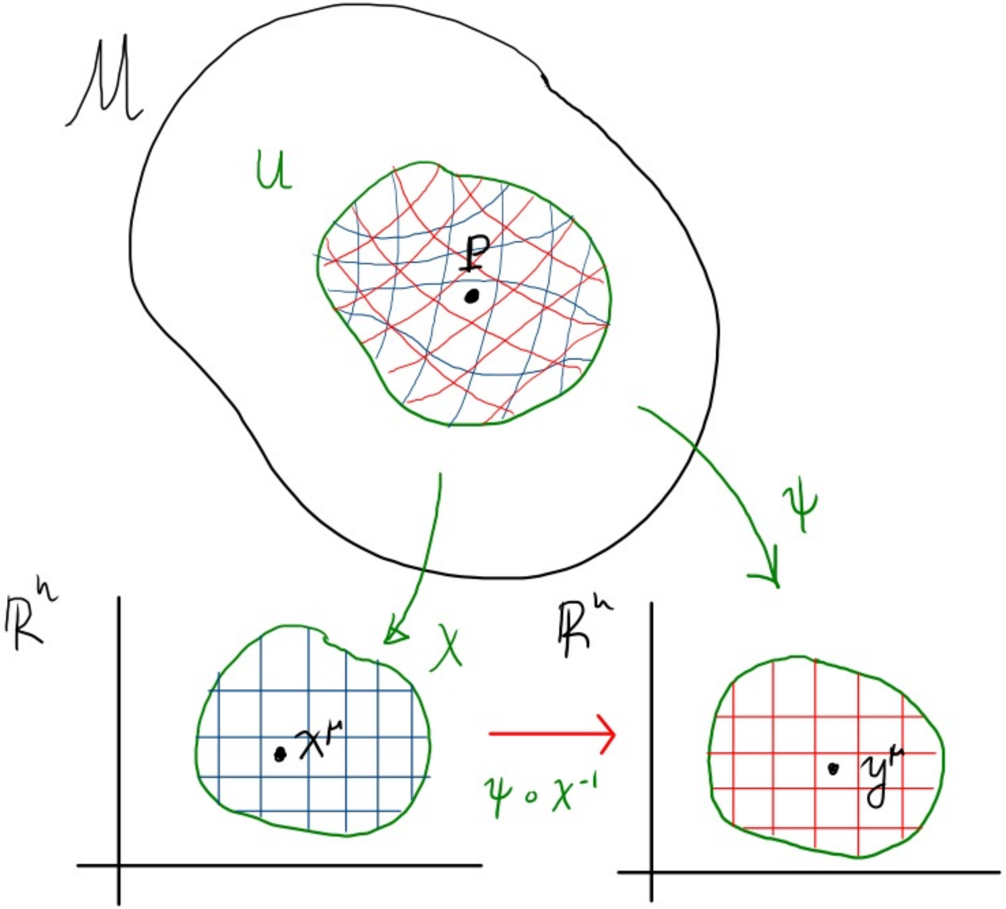


Observers agree on geometry (coordinate invariant observables)

