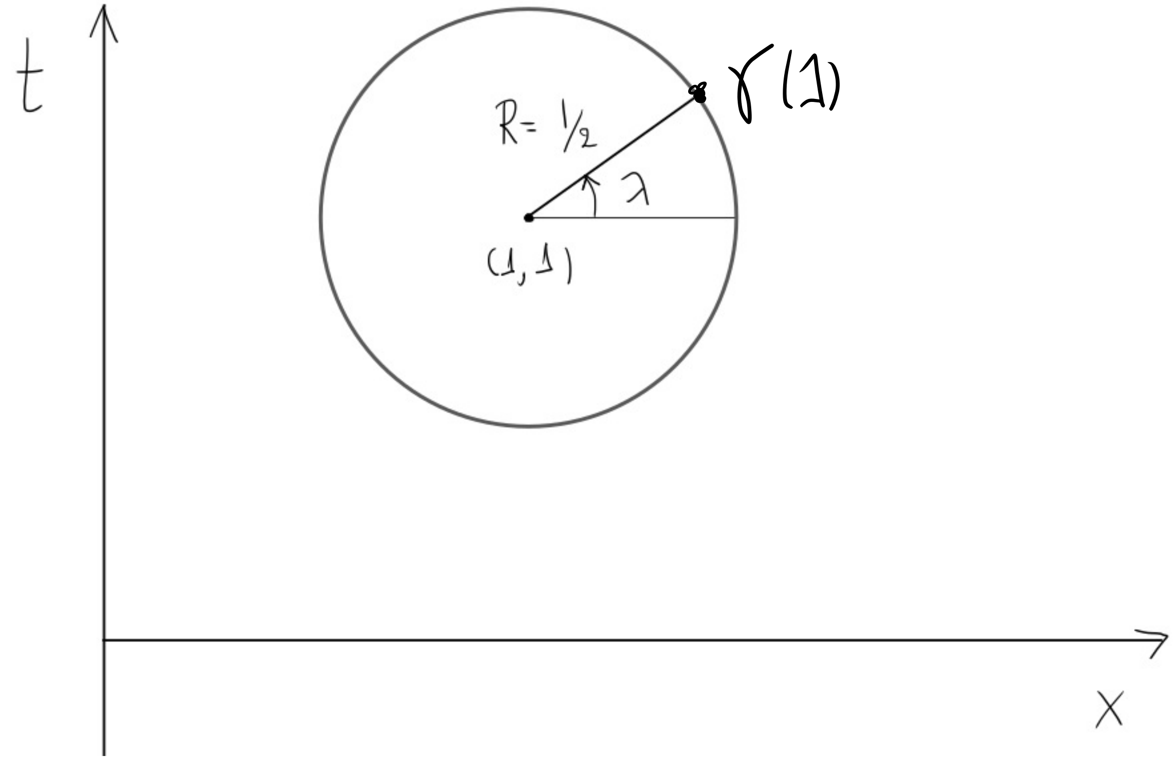
Θέμα 1ο – Καμπύλες



$$ds^2 = -x^2\,dt^2 + dx^2$$

$$x = 1 + \frac{1}{2}\cos\lambda \qquad t = 1 + \frac{1}{2}\sin\lambda \qquad 0 \le \lambda < 2\pi$$

$$dx = -\frac{1}{2}\sin\lambda\,d\lambda \qquad dt = \frac{1}{2}\cos\lambda\,d\lambda$$

$$ds^2 = -\left(1 + \frac{1}{2}\cos\lambda\right)^2\left(\frac{1}{2}\cos\lambda\,d\lambda\right)^2 + \left(-\frac{1}{2}\sin\lambda\,d\lambda\right)^2$$

