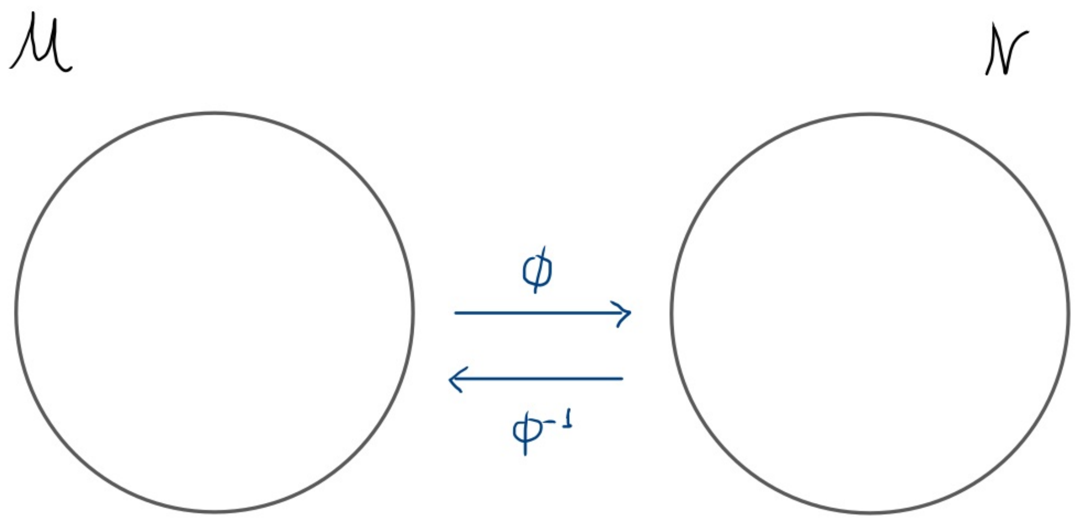
Plan:

- Diffeomorphisms (diffeos...)
  - → pullback and pushforward of any tensor
- One parameter family of diffeos...
  - ... and vector fields
- Lie derivative $\mathcal{L}_v$
  - → compute components of $\mathcal{L}_v T$ (in coord bases...)

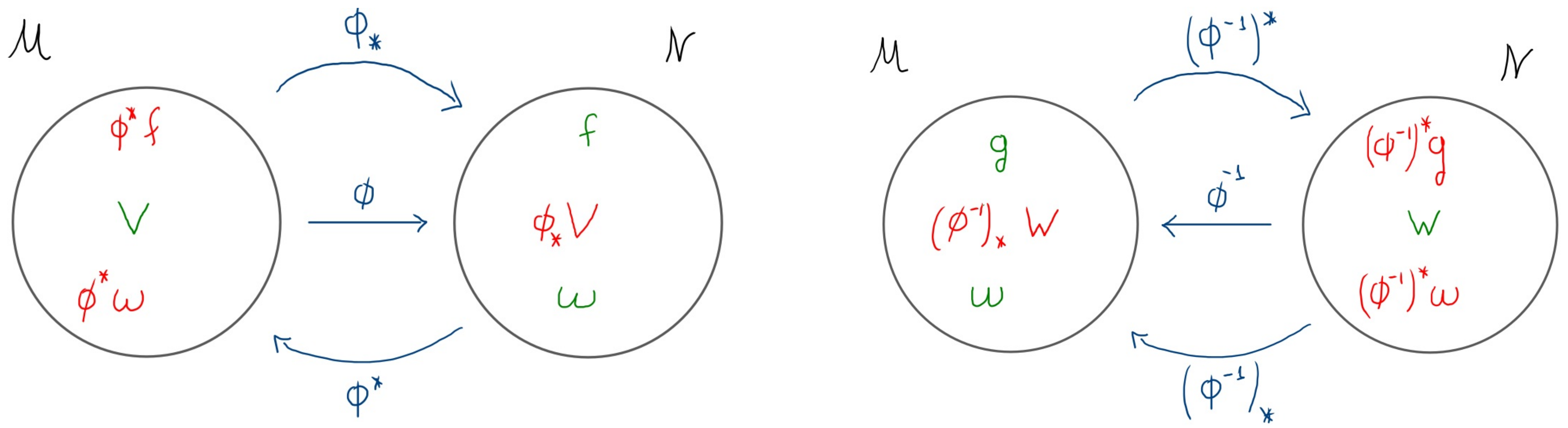
Diffeomorphisms



- $\phi : \mathcal{M} \to \mathcal{N}$, $\dim \mathcal{M} = \dim \mathcal{N} = n$, smooth
- if $\exists \; \phi^{-1} : \mathcal{N} \to \mathcal{M}$, smooth , then ϕ is a diffeomorphism
- $\mathcal{M}$ and $\mathcal{N}$ are diffeomorphic = "the same manifold"

(analogous to homeomorphic in topology)

Diffeomorphisms



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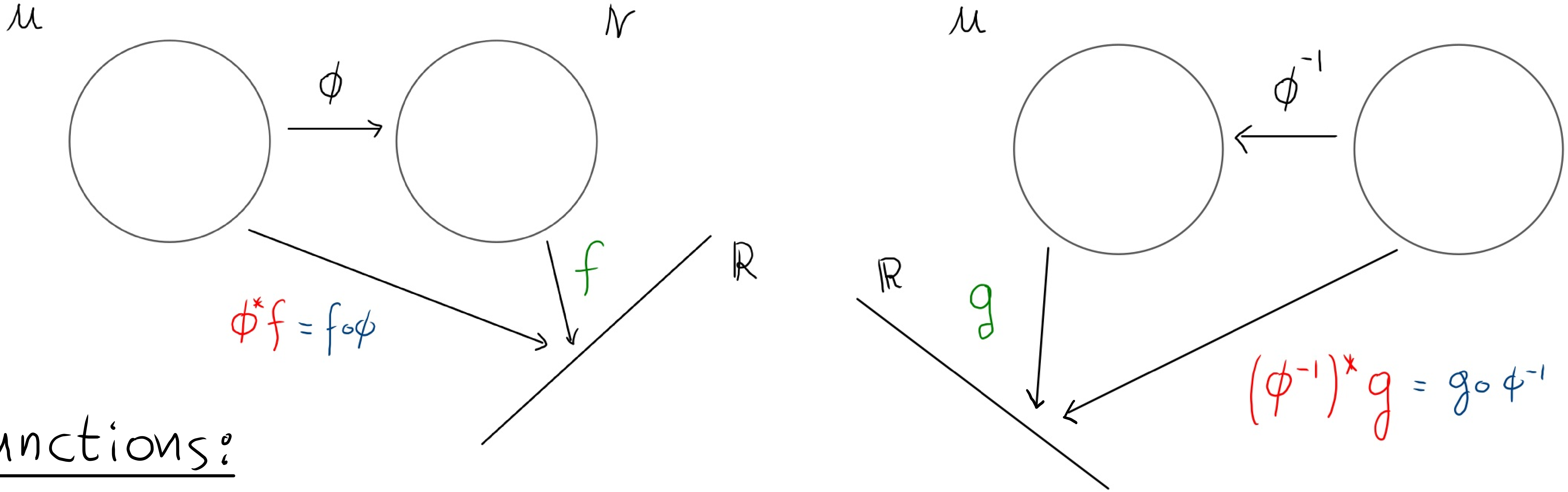
- $\mathcal{M}$ and $\mathcal{N}$ are diffeomorphic = " the same manifold"

  (analogous to homeomorphic in topology)
- we can use ϕ^{-1} to define ϕ^* and ϕ_* for any tensor

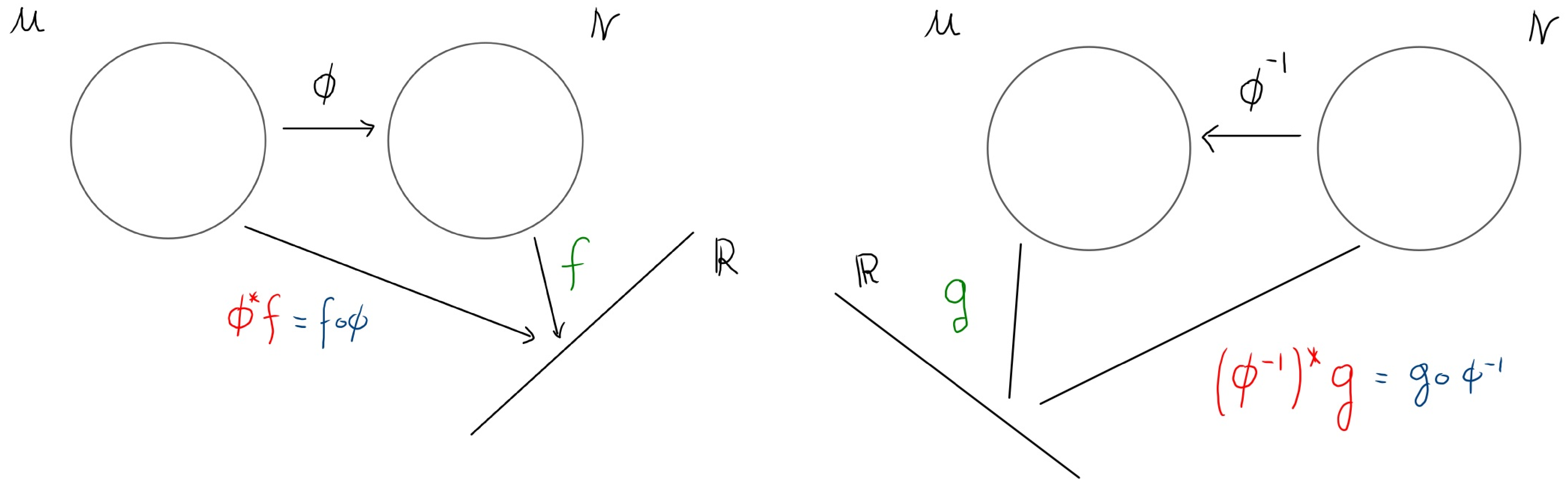
Diffeomorphisms



Functions:

- Define $\quad \phi_* g = (\phi^{-1})^* g = g \circ \phi^{-1} \qquad$ pushforward of g

Diffeomorphisms

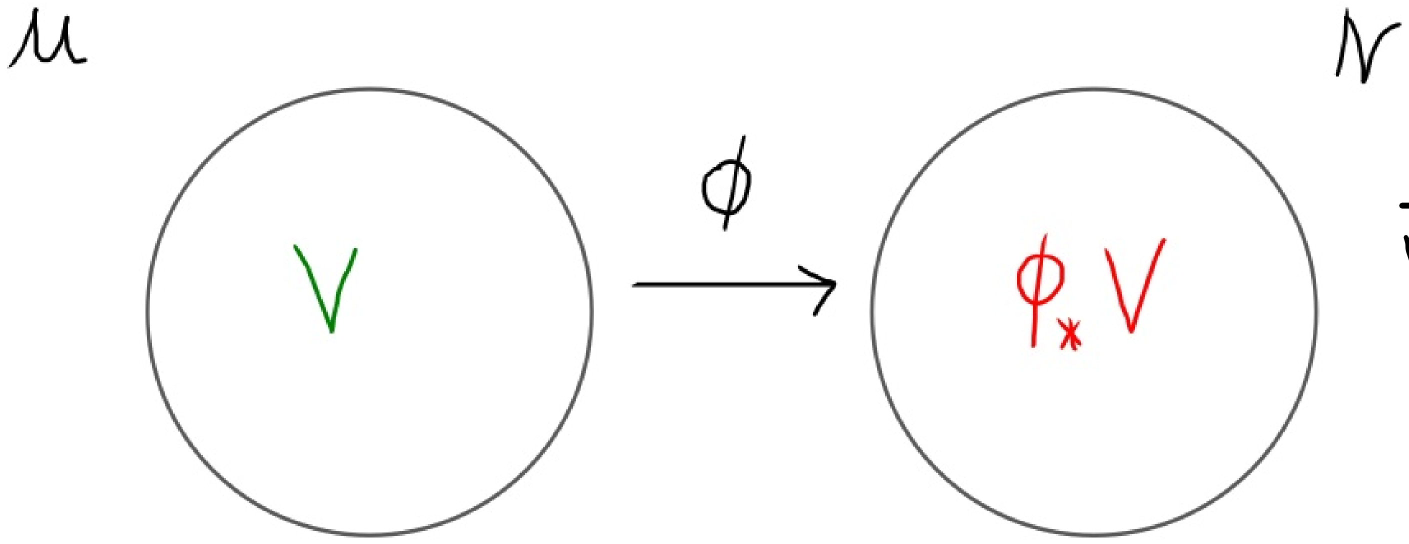


- Define $\phi_* g = (\phi^{-1})^* g = g \circ \phi^{-1}$ push forward of g

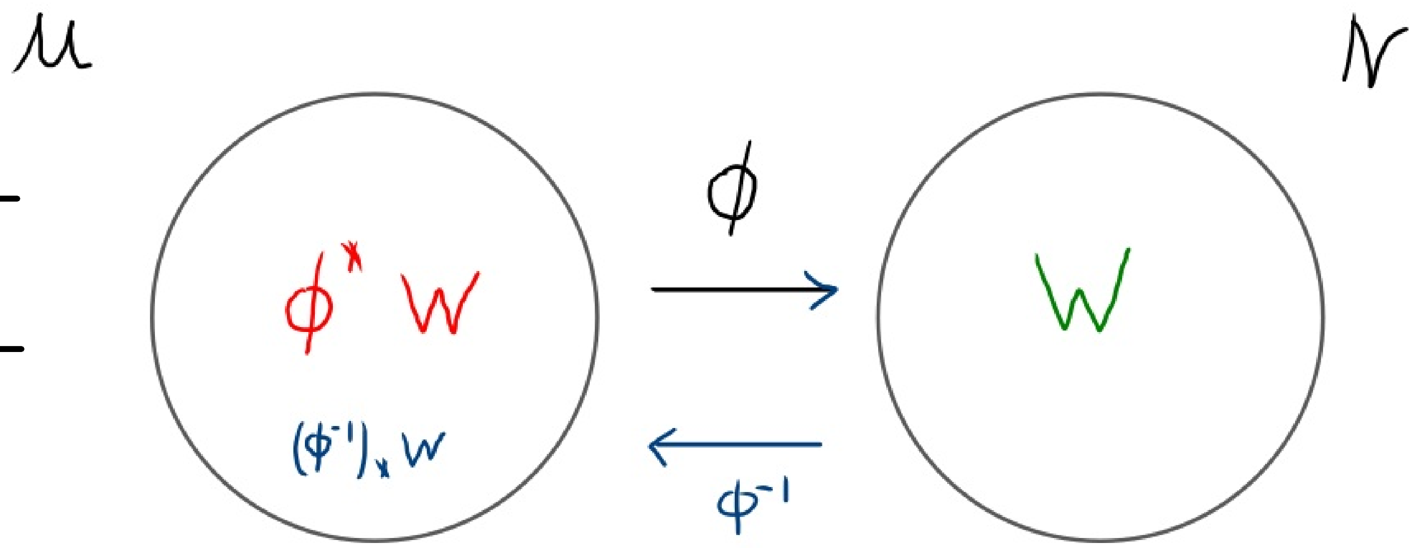
then for $g = \phi^* f \Rightarrow (\phi^{-1})^* \phi^* f = (\phi^{-1})^* f \circ \phi = f \circ \phi \circ \phi^{-1} = f$

$\Rightarrow (\phi^{-1})^* \phi^* = 1$

Diffeomorphisms



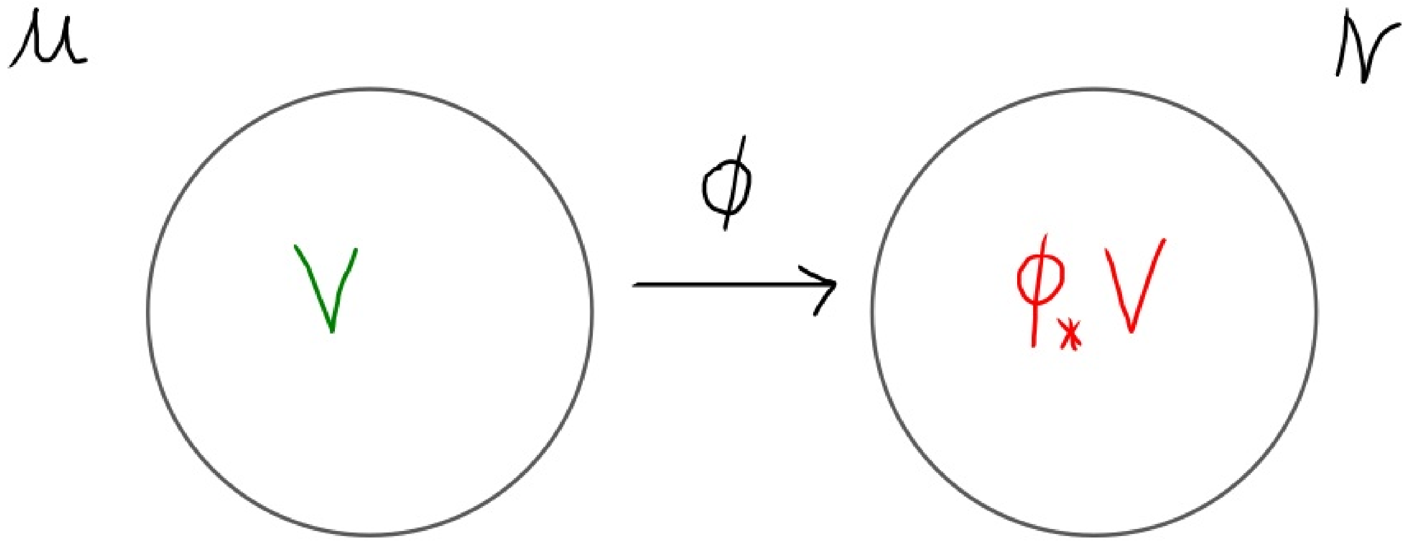
Vectors

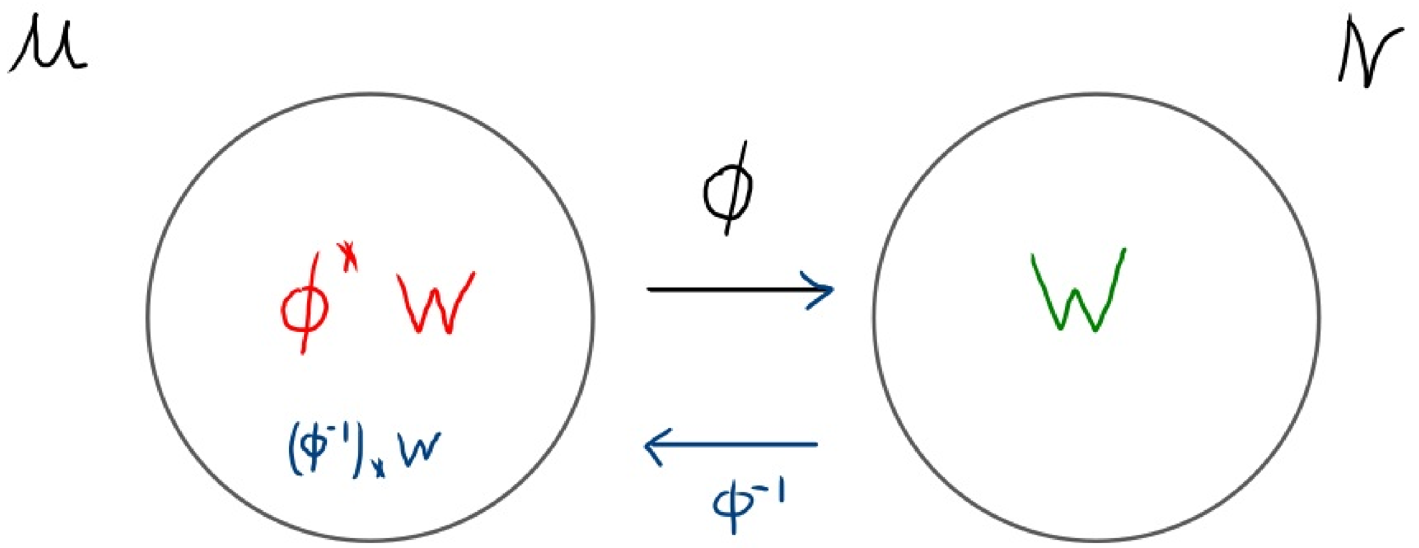


- Define $\phi^* W = (\phi^{-1})_* W$ pullback of W on $\mathcal{M}$

s.t. $\phi^* W(g) = (\phi^{-1})_* W(g) = W\Big((\phi^{-1})^* g\Big) = W(\phi_* g)$

Diffeomorphisms





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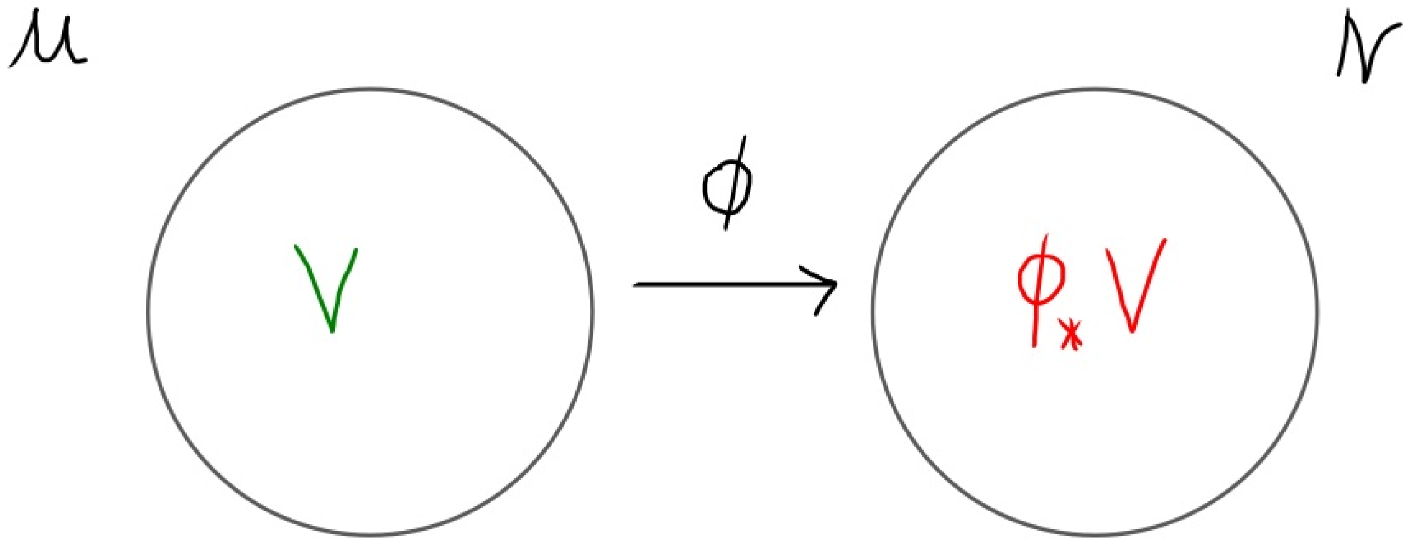
s.t. $\phi^* W(g) = (\phi^{-1})_* W(g) = W\left((\phi^{-1})^* g\right) = W(\phi_* g)$

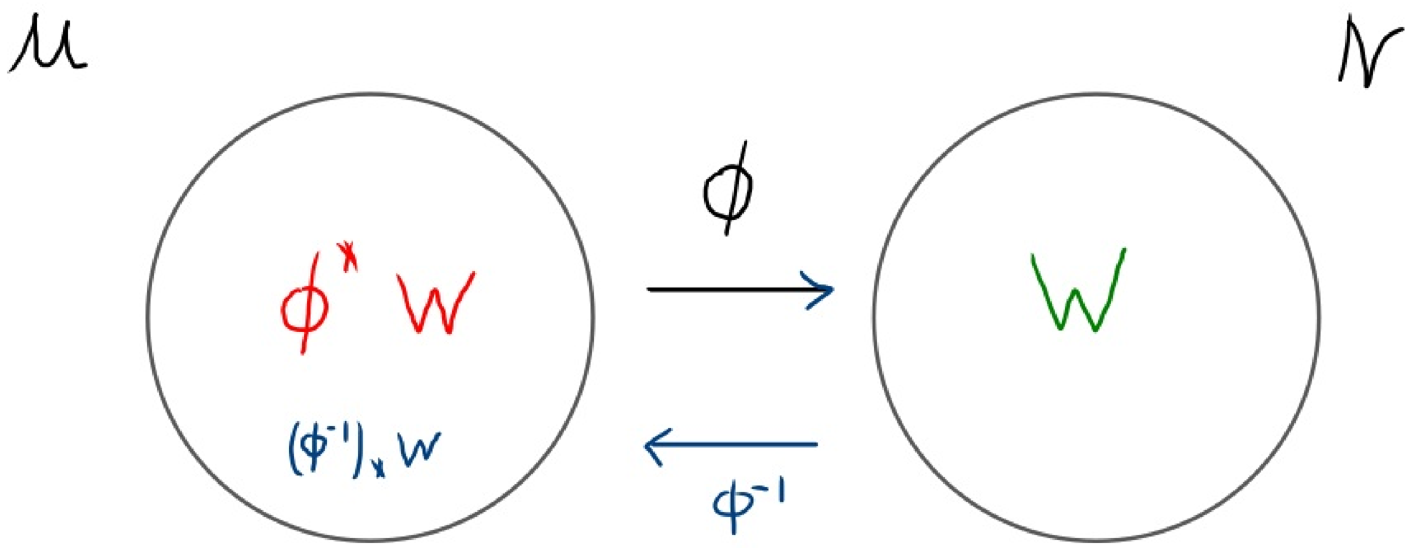
- for $W = \phi_* V$

$$[\phi^*(\phi_* V)](g) = [(\phi_{-1})_* (\phi_* V)](g) = \phi_* V\left((\phi_{-1})^* g\right) = V\left(\phi^*((\phi_{-1})^* g)\right) = V(g)$$

