Use of ctensor for calculating [redacted]

1 *Introduction and general instructions:*

Sources for reading about Maxima: see the Help menu!
https://wxmaxima-developers.github.io/wxmaxima/help.html
https://maxima.sourceforge.io/documentation.html
https://maxima.sourceforge.io/docs/manual/maxima_toc.html
file:///usr/share/doc/wxmaxima/wxmaxima.html#Introduction

Introductory videos by the instructor (in Greek):
https://youtu.be/RmF_MECumyI
https://youtu.be/kvtrETJotx8

Typing special characters:
[Esc] <char>

char	result				
L	Λ				
ee	ⅇ	p	π	ii	ⅈ (imaginary i)
inf	∞	hb	ℏ	in	∈
partial	∂	integral	∫		
^2	²	^3	³		
sq	√	impl	⇒	equiv	⇔
=>	⇒	<=>	⇔		

Ctrl+Tab or Ctrl+Shift triggers the auto-completion mechanism.
Ctrl+Shift+Delete deletes a complete cell.

.mac files read with e.g. read("test.mac");

2 *ctensor package*

Documentation:
https://maxima.sourceforge.io/docs/manual/maxima_126.html#ctensor

Read also: arXiv:cs/0503073, Viktor Toth, Tensor Manipulation with GPL Maxima https://arxiv.org/abs/cs/0503073

Tensors: (not all, see documentation)
lg [i,j] g_{i j}
ug [i,j] g^{i j}
mcs[i,j,k] Γ^k_{i j} = (1/2) g^{k m} (∂_i g_{j m} + ∂_ j g_{i m} - ∂_m g_{i j})
riem[i,j,k,m] -R^m_{i j k} = -[∂_j Γ^m_{k i} - ∂_k Γ^m_{j i} + Γ^m_{j n} Γ^n_{ki} - Γ^m_{k n} Γ^n_{ji}] (Carroll convention for R^m_{i j k})
lriem[i,j,k,m] -R_{m i j k}
ric [i j] R_{i j}
lein [i,j] G_{i j}
geod[i] Geodesic equations

Variables:
dim dimension of space
gdet determinant of metric
ct_coords[ ] list of coordinates, e.g. ct_coords[theta,phi]

Functions:
load("ctensor") loads the session
csetup() interactive initialization of the package
cmetric() computes inverse metric, gdet, after metric has been defined
ct_coordsys(coord): sets a predifined coordinate system, e.g. coord= exteriorschwarzschild, interiorschwarzschild, kerr_newman
christoff(mcs) compute and display mcs[i,j,k] = Γ^k_{i j}
riemann(true) compute and display riem[i,j,k,m] = -R^m_{i j k} (Carroll convention)
lriemann(true) compute and display lriem[i,j,k,m] = - R_{m i j k}
ricci(true) compute and display ric[i,j] = R_{i j}
scurvature() compute the scalar curvature
leinstein(true) compute G_{i j}
rinvariant() compute Kretschmann-invariant R_{i j k m} R^{i j k m}.
Must have calculated lriem and uriem. Call as: lriemann(false);uriemann(false); rinvariant()
cgeodesic(true) computes geodesic equations, stored in geod[i]

Utilities:
init_tensor() reinitializes the ctensor package
cdisplay(tensor) displays tensor as multidimensional array, e.g. cdisplay(mcs)

The routine:
load(ctensor); load the package
init_ctensor(); or reset the variables

3 Load the package:

```
load(ctensor);
```

/usr/share/maxima/5.45.1/share/tensor/ctensor.mac

4 A simple example: The two sphere S^2

First set the dimensionality of the manifold and the names of the coodinates:

```
dim:2;
ct_coords:[theta,phi];
```

$$2$$

$$[theta, phi]$$

Define the metric:

```
lg:matrix(
[a^2,0                    ],
[0   ,a^2 · sin(theta)^2]);
```

$$\begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin(theta)^2 \end{pmatrix}$$

Compute the inverse metric, the determinant and initialize various variables:

```
cmetric();
```

done

```
ug;
gdet;
```

$$\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2 \sin(theta)^2} \end{pmatrix}$$

$$a^4 \sin(theta)^2$$

Compute the Christoffel symbols: mcs[i,j,k] Γ^k_{i j} = (1/2) g^{k m} (∂_i g_{j m} + ∂_ j g_{i m} - ∂_m g_{i j})
and display the nonzero components. Compare with (3.154) Carroll:
Γ^2_{1 2} = cos(θ) / sin(θ)
Γ^1_{2 2} = -cos(θ) sin(θ)

```
christof(mcs);
```

$$mcs_{1,2,2} = \frac{\cos(theta)}{\sin(theta)}$$

$$mcs_{2,2,1} = -\cos(theta)\sin(theta)$$

done

riem[i,j,k,m] -R^m_{i j k} = -[∂_j Γ^m_{k i} - ∂_k Γ^m_{j i} + Γ^m_{j n} Γ^n_{ki} - Γ^m_{k n} Γ^n_{ji}]

