ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ
ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ
ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ
ΤΟΜΕΑΣ ΦΥΣΙΚΗΣ
ΗΡΩΩΝ ΠΟΛΥΤΕΧΝΕΙΟΥ 9 - ΖΩΓΡΑΦΟΥ
157 80 ΑΘΗΝΑ
ΤΗΛ. 210 772 3009, 772 3032 - FAX: 210 772 3025

NATIONAL TECHNICAL UNIVERSITY
OF ATHENS
SCHOOL OF APPLIED SCIENCES
DEPARTMENT OF PHYSICS
ZOGRAFOU CAMPUS
157 80 ATHENS - GREECE
TEL. +30210 772 3009, 772 3032 - FAX: +30210 772 3025

Problems in General Relativity

K.N. Anagnostopoulos

The Gödel Universe

Consider the Gödel spacetime:

$$ds^2 = -dt^2 - 2\frac{r^2}{\sqrt{2}a}dt d\phi + \frac{dr^2}{1+\left(\frac{r}{2a}\right)^2} + r^2\left[1-\left(\frac{r}{2a}\right)^2\right]d\phi^2 + dz^2\,. \tag{1}$$

1. Determine whether ∂_μ are timelike, null, or spacelike. From the kind of ∂_ϕ, discuss if it is possible to have (local future pointing) timelike geodesics moving in the negative t direction.

2. Show that $\xi_0 = \partial_t$, $\xi_2 = \partial_\phi$, $\xi_3 = \partial_z$, are Killing Vector Fields (KVF), and compute the corresponding conserved quantities k_0, k_2, and k_3 along a geodesic with tangent vector $u^\mu = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$.

3. Compute $\dot{t}$, $\dot{\phi}$ and $\dot{z}$ in terms of k_0, k_2, and k_3.

4. Show that $u^\mu u_\mu = \kappa$, $\kappa = 0, -1$ for null/timelike geodesics yield

$$\mathcal{E} = \frac{1}{2}(\dot{r})^2 + \frac{1}{2}A^2 r^2 + \frac{L^2}{2r^2}\,, \tag{2}$$

   where $\mathcal{E}$, A, and L are constants, which you should calculate. Find conditions for motion $r_1 \le r \le r_2$, and compute $r_{1,2}$ in terms of $\mathcal{E}$, A, and L. (Notice that the problem of the radial motion is similar to the 3-dimensional harmonic oscillator)

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)}e_a = u^\mu\partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

6. Compute k_0, k_2, and k_3 in terms of $u^{(a)}$, so that the $u^{(a)}$ can be used as initial conditions in the geodesic equations.

7. Free massless particle goes through the local inertial frame $\{e_a\}$ with 4-velocity $(u^{(0)}, u^{(1)}, 0, 0)$. Write down the geodesic equations for $(\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$ in terms of $u^{(0)}, u^{(1)}$.

8. Compute *all* the null vectors at a point with coordinate r (i.e. compute the lightcone). Give expressions for both $u^{(a)}$ and u^μ (Hint: You will need a 3-parameter family of vectors, start from $u^{(a)}$ which is easier).

9. Compute the Christoffel symbols of the Levi-Civita connection of the metric. The nonzero components of the inverse metric are

$$g^{tt} = -\frac{1-\left(\frac{r}{2a}\right)^2}{1+\left(\frac{r}{2a}\right)^2}\,,\quad g^{rr} = 1+\left(\frac{r}{2a}\right)^2\,,\quad g^{zz} = 1\,,\quad g^{t\phi} = -\frac{1}{\sqrt{2}a\left(1+\left(\frac{r}{2a}\right)^2\right)}\,, \tag{3}$$

$$g^{\phi\phi} = \frac{1}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)}\,,\quad g^{t\phi} = -\frac{1}{\sqrt{2}a\left(1+\left(\frac{r}{2a}\right)^2\right)}\,. \tag{4}$$

   The result is:

$$\Gamma^t_{rt} = \frac{r}{2a^2}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,,\quad \Gamma^t_{\phi r} = \frac{r^3}{4\sqrt{2}a^3}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,,\quad \Gamma^r_{rr} = -\frac{r}{4a^2}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,, \tag{5}$$

$$\Gamma^r_{\phi t} = \frac{r}{\sqrt{2}a}\left(1+\left(\frac{r}{2a}\right)^2\right)\,,\quad \Gamma^r_{\phi\phi} = r\left(1+\left(\frac{r}{2a}\right)^2\right)\left(2\left(\frac{r}{2a}\right)^2-1\right)\,, \tag{6}$$

$$\Gamma^\phi_{\phi r} = \frac{1}{r}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,,\quad \Gamma^\phi_{rt} = -\frac{1}{\sqrt{2}ar}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,. \tag{7}$$

10. Consider the massive particle moving on the trajectory with $t = 0$, $r = R$, $\phi = \omega\tau$, $z = 0$, where R, ω are constants. Determine when the 4-velocity of the particle is timelike, in which case we have a closed timelike curve (CTC).

11. Compute the relation $\omega = \omega(R)$.

12. Compute the 4-acceleration of the particle $a^\mu = \ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho$, where $\dot{x}^\mu = dx^\mu/d\tau$. Conclude that the particle is not falling freely.