$\Rightarrow \phi^* \phi_* = 1$

Diffeomorphisms





- Define $\phi^* W = (\phi^{-1})_* W$ pullback of W on $\mathcal{M}$

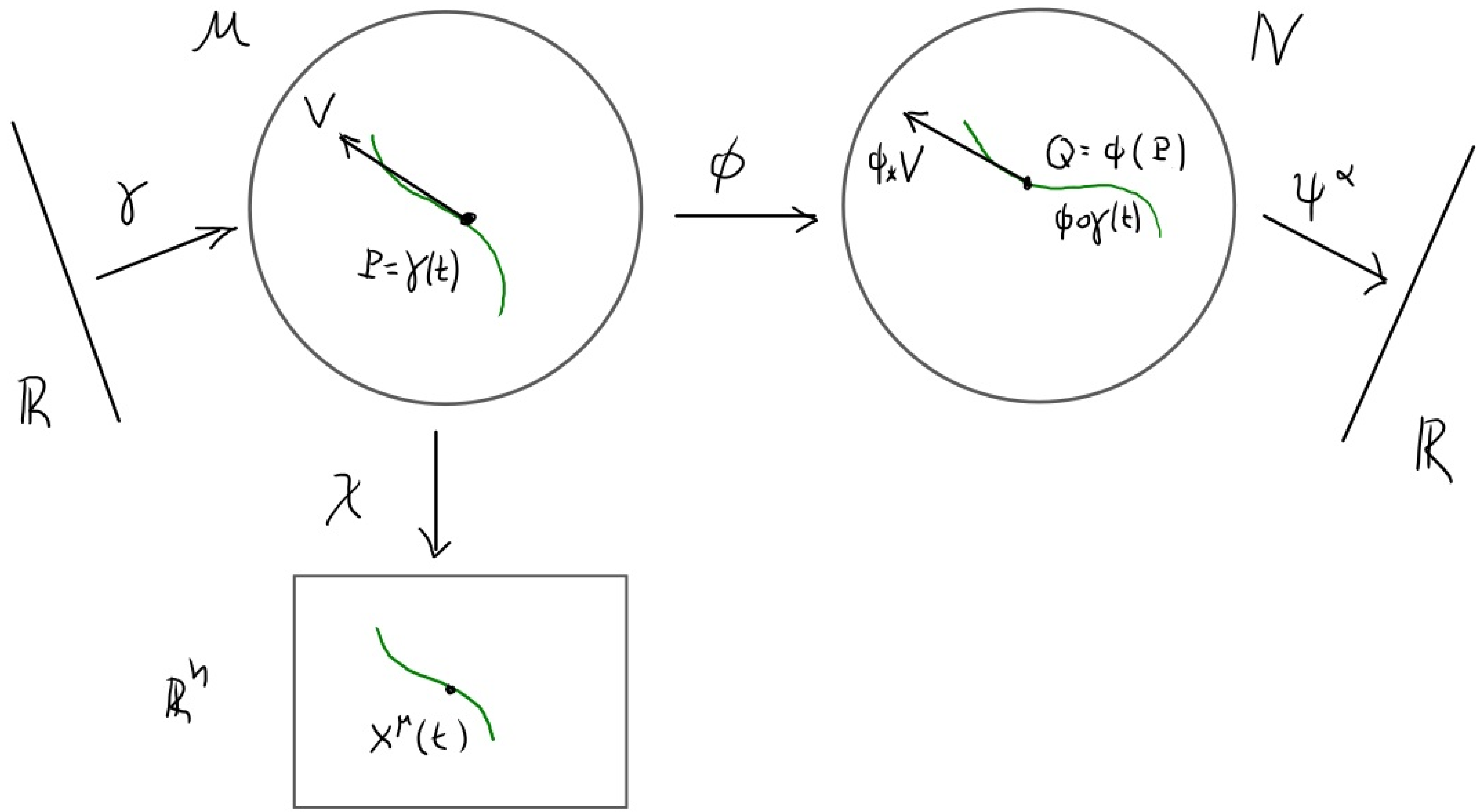
- Components: $(\phi_* V)^\alpha = \dfrac{\partial y^\alpha}{\partial x^\mu} V^\mu = (\phi_*)^\alpha{}_\mu V^\mu$

$$(\phi^* W)^\mu = \frac{\partial x^\mu}{\partial y^\alpha} W^\alpha = (\phi_*^{-1})^\mu{}_\alpha W^\alpha$$

inverse matrix

Proof: $(\phi_* V)^\alpha = \dfrac{\partial y^\alpha}{\partial x^\mu} V^\mu$

consider $\psi^\alpha : Q \mapsto y^\alpha(Q)$ (fixed α)



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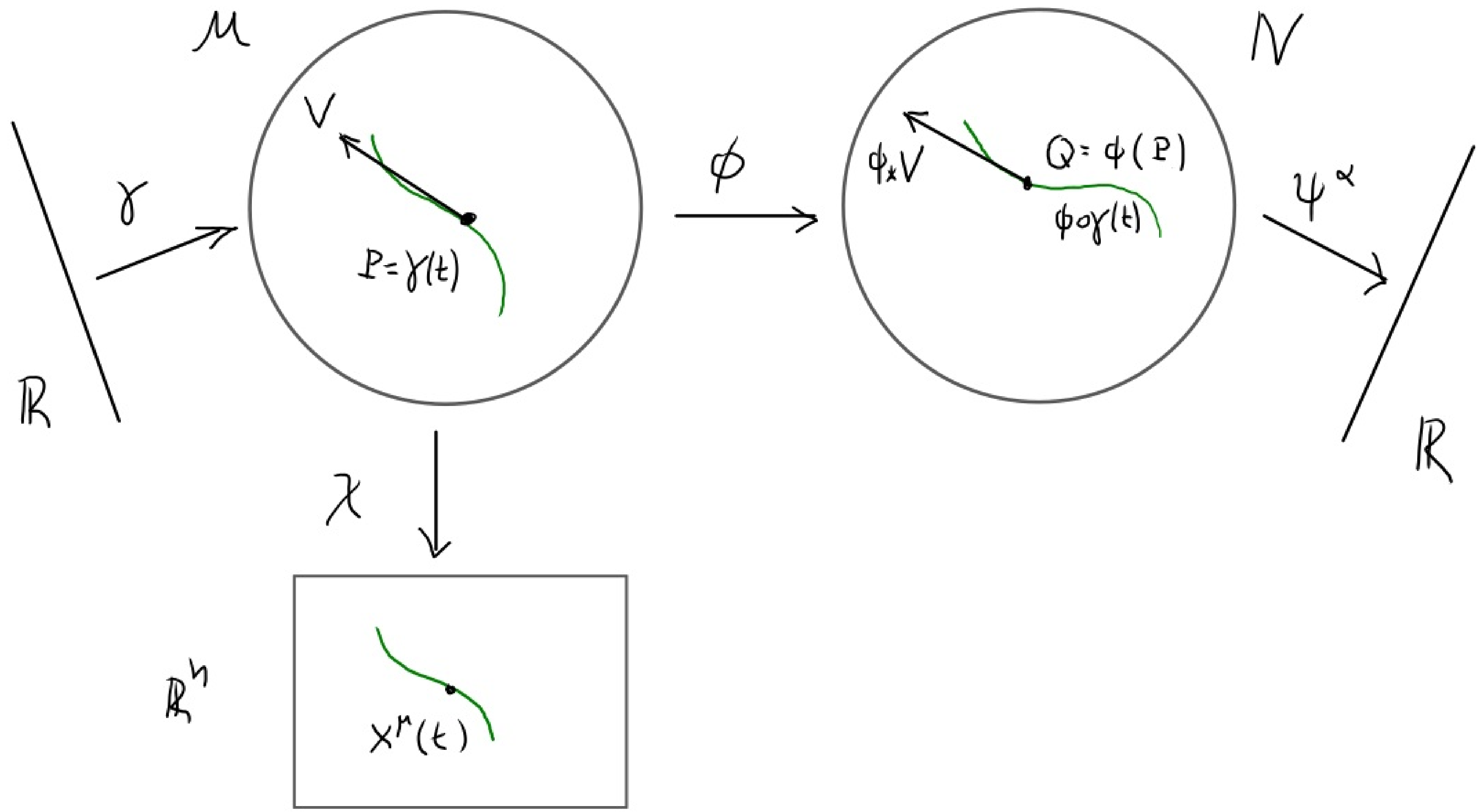
then:

$(\phi_* V)^\alpha = \phi_* V(\psi^\alpha)$



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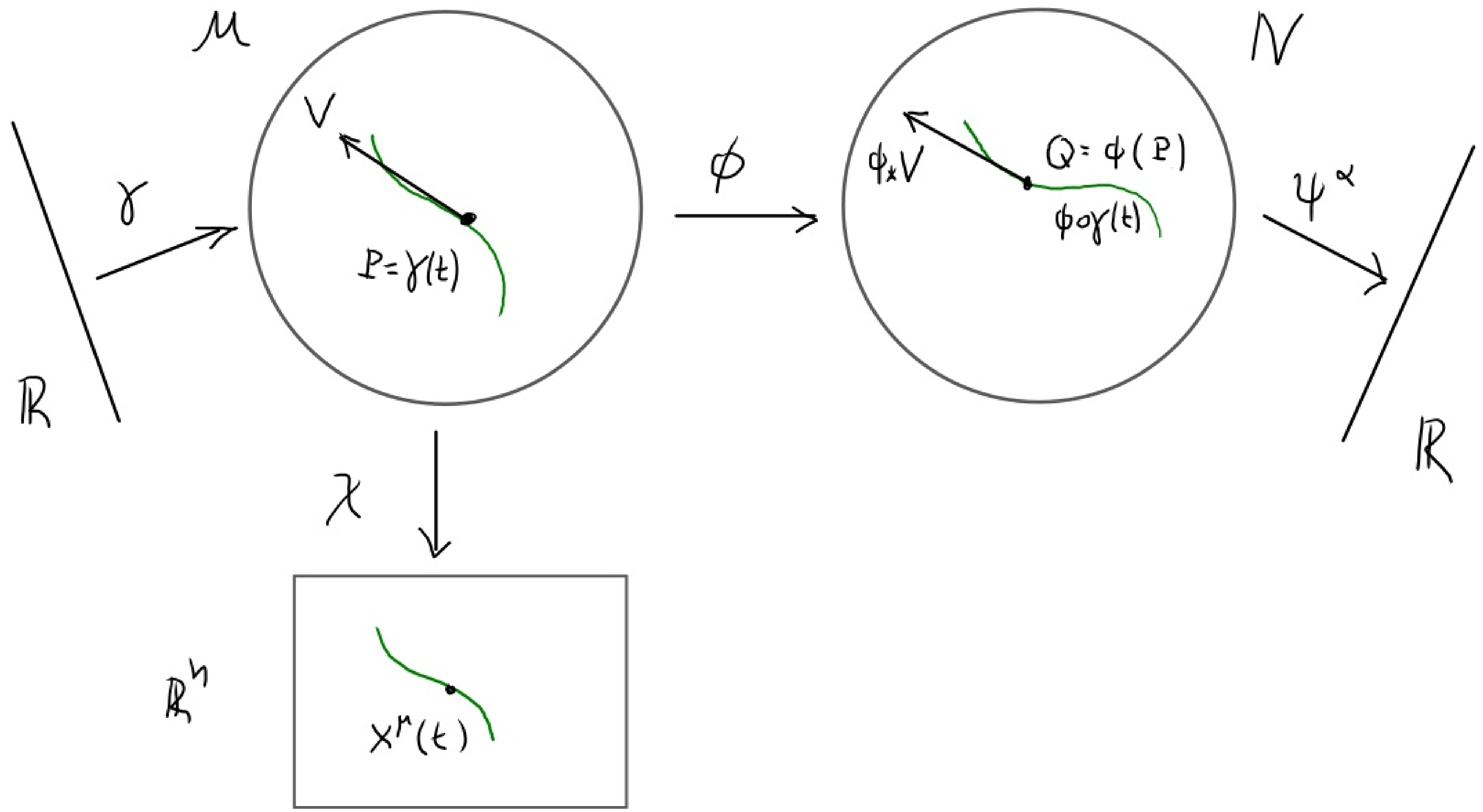
$$(\phi_* V)^\alpha = \phi_* V(\psi^\alpha)$$
$$= V(\phi^* \psi^\alpha) = V(\psi^\alpha \circ \phi)$$



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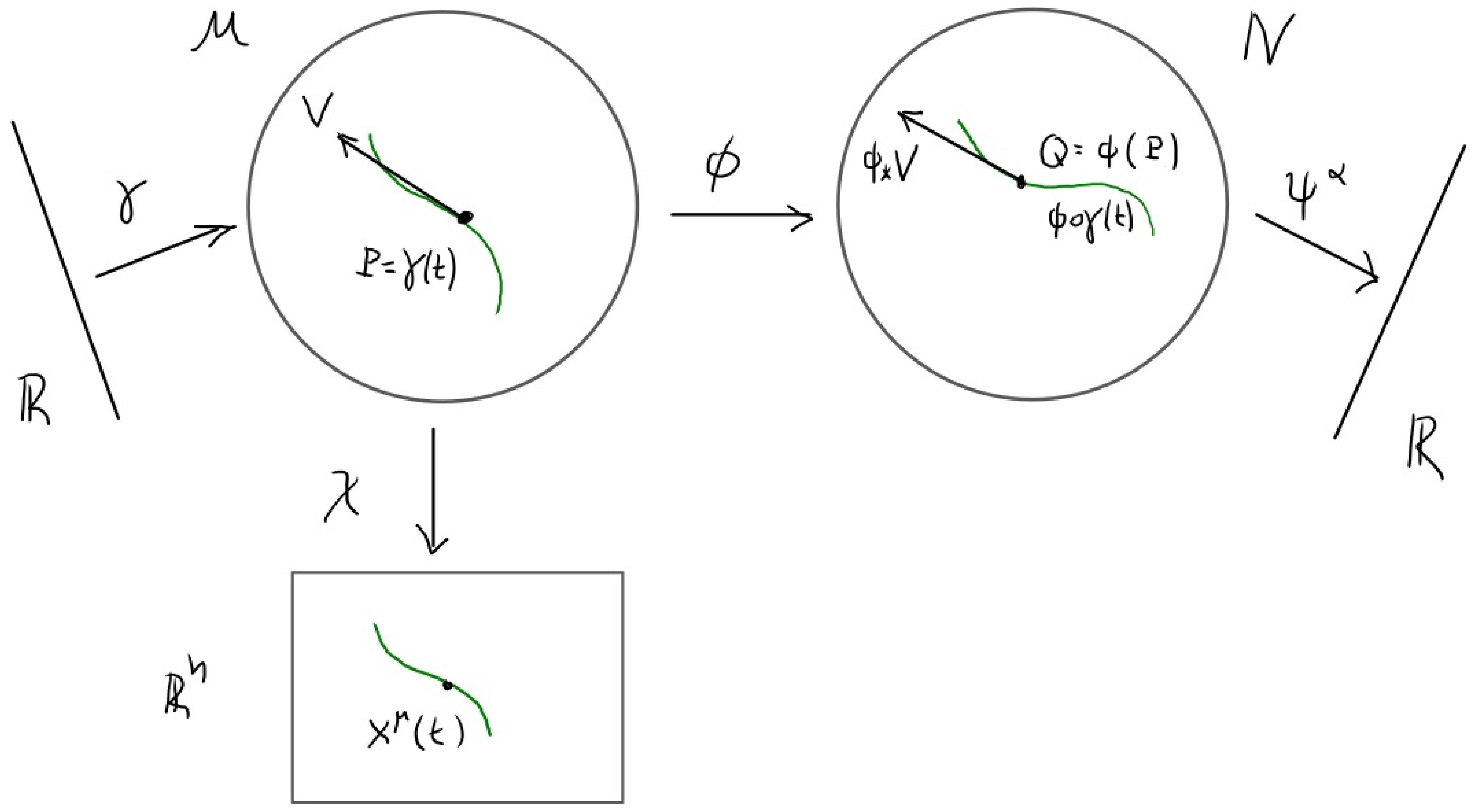
$$= \frac{d}{dt} \psi^\alpha \circ \phi \circ \gamma(t)$$



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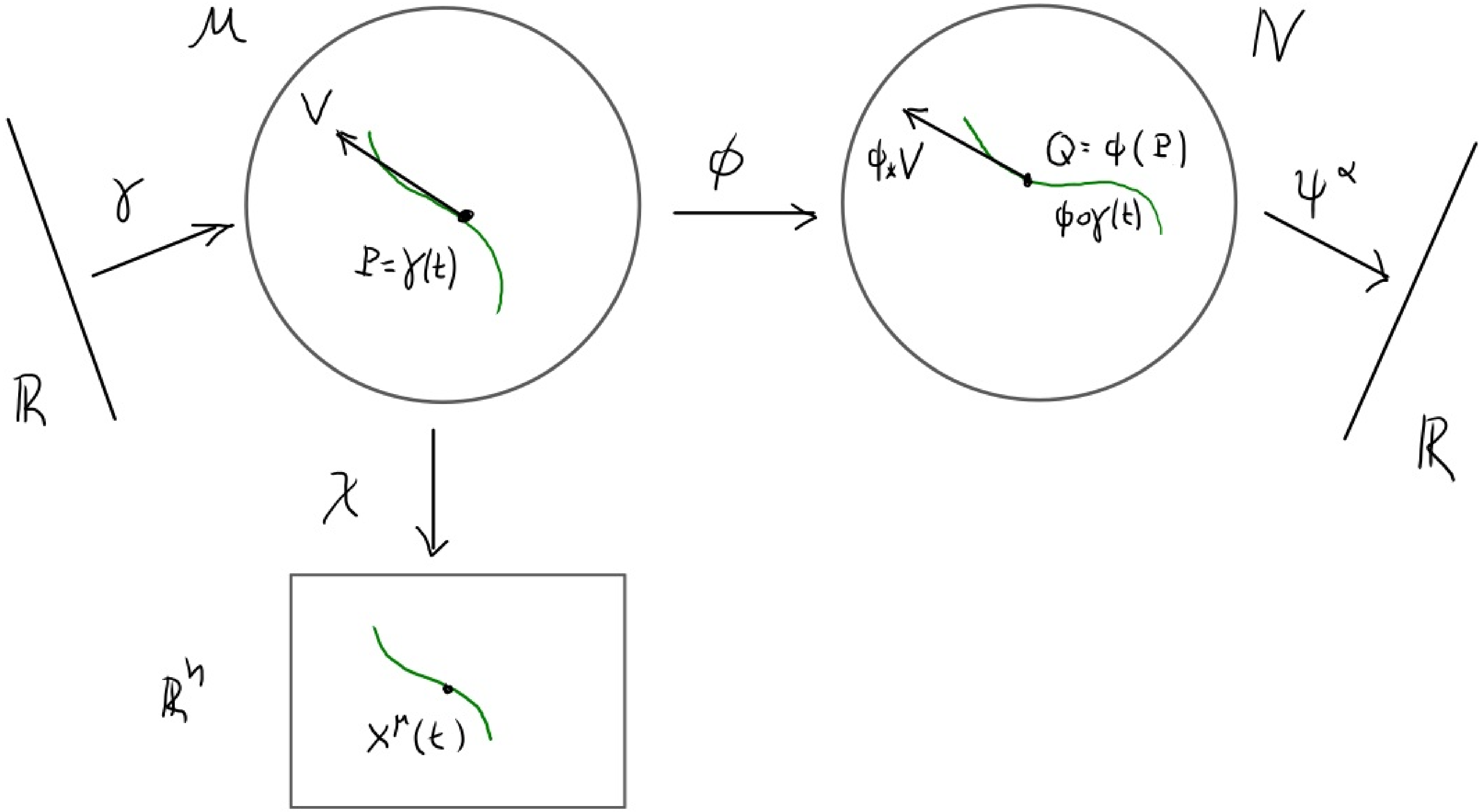

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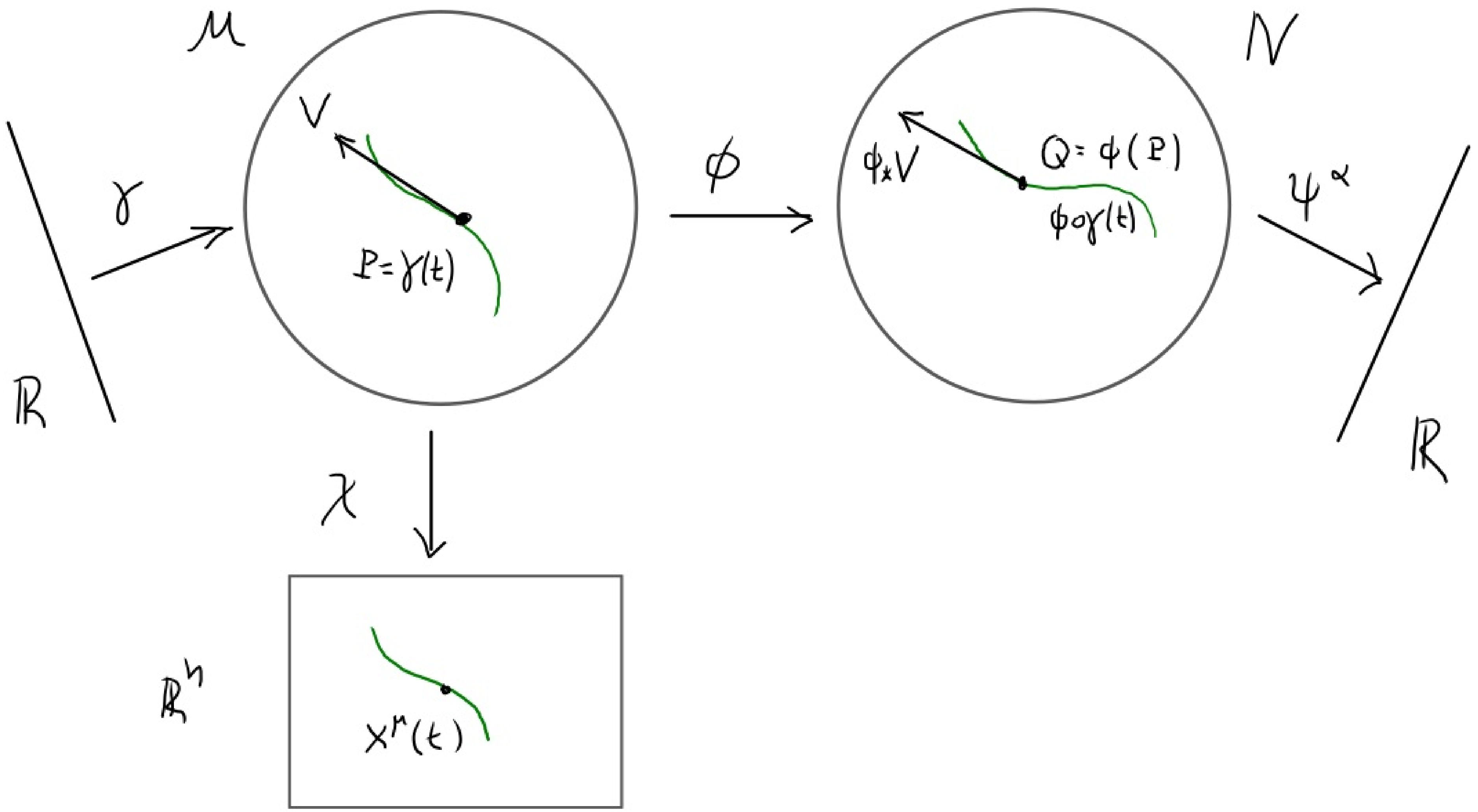