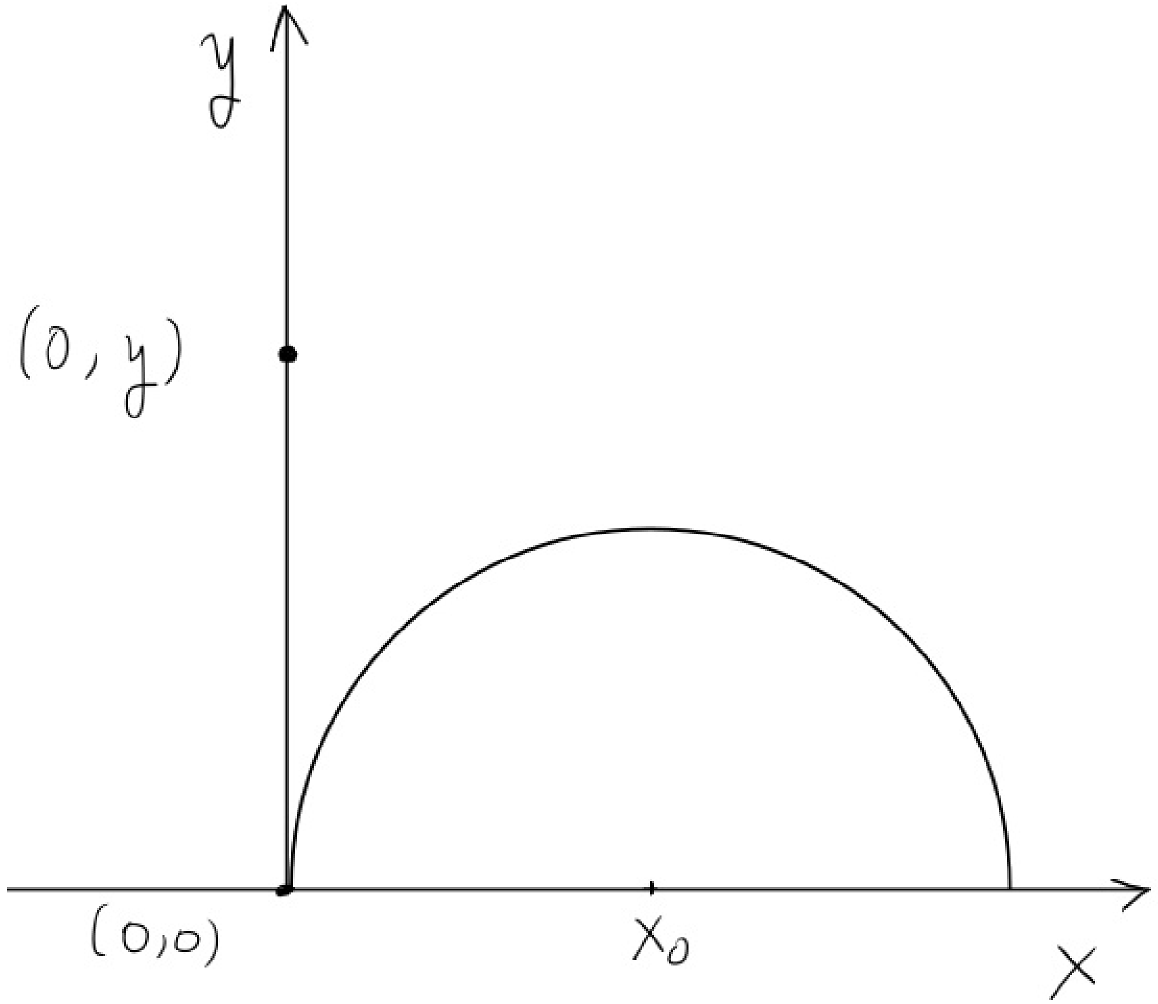
$$= \frac{1}{16}\left(4 - 8\cos^2\lambda - 4\cos^3\lambda - \cos^4\lambda\right)d\lambda^2 = f(\lambda)\,d\lambda^2$$

$$S = \int |ds^2|^{1/2} = \int_0^{2\pi} \sqrt{|f(\lambda)|}\,d\lambda$$

$$\lambda = 0,\quad f(0) = \frac{1}{16}(4 - 8 - 4 - 1) = -\frac{9}{16} < 0 \quad \text{timelike}$$

$$f\left(\frac{\pi}{3}\right) = \frac{23}{256} > 0 \quad \text{spacelike} \qquad f\left(\frac{\pi}{2}\right) = \frac{1}{4} > 0 \quad \text{spacelike}$$

Θέμα 2 - Το Υπερβολικό Επίπεδο



$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right) \qquad y>0$$

$$(g_{\mu\nu}) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} \qquad (g^{\mu\nu}) = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$$

