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Final Exam - General Theory of Relativity

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Problem 1

Consider the Schwarzschild metric in the $r > 2M$ region, and the coordinate transformation $(t, r, \theta, \phi) \to (t, \xi, \theta, \phi)$, so that

$$r - 2M = \frac{\xi^2}{8M}, \quad \xi > 0\,. \tag{1}$$

The Metric

Show that in the (t, ξ, θ, ϕ) coordinate system

$$ds^2 = -\frac{\kappa^2\xi^2}{\kappa^2\xi^2+1}dt^2 + (\kappa^2\xi^2+1)d\xi^2 + \frac{1}{4\kappa^2}(\kappa^2\xi^2+1)^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{2}$$

where

$$\kappa = \frac{1}{4M}\,. \tag{3}$$

Tangent Space

Consider the coordinate basis

$$\{\partial_\mu\} = \{\partial_0, \partial_1, \partial_2, \partial_3\} = \{\partial_t, \partial_\xi, \partial_\theta, \partial_\phi\}\,. \tag{4}$$

Determine the type (spacelike, timelike or null) of the coordinate basis vectors.

Compute the orthocanonical basis

$$\{\hat{e}_\mu\} = \{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{e}_t, \hat{e}_\xi, \hat{e}_\theta, \hat{e}_\phi\}\,. \tag{5}$$

Compute a null vector, and write it as a linear combination of the coordinate basis elements $\{\partial_\mu\}$.

Killing Vector Fields (KVF)

Show that the vector fields ∂_t and ∂_ϕ are KVF of the metric (2).

A massive particle is falling freely following a trajectory $(t(\tau), \xi(\tau), \theta(\tau), \phi(\tau))$. Write down the equations that give the respective conserved quantities during the particle's motion.

Curvature

The components of the Riemann tensor in the $\{\partial_\mu\}$ basis are:

$$R_{1010} = -\frac{4\kappa^4\xi^2}{(\kappa^2\xi^2+1)^3} \tag{6}$$

$$R_{2020} = \frac{\kappa^2\xi^2}{2(\kappa^2\xi^2+1)^2} \tag{7}$$

$$R_{2121} = -\frac{1}{2} \tag{8}$$

$$R_{3030} = \frac{\kappa^2\xi^2\sin^2\theta}{2(\kappa^2\xi^2+1)^2} \tag{9}$$

$$R_{3131} = -\frac{1}{2}\sin^2\theta \tag{10}$$

$$R_{3232} = \frac{(\kappa^2\xi^2+1)\sin^2\theta}{4\kappa^2}\,. \tag{11}$$

Compute the components of $R_{\hat\mu\hat\nu\hat\rho\hat\lambda}$ in the $\{\hat{e}_\mu\}$ basis.

Free fall, fixed ξ

An observer is falling freely, following a trajectory with fixed $\xi = \xi_0$.
Compute the angular velocity $\Omega = \frac{d\phi}{dt}$.
How much time elapses according to the observer (her proper time) during one revolution?
The geodesic equations are:

$$\ddot{t} + \frac{2}{\kappa^2\xi^3+\xi}\dot{t}\dot{\xi} = 0 \tag{12}$$

$$\ddot{\xi} - \frac{1}{2}\xi\left[\dot{\theta}^2 - \frac{2\kappa^2}{(\kappa^2\xi^2+1)^3}\left(\dot{t}^2 + (\kappa^2\xi^2+1)^2\dot{\xi}^2\right) + \sin^2\theta\,\dot{\phi}^2\right] = 0 \tag{13}$$

$$\ddot{\theta} + \frac{4\kappa^2\xi}{\kappa^2\xi^2+1}\dot{\theta}\dot{\xi} - \cos\theta\sin\theta\,\dot{\phi}^2 = 0 \tag{14}$$

$$\ddot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} + \frac{4\kappa^2\xi}{\kappa^2\xi^2+1}\dot{\xi}\dot{\phi} = 0\,. \tag{15}$$

where

$$\dot{t} = \frac{dt}{d\tau}\,, \quad \dot{\xi} = \frac{d\xi}{d\tau}\,, \quad \ldots \tag{16}$$

Problem 2

Consider the electromagnetic (EM) field, whose Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{17}$$

where

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{18}$$

The following questions should be answered for the case of a Minkowski (flat) metric. You may use equations which are valid for a more general, curved spacetime.

Energy-Momentum Tensor

Show that the energy-momentum tensor of the EM field can be written in the form:

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} \tag{19}$$

Given that

$$E_i = -F_{0i} \tag{20}$$
$$B_k = \frac{1}{2}\epsilon_{kij}F^{ij}\,, \tag{21}$$

compute the $T_{\mu\nu}$ components in terms of the E_i, B_i.

Compute the Lagrangian density (17) in terms of the E_i, B_i.

You may use the relations:

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{22}$$
$$\delta g = -g\, g_{\mu\nu}\delta g^{\mu\nu}\,. \tag{23}$$

Energy-Momentum Conservation

Show that when the equations of motion for (17) (Maxwell's equations) are satified, then

$$\partial_\mu T^{\mu\nu} = 0\,. \tag{24}$$

Problem 3

Consider the covariant derivative ∇_μ of the Levi-Civita connection compatible with the metric $g_{\mu\nu}$.

You may consider the following equations to be given:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda}V^\lambda \tag{25}$$
$$[\nabla_\mu, \nabla_\nu]\, V^\rho = R^\rho{}_{\lambda\mu\nu}V^\lambda \tag{26}$$
$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\rho{}_{\nu\lambda} - \partial_\nu\Gamma^\rho{}_{\mu\lambda} + \Gamma^\rho{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\lambda} - \Gamma^\rho{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\lambda}\,. \tag{27}$$

Connections

The vectors W^μ, U^μ are parallel transported along a curve, whose tangent vector is V^μ. Show that the inner products $W^\mu W_\mu$, $U^\mu U_\mu$, $W^\mu U_\mu$, remain constant along the curve.

Show that, if ω_μ is a one-form field, then

$$\nabla_\mu\omega_\nu = \partial_\mu\omega_\nu - \Gamma^\lambda{}_{\mu\nu}\omega_\lambda\,. \tag{28}$$

Curvature

Show that

$$[\nabla_\mu, \nabla_\nu]\,\omega_\rho = -R^\lambda{}_{\rho\mu\nu}\omega_\lambda \tag{29}$$
$$[\nabla_\mu, \nabla_\nu]\,F^\sigma{}_\rho = R^\sigma{}_{\lambda\mu\nu}F^\lambda{}_\rho - R^\lambda{}_{\rho\mu\nu}F^\sigma{}_\lambda\,. \tag{30}$$

Symmetries of the Riemann Tensor

Show that

$$R^\mu{}_{[\nu\rho\sigma]} = 0\,. \tag{31}$$