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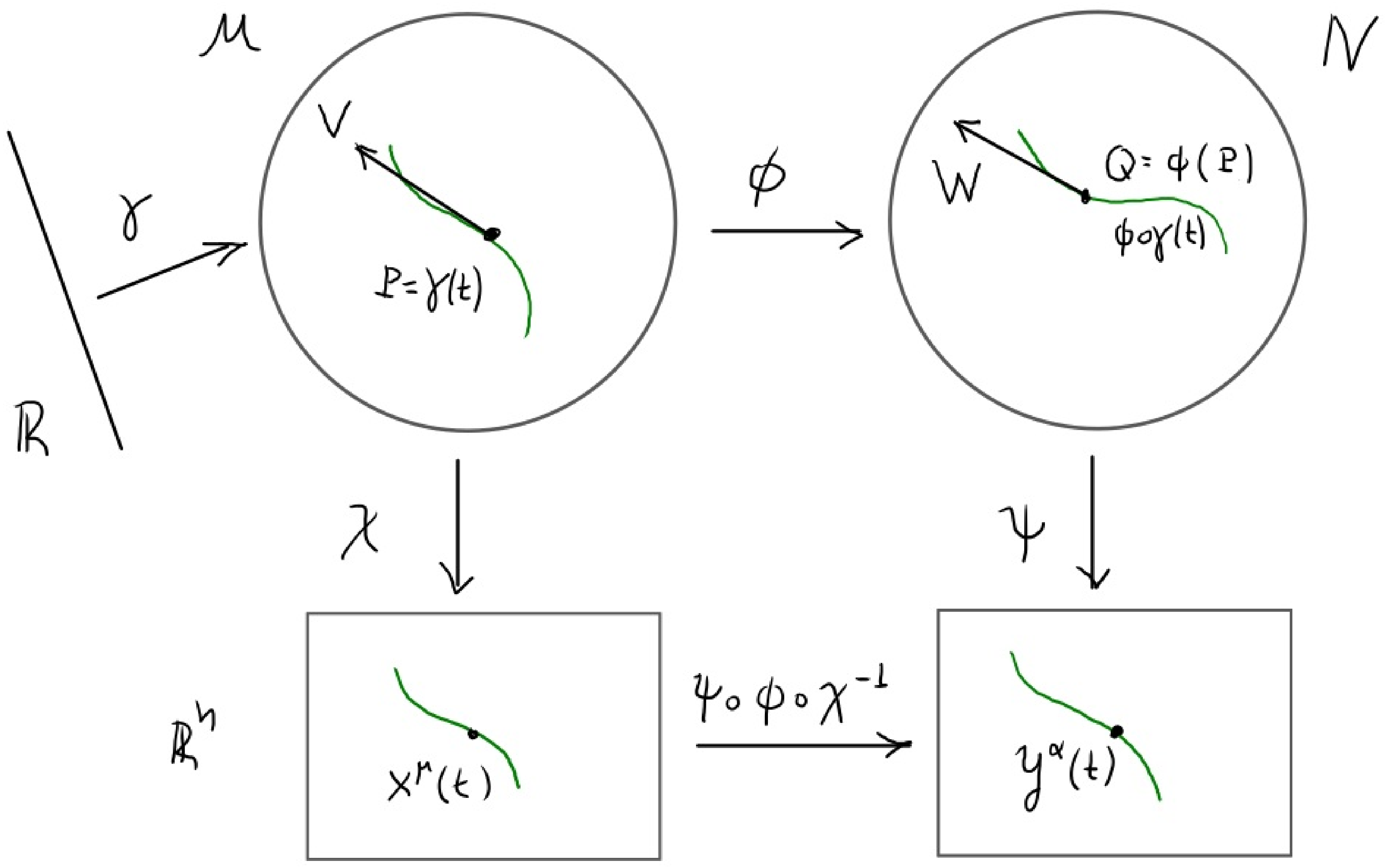
$$
\begin{aligned}
(\phi_* V)^\alpha &= \phi_* V(\psi^\alpha) \\
&= V(\phi^* \psi^\alpha) = V(\psi^\alpha \circ \phi) \\
&= \frac{d}{dt} \psi^\alpha \circ \phi \circ \gamma(t) \\
&= \frac{d}{dt} \underbrace{\psi^\alpha \circ \phi \circ}_{y^\alpha(x^\mu)} \chi^{-1} \circ \underbrace{\chi \circ \gamma}_{x^\mu(t)}(t) = \frac{\partial y^\alpha}{\partial x^\mu} \cdot \underbrace{\frac{d x^\mu}{dt}}_{V^\mu} = \frac{\partial y^\alpha}{\partial x^\mu} V^\mu
\end{aligned}
$$

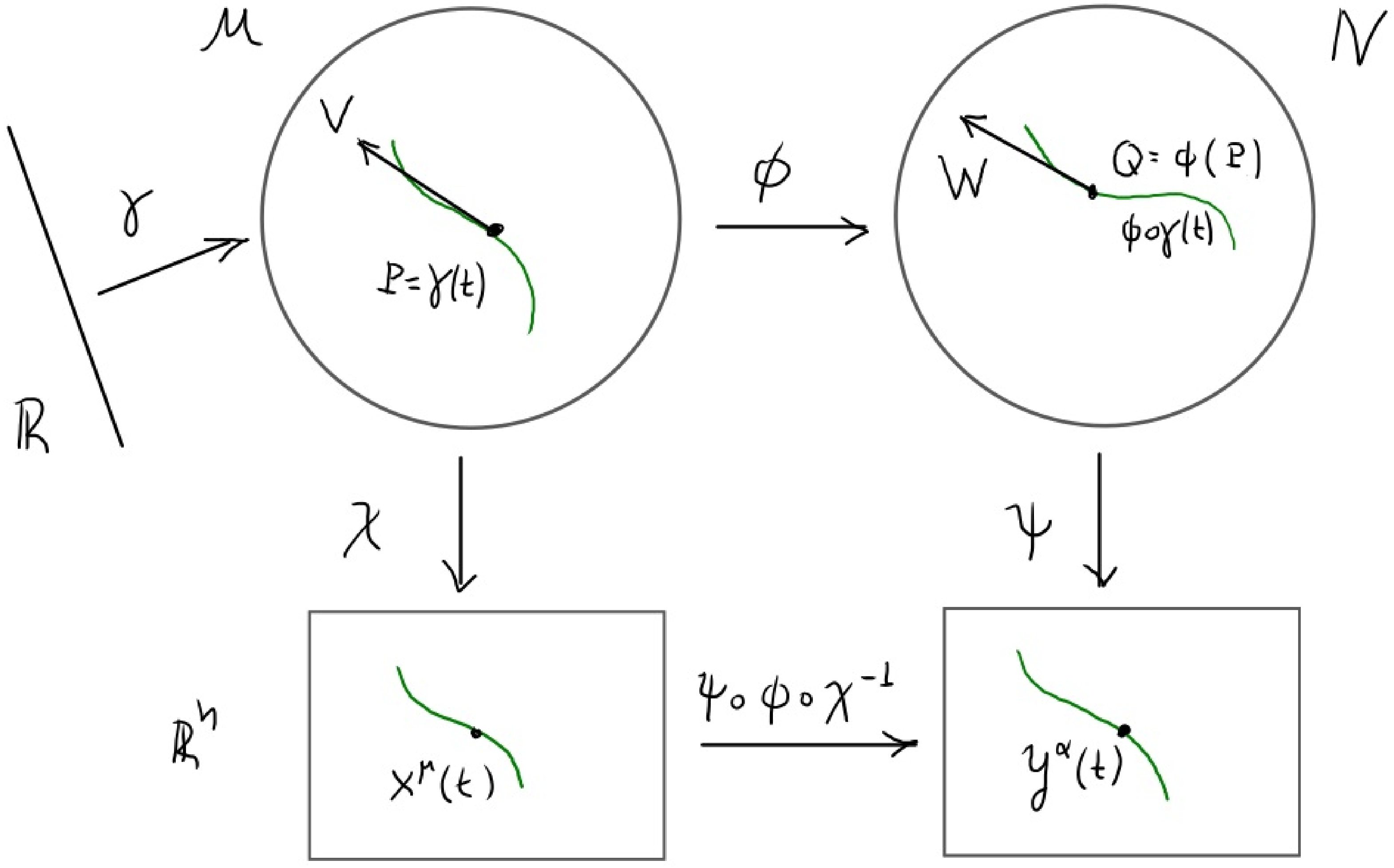
Proof: Image of curve $\gamma \to \phi \circ \gamma$ has tangent $W = \phi_* V$



Proof: Image of curve $\gamma \to \phi \circ \gamma$ has tangent $W = \phi_* V$

For any $f: N \to \mathbb{R}$

$$W(f) = W^\alpha \partial_\alpha f = \frac{dy^\alpha}{dt} \partial_\alpha f \quad (1)$$



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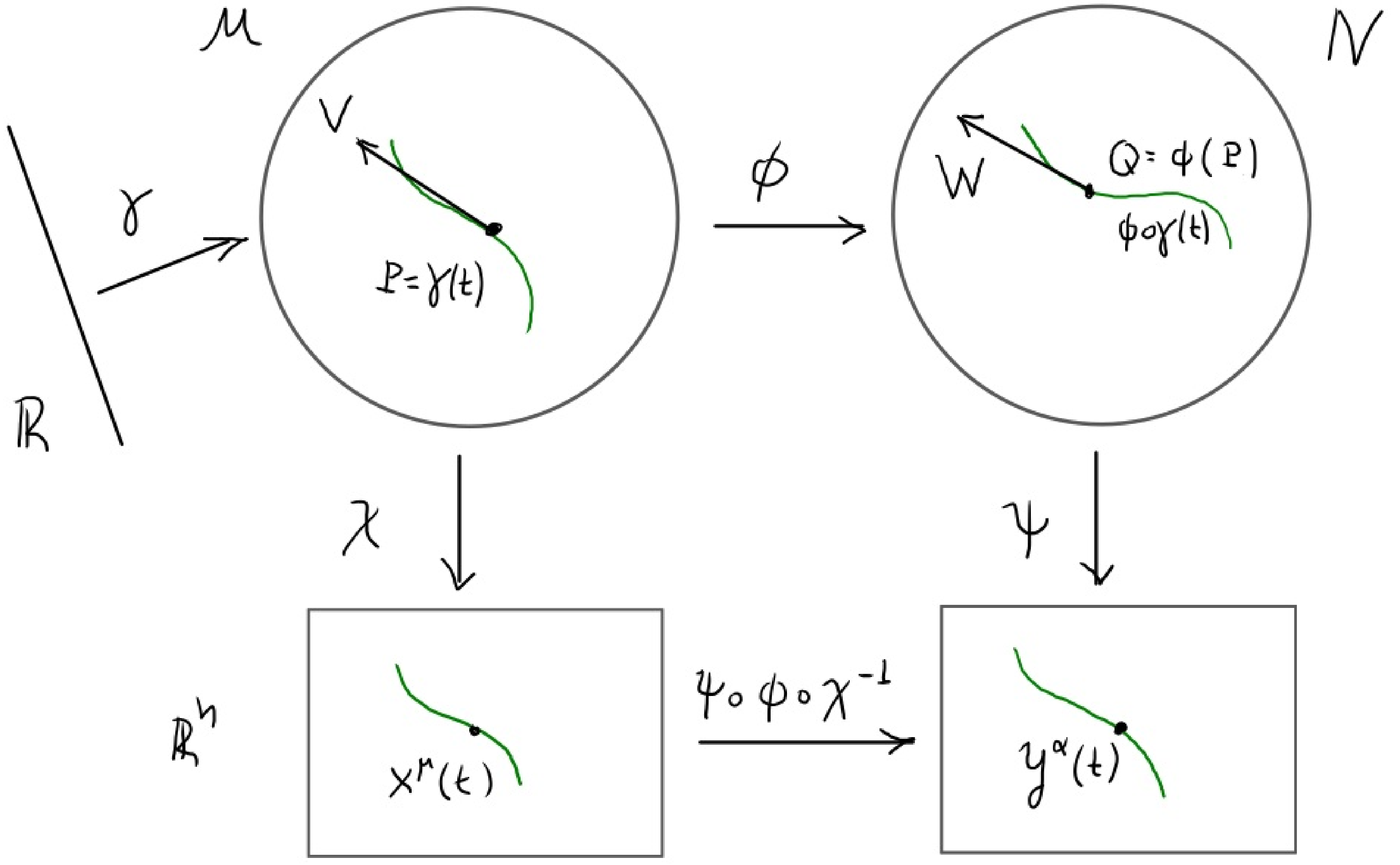
$$\phi_* V(f) = V(\phi^* f) \quad \text{(definition)}$$



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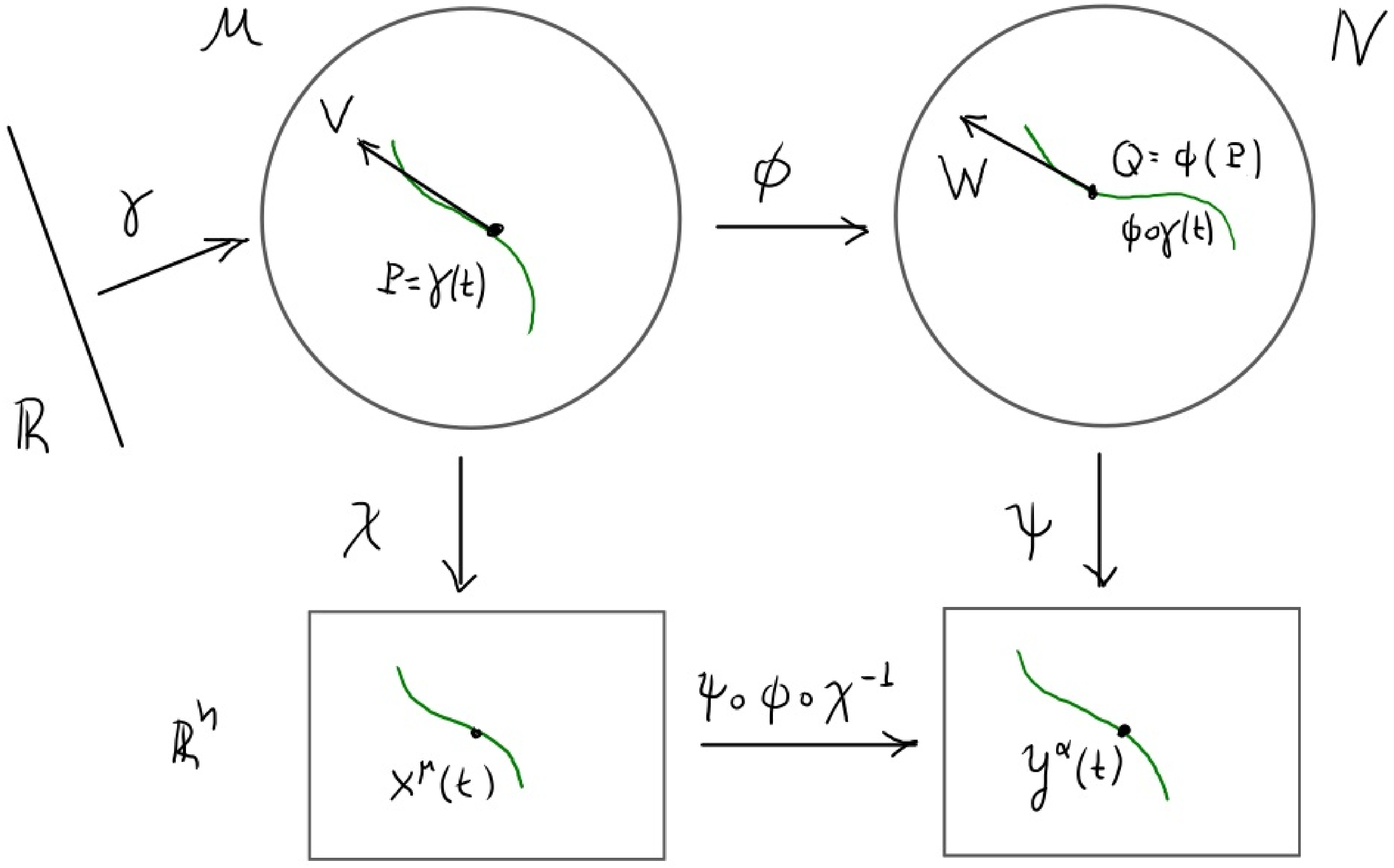

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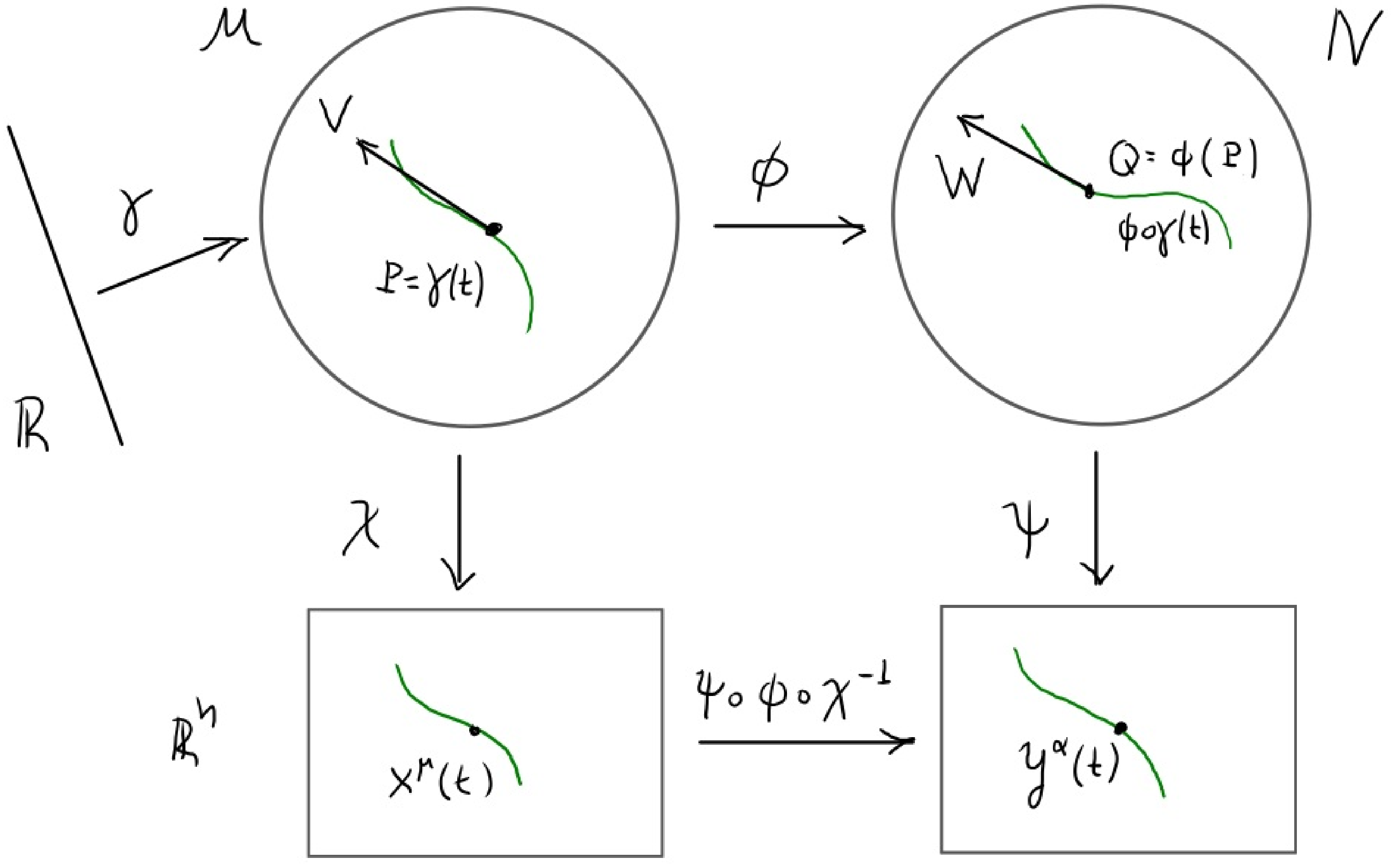

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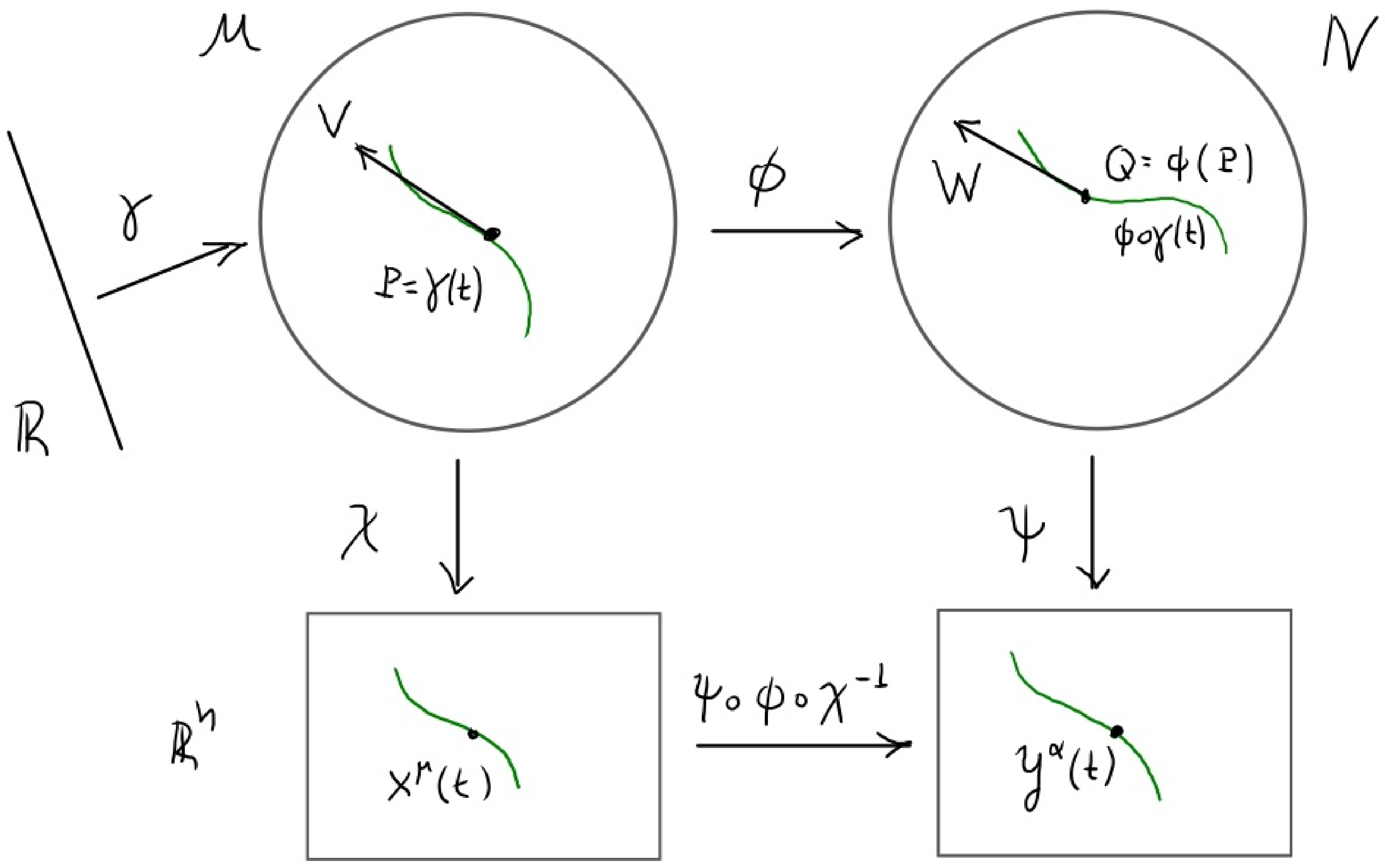

$$y^\alpha(t) = \psi\circ\phi\circ\gamma(t) = \underbrace{(\psi\circ\phi\circ\chi^{-1})}_{y^\alpha(x^\mu)}\circ\underbrace{(\chi\circ\gamma)}_{x^\mu(t)}(t) = y^\alpha(x^\mu(t))$$

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Therefore: $$\frac{dy^\alpha}{dt} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{dx^\mu}{dt} \quad (2)$$

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$$(1),(2) \Rightarrow W(f) = \left(\frac{\partial y^\alpha}{\partial x^\mu} \cdot \frac{dx^\mu}{dt}\right) \cdot \partial_\alpha f$$

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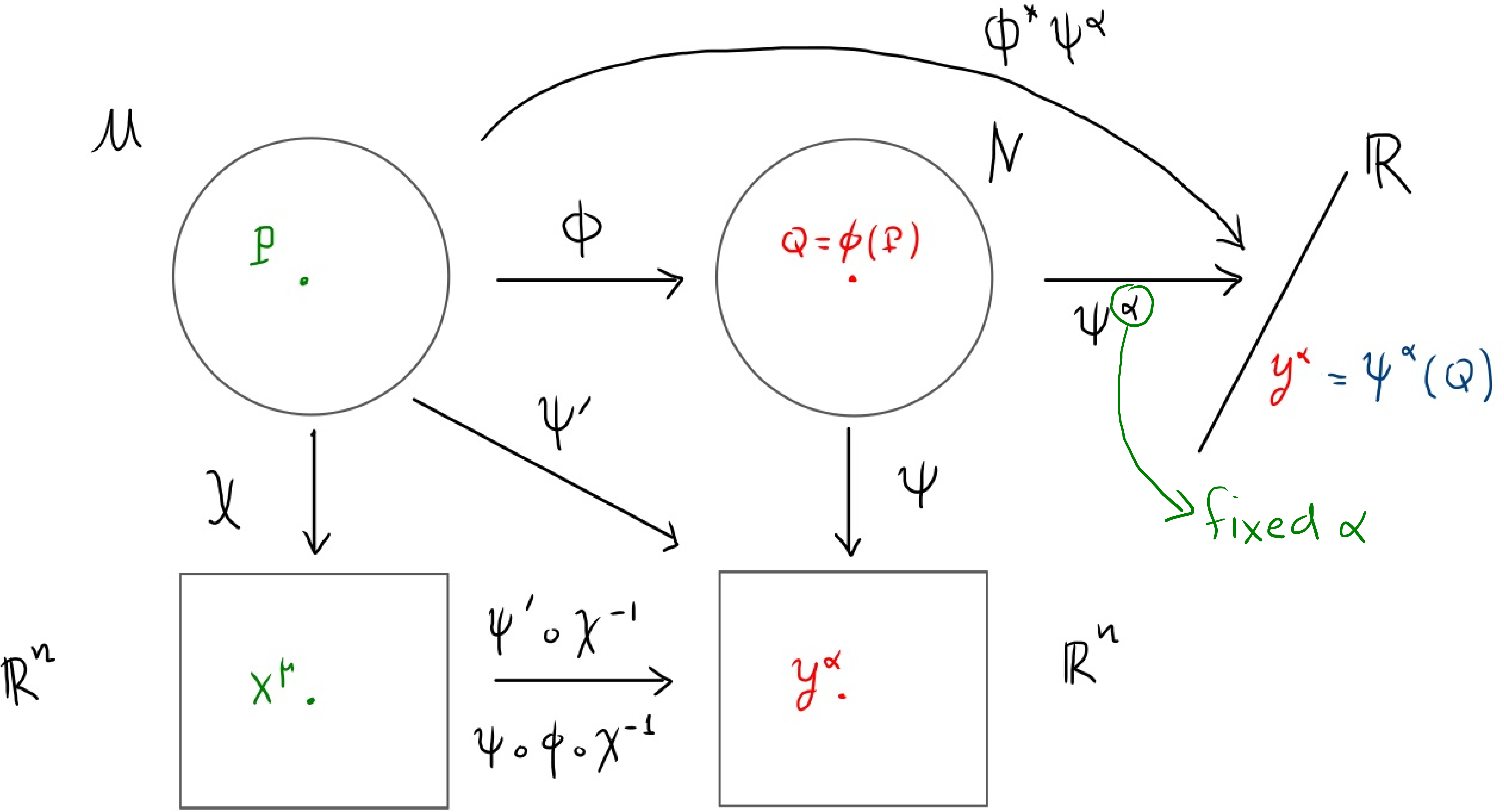
$$= \phi_* V(f)$$