Compute: riem[i,j,k,m] -R^m_{i j k} = -[∂_j Γ^m_{k i} - ∂_k Γ^m_{j i} + Γ^m_{j n} Γ^n_{ki} - Γ^m_{k n} Γ^n_{ji}]
Gives: R^2_{1 2 1} = 1
R^1_{2 2 1} = -sin(θ)^2 Compare with (3.155) Carroll

```
riemann(true);
```

$$riem_{1,2,1,2} = -1$$

$$riem_{2,2,1,1} = \sin(theta)^2$$

done

Compute: lriem[i,j,k,m] -R_{m i j k}
Gives: R_{1 2 2 1} = a^2 sin(θ)^2 Compare with (3.156) Carroll

```
lriemann(true);
```

$$lriem_{2,2,1,1} = a^2 \sin(theta)^2$$

done

Compute Ricci tensor: Compare with (3.157) Carroll

```
ricci(true);
```

$$ric_{1,1} = 1$$

$$ric_{2,2} = \sin(theta)^2$$

done

Compute Scalar Curvature: compare with (3.158) Carroll

```
scurvature();
```

$$\frac{2}{a^2}$$

Use cdisplay(tensor) to see components in a matrix like display, but be careful, the indices are not the same order as in Carroll.

```
cdisplay(lriem);
```

$$lriem_{1,1}=\begin{pmatrix}0 & 0\\ 0 & a^2\sin(theta)^2\end{pmatrix}$$

$$lriem_{1,2}=\begin{pmatrix}0 & -a^2\sin(theta)^2\\ 0 & 0\end{pmatrix}$$

$$lriem_{2,1}=\begin{pmatrix}0 & 0\\ -a^2\sin(theta)^2 & 0\end{pmatrix}$$

$$lriem_{2,2}=\begin{pmatrix}a^2\sin(theta)^2 & 0\\ 0 & 0\end{pmatrix}$$

done

```
cdisplay(ric);
```

$$ric=\begin{pmatrix}1 & 0\\ 0 & \sin(theta)^2\end{pmatrix}$$

done

```
cdisplay(mcs);
```

$$mcs_1=\begin{pmatrix}0 & 0\\ 0 & \frac{\cos(theta)}{\sin(theta)}\end{pmatrix}$$

$$mcs_2=\begin{pmatrix}0 & \frac{\cos(theta)}{\sin(theta)}\\ -\cos(theta)\sin(theta) & 0\end{pmatrix}$$

done

Compute the geodesic equations:

```
cgeodesic(true);
```

$$geod_1=\frac{d^2}{d\,s^2}\,theta-\left(\frac{d}{d\,s}\,phi\right)^2\cos(theta)\sin(theta)$$

$$geod_2=$$

$$\frac{2\left(\frac{d}{d\,s}\,phi\right)\cos(theta)\left(\frac{d}{d\,s}\,theta\right)+\left(\frac{d^2}{d\,s^2}\,phi\right)\sin(theta)}{\sin(theta)}$$

done

They are stored in the array geod[i]:

```
geod[1];geod[2];
```

$$\frac{d^2}{d s^2} theta - \left(\frac{d}{d s} phi\right)^2 \cos(theta) \sin(theta)$$

$$\frac{2 \left(\frac{d}{d s} phi\right) \cos(theta) \left(\frac{d}{d s} theta\right) + \left(\frac{d^2}{d s^2} phi\right) \sin(theta)}{\sin(theta)}$$

5 Friedmann metric

First, reinitialize ctensor:

```
init_ctensor();
```

done

```
dim:4;
ct_coords:[t,chi,theta,phi];
```

4

$[t, chi, theta, phi]$

The metric:

```
lg:matrix(
[−1,0      ,0                  ,0                              ],
[ 0,a(t)^2,0                  ,0                              ],
[ 0,0      ,a(t)^2·sin(chi)^2  ,0                              ],
[ 0,0      ,0                  , a(t)^2·sin(chi)^2·sin(theta)^2 ]);
```