Η μετρική είναι διαγώνια, άρα χρησιμοποιούμε την

$$g_{\mu\lambda}\Gamma^{\lambda}{}_{\mu\rho} = \Gamma_{\mu\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\mu\rho} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$$

$$g_{11}\Gamma^{1}{}_{21} = \Gamma_{121} = \frac{1}{2}\left(\partial_2 g_{11} + \cancel{\partial_1 g_{12}} - \cancel{\partial_1 g_{21}}\right) \Rightarrow \frac{1}{y^2}\Gamma^{1}{}_{21} = \frac{1}{2}\partial_y\left(\frac{1}{y^2}\right) = \frac{1}{2}\left(-\frac{2}{y^3}\right) \Rightarrow \Gamma^{1}{}_{21} = -\frac{1}{y}$$

$$g_{22}\Gamma^{2}{}_{11} = \Gamma_{211} = \frac{1}{2}\left(\cancel{\partial_1 g_{21}} + \cancel{\partial_1 g_{21}} - \partial_2 g_{11}\right) \Rightarrow \frac{1}{y^2}\Gamma^{2}{}_{11} = -\frac{1}{2}\partial_y\left(\frac{1}{y^2}\right) = -\frac{1}{2}\left(-\frac{2}{y^3}\right) \Rightarrow \Gamma^{2}{}_{11} = \frac{1}{y}$$

$$g_{22}\Gamma^{2}{}_{22} = \Gamma_{222} = \frac{1}{2}\left(\partial_2 g_{22} + \cancel{\partial_2 g_{22}} - \cancel{\partial_2 g_{22}}\right) \Rightarrow \frac{1}{y^2}\Gamma^{2}{}_{22} = \frac{1}{2}\partial_y\left(\frac{1}{y^2}\right) = \frac{1}{2}\left(-\frac{2}{y^3}\right) \Rightarrow \Gamma^{2}{}_{22} = -\frac{1}{y}$$

Εξισώσεις γεωδαισιακών: $\ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu \dot{x}^\rho = 0$ $\qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau}$

$\mu=1$ $\quad \ddot{x} + 2\Gamma^1{}_{21}\dot{y}\dot{x} = 0 \Rightarrow \ddot{x} - 2\dot{x}\frac{\dot{y}}{y} = 0$

$\mu=2$ $\quad \ddot{y} + \Gamma^2{}_{11}\dot{x}\dot{x} + \Gamma^2{}_{22}\dot{y}\dot{y} = 0 \Rightarrow \ddot{y} + \frac{1}{y}(\dot{x})^2 - \frac{1}{y}(\dot{y})^2 = 0$

Killing Vector Field (KVF)

Επειδή $\partial_x g_{\mu\nu} = 0$ το $\xi = \partial_x$ είναι KVF

Άρα η ποσότητα $\xi^\mu u_\mu = g_{\mu\nu}\xi^\mu u^\nu$ διατηρείται πάνω στη γεωδαισιακή, δηλ. $\frac{d}{d\tau}(\xi^\mu u_\mu) = u^\nu \nabla_\nu (\xi^\mu u_\mu) = 0$. Αλλά

$\xi^\mu u_\mu = g_{\mu\nu}\xi^\mu u^\nu = g_{11}\xi^1 u^1 = \frac{1}{y^2}\cdot 1 \cdot \dot{x}$

Ορίζουμε $\xi^\mu u_\mu \equiv \frac{1}{R}$ πάνω στη γεωδαισιακή, οπότε

$$\frac{1}{y^2}\dot{x} = \frac{1}{R} \Rightarrow \dot{x} = \frac{y^2}{R} \quad (1) \qquad \boxed{\dot{x} \neq 0}$$

Αλλά, αν τ είναι affine parameter, τότε

$$u_\mu u^\mu = 1 \Rightarrow g_{\mu\nu} u^\mu u^\nu = 1 \Rightarrow g_{11}(\dot{x})^2 + g_{22}(\dot{y})^2 = 1 \Rightarrow \frac{1}{y^2}(\dot{x})^2 + \frac{1}{y^2}(\dot{y})^2 = 1$$

$$\Rightarrow (\dot{x})^2 + (\dot{y})^2 = y^2 \quad (2)$$

$$(2) \overset{(1)}{\Rightarrow} \frac{1}{R^2} y^4 + \dot{y}^2 = y^2 \Rightarrow \dot{y}^2 = y^2\left(1 - \frac{y^2}{R^2}\right) \Rightarrow \dot{y} = \pm y \left(1 - \frac{y^2}{R^2}\right)^{1/2}$$