Note:
We never used ϕ^{-1}

Diffeomorphisms

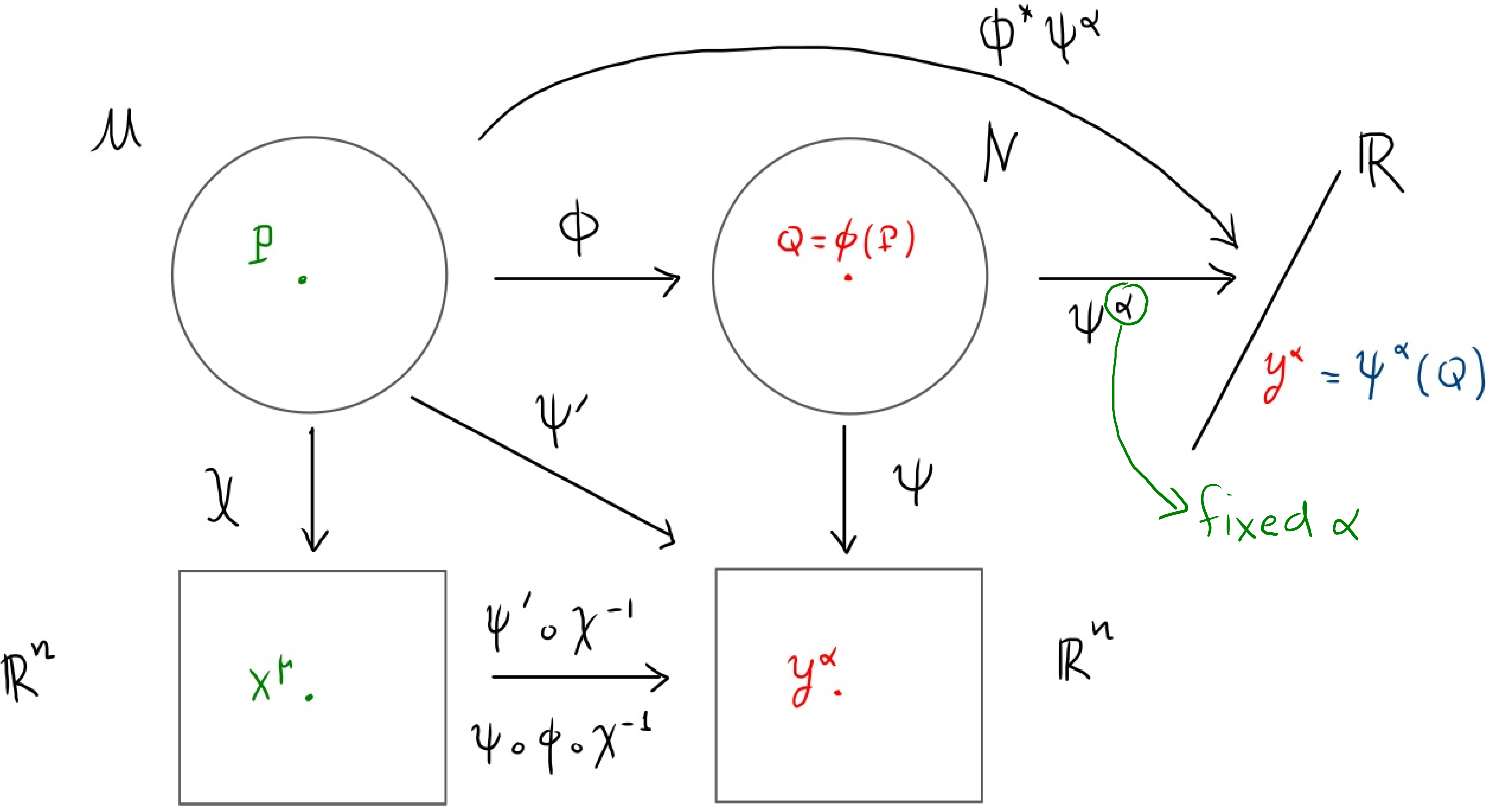


Diffeomorphic manifolds are "the same"

$\Updownarrow$

Diffeomorphisms equivalent to coordinate transformations

Diffeomorphisms



fixed α

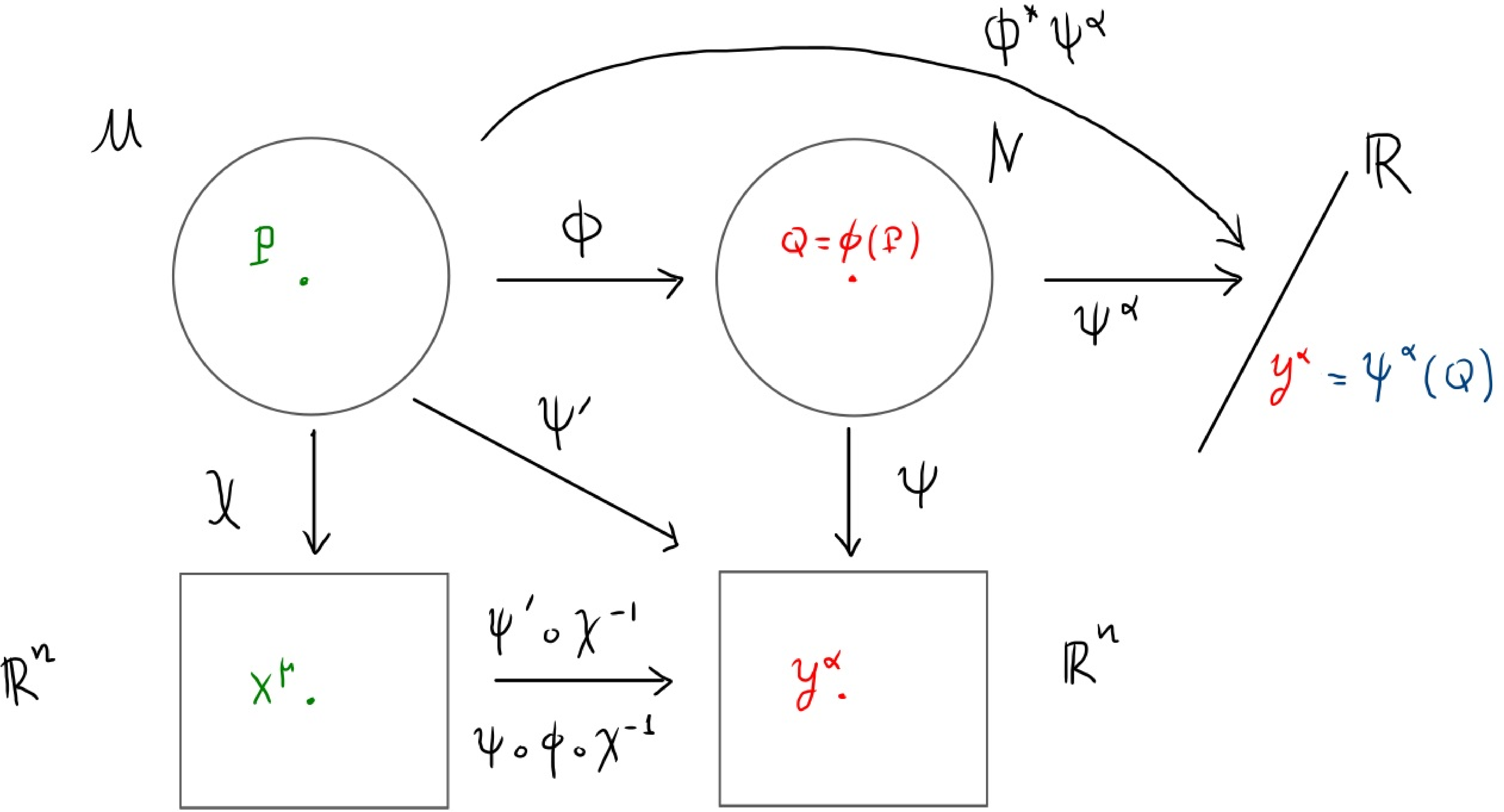
$\psi^\alpha : N \to \mathbb{R}$

$Q \mapsto y^\alpha(Q)$

$\phi^* \psi^\alpha : M \to \mathbb{R}$

$P \mapsto y^\alpha(P) \equiv y^\alpha(Q)$

Diffeomorphisms



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$Q \mapsto y^\alpha(Q)$

$\phi^* \psi^\alpha : \mathcal{M} \to \mathbb{R}$

$P \mapsto y^\alpha(P) = y^\alpha(Q)$

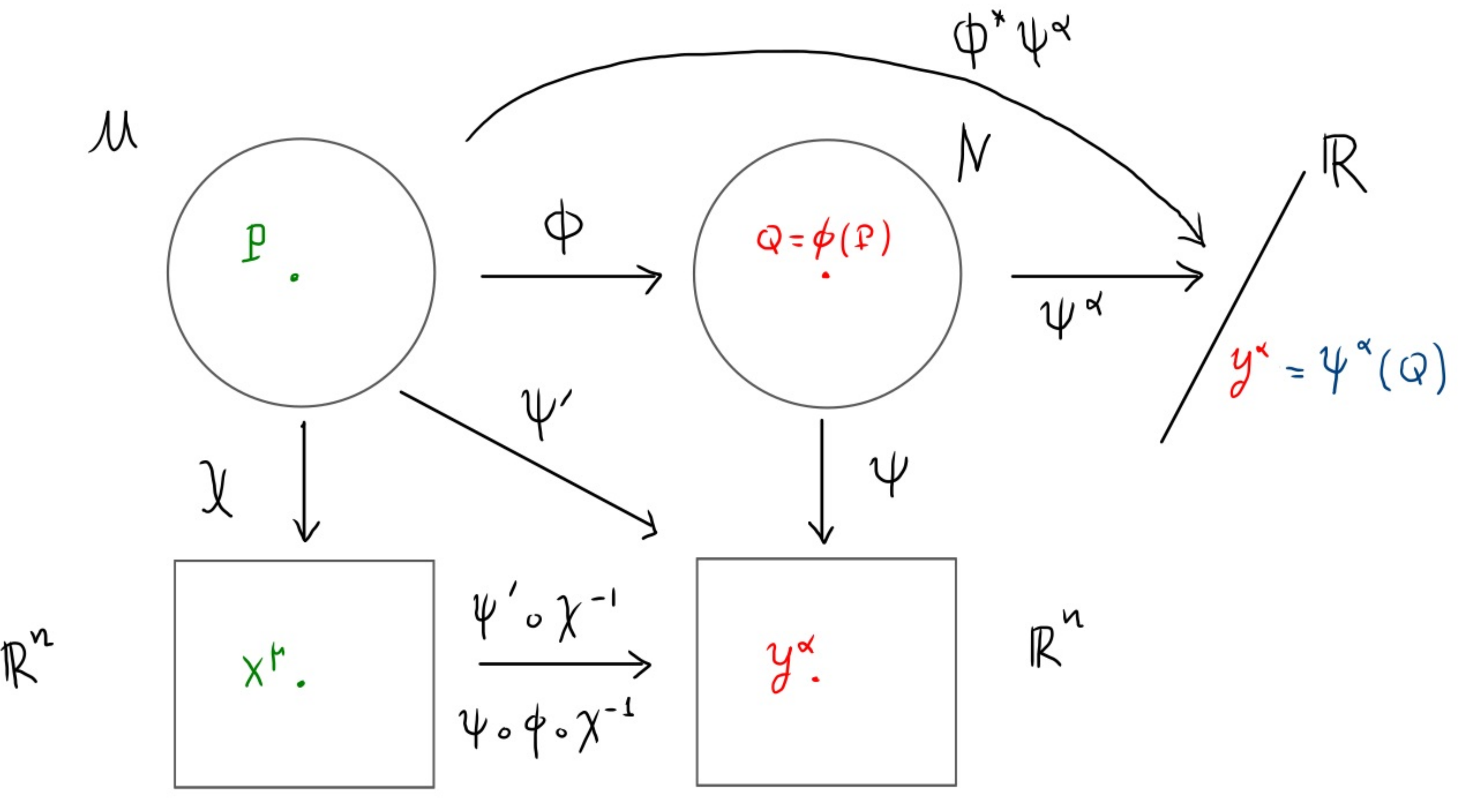
Define chart at P:

$$\psi' = \{ \phi^* \psi^\alpha \}$$

Possible because invertible:

$$\psi' = \psi \circ \phi \Rightarrow (\psi')^{-1} = \phi^{-1} \circ \psi^{-1}$$

Diffeomorphisms



Then $y^\alpha(x^\mu) = \psi' \circ \chi^{-1}(x^\mu)$ is a coordinate transformation
(passive transformation)

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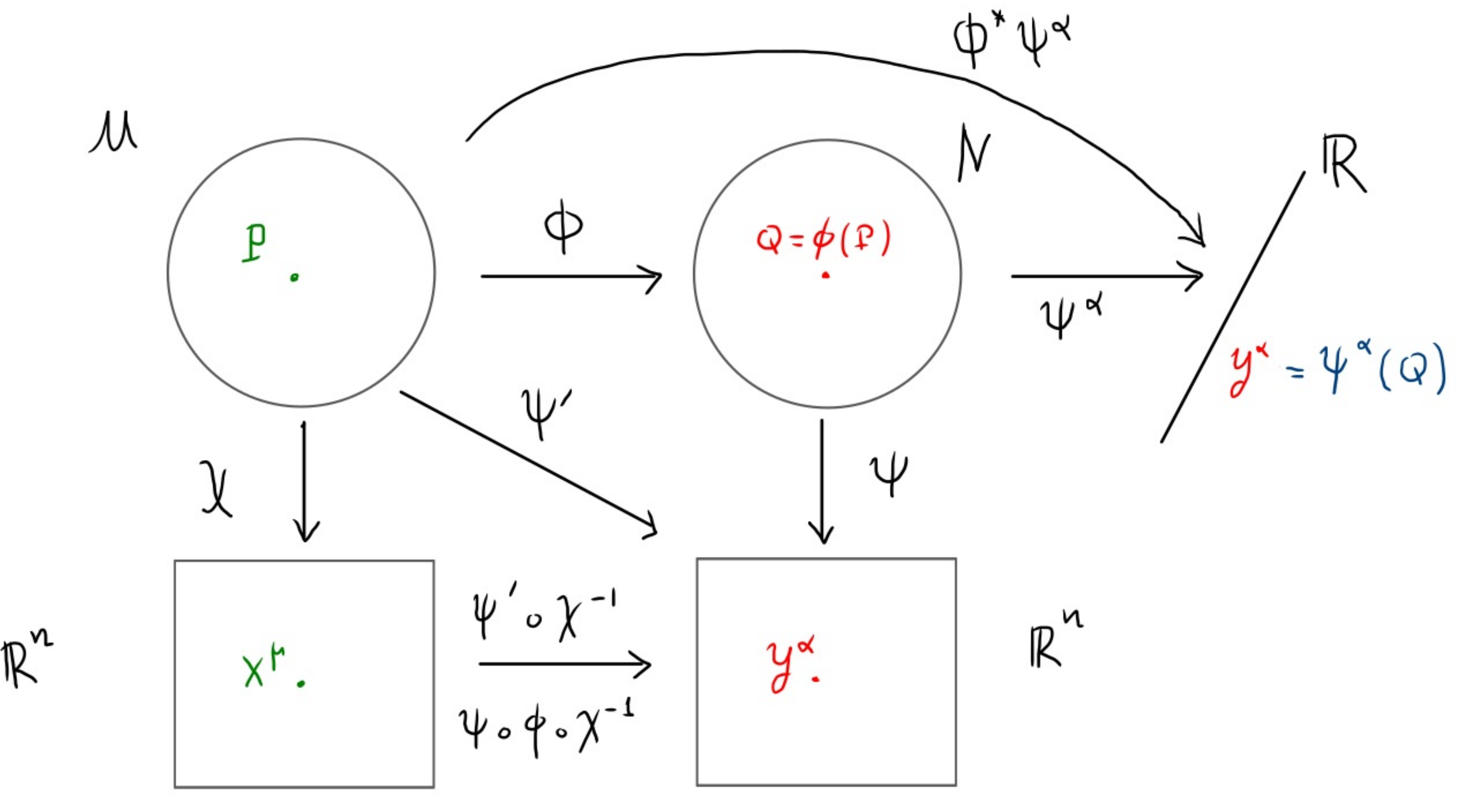

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(passive transformation)

$y^\alpha(x^\mu) = \psi \circ \phi \circ \chi^{-1}(x^\mu)$ maps coordinates of P and Q
$= \psi(\phi(P))$ (active transformation)

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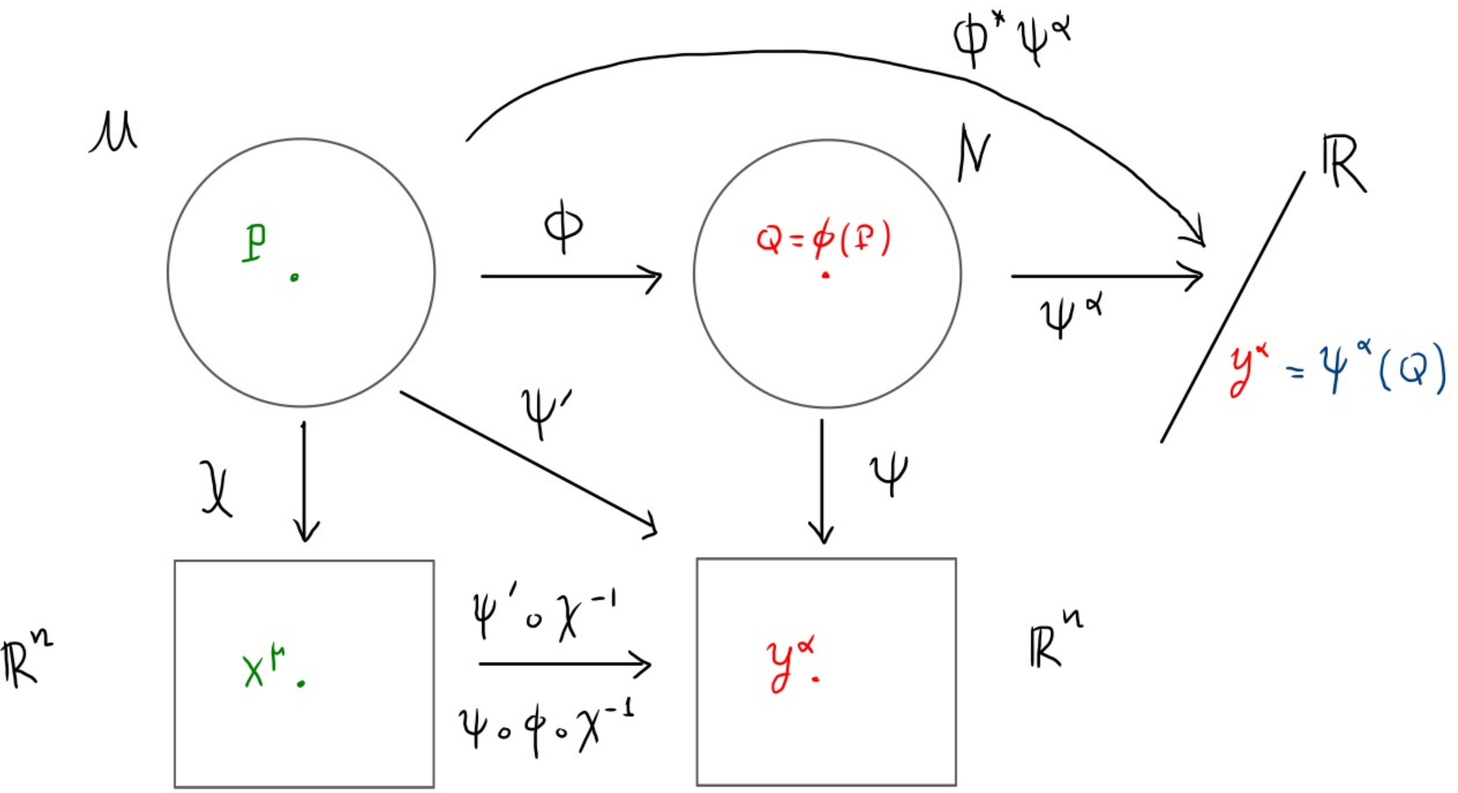
Diffeomorphisms



In a sense:

(coordinate transformations)

$\Updownarrow$

(diffeomorphisms)

$\mathcal{M}$ and $\mathcal{N}$ are the "same"

$\psi^\alpha : \mathcal{N} \to \mathbb{R}$ ← fixed α

$Q \mapsto y^\alpha(Q)$

$\phi^* \psi^\alpha : \mathcal{M} \to \mathbb{R}$

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Define chart at P:

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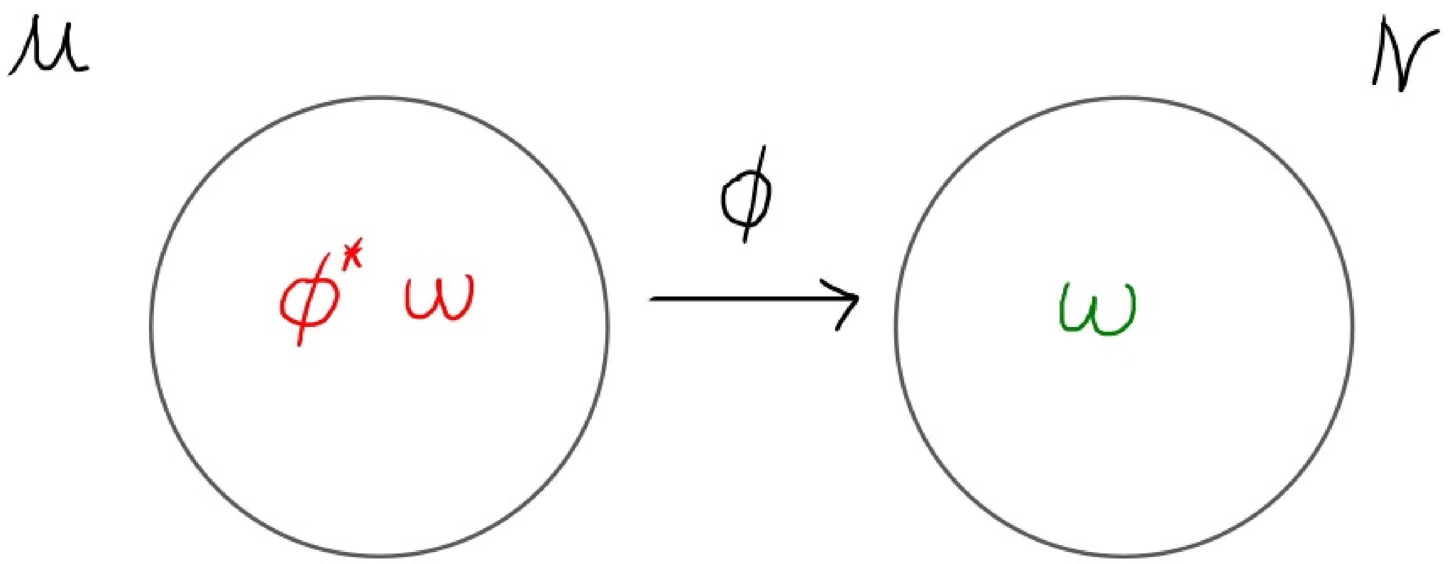
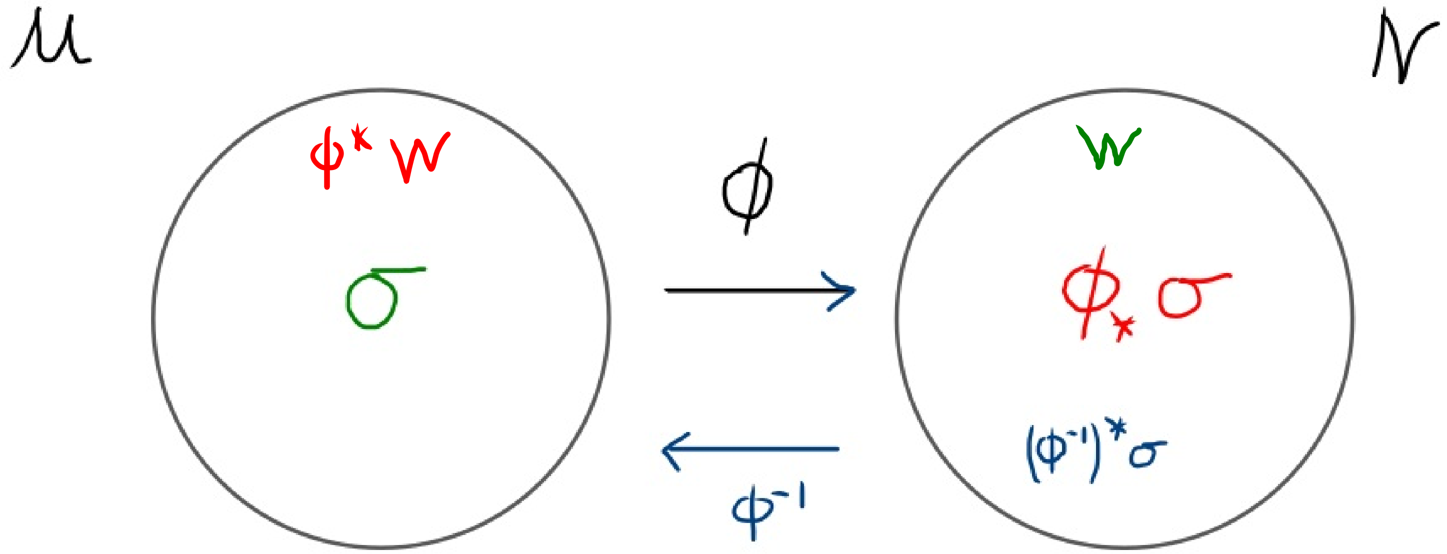
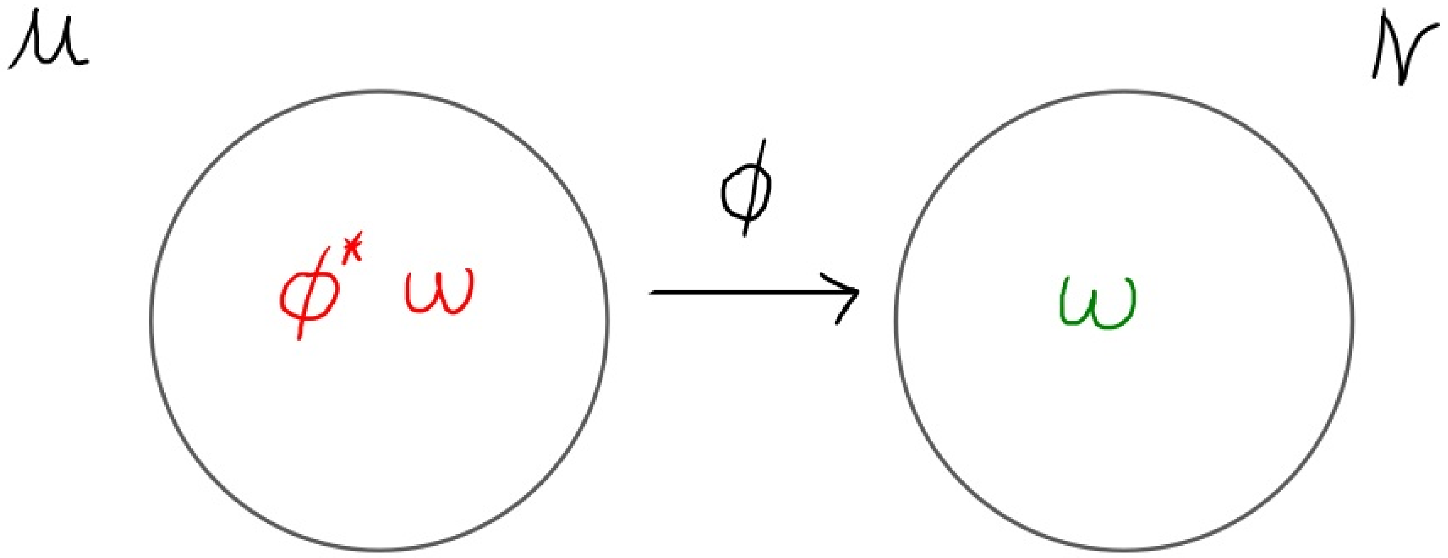