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & \sin(chi)^2 a(t)^2 & 0 \\ 0 & 0 & 0 & \sin(chi)^2 a(t)^2 \sin(theta)^2 \end{pmatrix}$$

```
cmetric();
```

done

```
christof(mcs);
```

$$mcs_{1,2,2}=\frac{a(t)_t}{a(t)}$$

$$mcs_{1,3,3}=\frac{a(t)_t}{a(t)}$$

$$mcs_{1,4,4}=\frac{a(t)_t}{a(t)}$$

$$mcs_{2,2,1}=a(t)\,(a(t)_t)$$

$$mcs_{2,3,3}=\frac{\cos(chi)}{\sin(chi)}$$

$$mcs_{2,4,4}=\frac{\cos(chi)}{\sin(chi)}$$

$$mcs_{3,3,1}=\sin(chi)^2\,a(t)\,(a(t)_t)$$

$$mcs_{3,3,2}=-\cos(chi)\,\sin(chi)$$

$$mcs_{3,4,4}=\frac{\cos(theta)}{\sin(theta)}$$

$$mcs_{4,4,1}=\sin(chi)^2\,a(t)\,(a(t)_t)\,\sin(theta)^2$$

$$mcs_{4,4,2}=-\cos(chi)\,\sin(chi)\,\sin(theta)^2$$

$$mcs_{4,4,3}=-\cos(theta)\,\sin(theta)$$

done

riemann(true);

$$riem_{1,2,1,2} = \frac{a(t)_{tt}}{a(t)}$$

$$riem_{1,3,1,3} = \frac{a(t)_{tt}}{a(t)}$$

$$riem_{1,4,1,4} = \frac{a(t)_{tt}}{a(t)}$$

$$riem_{2,2,1,1} = a(t)\,(a(t)_{tt})$$

$$riem_{2,3,2,3} = -(a(t)_t)^2 - 1$$

$$riem_{2,4,2,4} = -(a(t)_t)^2 - 1$$

$$riem_{3,3,1,1} = \sin(chi)^2\, a(t)\,(a(t)_{tt})$$

$$riem_{3,3,2,2} = \sin(chi)^2\,(a(t)_t)^2 + \sin(chi)^2$$

$$riem_{3,4,3,4} = -\sin(chi)^2\,(a(t)_t)^2 + \cos(chi)^2 - 1$$

$$riem_{4,4,1,1} = \sin(chi)^2\, a(t)\,(a(t)_{tt})\,\sin(theta)^2$$

$$riem_{4,4,2,2} = \left(\sin(chi)^2\,(a(t)_t)^2 + \sin(chi)^2\right)\sin(theta)^2$$

$$riem_{4,4,3,3} = \left(\sin(chi)^2\,(a(t)_t)^2 - \cos(chi)^2 + 1\right)\sin(theta)^2$$

done

ricci(true);

$$ric_{1,1} = -\frac{3\,(a(t)_{tt})}{a(t)}$$

$$ric_{2,2} = a(t)\,(a(t)_{tt}) + 2\,(a(t)_t)^2 + 2$$

$$ric_{3,3} = \sin(chi)^2\, a(t)\,(a(t)_{tt}) + 2\sin(chi)^2\,(a(t)_t)^2 + \sin(chi)^2 - \cos(chi)^2 + 1$$

$$ric_{4,4} = \sin(chi)^2\, a(t)\,(a(t)_{tt})\,\sin(theta)^2 + 2\sin(chi)^2\,(a(t)_t)^2\,\sin(theta)^2 + \sin(chi)^2\,\sin(theta)^2 - \cos(chi)^2\,\sin(theta)^2 + \sin(theta)^2$$

done

```
scurvature();
```

$$\left(6\sin(chi)^2\,a(t)\,(a(t)_{tt})+6\sin(chi)^2\,(a(t)_t)^2+4\sin(chi)^2-2\cos(chi)^2+2\right)$$
$$/(\sin(chi)^2\,a(t)^2)$$

```
leinstein(true);
```

$$lein_{1,1}=\frac{3\sin(chi)^2\,(a(t)_t)^2+2\sin(chi)^2-\cos(chi)^2+1}{\sin(chi)^2\,a(t)^2}$$

$$lein_{2,2}=-\frac{2\sin(chi)^2\,a(t)\,(a(t)_{tt})+\sin(chi)^2\,(a(t)_t)^2-\cos(chi)^2+1}{\sin(chi)^2}$$

$$lein_{3,3}=-\sin(chi)^2\left(2\,a(t)\,(a(t)_{tt})+(a(t)_t)^2+1\right)$$

$$lein_{4,4}=-\sin(chi)^2\left(2\,a(t)\,(a(t)_{tt})+(a(t)_t)^2+1\right)\sin(theta)^2$$