13. The vectors with components in the coordinate basis below are KVFs:

$$\xi_1 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}\left(\frac{r}{\sqrt{2}}\cos\phi, a\left(1+\left(\frac{r}{2a}\right)^2\right)\sin\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right)\cos\phi, 0\right) \tag{8}$$

$$\xi_4 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}\left(\frac{r}{\sqrt{2}}\sin\phi, -a\left(1+\left(\frac{r}{2a}\right)^2\right)\cos\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right)\sin\phi, 0\right) \tag{9}$$

   Compute the corresponding conserved quantities k_1 and k_4 along a geodesic with tangent vector $u^\mu = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$.

14. The KVF ξ_4 deforms isometrically the constant-t, circular, closed CTC, to a closed CTC on which the coordinate t varies. Show that $\mathcal{L}_\xi(u^\mu u_\mu) = 0$, so that the timelike kind of the curve does not change under this deformation.

15. Verify that $\nabla_t\xi_{1r} + \nabla_r\xi_{1t} = 0$ for the KVF ξ_1.

1. Determine whether ∂_μ are timelike, null, or spacelike. From the kind of ∂_ϕ, discuss if it is possible to have (local future pointing) timelike geodesics moving in the negative t direction.

$$\partial_t \cdot \partial_t = g_{tt} = -1 < 0$$ timelike

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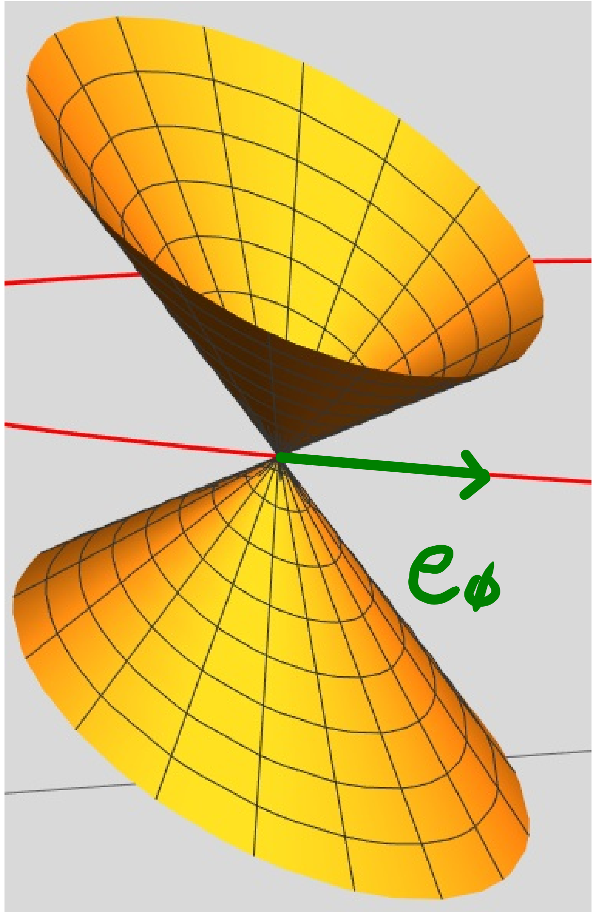
$$\partial_\phi \cdot \partial_\phi = g_{\phi\phi} = r^2\left[1-\left(\frac{r}{2a}\right)^2\right]$$