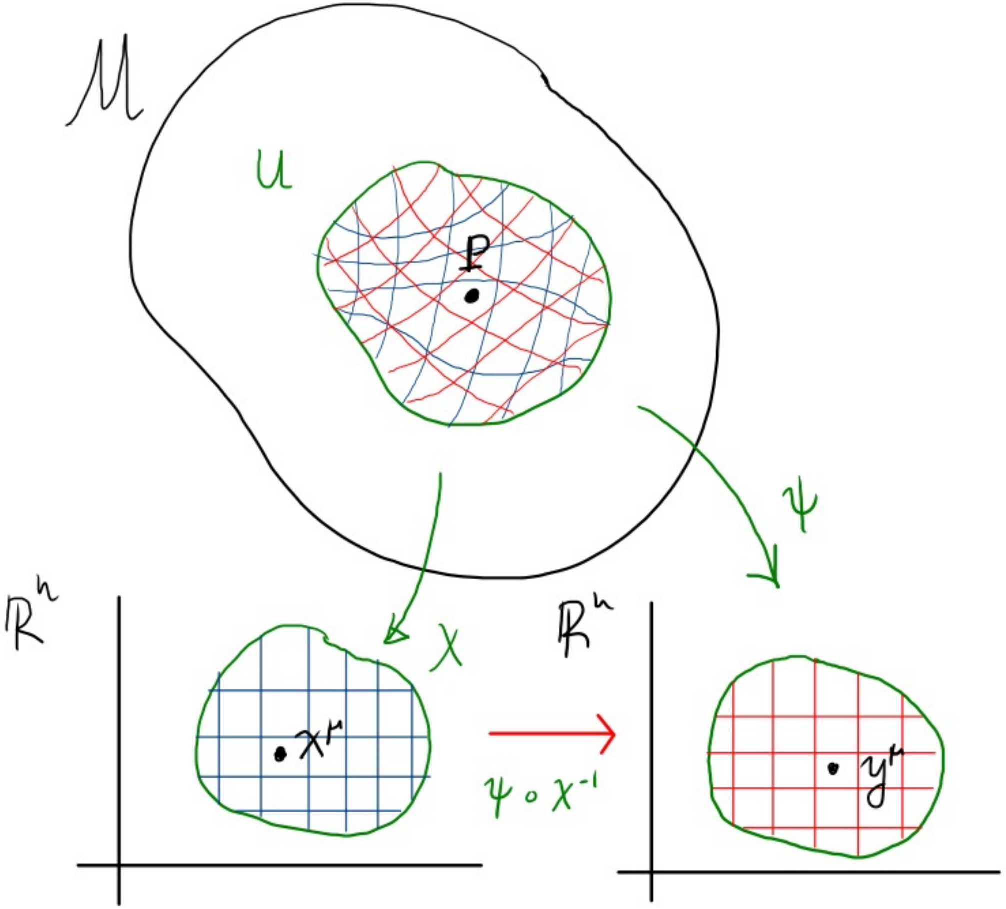
Differentiable Manifords

- look locally like $\mathbb{R}^n$
- coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$

- differentiable structure

$$\frac{\partial y^\mu}{\partial x^\nu}$$



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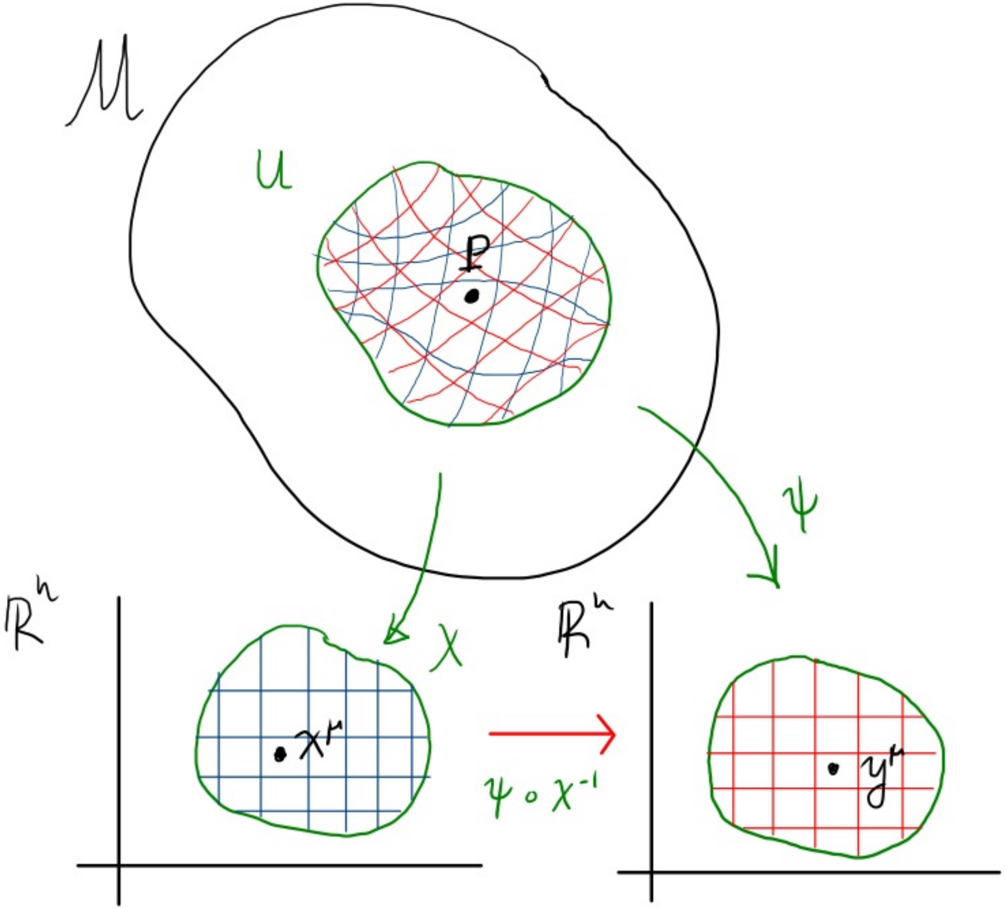