Τότε: $\frac{dx}{dy} = \frac{dx/d\tau}{dy/d\tau} = \frac{y^2/R}{\pm y \left(1 - \frac{y^2}{R^2}\right)^{1/2}} = \pm \frac{y}{R}\left(1 - \frac{y^2}{R^2}\right)^{-1/2}$

$$\Rightarrow x = \pm \int dy \, \frac{y}{R}\left(1-\frac{y^2}{R^2}\right)^{-1/2} = \mp R\left(1-\frac{y^2}{R^2}\right)^{1/2} + x_0$$

$$\Rightarrow (x-x_0)^2 + y^2 = R^2$$ ημικύκλιο κέντρου $(x_0, 0)$, ακτίνας R

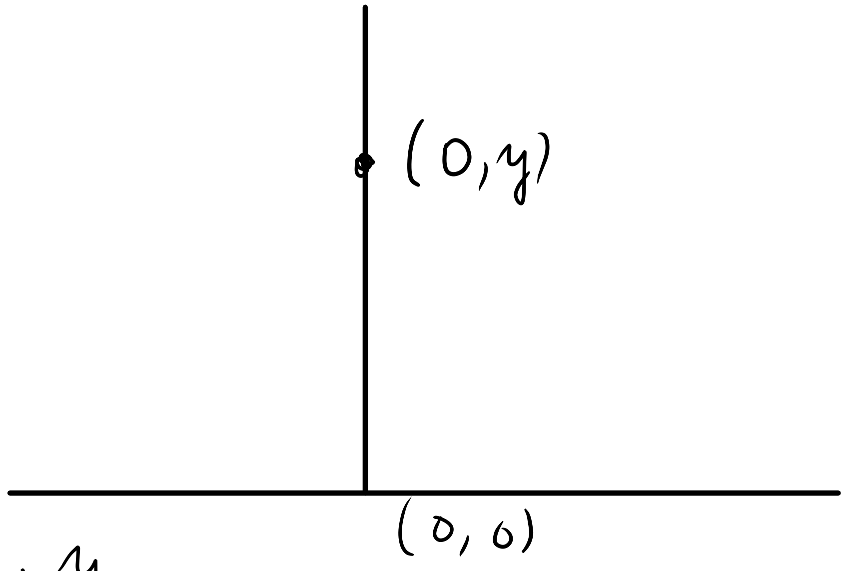
Αν $\dot{x} = 0 \Rightarrow R = \infty$, και από την 1η σχέση βλέπουμε ότι $x = x_0$

(ημιευθείες παράλληλες στον άξονα y)

Στο 1ο ερώτημα ζητείται το μήκος της

καμπύλης $\quad y = \tau \quad x = 0$

$$dy = d\tau \quad dx = 0$$



$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{22}\,dy\,dy = \frac{1}{\tau^2}d\tau^2 \Rightarrow s = \int_0^y \frac{d\tau}{\tau} = \ln\tau \Big|_0^y = \infty$$

Παρατήρηση: Όλες οι γεωδαισιακές που περνάν από το (x,y) έχουν άπειρο μήκος μέχρι να "φτάσουν" στον άξονα x: Θεωρείστε το ημικύκλιο

$$\left.\begin{array}{l} x = x_0 + R\cos\tau \\ y = R\sin\tau \end{array}\right\} \Rightarrow \left.\begin{array}{l} dx = -R\sin\tau\, d\tau \\ dy = R\cos\tau\, d\tau \end{array}\right\} \Rightarrow ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right) = \frac{1}{R^2\sin^2\tau}\left(R^2\sin^2\tau\, d\tau^2 + R^2\cos^2\tau\, d\tau^2\right)$$

$0 < \tau < \pi$

$$= \frac{1}{\sin^2\tau}\, d\tau^2$$

Για αρκετά μικρό τ:

$$s = \int_0^{\tau_0} \frac{d\tau}{\sin\tau} = \ln\tan\frac{\tau}{2}\Big|_0^{\tau_0} = +\infty$$

Θέμα 3 - Τανυστής Ενέργειας-Ορμής

$$\left.\begin{aligned} \nabla_\mu A_\nu &= \partial_\mu A_\nu - \Gamma^\rho{}_{\mu\nu} A_\rho \\ \nabla_\nu A_\mu &= \partial_\nu A_\mu - \Gamma^\rho{}_{\nu\mu} A_\rho \end{aligned}\right\} \Rightarrow \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - \cancel{(\Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu})} A_\rho$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu$$