→ spacelike $r < 2a$

→ null $r = 2a$

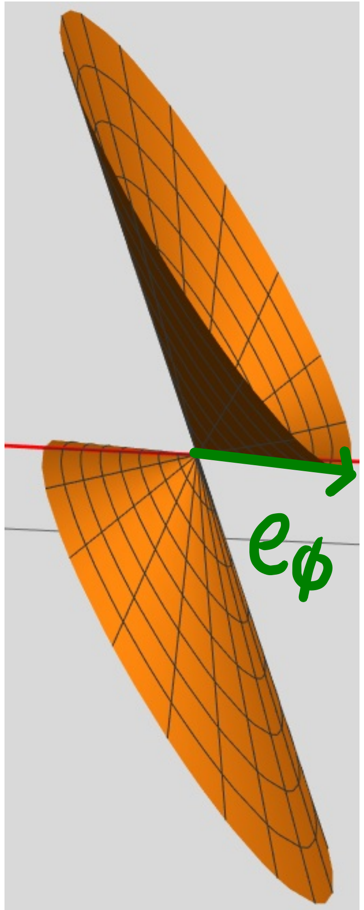
→ timelike $r > 2a$



$r = a$

space like

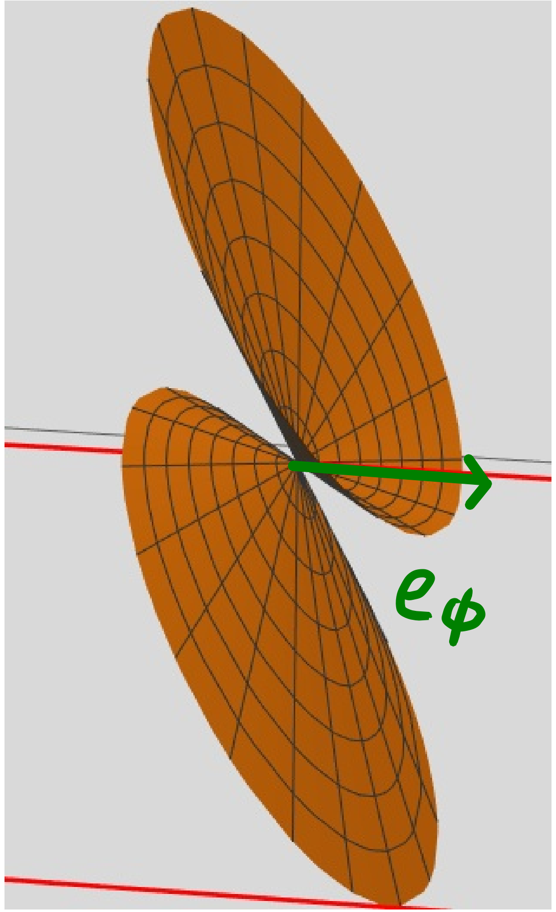
outside the lightcone



$r = 2a$

null

on the lightcone

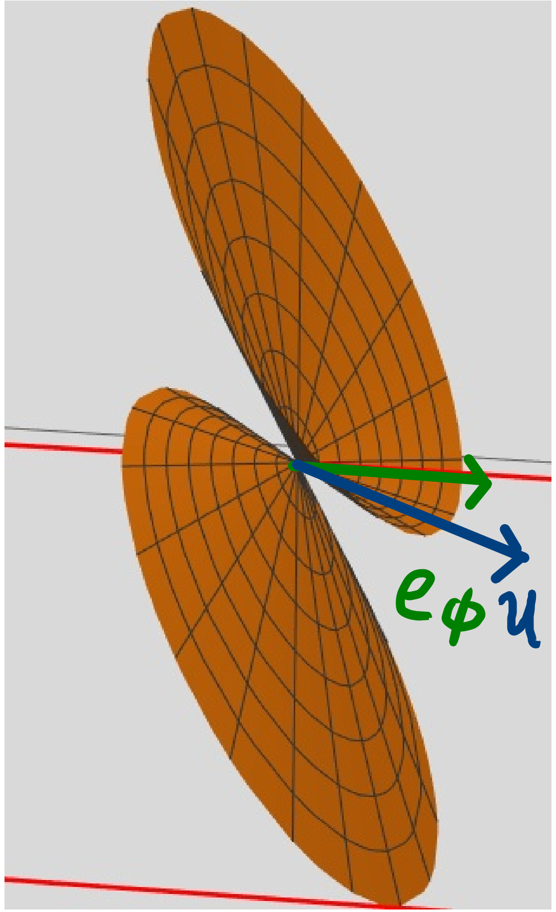


$r = 4a$

timelike

inside the lightcone

- u is timelike
- points in $-t$ direction

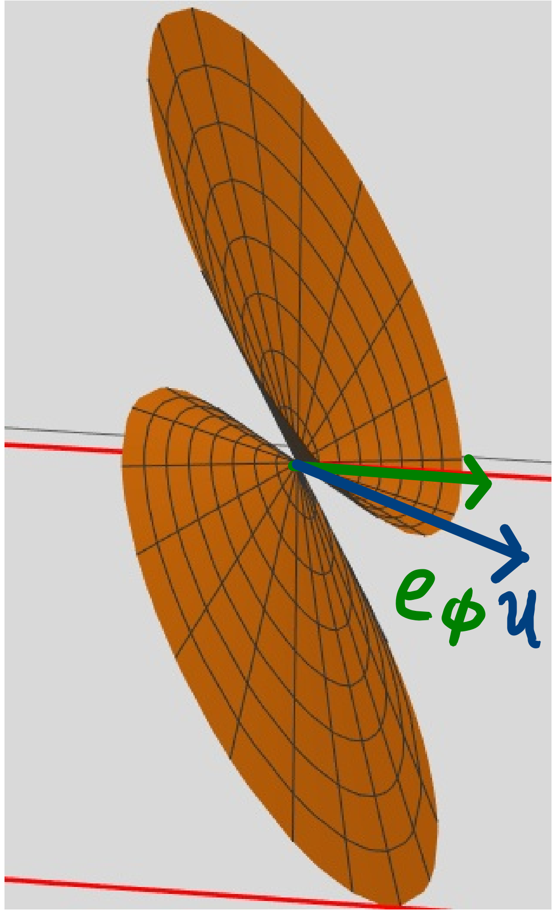


$r = 4a$

timelike

inside the lightcone

- u is timelike
- points in $-t$ direction
- the timelike geodesic with u as tangent vector, visits events with decreasing coordinate time t



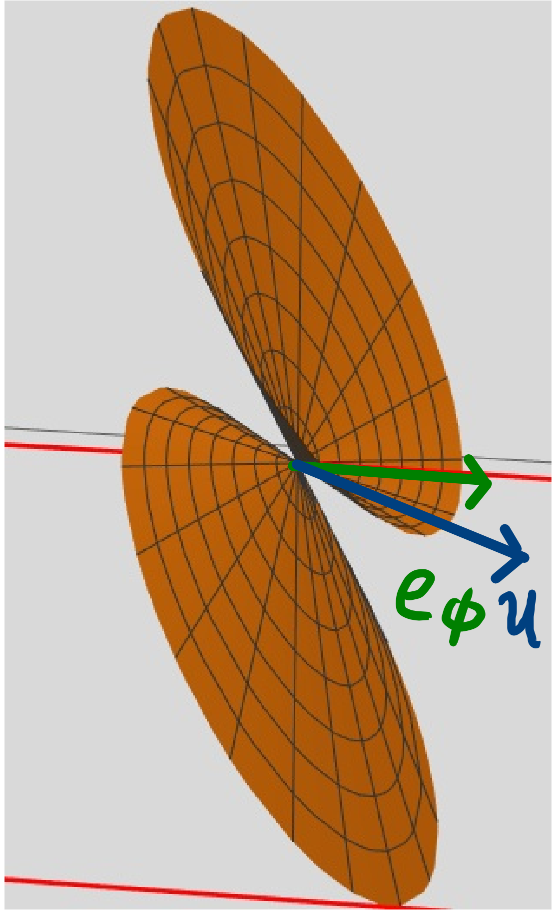
$r = 4a$

timelike

inside the lightcone

- u is timelike
- points in $-t$ direction
- the timelike geodesic with u as tangent vector, visits events with decreasing coordinate time t

- ∂_t always timelike!
- For $r > 2a$, $\exists$ timelike curves with increasing and decreasing t!