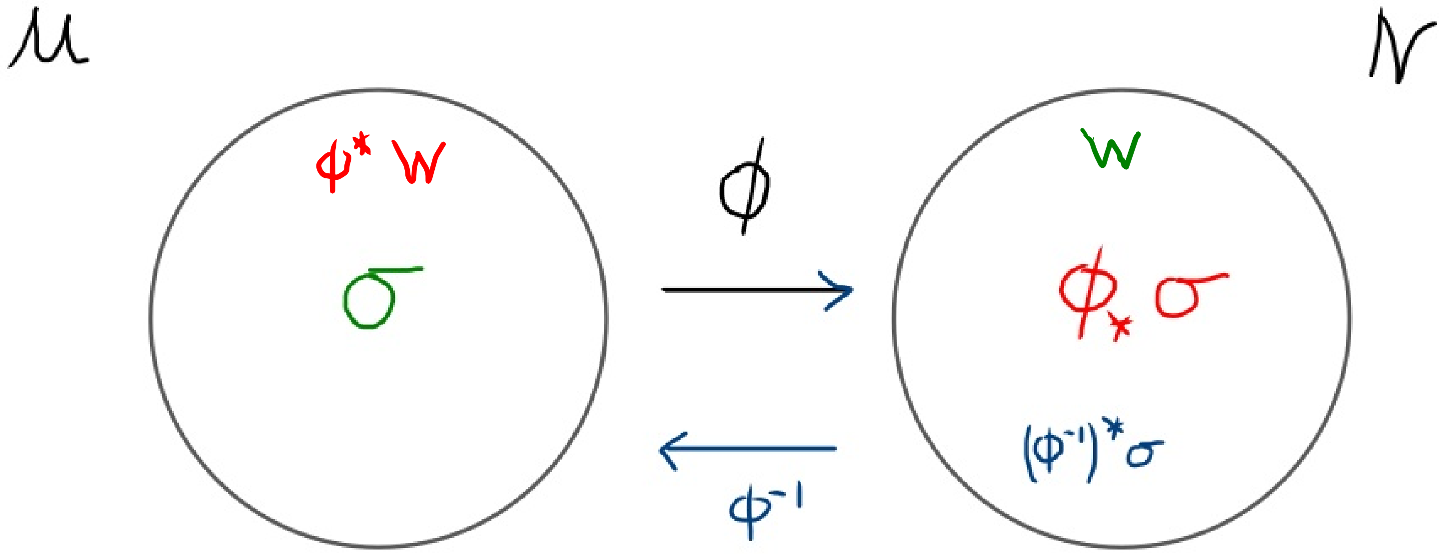


One forms:

- Define $\phi_* \sigma = (\phi^{-1})^* \sigma$ push forward of σ

then $\phi_* \sigma (W) = (\phi^{-1})^* \sigma (W) = \sigma\left((\phi^{-1})_* W \right) = \sigma(\phi^* W)$





One forms:

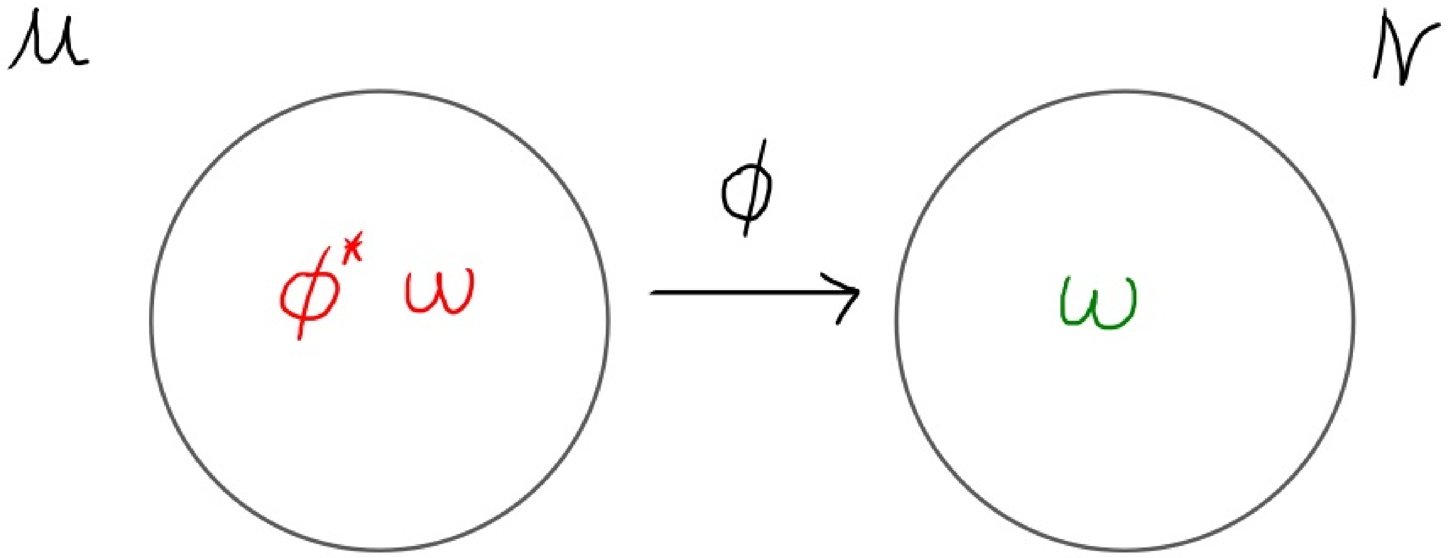
- Define $\phi_* \sigma = (\phi^{-1})^* \sigma$ push forward of σ

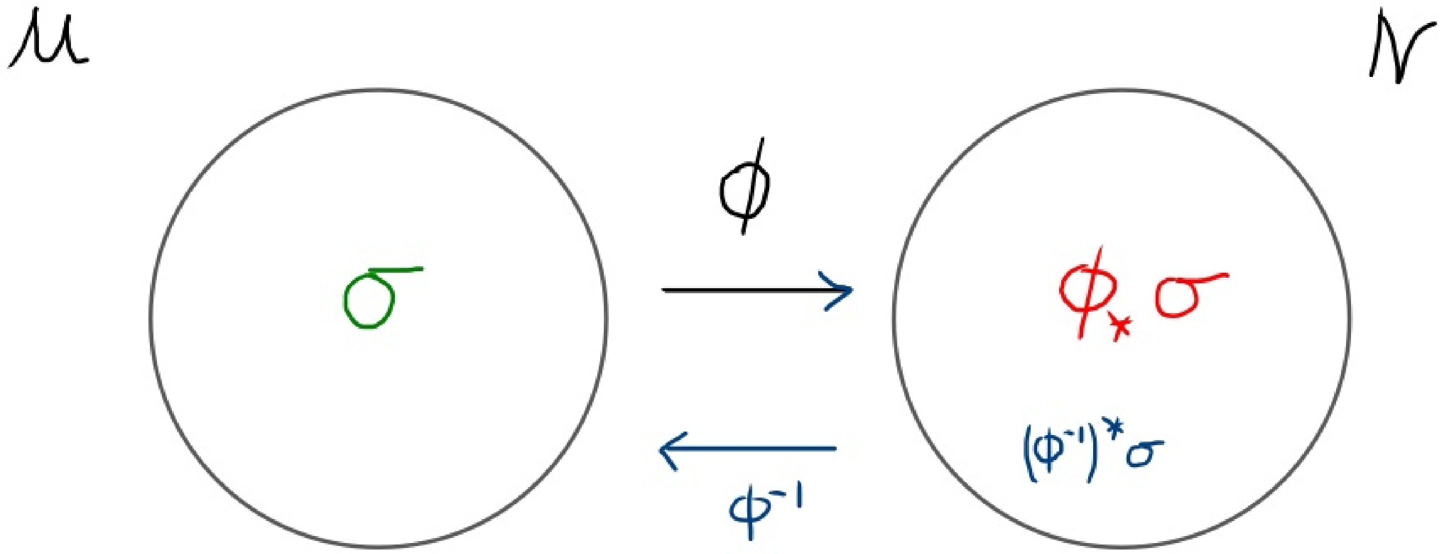
then $\phi_* \sigma (W) = (\phi^{-1})^* \sigma (W) = \sigma\left((\phi^{-1})_* W \right) = \sigma(\phi^* W)$

then $\left(\phi_* (\phi^* \omega)\right)(W) = \left((\phi^{-1})^* (\phi^* \omega)\right)(W) = \phi^* \omega\left((\phi^{-1})_* W \right) = \omega\left(\phi_* (\phi^{-1})_* W \right) = \omega(W)$

$\Rightarrow \phi_* (\phi^* \omega) = \omega$

$\Rightarrow \phi_* \phi^* = 1$





One forms:

- Define $\phi_* \sigma = (\phi^{-1})^* \sigma$ push forward of σ

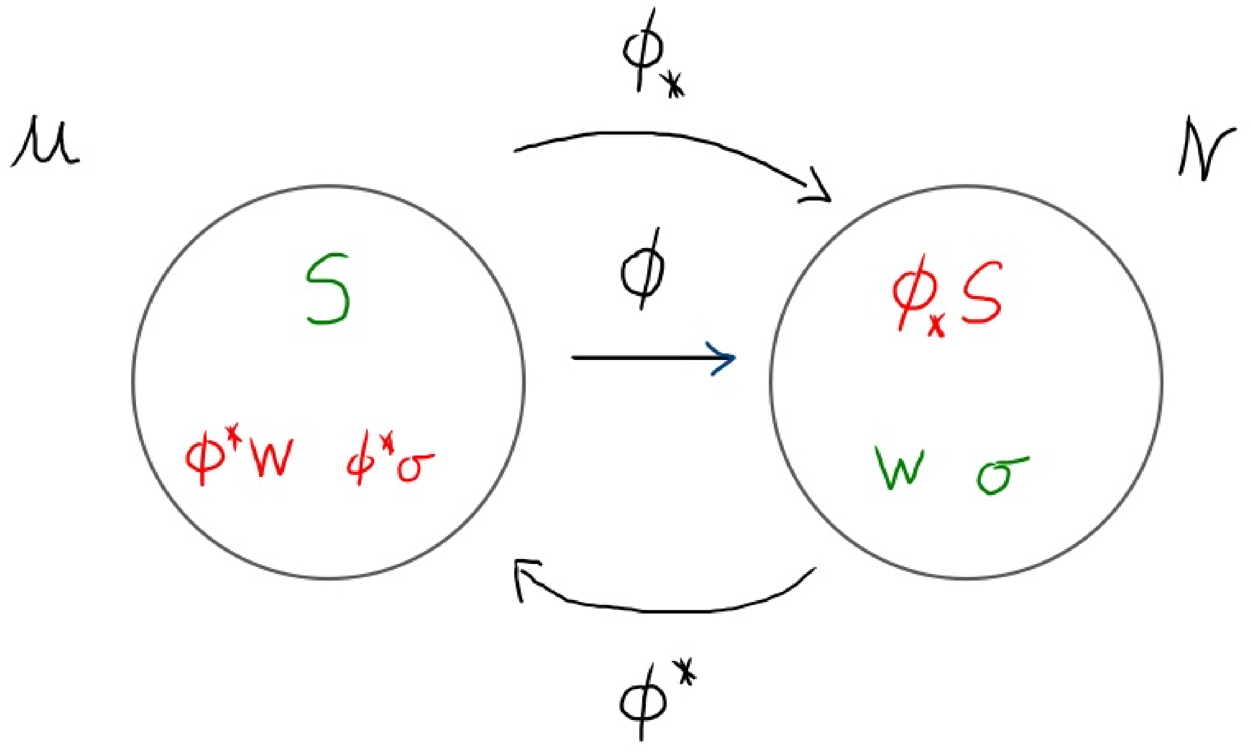
Components: $$(\phi_* \sigma)_\alpha = \left((\phi^{-1})^* \sigma\right)_\alpha = \frac{\partial x^\mu}{\partial y^\alpha} \sigma_\mu = (\phi^{-1})^*{}_\alpha{}^\mu \, \sigma_\mu$$

Summary:

$(\phi_* V)^\alpha = \dfrac{\partial y^\alpha}{\partial x^\mu} V^\mu$	$(\phi_* \sigma)_\alpha = \dfrac{\partial x^\mu}{\partial y^\alpha} \sigma_\mu$	pushforward
$(\phi^* W)^\mu = \dfrac{\partial x^\mu}{\partial y^\alpha} W^\alpha$	$(\phi^* \omega)_\mu = \dfrac{\partial y^\alpha}{\partial x^\mu} \omega_\alpha$	pullback

$\Rightarrow$ we can define pullback / pusforward for all tensors!

$(\exists\, \phi^{-1})$

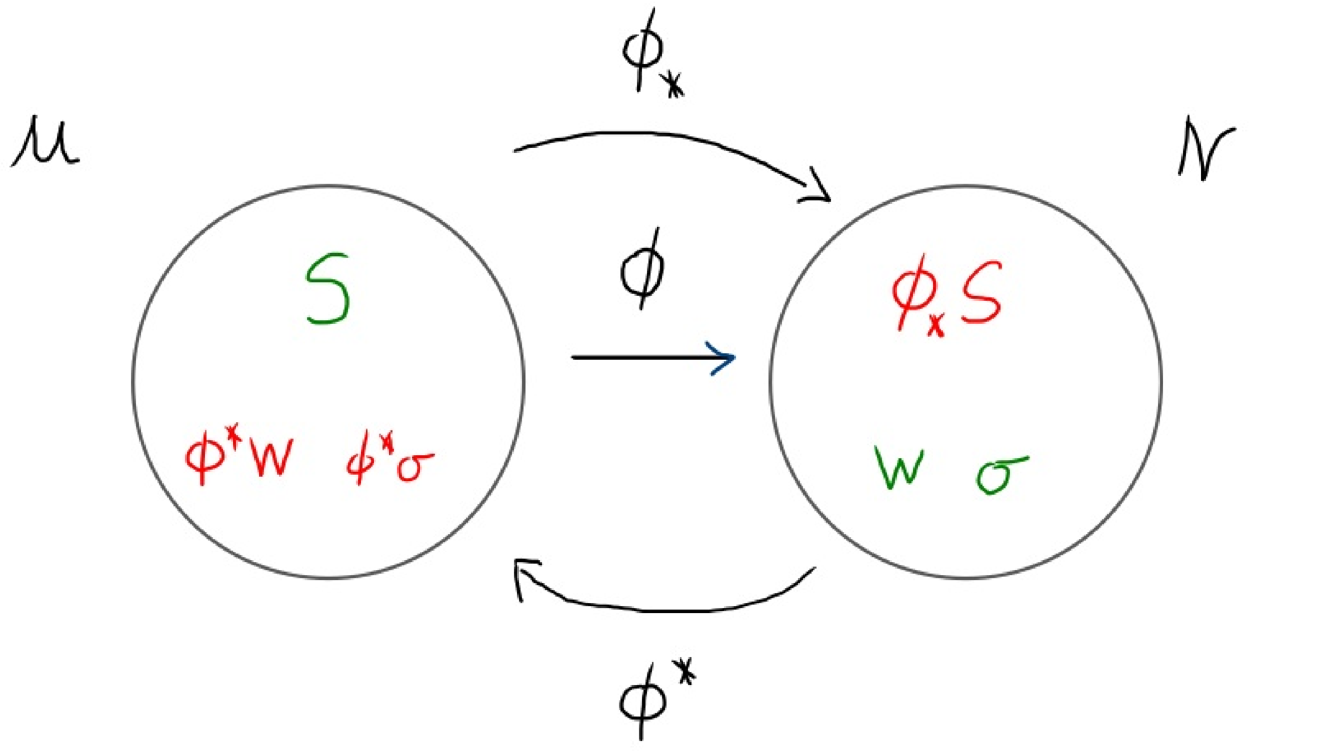


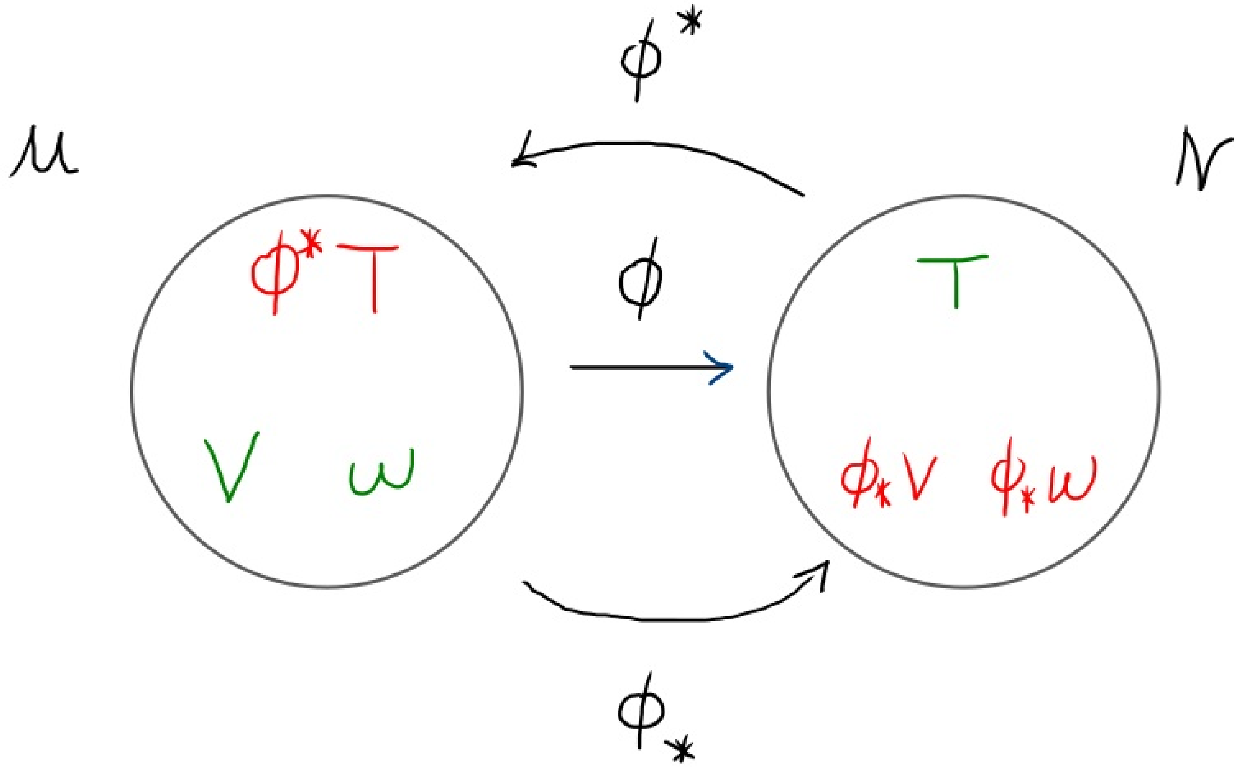
Mixed Tensors (1,1)

$$\phi_* S(\sigma, w) = S(\phi^* \sigma, \phi^* w)$$

push forward

$$(\phi_* S)^{\alpha}{}_{\beta} = \frac{\partial y^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial y^{\beta}} S^{\mu}{}_{\nu}$$