Inherited from $\mathbb{R}^n$ $\Rightarrow$ derivatives, integrals, analysis, all geometric!

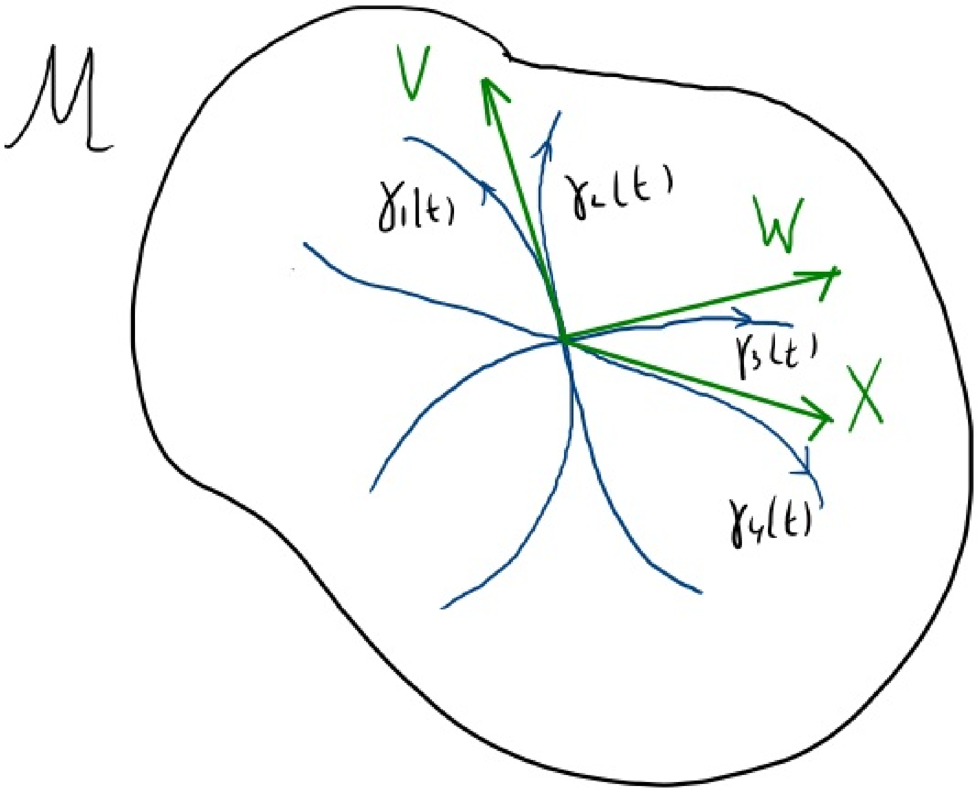
Differentiable Manifords

- look locally like $\mathbb{R}^n$
- coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$

- differentiable structure
- differentiable fields

curves → vectors → tensor fields



Differentiable Manifords

- look locally like $\mathbb{R}^n$
- coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$