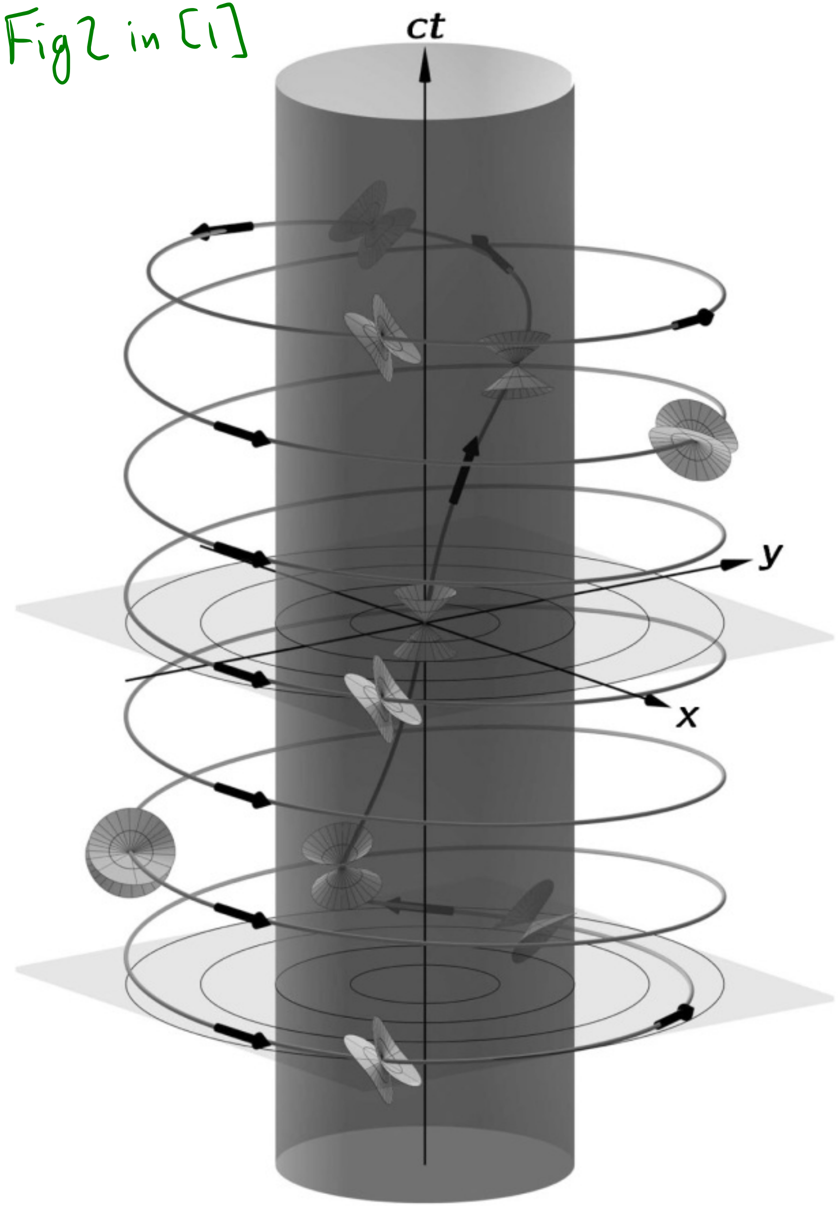
$r = 4a$

timelike

inside the lightcone

- Example of a Closed Timelike Curve (CTC):
    - Start @ origin, move towards $r = 2a$

Fig 2 in [1]



[1] Grave et al, Phys. Rev. D 80, 103002 (2009)

- Example of a Closed Timelike Curve (CTC):
  - Start @ origin, move towards r=2a
  - Cross r>2a, move in negative t direction

Fig 2 in [1]