Άρα, κάτω από μεταβολή $\delta g^{\mu\nu}$ έχω $\delta F_{\mu\nu} = 0$

$$\delta S = \delta\left(\int \sqrt{-g}\left(-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\right)\right)$$

$$= -\frac{1}{4}\int \delta\sqrt{-g}\, F^2 - \frac{1}{4}\int \sqrt{-g}\, \delta g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} - \frac{1}{4}\int \sqrt{-g}\, g^{\mu\rho}\, \delta g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$= -\frac{1}{4}\int \sqrt{-g}\left(-\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu}\right) F^2 - \frac{1}{4}\int \sqrt{-g}\, \overset{\rho\to\nu}{\delta g^{\mu\nu}}\, \overset{\nu\to\rho}{g^{\rho\sigma}} F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4}\int \sqrt{-g}\, \overset{\sigma\to\mu}{g^{\sigma\rho}}\, \overset{\mu\to\sigma}{\delta g^{\nu\mu}}\, \underbrace{F_{\sigma\nu}}_{=-F_{\nu\sigma}}\, \underbrace{F_{\rho\mu}}_{=-F_{\mu\rho}}$$

$$= -\frac{1}{2}\int \sqrt{-g}\left[g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^2\right]\delta g^{\mu\nu}$$

$$\Rightarrow T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F^2$$

Παίρνουμε τώρα $g_{\mu\nu} = \eta_{\mu\nu} \Rightarrow F^{0i} = -F_{0i} = E^i \qquad F^{ij} = \epsilon^{ijk}B^k = F_{ij}$

$$T_{00} = F_{0\rho}F_0{}^{\rho} - \frac{1}{4}\eta_{00}F^2 = F_{0i}F_{0i} + \frac{1}{4}F^2$$

$$F^2 = F_{\mu\nu}F^{\mu\nu} = F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} = 2F_{0i}F^{0i} + F_{ij}F^{ij}$$

$$= 2(-E_i)E_i + \epsilon_{ijk}B_k\,\epsilon_{ijl}B_l = -2E_iE_i + (\delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj})B_kB_l$$

$$= -2E^2 + (3\delta_{kl} - \delta_{kl})B_kB_l = -2E^2 + 2\delta_{kl}B_kB_l = -2(E^2 - B^2)$$

$$T^{00} = T_{00} = \mathcal{E}^2 - \frac{1}{4}2(\mathcal{E}^2 - B^2) = \frac{\mathcal{E}^2 + B^2}{2}$$

$$T_{0i} = F_{0\rho} F_i{}^\rho - \frac{1}{4}\cancel{\eta_{0i}} F^2 = F_{0j} F_i{}^j = (-\mathcal{E}_j)\epsilon_{ijk} B_k = -\epsilon_{ijk}\mathcal{E}_j B_k = -(\vec{\mathcal{E}} \times \vec{B})_i$$

$$T^{0i} = -T_{0i} = +(\vec{\mathcal{E}} \times \vec{B})_i$$

$$T_{ij} = F_{i\rho} F_j{}^\rho - \frac{1}{4}\eta_{ij} F^2 = F_{i0} F_j{}^0 + F_{ik} F_j{}^k - \frac{1}{4}\delta_{ij} F^2$$

$$= \mathcal{E}_i(-\mathcal{E}_j) + \epsilon_{ik\ell} B_\ell \, \epsilon_{jkm} B_m - \frac{1}{4}\delta_{ij} F^2$$

$$= -\mathcal{E}_i \mathcal{E}_j + (\delta_{\ell m}\delta_{ij} - \delta_{\ell j}\delta_{im}) B_\ell B_m - \frac{1}{4}\delta_{ij} F^2$$

$$= -\mathcal{E}_i \mathcal{E}_j + \delta_{ij}(\delta_{\ell m} B_\ell B_m) - B_i B_j + \frac{1}{4}\delta_{ij} 2(\mathcal{E}^2 - B^2)$$

$$= \left(-\mathcal{E}_i \mathcal{E}_j + \frac{1}{2}\delta_{ij}\mathcal{E}^2\right) + \left(-B_i B_j + \frac{1}{2}B^2\right) = T^{ij}$$