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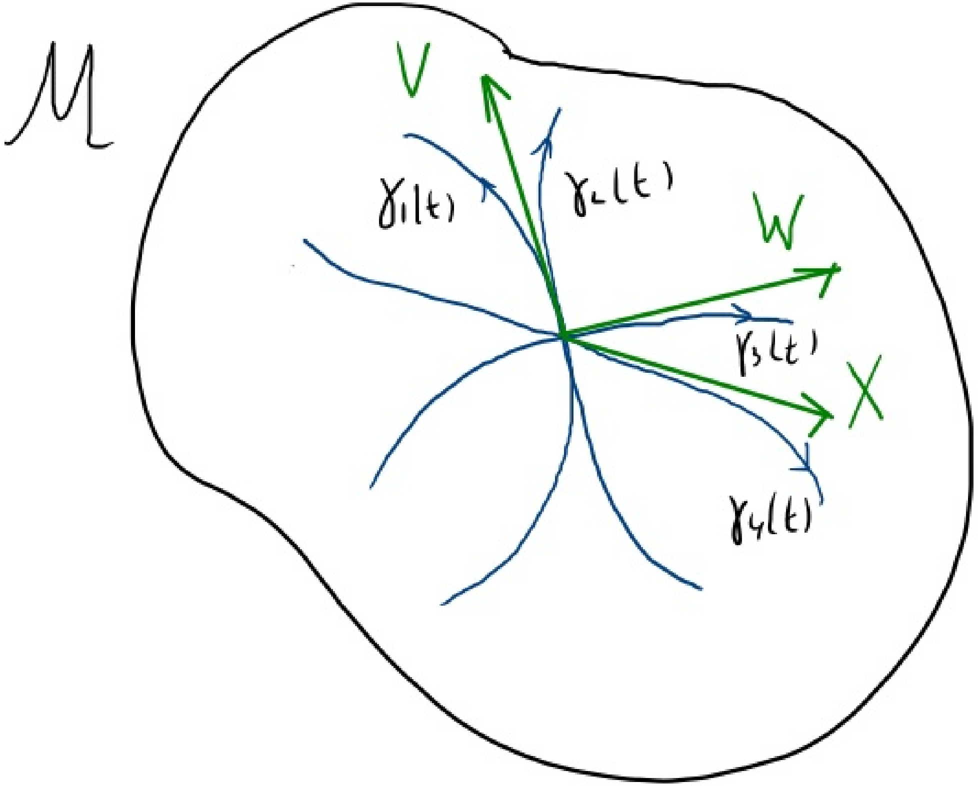

curves → vectors → tensor fields

Geometric Objects

(can do physics!)

Differentiable Manifords

The map:

Topological spaces, locally $\mathbb{R}^n$

Differentiable Manifords

The map:

Topological spaces, locally $\mathbb{R}^n$

↓

Local coordinate systems

Differentiable Manifolds

The map:

Topological spaces, locally $\mathbb{R}^n$

$\downarrow$

Local coordinate systems

$\downarrow$

Differentiable coordinate transformations

Differentiable Manifords

The map:

Topological spaces, locally $\mathbb{R}^n$

$\downarrow$

Local coordinate systems

$\downarrow$

Differentiable coordinate transformations

$\downarrow$

Differential Structure

Differentiable Manifords

The map:

↓

Differential Structure

↓

Fields

Differentiable Manifords

The map:

Differential Structure

Fields

vector

(rate of change of
functions on curves)

Differentiable Manifords

The map:

↓

Differential Structure

↓

Fields

↙

vector ⟶ 1-form

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vector ⟶ 1-form

↓

tensor

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vector → 1-form → n-form

exterior derivative

integrals

vector ↓ Lie Derivatives

1-form ↓ tensor

Differentiable Manifords

The map:

↓

Differential Structure

↓

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vector ⟶ 1-form ⟶ n-form

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Only using the maniford structure!

Differentiable Manifords

Additional Structure:

- Affine Connection

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  - parallel transport
  - geodesics
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Compare objects at different points

Differentiable Manifords

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  - geodesics } straightest curves
  - curvature

Differentiable Manifords

Additional Structure:

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  - geodesics
  - curvature } geodesic deviation (relative acceleration of freely falling particles

Differentiable Manifolds

Additional Structure:

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No metric necessary!!

e.g. gauge theories: $F_{\mu\nu}$ is curvature
D_μ is covariant derivative

Differentiable Manifords

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- **Metric**
  - distances
  - angles, inner product
  - causal stucture (GR)
  - singles out affine connection and curvature

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→ Now we can have spacetime and do GR!

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→ Now we can have spacetime and do GR!

→ Infinite metrics, choose using dynamics of GR

Examples of Manifolds

- $\mathbb{R}^n$: $\mathbb{R}$ (line), $\mathbb{R}^2$ (plane), $\mathbb{R}^3$ (space), ...

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- $M = M_1 \times M_2 = \{ P = (P_1, P_2) \mid P_1 \in M_1 , P_2 \in M_2 \}$