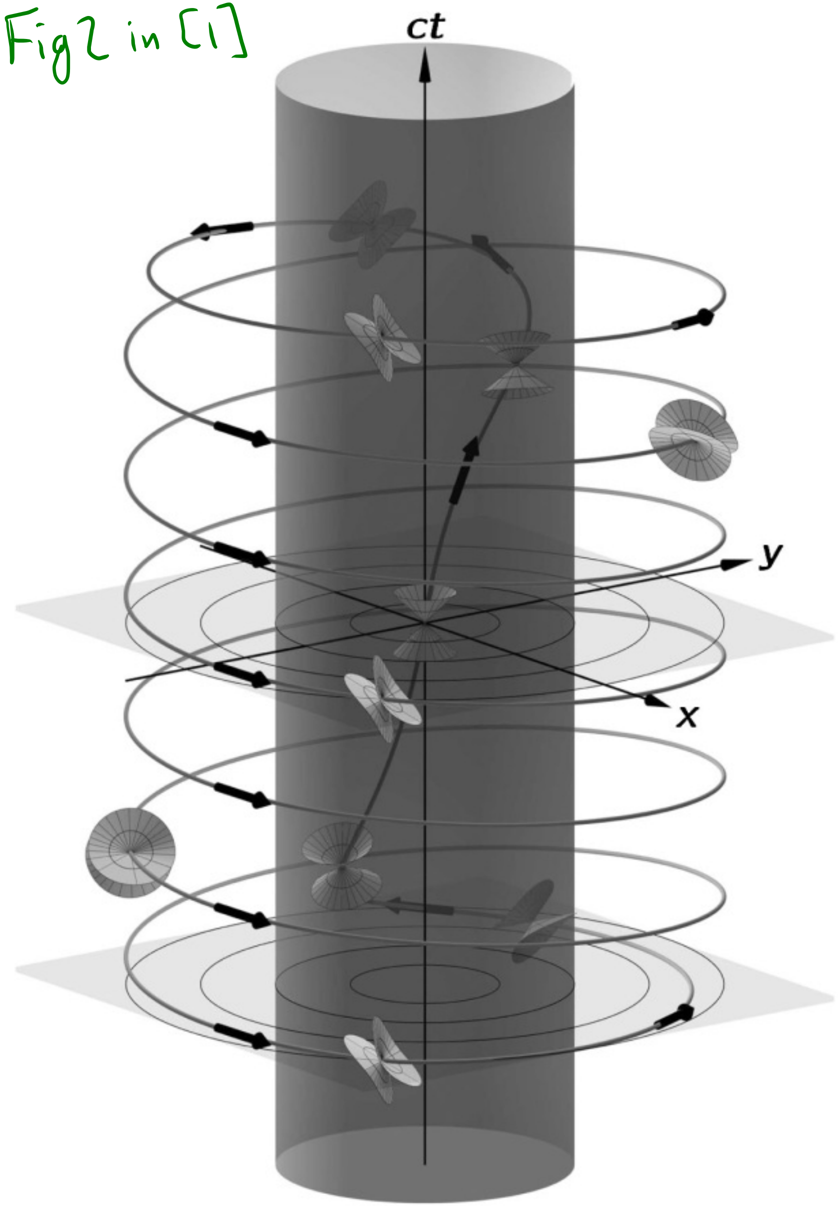


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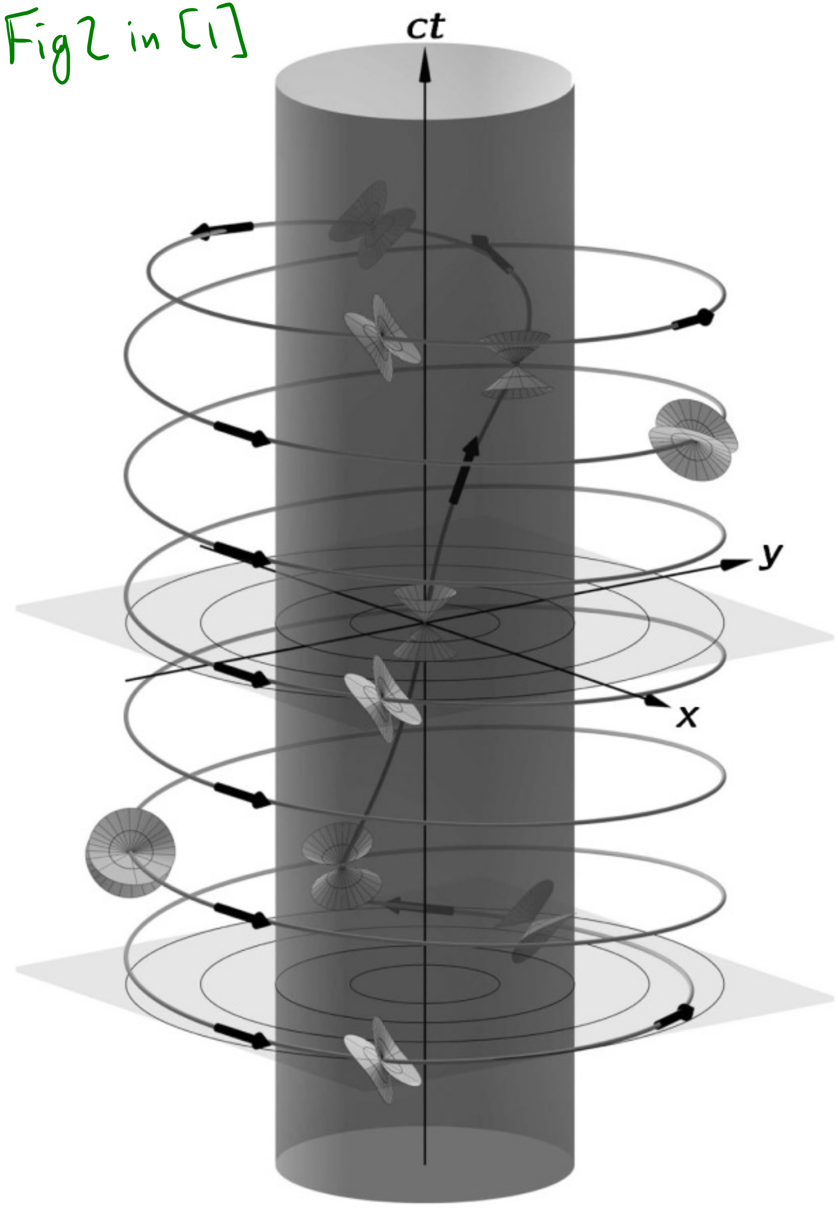
Fig2 in [1]