Mixed Tensors (1,1)

$$\phi_* S(\sigma, w) = S(\phi^*\sigma, \phi^* w)$$

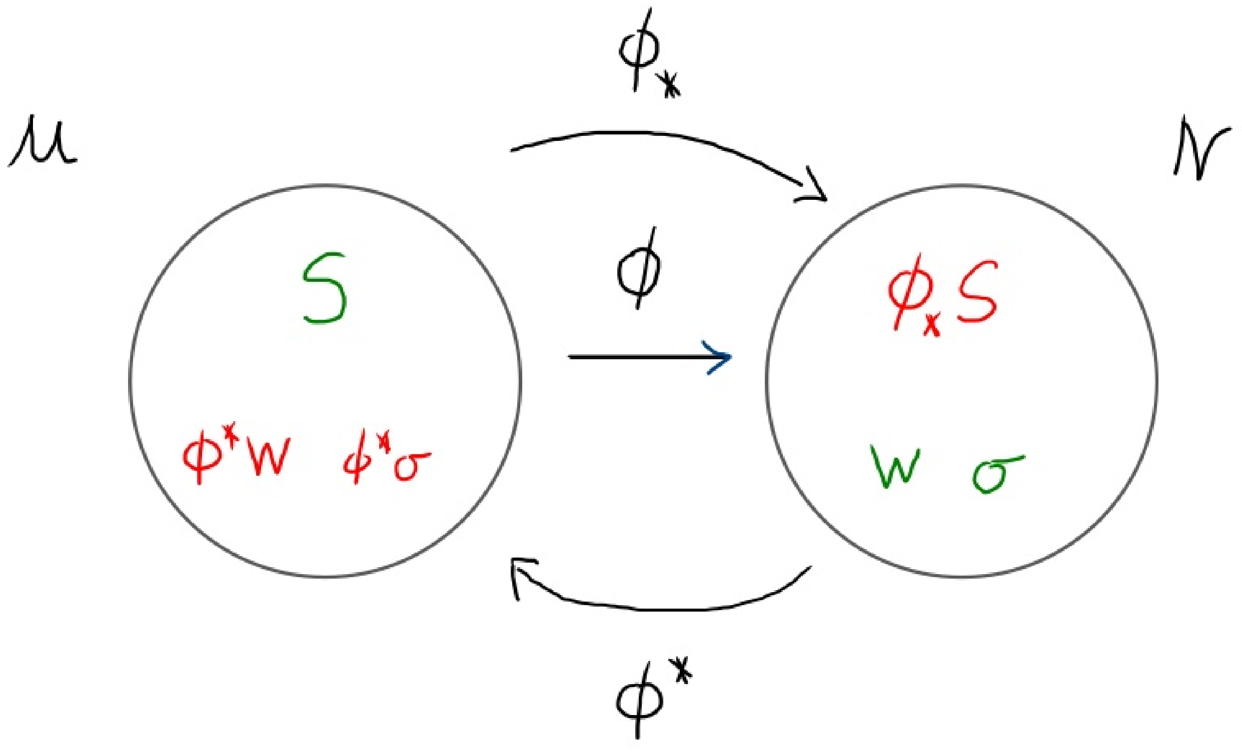
push forward

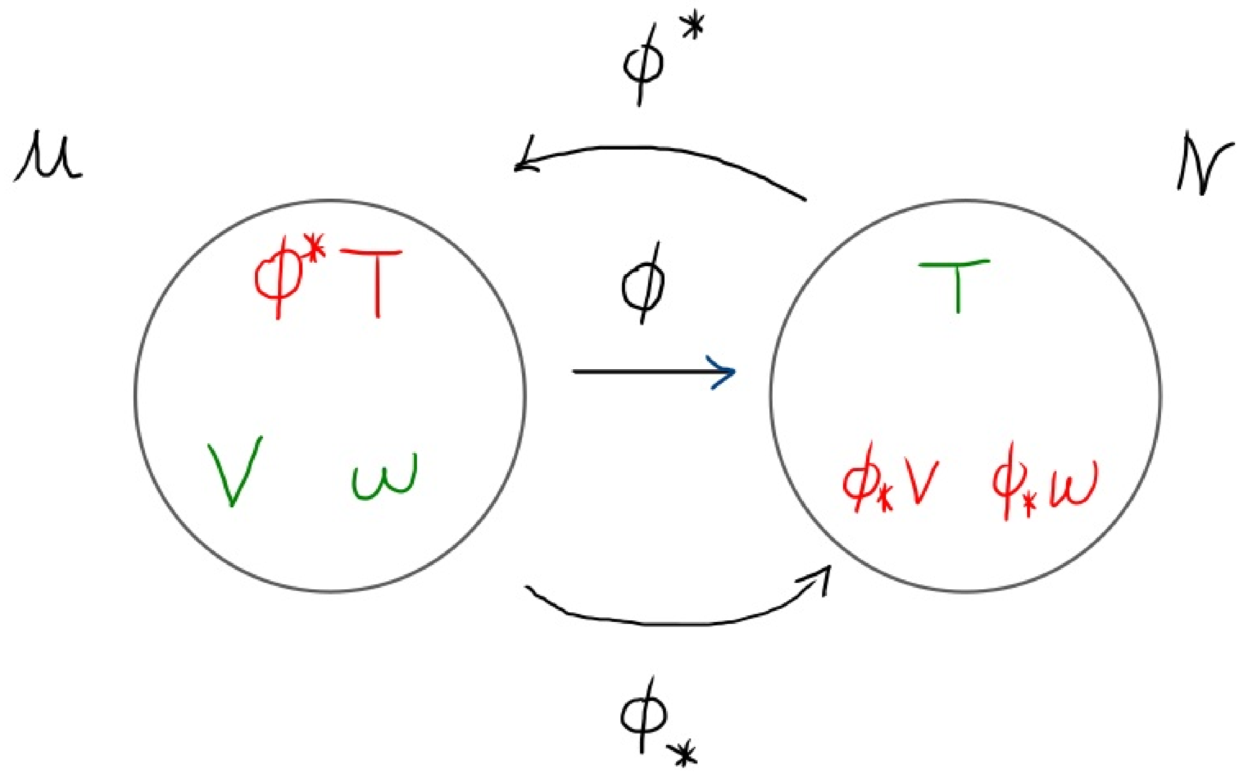
$$(\phi_* S)^\alpha{}_\beta = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial y^\beta} S^\mu{}_\nu$$

$$\phi^* T(\omega, V) = T(\phi_*\omega, \phi_* V)$$

pullback

$$(\phi^* T)^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial y^\beta}{\partial x^\nu} T^\alpha{}_\beta$$





(ℓ, k) tensors

$$(\phi_* S)^{\alpha_1 \cdots \alpha_\ell}{}_{\beta_1 \cdots \beta_k} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_\ell}}{\partial x^{\mu_\ell}} \frac{\partial x^{\nu_1}}{\partial y^{\beta_1}} \cdots \frac{\partial x^{\nu_k}}{\partial y^{\beta_k}} S^{\mu_1 \cdots \mu_\ell}{}_{\nu_1 \cdots \nu_k}$$

$$(\phi^* T)^{\mu_1 \cdots \mu_\ell}{}_{\nu_1 \cdots \nu_k} = \frac{\partial x^{\mu_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial x^{\mu_\ell}}{\partial y^{\alpha_\ell}} \frac{\partial y^{\beta_1}}{\partial x^{\nu_1}} \cdots \frac{\partial y^{\beta_k}}{\partial x^{\nu_k}} T^{\alpha_1 \cdots \alpha_\ell}{}_{\beta_1 \cdots \beta_k}$$

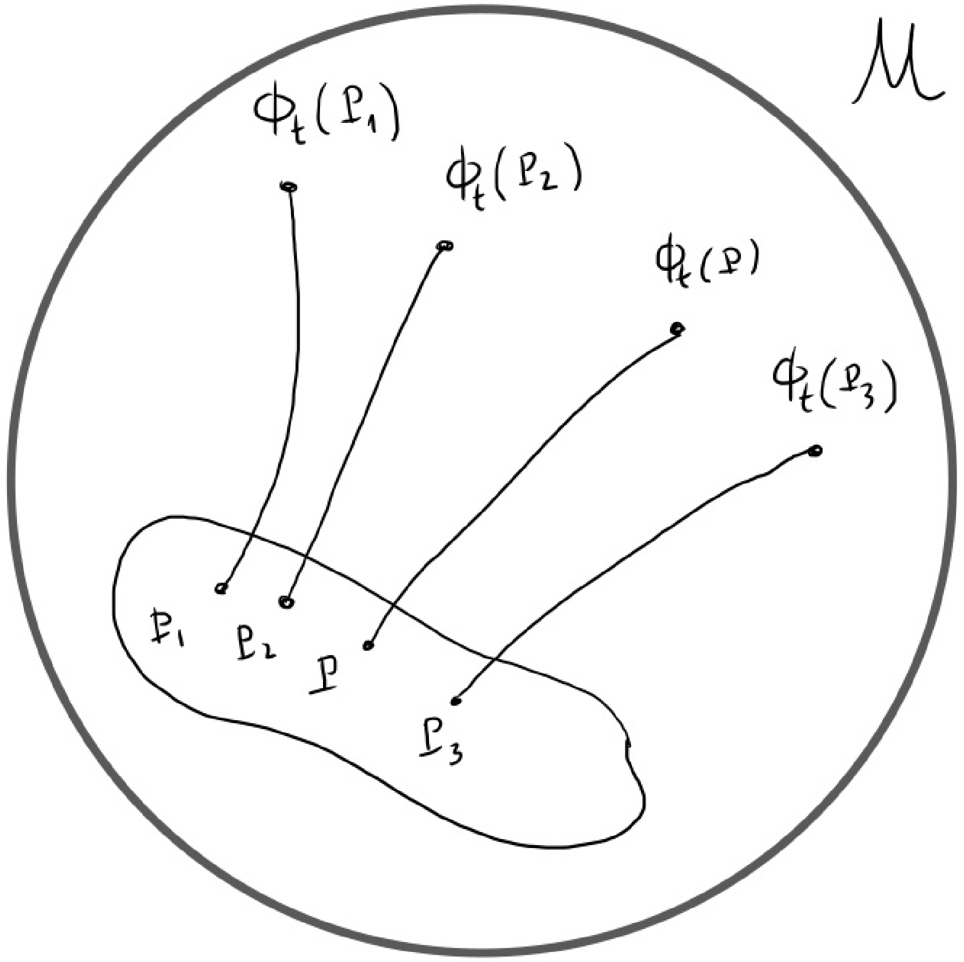
One parameter family of diffeomorphisms

$\phi_t : \mathcal{M} \longrightarrow \mathcal{M}$ $\qquad t \in \mathbb{R}$

s.t.: $\phi_s \circ \phi_t = \phi_{s+t}$

$\phi_0 = 1$

$\phi_t^{-1} = \phi_{-t}$



continuity: $\phi_t(P) = \lim_{\epsilon \to 0} \phi_{t+\epsilon}(P)$

differentiability w.r.t. to t

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[f \circ \phi_{t+\epsilon}(P) - f \circ \phi_t(P) \right]$$

One parameter family of diffeomorphisms

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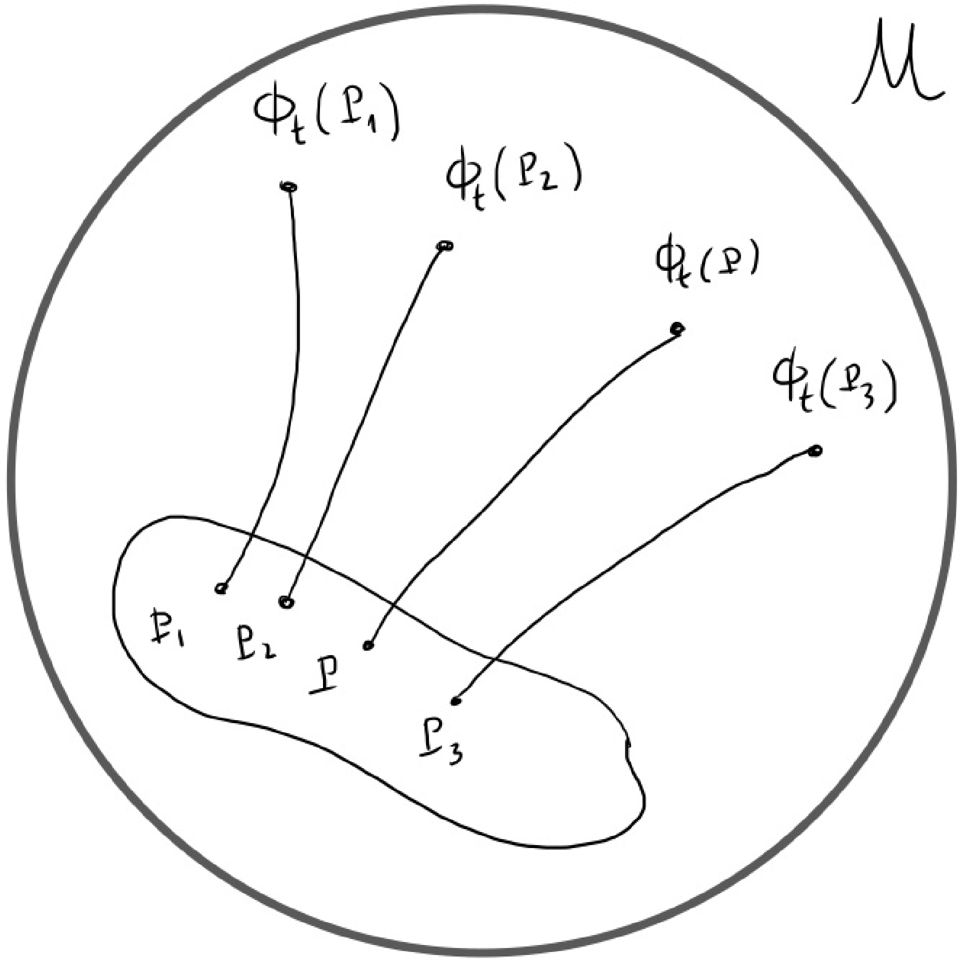
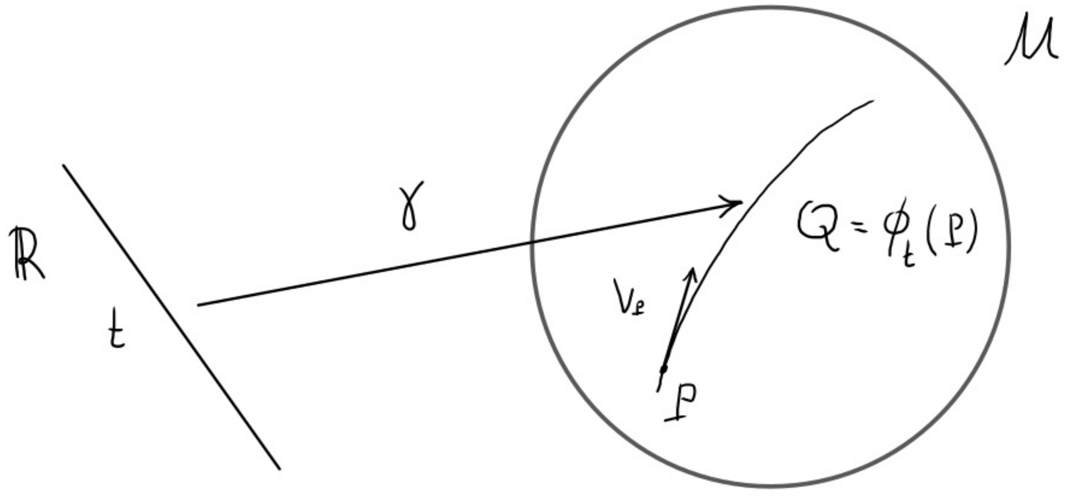


- Starting from p, $\phi_t(p)$ defines a curve:

$$\gamma(t) = \phi_t(p)$$

↳ the orbit of p under ϕ_t



$\gamma : \mathbb{R} \to \mathcal{M}$

$t \mapsto \gamma(t) = \phi_t(p)$

One parameter family of diffeomorphisms

$\phi_t : M \longrightarrow M$

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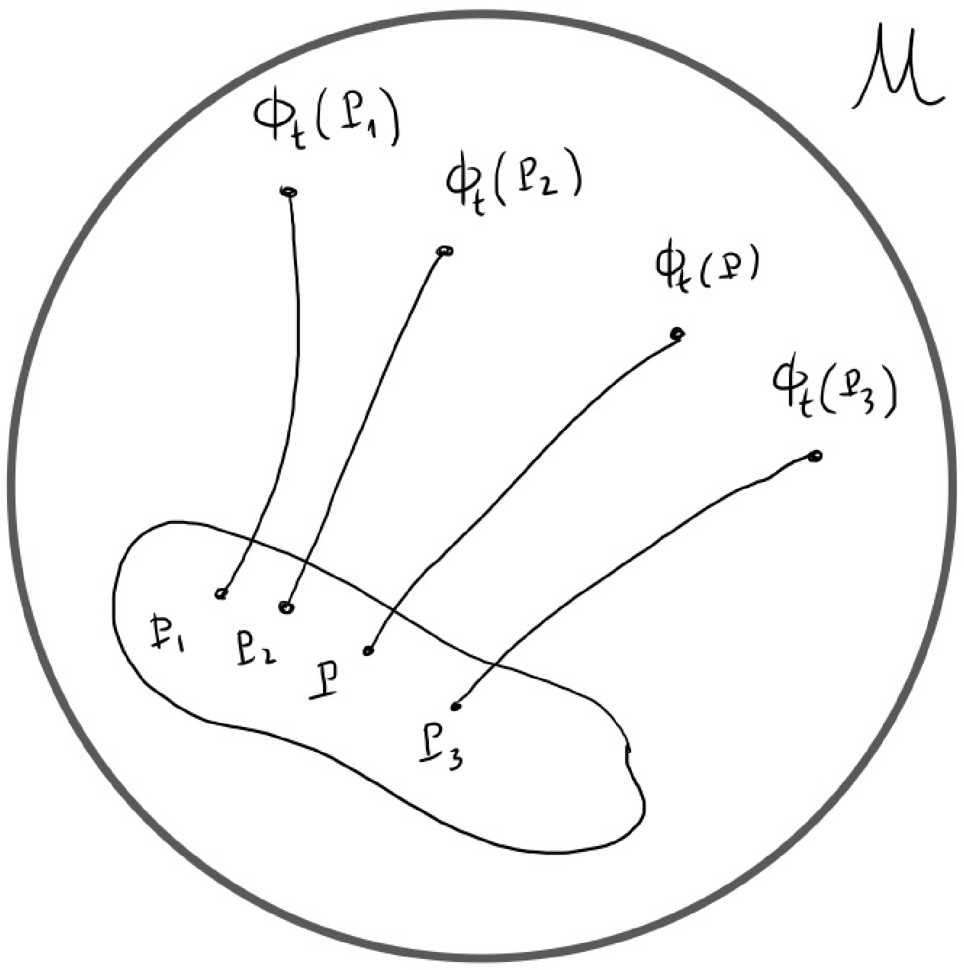
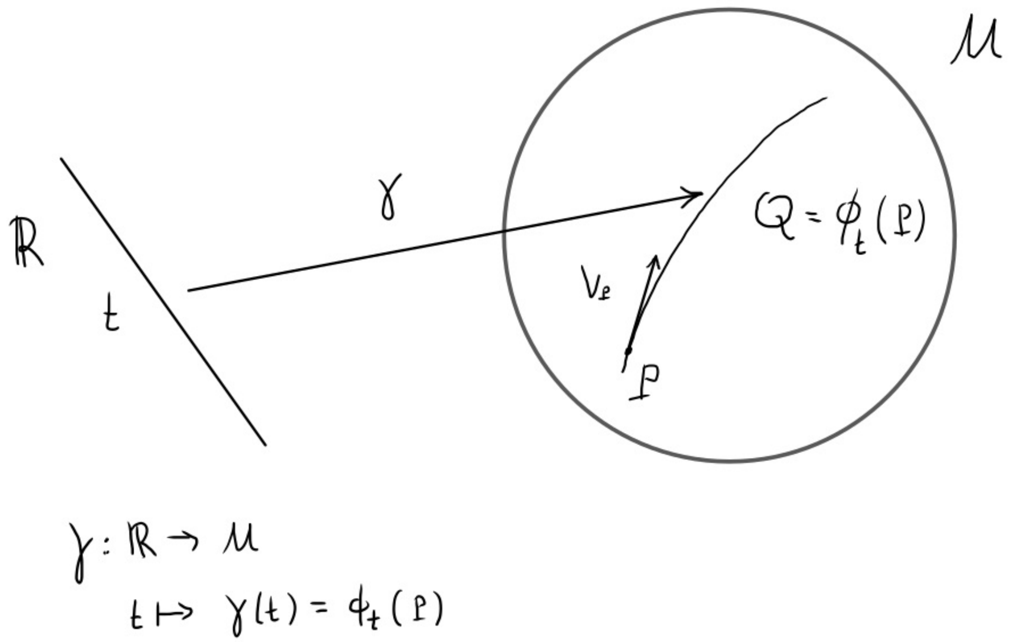


- Starting from $\underline{P}$, $\phi_t(P)$ defines a curve
- defines a vector at $\underline{P}$: $V_P = \frac{d}{dt}$



One parameter family of diffeomorphisms

$$\phi_t : M \longrightarrow M$$

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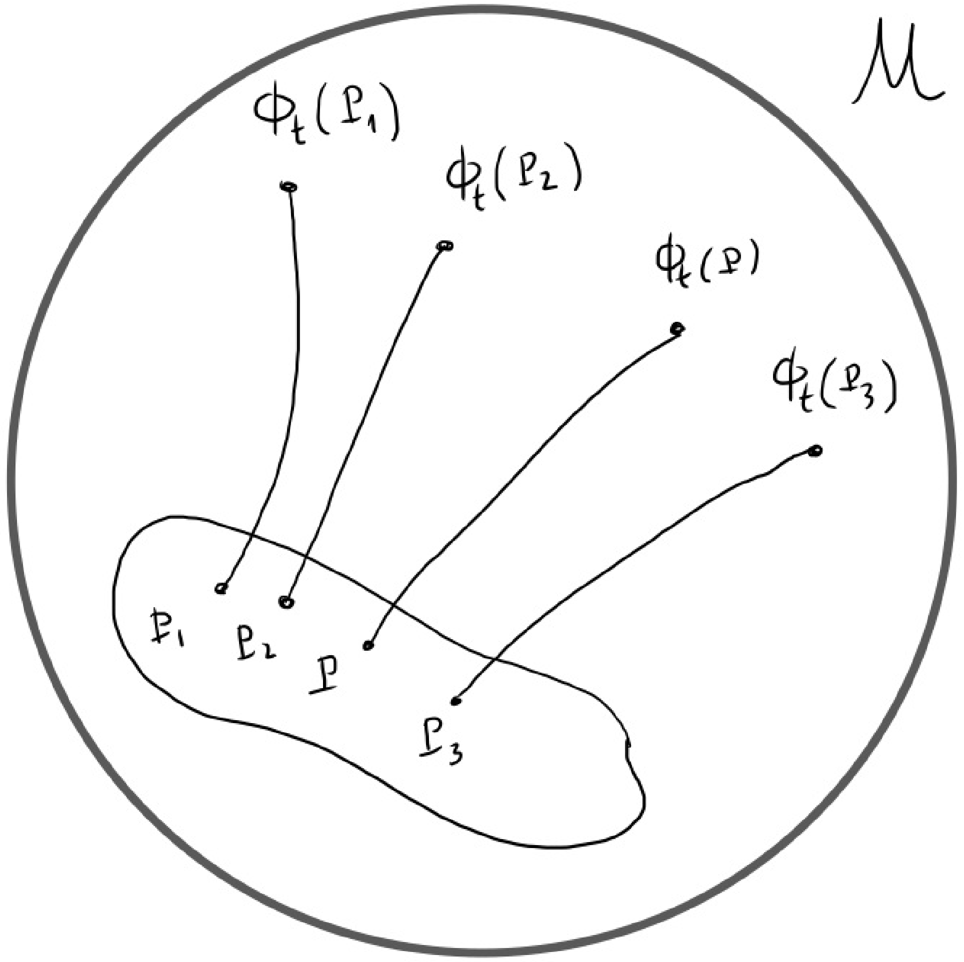
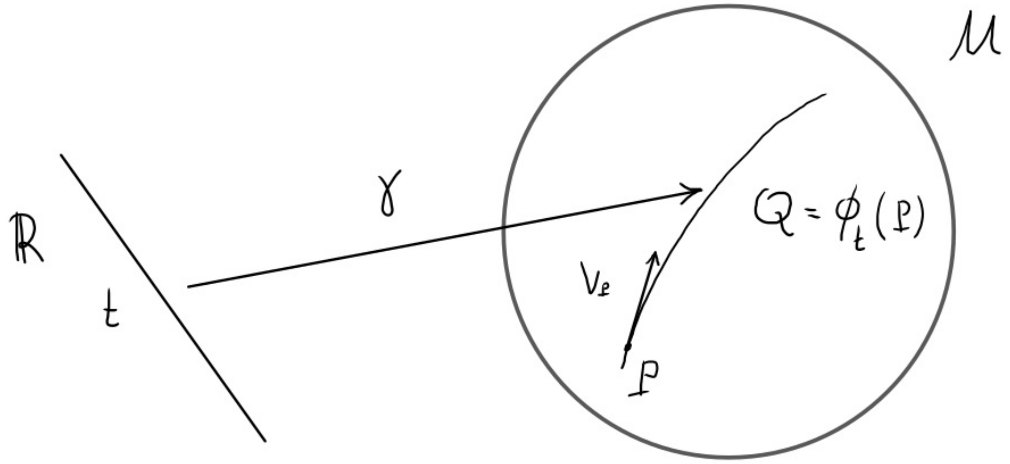


- Starting from P, $\phi_t(P)$ defines a curve
- defines a vector at P : $V_P = \frac{d}{dt}$
- defines vector $V = \frac{d}{dt}$ at all points of $\phi_t(P)$



$\gamma : \mathbb{R} \to M$

$t \mapsto \gamma(t) = \phi_t(P)$

One parameter family of diffeomorphisms

$$\phi_t : \mathcal{M} \longrightarrow \mathcal{M}$$

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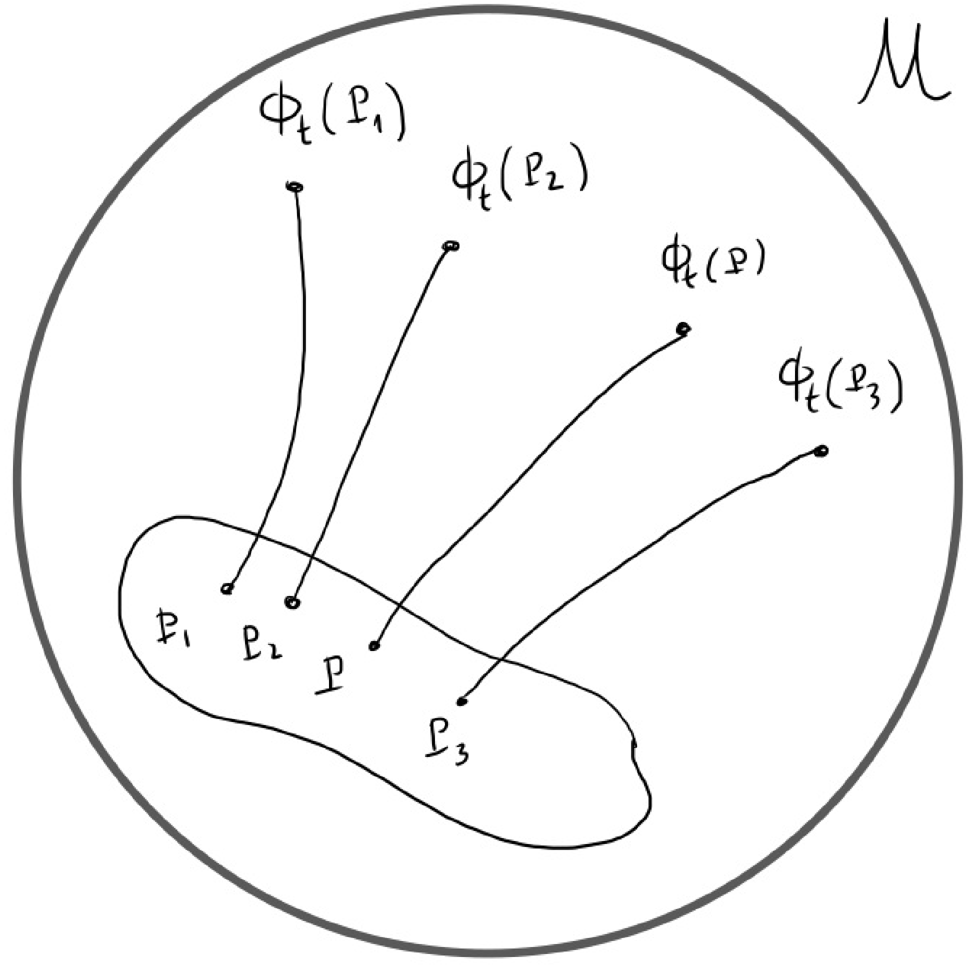