Θέμα 4: Καμπυλότητα

Αφού δεν έχω στρέψη: $[\nabla_\mu, \nabla_\nu](T^\alpha{}_{\beta\gamma}\, \omega_\alpha V^\beta W^\gamma) = 0$

$$\nabla_\mu \nabla_\nu T^\alpha{}_{\beta\gamma}\, \omega_\alpha V^\beta W^\gamma =$$

$$\nabla_\mu \nabla_\nu T\, \omega\, V\, W + \cancel{\nabla_\nu T \nabla_\mu \omega\, V\, W} + \cancel{\nabla_\nu T\, \omega \nabla_\mu V\, W} + \cancel{\nabla_\nu T\, \omega\, V \nabla_\mu W}$$

$$\cancel{\nabla_\mu T \nabla_\nu \omega\, V\, W} + T \nabla_\mu \nabla_\nu \omega\, V\, W + \cancel{T \nabla_\nu \omega \nabla_\mu V\, W} + \cancel{T \nabla_\nu \omega\, V \nabla_\mu W}$$

$$\cancel{\nabla_\mu T\, \omega \nabla_\nu V\, W} + \cancel{T \nabla_\mu \omega \nabla_\nu V\, W} + T\, \omega \nabla_\mu \nabla_\nu V\, W + \cancel{T\, \omega \nabla_\nu V \nabla_\mu W}$$

$$\cancel{\nabla_\mu T\, \omega\, V \nabla_\nu W} + \cancel{T \nabla_\mu \omega\, V \nabla_\nu W} + \cancel{T\, \omega \nabla_\mu V \nabla_\nu W} + T\, \omega\, V \nabla_\mu \nabla_\nu W$$

$$\nabla_\nu \nabla_\mu T^\alpha{}_{\beta\gamma}\, \omega_\alpha V^\beta W^\gamma =$$

$$\nabla_\nu \nabla_\mu T\, \omega\, V\, W + \cancel{\nabla_\mu T \nabla_\nu \omega\, V\, W} + \cancel{\nabla_\mu T\, \omega \nabla_\nu V\, W} + \cancel{\nabla_\mu T\, \omega\, V \nabla_\nu W}$$

$$\cancel{\nabla_\nu T \nabla_\mu \omega\, V\, W} + T \nabla_\nu \nabla_\mu \omega\, V\, W + \cancel{T \nabla_\mu \omega \nabla_\nu V\, W} + \cancel{T \nabla_\mu \omega\, V \nabla_\nu W}$$

$$\cancel{\nabla_\nu T\, \omega \nabla_\mu V\, W} + \cancel{T \nabla_\nu \omega \nabla_\mu V\, W} + T\, \omega \nabla_\nu \nabla_\mu V\, W + \cancel{T\, \omega \nabla_\mu V \nabla_\nu W}$$

$$\cancel{\nabla_\nu T\, \omega\, V \nabla_\mu W} + \cancel{T \nabla_\nu \omega\, V \nabla_\mu W} + \cancel{T\, \omega \nabla_\nu V \nabla_\mu W} + T\, \omega\, V \nabla_\nu \nabla_\mu W$$

$$\Rightarrow [\nabla_\mu, \nabla_\nu](T\omega V W) = ([\nabla_\mu, \nabla_\nu] T)\omega V W + T([\nabla_\mu, \nabla_\nu]\omega) V W + T\omega([\nabla_\mu, \nabla_\nu] V) W$$
$$+ T\omega V([\nabla_\mu, \nabla_\nu] W)$$

$$= 0 \quad \Rightarrow$$

$$([\nabla_\mu, \nabla_\nu] T^\alpha{}_{\beta\gamma})\omega_\alpha V^\beta W^\gamma = + T^\alpha{}_{\beta\gamma} R^\lambda{}_{\alpha\mu\nu}\omega_\lambda V^\beta W^\gamma - T^\alpha{}_{\beta\gamma}\omega_\alpha R^\beta{}_{\lambda\mu\nu} V^\lambda W^\gamma$$
$$- T^\alpha{}_{\beta\gamma}\omega_\alpha V^\beta R^\gamma{}_{\lambda\mu\nu} W^\lambda$$

$$= \left(R^\alpha{}_{\lambda\mu\nu} T^\lambda{}_{\beta\gamma} - R^\lambda{}_{\beta\mu\nu} T^\alpha{}_{\lambda\gamma} - R^\lambda{}_{\gamma\mu\nu} T^\alpha{}_{\beta\lambda} \right)\omega_\alpha V^\beta W^\gamma$$

$$\Rightarrow$$
$$[\nabla_\mu, \nabla_\nu] T^\alpha{}_{\beta\gamma} = R^\alpha{}_{\lambda\mu\nu} T^\lambda{}_{\beta\gamma} - R^\lambda{}_{\beta\mu\nu} T^\alpha{}_{\lambda\gamma} - R^\lambda{}_{\gamma\mu\nu} T^\alpha{}_{\beta\lambda}$$