- Example of a Closed Timelike Curve (CTC):
  - Start @ origin, move towards $r=2a$
  - Cross $r>2a$, move in negative t direction
  - Move back in $r<2a$ region, start moving in positive t direction towards origin

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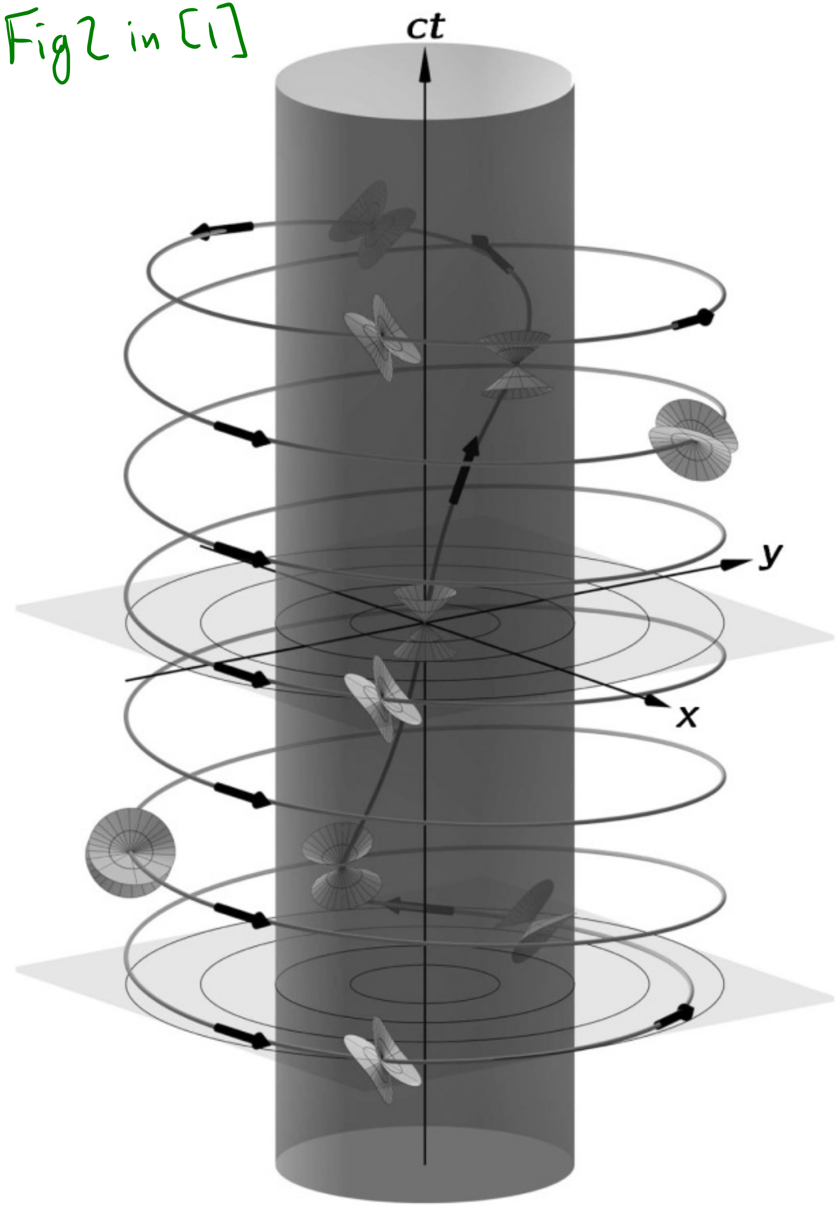

- Example of a Closed Timelike Curve (CTC):

  – not a geodesic: must accelerate

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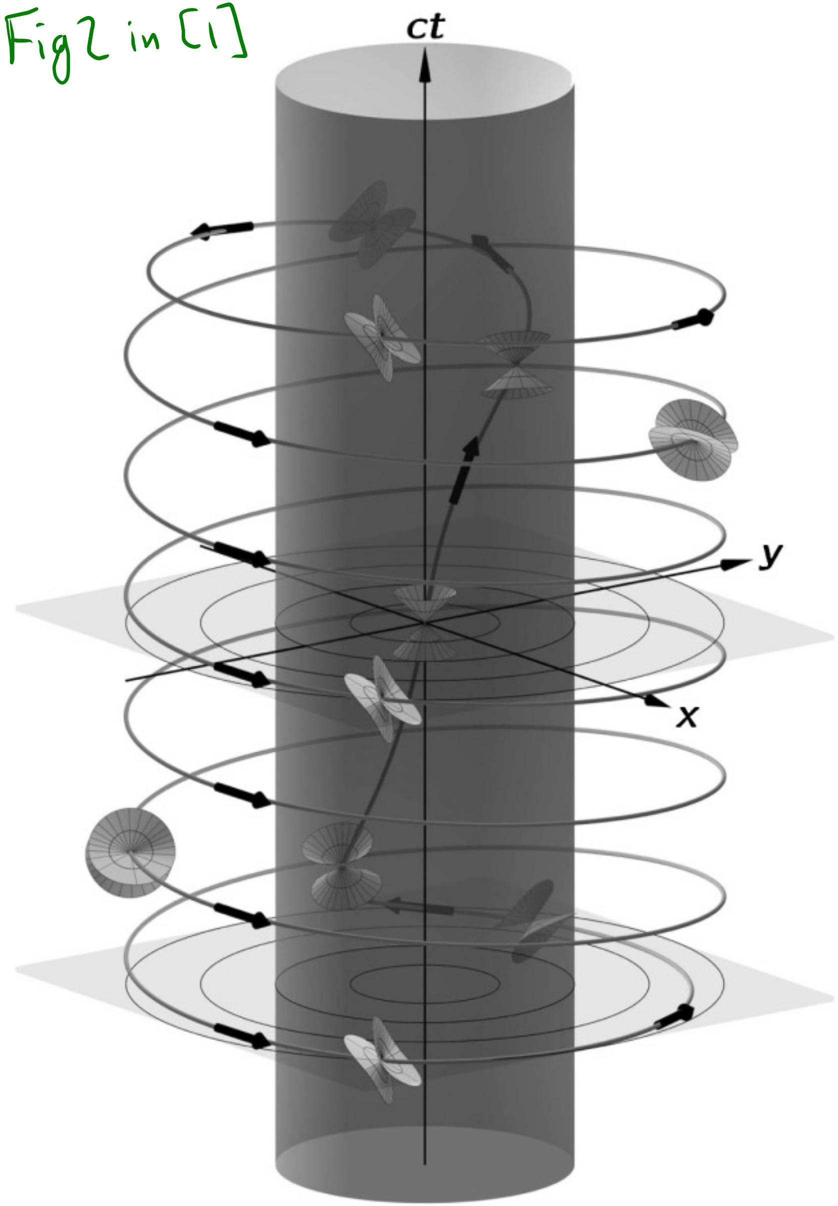
[1] Grave et al, Phys. Rev. D 80, 103002 (2009)