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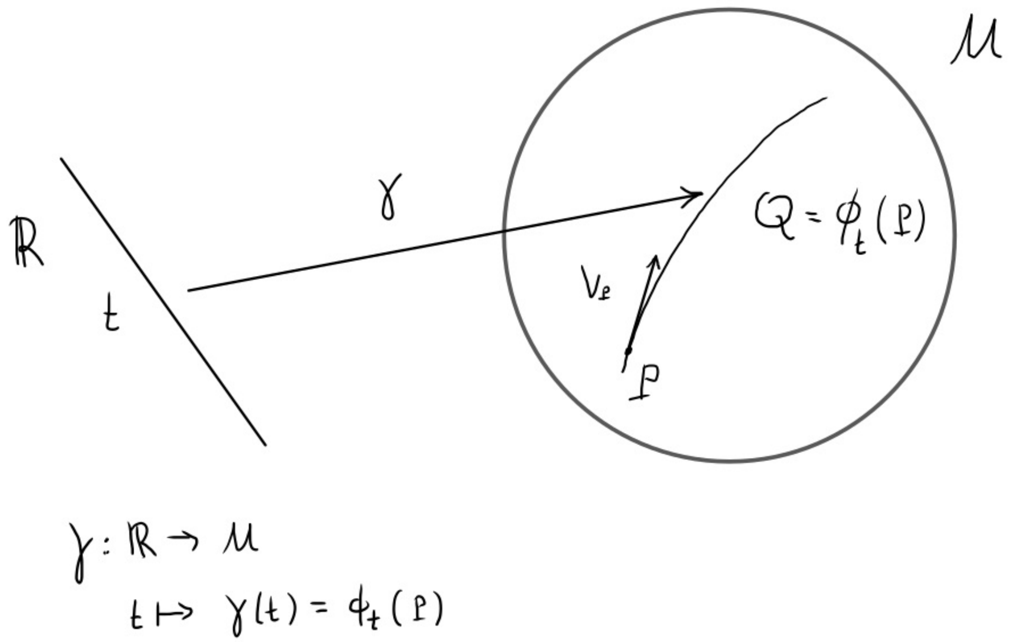
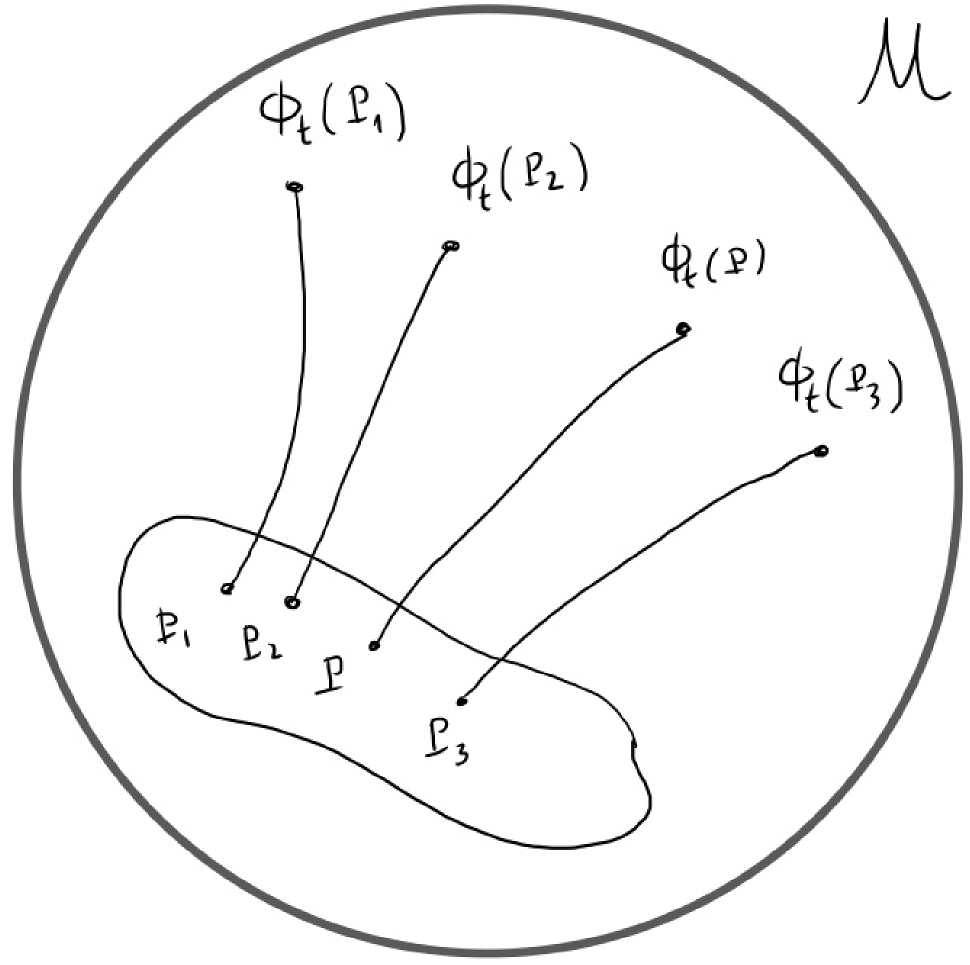
- such curves can start from every point of $\mathcal{M}$
  - $\Rightarrow$ they "fill" $\mathcal{M}$
  - $\Rightarrow$ they never intersect (ϕ_t is a diffeo)



One parameter family of diffeomorphisms

- the congruence of curves defines a vector at every $P \in \mathcal{M}$ $\Rightarrow$ a vector field $V = \frac{d}{dt}$



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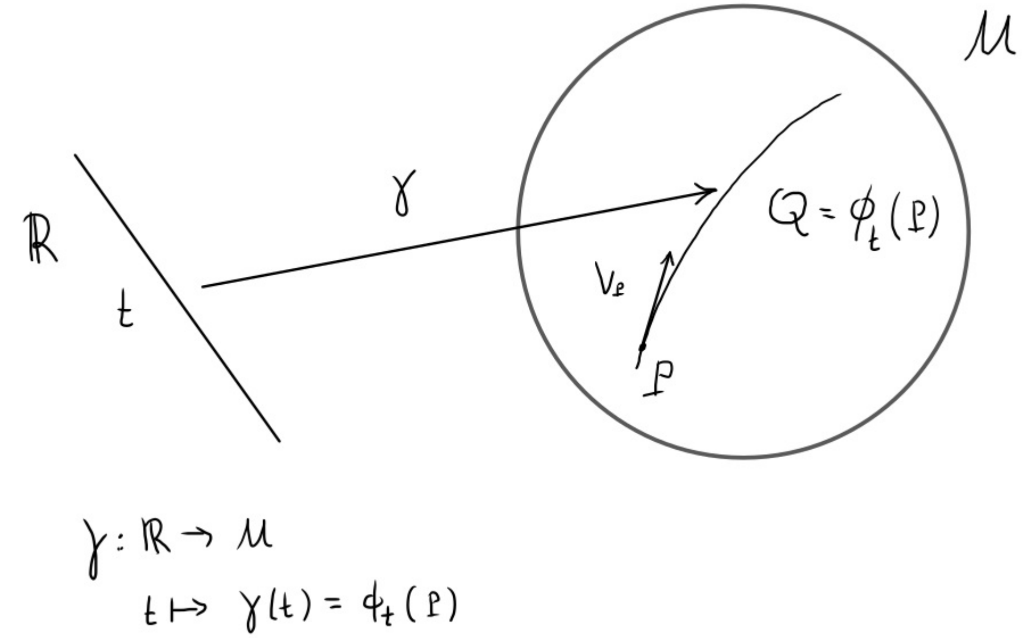
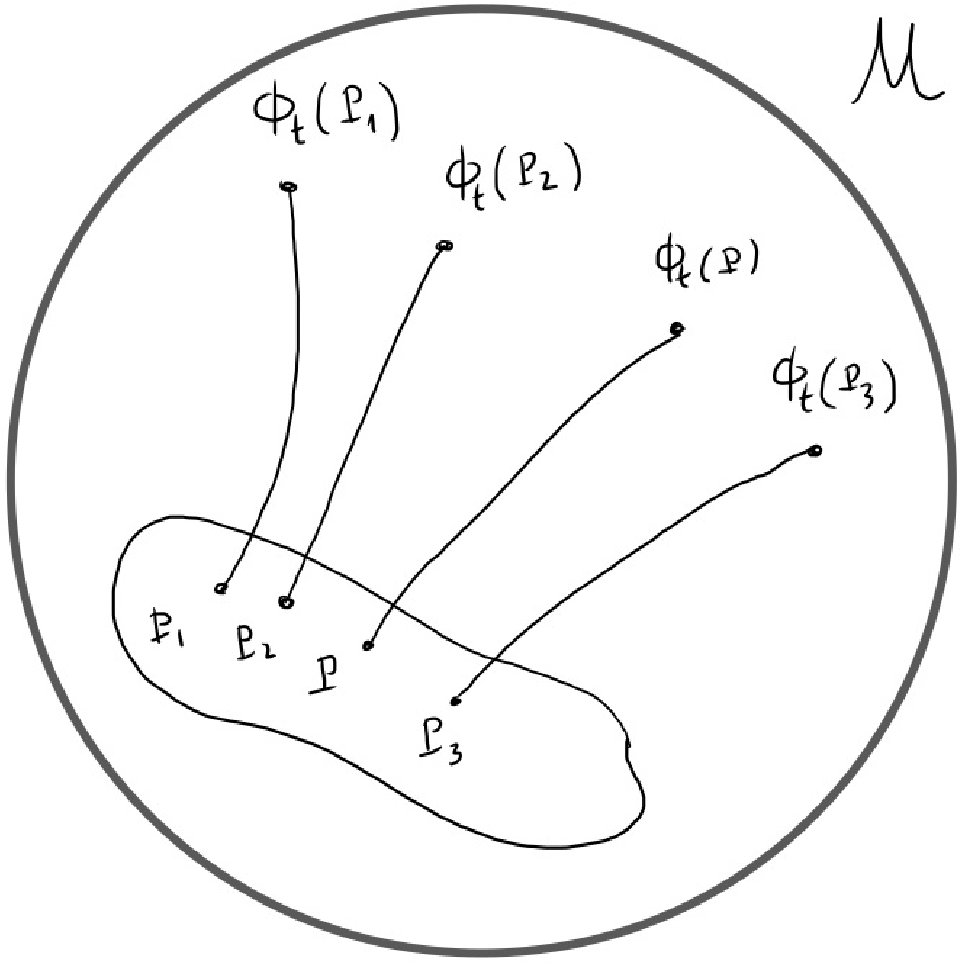


a congruence

One parameter family of diffeomorphisms

- the congruence of curves defines a vector at every $P \in \mathcal{M}$ $\Rightarrow$ a vector field $V = \frac{d}{dt}$



- vector fields that don't vanish define ϕ_t : move a point P a parameter t on integral curve of V that passes through P (unique)

- Starting from P, $\phi_t(P)$ defines a curve
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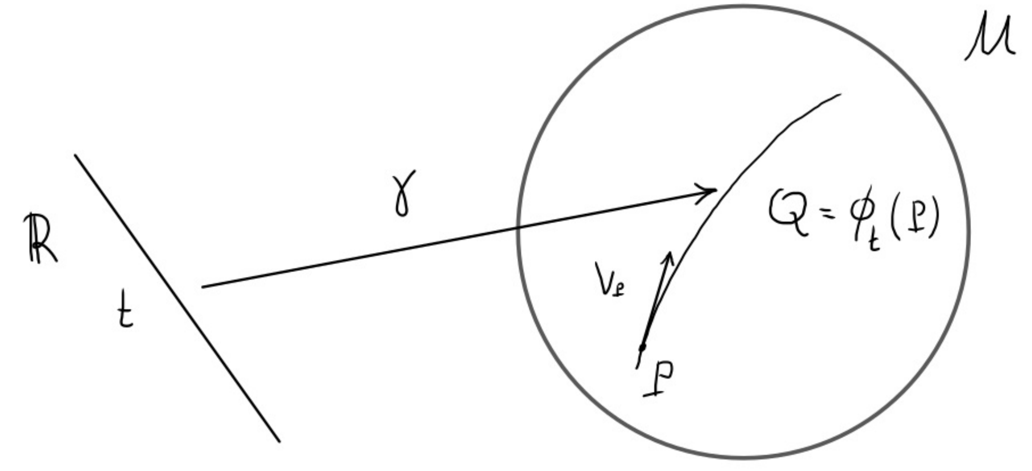
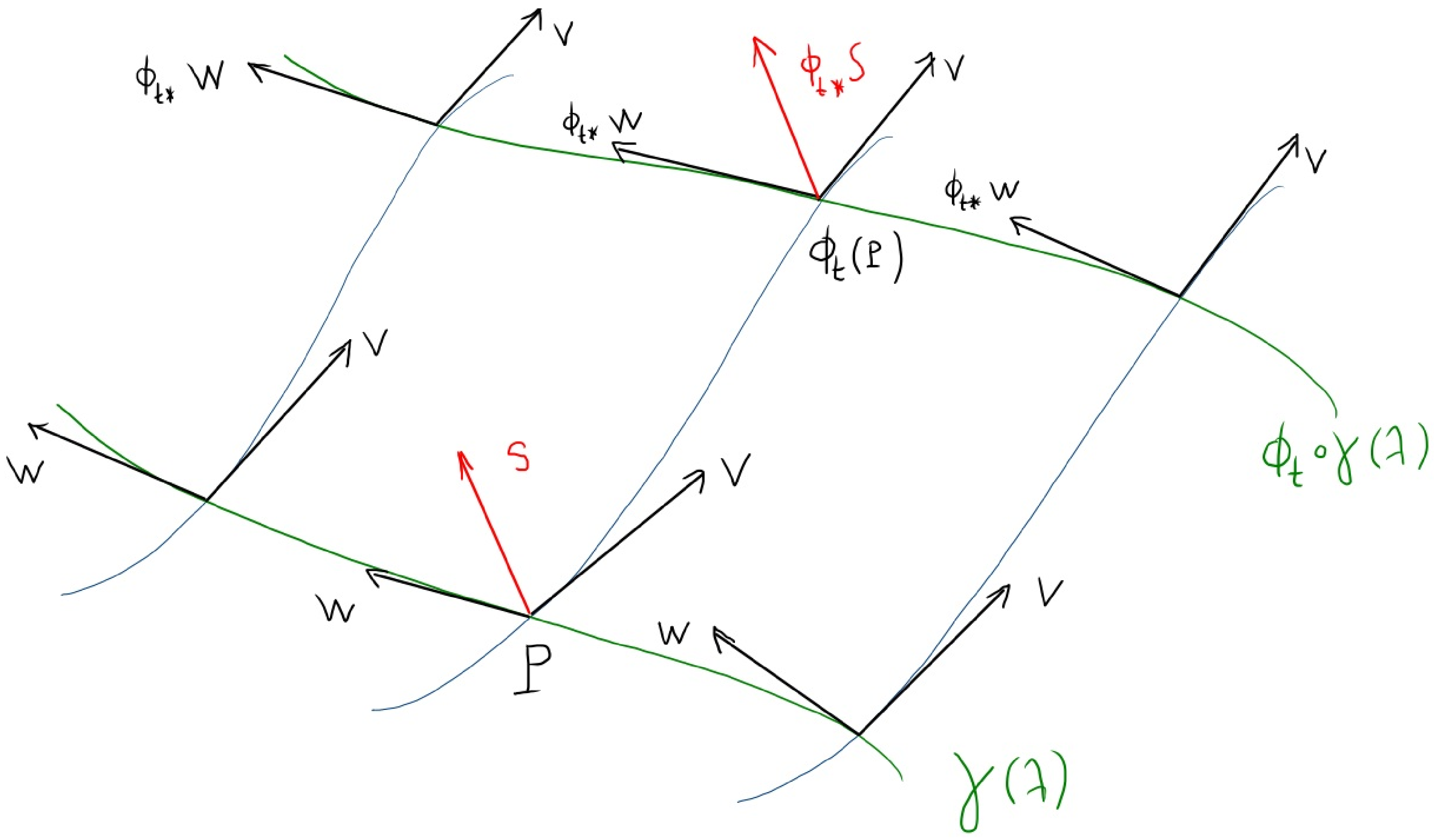


$\gamma : \mathbb{R} \to \mathcal{M}$
$t \mapsto \gamma(t) = \phi_t(P)$

a congruence

Dragging

- ϕ_t a diffeo $\Rightarrow$ use ϕ_t^* to pullback
  ϕ_{t*} pushforward



Dragging

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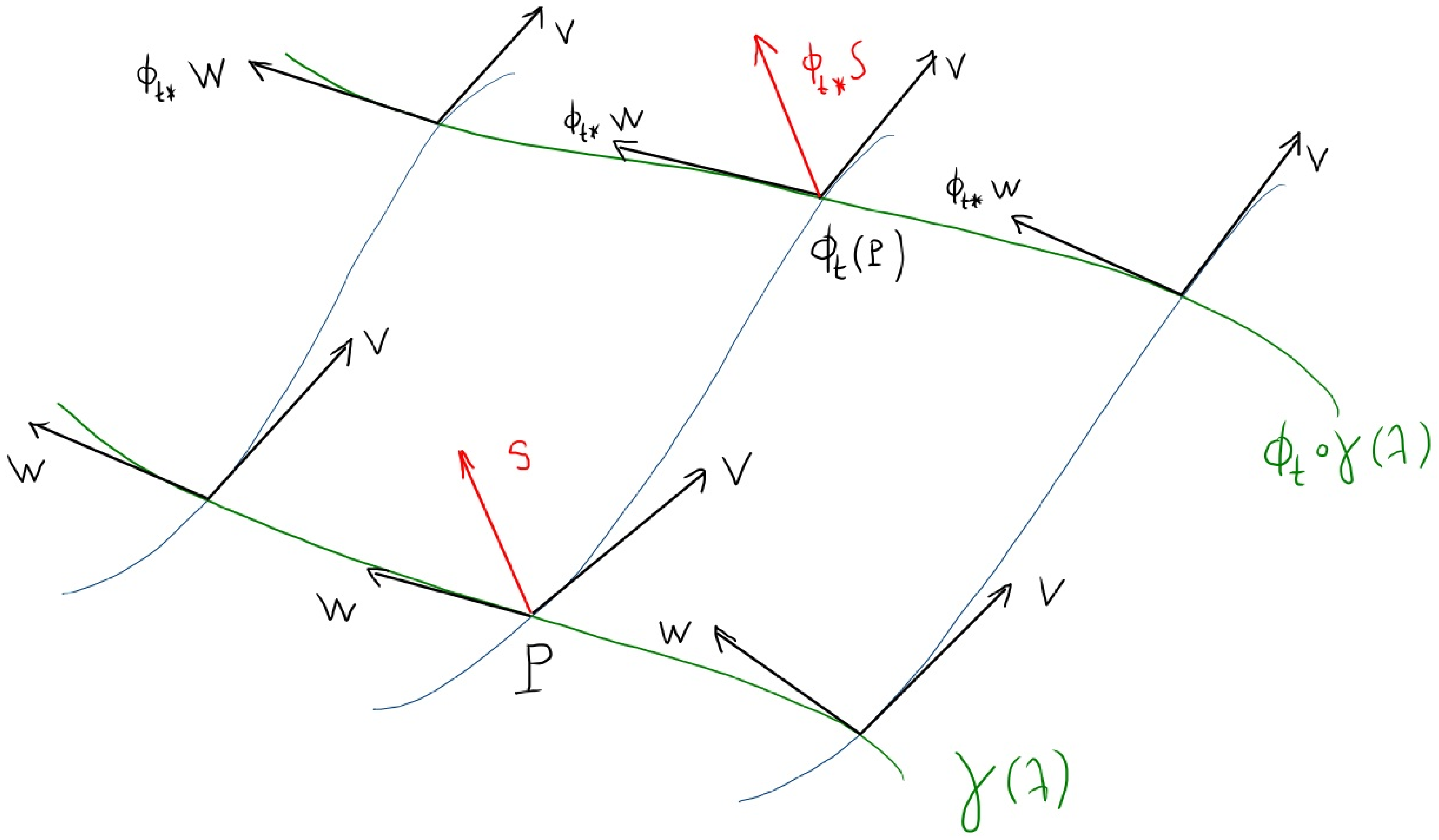
- curve $\gamma(\lambda)$ dragged to $\phi_t \circ \gamma(\lambda)$



Dragging

- ϕ_t a diffeo $\Rightarrow$ use ϕ_t^* to pullback
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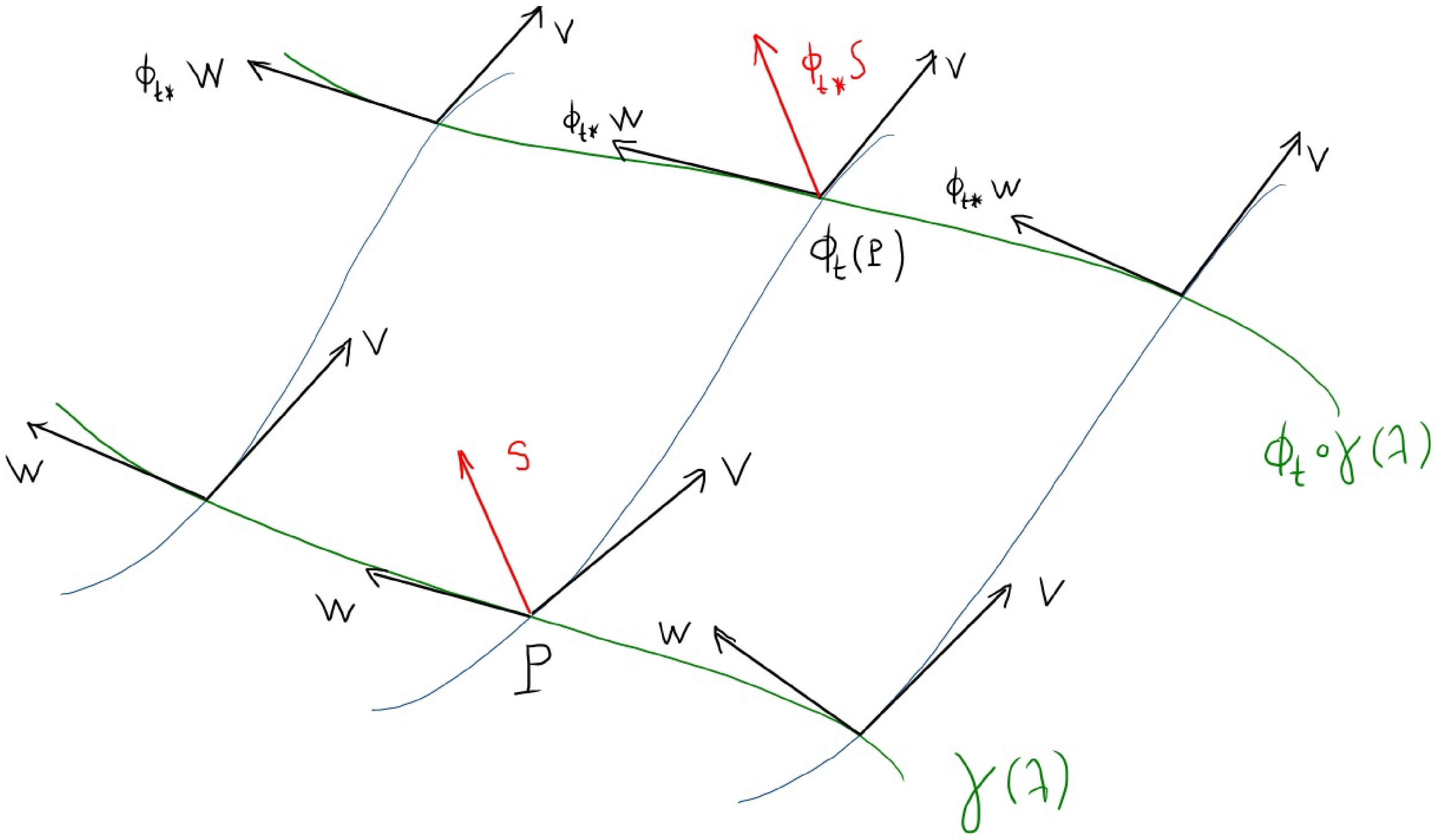
- curve $\gamma(\lambda)$ dragged to $\phi_t \circ \gamma(\lambda)$
- tensors S dragged to $\phi_{t*} S$



Notice that W tangent of γ

$\phi_{t*} W$ $\qquad\qquad$ $\phi_t \circ \gamma$

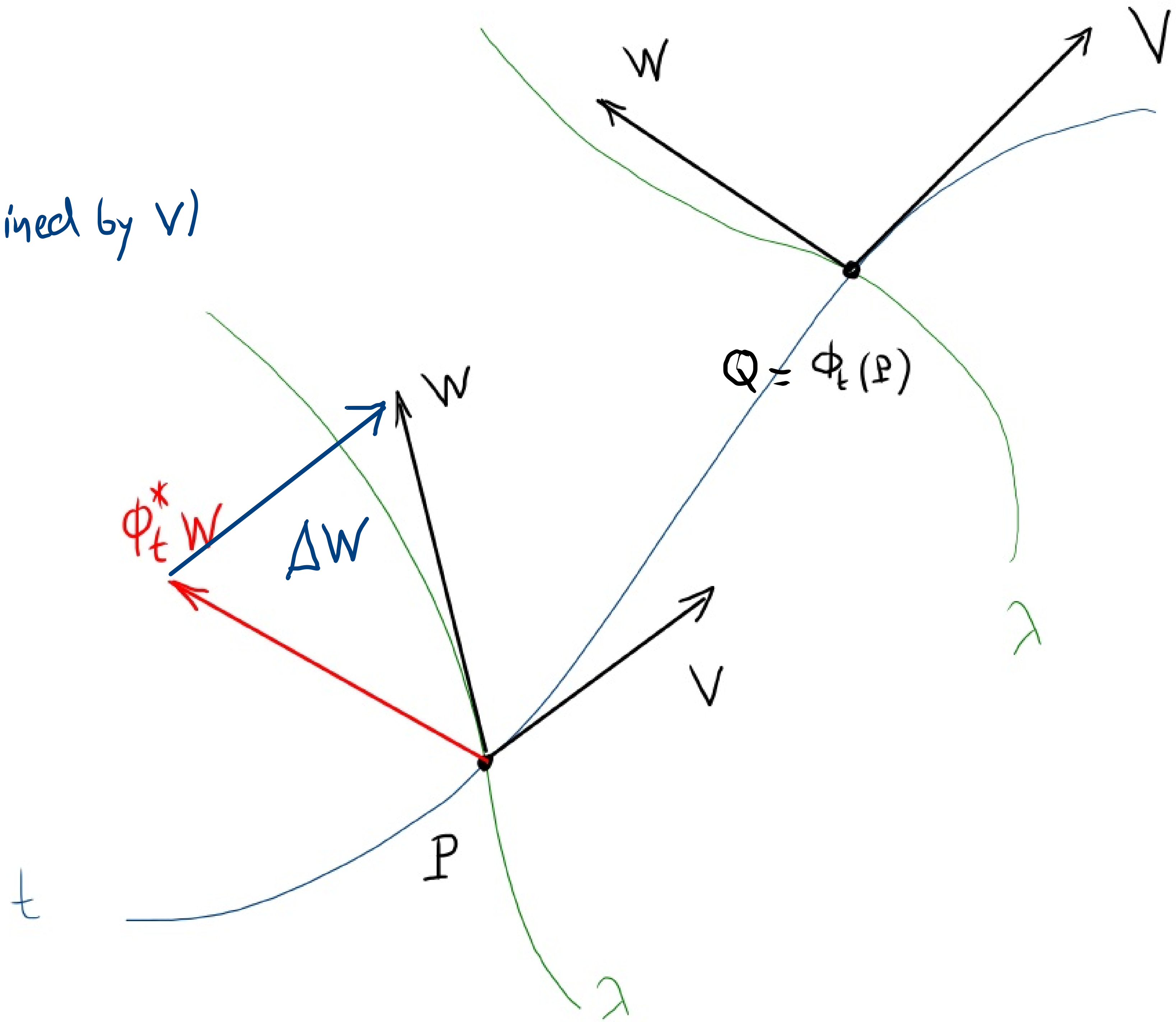
(we proved that before, see previous slides ...)