done

```
cgeodesic(true);
```

$$geod_1=\sin(chi)^2\,a(t)\,(a(t)_t)\,(theta_s)^2+\sin(chi)^2\,(phi_s)^2$$
$$a(t)\,(a(t)_t)\,\sin(theta)^2+(chi_s)^2\,a(t)\,(a(t)_t)+t_{ss}$$

$$geod_2=-$$
$$\left(\cos(chi)\sin(chi)\,a(t)\,(theta_s)^2+\cos(chi)\sin(chi)\,(phi_s)^2\,a(t)\sin(theta)^2-2\,(chi_s)\,(t_s)\,(a(\right.$$
$$/a(t)$$

$$geod_3=$$
$$\left(\sin(chi)\,a(t)\,(theta_{ss})+2\sin(chi)\,(t_s)\,(a(t)_t)\,(theta_s)+2\cos(chi)\,(chi_s)\,a(t)\,(theta_s)-\sin(\right.$$
$$/(\sin(chi)\,a(t))$$

$$geod_4=$$
$$(2\sin(chi)\,(phi_s)\,a(t)\cos(theta)\,(theta_s)+2\sin(chi)\,(phi_s)\,(t_s)\,(a(t)_t)\sin(theta)+\sin(chi)\,($$
$$/(\sin(chi)\,a(t)\sin(theta))$$

done

```
uriemann(false);lriemann(false);rinvariant();
```

done

done

$$\frac{12\,(\mathrm{a}(t)_{t\,t})^2}{\mathrm{a}(t)^2}+$$

$$\left(4\left(\sin(chi)^2\,(\mathrm{a}(t)_t)^2-\cos(chi)^2+1\right)\left(\sin(chi)^4\,\mathrm{a}(t)^2\,(\mathrm{a}(t)_t)^2+(1-\cos(chi)^2)\,\sin(chi)^2\,\mathrm{a}(t)^2\right.\right.$$

$$/(\sin(chi)^6\,\mathrm{a}(t)^6)+$$

$$\frac{6\left((\mathrm{a}(t)_t)^2+1\right)\left(\sin(chi)^2\,\mathrm{a}(t)^2\,(\mathrm{a}(t)_t)^2+\sin(chi)^2\,\mathrm{a}(t)^2\right)}{\sin(chi)^2\,\mathrm{a}(t)^6}-$$

$$\frac{2\left((\mathrm{a}(t)_t)^2+1\right)\left(-\sin(chi)^2\,\mathrm{a}(t)^2\,(\mathrm{a}(t)_t)^2-\sin(chi)^2\,\mathrm{a}(t)^2\right)}{\sin(chi)^2\,\mathrm{a}(t)^6}$$

6 Schwarzschild Metric

```
init_ctensor();
```

done

The Schwarzschild coordinate system (r,t,θ,φ) is predefined:

```
ct_coordsys(exteriorschwarzschild);
```

done

```
cmetric();
```

done

```
lg;ug;gdet;
```

$$\begin{pmatrix} \frac{2\,m-r}{r} & 0 & 0 & 0 \\ 0 & \frac{r}{r-2\,m} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\sin(theta)^2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{r}{2\,m-r} & 0 & 0 & 0 \\ 0 & \frac{r-2\,m}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\,\sin(theta)^2} \end{pmatrix}$$

$$-r^4\,\sin(theta)^2$$

Compare with Carroll (5.52), p 206

```
christof(mcs);
```

$$mcs_{1,1,2}=\frac{m\,r-2\,m^2}{r^3}$$

$$mcs_{1,2,1}=\frac{m}{r^2-2\,m\,r}$$

$$mcs_{2,2,2}=-\frac{m}{r^2-2\,m\,r}$$

$$mcs_{2,3,3}=\frac{1}{r}$$

$$mcs_{2,4,4}=\frac{1}{r}$$

$$mcs_{3,3,2}=2\,m-r$$

$$mcs_{3,4,4}=\frac{\cos(theta)}{\sin(theta)}$$

$$mcs_{4,4,2}=(2\,m-r)\,\sin(theta)^2$$

$$mcs_{4,4,3}=-\cos(theta)\,\sin(theta)$$

done

```
riemann(true);
```

$$riem_{1,2,1,2} = \frac{2\,m\,(m\,r-2\,m^2)}{r^3\,(r^2-2\,m\,r)} + \frac{3\,(m\,r-2\,m^2)}{r^4} - \frac{m}{r^3}$$