- Example of a Closed Timelike Curve (CTC):

  - not a geodesic: must accelerate

  - no closed timelike geodesic

Fig2 in [1]



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2. Show that $\xi_0 = \partial_t$, $\xi_2 = \partial_\phi$, $\xi_3 = \partial_z$, are Killing Vector Fields (KVF), and compute the corresponding conserved quantities k_0, k_2, and k_3 along a geodesic with tangent vector $u^\mu = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$.

$\partial_t g_{\mu\nu} = 0$, $\partial_\varphi g_{\mu\nu} = 0$, $\partial_z g_{\mu\nu} = 0$ $\Rightarrow$

$\xi_0 = \partial_t$, $\xi_2 = \partial_\varphi$, $\xi_3 = \partial_z$ are KVFs

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Consider a geodesic with $u^\mu = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$ tangent vector

$\dot{t} = \frac{dt}{d\tau}$ (timelike) $= \frac{dt}{d\lambda}$ (null)

$\dot{r} = \frac{dr}{d\tau}$ $\frac{dr}{d\lambda}$

$\dot{\phi} = \frac{d\varphi}{d\tau}$ $\frac{d\varphi}{d\lambda}$

$\dot{z} = dz/d\tau$ $dz/d\lambda$

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$k_0 = (\partial_t)^\mu u_\mu$ is conserved along the geodesic

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Consider a geodesic with $u^\mu = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$ tangent vector

$k_0 = (\partial_t)^\mu u_\mu = g_{\mu\nu} (\partial_t)^\mu u^\nu$

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$$= g_{tt} (\partial_t)^t u^t + g_{t\phi} (\partial_t)^t u^\varphi$$

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$= g_{tt} \cdot 1 \cdot \dot{t} + g_{t\phi} \cdot 1 \cdot \dot{\phi}$

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$$k_0 = -\dot{t} - \frac{r^2}{\sqrt{2}\,a}\dot{\phi}$$

$$k_2 = -\frac{r^2}{\sqrt{2}\,a}\dot{t} + r^2\left[1 - \left(\frac{r}{2a}\right)^2\right]\dot{\phi}$$

$$k_3 = (\partial_z)^\mu u_\mu = g_{\mu\nu}(\partial_z)^\mu u^\nu$$

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$$k_3 = (\partial_z)^\mu u_\mu = g_{\mu\nu}(\partial_z)^\mu u^\nu = g_{zz}(\partial_z)^z u^z$$

2. Show that $\xi_0 = \partial_t$, $\xi_2 = \partial_\phi$, $\xi_3 = \partial_z$, are Killing Vector Fields (KVF), and compute the corresponding conserved quantities k_0, k_2, and k_3 along a geodesic with tangent vector $u^\mu = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$.

$$\partial_t g_{\mu\nu} = 0 \ , \quad \partial_\varphi g_{\mu\nu} = 0 \ , \quad \partial_z g_{\mu\nu} = 0 \quad \Rightarrow$$

$$\xi_0 = \partial_t \ , \quad \xi_2 = \partial_\varphi \ , \quad \xi_3 = \partial_z \quad \text{are KVFs}$$

Consider a geodesic with $u^\mu = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$ tangent vector

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$$k_3 = (\partial_z)^\mu u_\mu = g_{\mu\nu}(\partial_z)^\mu u^\nu = g_{zz}(\partial_z)^z u^z = 1 \cdot 1 \cdot \dot{z}$$

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$$k_3 = \dot{z}$$

3. Compute $\dot{t}$, $\dot{\phi}$ and $\dot{z}$ in terms of k_0, k_2, and k_3.

$$k_0 = - \quad \dot{t} - \frac{r^2}{\sqrt{2}\,a} \quad \dot{\phi}$$

a 2x2 linear system

$$k_2 = \frac{-r^2}{\sqrt{2}\,a} \dot{t} + r^2 \left[1 - \left(\frac{r}{2a} \right)^2 \right] \dot{\phi}$$