Lie Derivatives

Consider vector fields V and W and their t- and λ- integral curves (ϕ_t defined by V)

- pullback via ϕ_t: compare tensors at same point

$$\Delta W = \phi_t^* W - W$$



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Consider vector fields V and W and their t- and λ- integral curves

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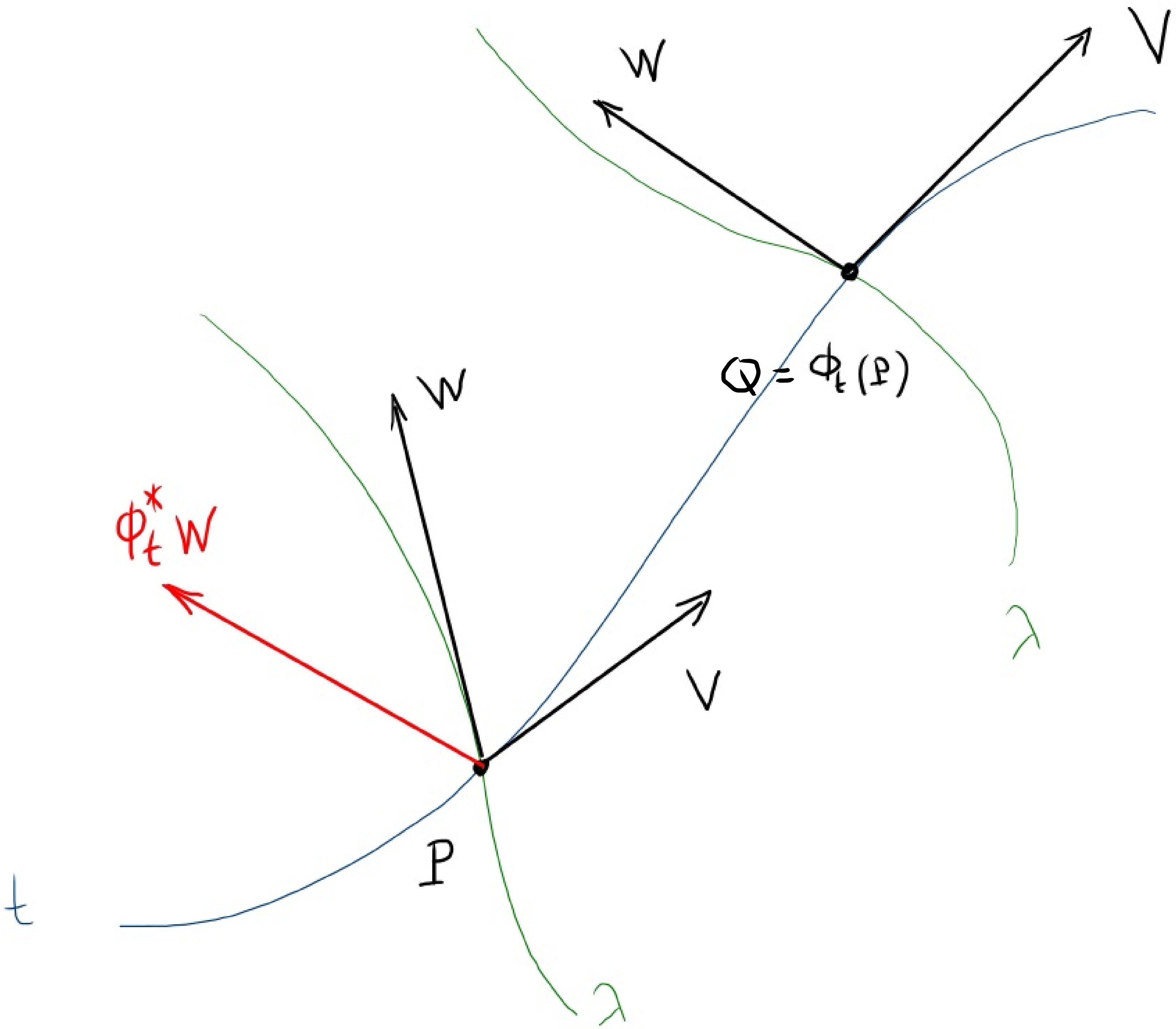
- if V vector field generating ϕ_t

$$\mathcal{L}_V W = \lim_{t \to 0} \frac{1}{t} \left[\phi_t^* W - W \right]$$



Lie Derivatives

- $\mathcal{L}_V$ is a derivative: (Lie derivative)

$$\mathcal{L}_V(\alpha T + \beta S) = \alpha \mathcal{L}_V T + \beta \mathcal{L}_V S$$

$$\mathcal{L}_V(S \otimes T) = \mathcal{L}_V T \otimes S + T \otimes \mathcal{L}_V S$$

(Leibnitz rule)

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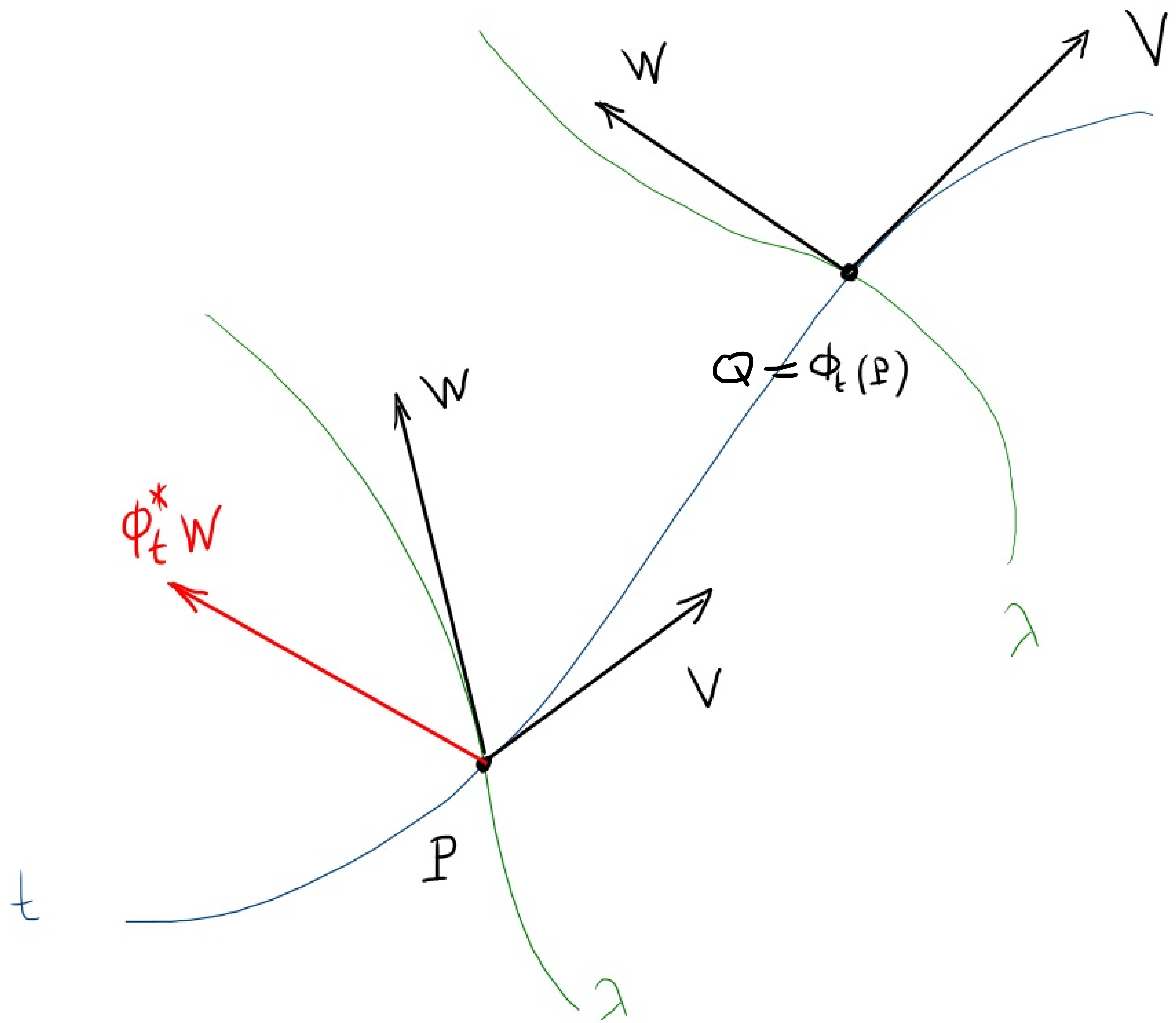

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Proof: consider 1-forms

$$\phi^*(\omega \otimes \chi) = \phi^*\omega \otimes \phi^*\chi$$

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$$\phi^*(\omega \otimes \chi)(V,W) = \omega \otimes \chi(\phi_* V, \phi_* W)$$

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$$\phi^*\omega \otimes \phi^*\chi(V,W) = \phi^*\omega(V) \cdot \phi^*\chi(W)$$

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$$\phi^*\omega\otimes\phi^*\chi(V,W) = \phi^*\omega(V)\cdot\phi^*\chi(W) = \omega(\phi_*V)\cdot\chi(\phi_*W)$$

q.e.d

Lie Derivatives

$$\phi_t^*(\omega\otimes\chi) - \omega\otimes\chi = \phi_t^*\omega\otimes\phi_t^*\chi - \omega\otimes\chi$$

Proof: consider 1-forms

$\phi^*(\omega\otimes\chi) = \phi^*\omega \otimes \phi^*\chi$ Indeed:

$$\phi^*(\omega\otimes\chi)(V,W) = \omega\otimes\chi(\phi_*V, \phi_*W) = \omega(\phi_*V)\cdot\chi(\phi_*W)$$

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Lie Derivatives

$$\phi_t^*(\omega\otimes\chi)-\omega\otimes\chi=\phi_t^*\omega\otimes\phi_t^*\chi-\omega\otimes\chi$$

$$=\phi_t^*\omega\otimes\phi_t^*\chi-\phi_t^*\omega\otimes\chi+\phi_t^*\omega\otimes\chi-\omega\otimes\chi$$

$$=\phi_t^*\omega\otimes[\phi_t^*\chi-\chi]+[\phi_t^*\omega-\omega]\otimes\chi$$

Proof: consider 1-forms

$\phi^*(\omega\otimes\chi)=\phi^*\omega\otimes\phi^*\chi$ Indeed:

$$\phi^*(\omega\otimes\chi)(V,W)=\omega\otimes\chi(\phi_*V,\phi_*W)=\omega(\phi_*V)\cdot\chi(\phi_*W)$$

$$\phi^*\omega\otimes\phi^*\chi(V,W)=\phi^*\omega(V)\cdot\phi^*\chi(W)=\omega(\phi_*V)\cdot\chi(\phi_*W)$$ q.e.d

Lie Derivatives

$$\phi_t^*(\omega\otimes\chi)-\omega\otimes\chi=\phi_t^*\omega\otimes\phi_t^*\chi-\omega\otimes\chi$$

$$=\phi_t^*\omega\otimes\phi_t^*\chi-\phi_t^*\omega\otimes\chi+\phi_t^*\omega\otimes\chi-\omega\otimes\chi$$

$$=\phi_t^*\omega\otimes\left[\phi_t^*\chi-\chi\right]+\left[\phi_t^*\omega-\omega\right]\otimes\chi$$

Therefore:

$$\mathcal{L}_V(\omega\otimes\chi)=\lim_{t\to 0}\frac{1}{t}\left[\phi_t^*(\omega\otimes\chi)-\omega\otimes\chi\right]$$

Lie Derivatives

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Lie Derivatives

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Therefore:

$$\mathcal{L}_V(\omega\otimes\chi)=\lim_{t\to 0}\frac{1}{t}\left[\phi_t^*(\omega\otimes\chi)-\omega\otimes\chi\right]$$

$$=\lim_{t\to 0}\frac{1}{t}\left[\phi_t^*\omega\otimes(\phi_t^*\chi-\chi)\right]+\lim_{t\to 0}\frac{1}{t}\left[(\phi_t^*\omega-\omega)\otimes\chi\right]$$

$$=\omega\otimes\mathcal{L}_V\chi+\mathcal{L}_V\omega\otimes\chi$$

$\phi_0^*\omega=\omega$

Compute $\mathcal{L}_V$

- $\mathcal{L}_V W = [V, W]$ (see proof @ next video)

Compute $\mathcal{L}_V$

- $\mathcal{L}_V W = [V, W]$ (see proof @ next video)

$$\Rightarrow (\mathcal{L}_V W)^\mu = [V, W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$ in coordinate basis only!

Compute $\mathcal{L}_V$

- $\mathcal{L}_V W = [V, W]$ (see proof @ next video)

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- $\mathcal{L}_V f = \dfrac{df}{dt} = V(f) = V^\mu \partial_\mu f = \dfrac{dx^\mu}{dt} \partial_\mu f$

Compute $\mathcal{L}_V$

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- $\mathcal{L}_V (\omega(W)) = \mathcal{L}_V \omega (W) + \omega(\mathcal{L}_V W)$ Leibnitz-like rule!

Compute $\mathcal{L}_V$

- $\mathcal{L}_V W = [V,W]$ (see proof @ next video)

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$$\Rightarrow W^\mu (\mathcal{L}_V \omega)_\mu = W^\mu [V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu]$$

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$$\Rightarrow W^\mu (\mathcal{L}_V \omega)_\mu = W^\mu [V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu] \Rightarrow \boxed{(\mathcal{L}_V \omega)_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu}$$

Compute $\mathcal{L}_V$

- $\mathcal{L}_V W = [V, W]$ (see proof @ next video)

$$\Rightarrow (\mathcal{L}_V W)^\mu = [V,W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$ in coordinate basis only!

- $\mathcal{L}_V f = \frac{df}{dt} = V(f) = V^\mu \partial_\mu f = \frac{dx^\mu}{dt} \partial_\mu f$

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Compute $\mathcal{L}_V$

- $\mathcal{L}_V W = [V, W]$

$\Rightarrow (\mathcal{L}_V W)^\mu = [V,W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu \Rightarrow (\mathcal{L}_V W)^\mu = V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu$

- $\mathcal{L}_V f = \dfrac{df}{dt} = V(f) = V^\mu \partial_\mu f = \dfrac{dx^\mu}{dt} \partial_\mu f \qquad \Rightarrow \mathcal{L}_V f = V^\nu \nabla_\nu f$

- $\mathcal{L}_V(\omega(W)) = \mathcal{L}_V \omega(W) + \omega(\mathcal{L}_V W)$

$\Rightarrow (\mathcal{L}_V \omega)_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu \qquad \Rightarrow (\mathcal{L}_V \omega)_\mu = V^\nu \nabla_\nu \omega_\mu + \omega_\nu \nabla_\mu V^\nu$

Properties:

- $\mathcal{L}_v(fW) = \mathcal{L}_v f \cdot W + f \cdot \mathcal{L}_v W$ (Leibnitz rule)

- $[\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]}$

- $[[\mathcal{L}_{v_1}, \mathcal{L}_{v_2}], \mathcal{L}_{v_3}] + [[\mathcal{L}_{v_3}, \mathcal{L}_{v_1}], \mathcal{L}_{v_2}] + [[\mathcal{L}_{v_2}, \mathcal{L}_{v_3}], \mathcal{L}_{v_1}] = 0$

  $3 \to 1 \to 2 \to 3$

  Jacobi-like identity

- $\mathcal{L}_v(S \otimes T) = \mathcal{L}_v S \otimes T + S \otimes \mathcal{L}_v T$

- $\mathcal{L}_v[T(\omega,\dots,W,\dots)] = \mathcal{L}_v T(\omega,\dots,W,\dots) + T(\mathcal{L}_v\omega,\dots,W,\dots) + \dots + T(\omega,\dots,\mathcal{L}_v W,\dots) + \dots$

  $\hookrightarrow$ use this to compute components

Example: (0,2) tensor

$$\mathcal{L}_v[g(X,Y)] = \mathcal{L}_v g(X,Y) + g(\mathcal{L}_v X, Y) + g(X, \mathcal{L}_v Y)$$

Example:

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho)$$

Example:

$$\mathcal{L}_V [g(X,Y)] = \mathcal{L}_V g\,(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V [g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho)$$

$$= V^\mu \underline{\partial_\mu g_{\nu\rho}} X^\nu Y^\rho + V^\mu g_{\nu\rho} \underline{\partial_\mu X^\nu} Y^\rho + V^\mu g_{\nu\rho} X^\nu \underline{\partial_\mu Y^\rho}$$

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Example:

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$$g(X, \mathcal{L}_V Y) = g_{\nu\rho} X^\nu (\mathcal{L}_V Y)^\rho = g_{\nu\rho} X^\nu (V^\mu \partial_\mu Y^\rho - Y^\mu \partial_\mu V^\rho)$$

Example:

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho)$$
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Example:

$$(\mathcal{L}_V g)_{\nu\rho} = V^\mu \partial_\mu g_{\nu\rho} + g_{\underline{\mu}\rho}\, \partial_\nu V^\mu + g_{\nu\underline{\mu}}\, \partial_\rho V^\mu$$

$$\mathcal{L}_V [g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho)$$

$$= V^\mu \underline{\partial_\mu g_{\nu\rho}}\, X^\nu Y^\rho + V^\mu g_{\nu\rho}\, \cancel{\underline{\partial_\mu X^\nu}}\, Y^\rho + V^\mu g_{\nu\rho} X^\nu\, \cancel{\underline{\partial_\mu Y^\rho}}$$

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$$g(X, \mathcal{L}_V Y) = g_{\nu\rho} X^\nu (\mathcal{L}_V Y)^\rho = g_{\nu\rho} X^\nu (\cancel{V^\mu \partial_\mu Y^\rho} - Y^\mu \partial_\mu V^\rho) \longrightarrow -X^\nu Y^\rho g_{\nu\mu} \partial_\rho V^\mu$$

Example: (2,0) tensor

$$\mathcal{L}_V(F(\omega,\phi)) = \mathcal{L}_V F(\omega,\phi) + F(\mathcal{L}_V\omega,\phi) + F(\omega,\mathcal{L}_V\phi)$$

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$$= \omega_\nu\phi_\rho V^\mu\partial_\mu F^{\nu\rho} + V^\mu F^{\nu\rho}\partial_\mu\omega_\nu\,\phi_\rho + V^\mu F^{\nu\rho}\omega_\nu\partial_\mu\phi_\rho$$

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$$F(\mathcal{L}_V\omega,\phi) = F^{\nu\rho}\left(\mathcal{L}_V\omega\right)_\nu\phi_\rho = F^{\nu\rho}\left(V^\mu\partial_\mu\omega_\nu + \omega_\mu\partial_\nu V^\mu\right)\phi_\rho$$

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$$\mathcal{L}_V(F(\omega,\phi)) = \mathcal{L}_V F(\omega,\phi) + F(\mathcal{L}_V\omega,\phi) + F(\omega,\mathcal{L}_V\phi)$$

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$$F(\omega,\mathcal{L}_V\phi) = F^{\nu\rho}\omega_\nu(\mathcal{L}_V\phi)_\rho = F^{\nu\rho}\omega_\nu(\cancel{V^\mu\partial_\mu}\phi_\rho + \phi_\mu\partial_\rho V^\mu) \longrightarrow \omega_\nu\phi_\rho F^{\nu\mu}\partial_\mu V^\rho$$

Example: $(2,0)$ tensor

$$\Rightarrow (\mathcal{L}_V F)^{\nu\rho} = V^\mu \partial_\mu F^{\nu\rho} - F^{\mu\rho} \partial_\mu V^\nu - F^{\nu\mu} \partial_\mu V^\rho$$

$$\mathcal{L}_V\big(F(\omega,\phi)\big) = \mathcal{L}_V\left(F^{\nu\rho}\omega_\nu \phi_\rho\right) = V^\mu \partial_\mu\left(F^{\nu\rho}\omega_\nu\phi_\rho\right) =$$

$$= \omega_\nu \phi_\rho V^\mu \partial_\mu F^{\nu\rho} + V^\mu F^{\nu\rho} \cancel{\partial_\mu \omega_\nu} \phi_\rho + V^\mu F^{\nu\rho} \cancel{\omega_\nu} \partial_\mu \phi_\rho$$

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For (ℓ, k) tensors:

$$
\begin{aligned}
(\mathcal{L}_V T)^{\mu_1 \cdots \mu_\ell}{}_{\nu_1 \cdots \nu_k} &= V^\mu \partial_\mu T^{\mu_1 \cdots \mu_\ell}{}_{\nu_1 \cdots \nu_k} \\
&- T^{\rho \cdots \mu_\ell}{}_{\nu_1 \cdots \nu_k} \partial_\rho V^{\mu_1} - \cdots - T^{\mu_1 \cdots \rho}{}_{\nu_1 \cdots \nu_k} \partial_\rho V^{\mu_\ell} \\
&+ T^{\mu_1 \cdots \mu_\ell}{}_{\rho \cdots \nu_k} \partial_{\nu_1} V^\rho + \cdots + T^{\mu_1 \cdots \mu_\ell}{}_{\nu_1 \cdots \rho} \partial_{\nu_k} V^\rho
\end{aligned}
$$

For (l,k) tensors:

$$(\mathcal{L}_V T)^{\mu_1 \cdots \mu_l}{}_{\nu_1 \cdots \nu_k} = V^\mu \partial_\mu T^{\mu_1 \cdots \mu_l}{}_{\nu_1 \cdots \nu_k}$$

$$- T^{\rho \cdots \mu_l}{}_{\nu_1 \cdots \nu_k} \partial_\rho V^{\mu_1} - \cdots - T^{\mu_1 \cdots \rho}{}_{\nu_1 \cdots \nu_k} \partial_\rho V^{\mu_l}$$

$$+ T^{\mu_1 \cdots \mu_l}{}_{\rho \cdots \nu_k} \partial_{\nu_1} V^\rho + \cdots + T^{\mu_1 \cdots \mu_l}{}_{\nu_1 \cdots \rho} \partial_{\nu_k} V^\rho$$

Same formula is true if $\partial \rightarrow \nabla$