$$riem_{1,3,1,3} = -\frac{m\,r-2\,m^2}{r^4}$$

$$riem_{1,4,1,4} = -\frac{m\,r-2\,m^2}{r^4}$$

$$riem_{2,2,1,1} = \frac{2\,m}{r^3-2\,m\,r^2}$$

$$riem_{2,3,2,3} = \frac{m}{r\,(r^2-2\,m\,r)}$$

$$riem_{2,4,2,4} = \frac{m}{r\,(r^2-2\,m\,r)}$$

$$riem_{3,3,1,1} = -\frac{m}{r}$$

$$riem_{3,3,2,2} = -\frac{m}{r}$$

$$riem_{3,4,3,4} = -\frac{2\,m-r}{r} - 1$$

$$riem_{4,4,1,1} = -\frac{m\,\sin(theta)^2}{r}$$

$$riem_{4,4,2,2} = -\frac{m\,\sin(theta)^2}{r}$$

$$riem_{4,4,3,3} = \frac{2\,m\,\sin(theta)^2}{r}$$

done

You may simplify any of the above expressions: (use the "Simplify" menu of wxMaxima in this window to see other options)

```
ratsimp(riem[1,2,1,2]);
```

$$\frac{2\,m\,r-4\,m^2}{r^4}$$

Observe that R_{i j} = 0

```
ricci(true);
```

$$ric_{1,1}=-\frac{2\,m\,(m\,r-2\,m^2)}{r^3\,(r^2-2\,m\,r)}-\frac{m\,r-2\,m^2}{r^4}+\frac{m}{r^3}$$

$$ric_{2,2}=-\frac{2\,m}{r\,(r^2-2\,m\,r)}+\frac{m\,(2\,r-2\,m)}{(r^2-2\,m\,r)^2}-\frac{2\,m^2}{(r^2-2\,m\,r)^2}$$

done

We need to simplify in order to check that the above results are zero:

```
ratsimp(ric[1,1]);ratsimp(ric[2,2]);
```

0

0

```
scurvature();
```

0

Compare with Carroll (5.50) p 205: R^2=48 m^2/r^6

```
uriemann(false);lriemann(true);rinvariant();
```

done

$$lriem_{2,2,1,1}=-\frac{2\,m}{r^3}$$

$$lriem_{3,3,1,1}=\frac{m\,r-2\,m^2}{r^2}$$

$$lriem_{3,3,2,2}=-\frac{m}{r-2\,m}$$

$$lriem_{4,4,1,1}=\frac{(m\,r-2\,m^2)\sin(theta)^2}{r^2}$$

$$lriem_{4,4,2,2}=-\frac{m\sin(theta)^2}{r-2\,m}$$

$$lriem_{4,4,3,3}=2\,m\,r\sin(theta)^2$$

done

$$\frac{8\,m\,(m\,r-2\,m^2)}{r^2\,(r^5-2\,m\,r^4)}+\frac{8\,m\,(m\,r-2\,m^2)}{r^6\,(r-2\,m)}+\frac{32\,m^2}{r^6}$$

```
cgeodesic(true);
```

$$geod_1 = \frac{r^2 (t_{ss}) - 2\, m\, r\, (t_{ss}) + 2\, m\, (r_s)\, (t_s)}{r\, (r - 2\, m)}$$

$geod_2 = -$

$(r^5 (theta_s)^2 - 4\, m\, r^4 (theta_s)^2 + 4\, m^2\, r^3 (theta_s)^2 + (phi_s)^2\, r^5 \sin(theta)^2 - 4\, m\, (phi_s)^2\, r^4 \sin($

$/(r^3 (r - 2\, m))$

$geod_3 =$

$$\frac{r\, (theta_{ss}) + 2\, (r_s)\, (theta_s) - (phi_s)^2\, r \cos(theta) \sin(theta)}{r}$$

$geod_4 =$

$(2\, (phi_s)\, r \cos(theta)\, (theta_s) + 2\, (phi_s)\, (r_s) \sin(theta) + (phi_{ss})\, r \sin(theta))$

$/(r \sin(theta))$

done

Simplyfy some of the expressions above:

```
expand(geod[3]);expand(geod[4]);
```

$$theta_{ss} + \frac{2\, (r_s)\, (theta_s)}{r} - (phi_s)^2 \cos(theta) \sin(theta)$$

$$\frac{2\, (phi_s) \cos(theta)\, (theta_s)}{\sin(theta)} + \frac{2\, (phi_s)\, (r_s)}{r} + phi_{ss}$$

```
trigreduce(expand(geod[3]));trigreduce(expand(geod[4]));
```

$$-\frac{(phi_s)^2 \sin(2\, theta)}{2} + theta_{ss} + \frac{2\, (r_s)\, (theta_s)}{r}$$

$$2\, (phi_s) \cot(theta)\, (theta_s) + \frac{2\, (phi_s)\, (r_s)}{r} + phi_{ss}$$