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$$\left.\begin{aligned} k_0 &= -\dot{t} - \frac{r^2}{\sqrt{2}\,a}\dot{\phi} \\ k_2 &= -\frac{r^2}{\sqrt{2}\,a}\dot{t} + r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\dot{\phi} \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{t} &= k_0 - \frac{1}{2a}\,\frac{4k_0 a - \sqrt{2}\,k_2}{1+\left(\frac{r}{2a}\right)^2} \\ \dot{\phi} &= \frac{1}{2ar^2}\,\frac{2ak_2 - \sqrt{2}\,k_0 r^2}{1+\left(\frac{r}{2a}\right)^2} \end{aligned}$$

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$$k_3 = \dot{z} \qquad\qquad \dot{z} = k_3$$

4. Show that $u^\mu u_\mu = \kappa$, $\kappa = 0, -1$ for null/timelike geodesics yield

$$\mathcal{E} = \frac{1}{2}(\dot{r})^2 + \frac{1}{2}A^2 r^2 + \frac{L^2}{2r^2}, \tag{2}$$

where $\mathcal{E}$, A, and L are constants, which you should calculate. Find conditions for motion $r_1 \le r \le r_2$, and compute $r_{1,2}$ in terms of $\mathcal{E}$, A, and L. (Notice that the problem of the radial motion is similar to the 3-dimensional harmonic oscillator)

Consider geodesics with $u^\mu u_\mu = \kappa$ $\quad \kappa = \begin{cases} 0 & \text{null} \\ -1 & \text{timelike} \end{cases}$

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Consider geodesics with $u^\mu u_\mu = \kappa \Rightarrow$

$$g_{tt}(\dot{t})^2 + g_{rr}(\dot{r})^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{\phi\phi}(\dot{\phi})^2 + g_{zz}(\dot{z})^2 = \kappa$$

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$$-(\dot{t})^2 + \frac{1}{1+\left(\frac{r}{2a}\right)^2}(\dot{r})^2 - \frac{2r^2}{\sqrt{2}\,a}\dot{t}\dot{\phi} + r^2\left(1-\left(\frac{r}{2a}\right)^2\right)(\dot{\phi})^2 + (\dot{z})^2 = \kappa$$

$$-(\dot{t})^2 + \frac{1}{1+\left(\frac{r}{2a}\right)^2}(\dot{r})^2 - \frac{2r^2}{\sqrt{2}a}\left(k_0 - \frac{1}{2a}\frac{4k_0 a - \sqrt{2}k_2}{1+\left(\frac{r}{2a}\right)^2}\right)\left(\frac{1}{2ar^2}\frac{2ak_2 - \sqrt{2}k_0 r^2}{1+\left(\frac{r}{2a}\right)^2}\right)$$

$$+ r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\left[\frac{1}{2ar^2}\frac{2ak_2 - \sqrt{2}k_0 r^2}{1+\left(\frac{r}{2a}\right)^2}\right] + k_3^2 = K$$

$$-(\dot{t})^2 + \frac{1}{1+\left(\frac{r}{2a}\right)^2}(\dot{r})^2 - \frac{2r^2}{\sqrt{2}\,a}\dot{t}\,\dot{\phi} + r^2\left(1-\left(\frac{r}{2a}\right)^2\right)(\dot{\phi})^2 + (\dot{z})^2 = K$$

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$$(\dot{r})^2 = \left[k_0^2 - k_3^2 + \frac{\sqrt{2}k_0k_2}{a} + K\right] - \frac{k_2^2}{r^2} - \frac{k_0^2+k_3^2-K}{4a^2}r^2$$

$$-(\dot t)^2 + \frac{1}{1+\left(\frac{r}{2a}\right)^2}(\dot r)^2 - \frac{2r^2}{\sqrt{2}\,a}\left(k_0 - \frac{1}{2a}\,\frac{4k_0 a - \sqrt{2}k_2}{1+\left(\frac{r}{2a}\right)^2}\right)\left(\frac{1}{2ar^2}\,\frac{2ak_2 - \sqrt{2}k_0 r^2}{1+\left(\frac{r}{2a}\right)^2}\right)$$

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$$(\dot r)^2 = \underbrace{\left[k_0^2 - k_3^2 + \frac{\sqrt{2}k_0k_2}{a} + k\right]}_{2\mathcal{E}} - \underbrace{\frac{k_2^2}{r^2}}_{\frac{L^2}{2}} - \underbrace{\frac{k_0^2+k_3^2-k}{4a^2}}_{A^2}\, r^2$$

$$-(\dot{t})^2 + \frac{1}{1+\left(\frac{r}{2a}\right)^2}(\dot{r})^2 - \frac{2r^2}{\sqrt{2}a}\left(k_0 - \frac{1}{2a}\frac{4k_0 a - \sqrt{2}k_2}{1+\left(\frac{r}{2a}\right)^2}\right)\left(\frac{1}{2ar^2}\frac{2ak_2 - \sqrt{2}k_0 r^2}{1+\left(\frac{r}{2a}\right)^2}\right)$$

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$$-(\dot t)^2 + \frac{1}{1+\left(\frac{r}{2a}\right)^2}(\dot r)^2 - \frac{2r^2}{\sqrt{2}\,a}\left(k_0 - \frac{1}{2a}\,\frac{4k_0 a - \sqrt{2}k_2}{1+\left(\frac{r}{2a}\right)^2}\right)\left(\frac{1}{2ar^2}\,\frac{2ak_2 - \sqrt{2}k_0 r^2}{1+\left(\frac{r}{2a}\right)^2}\right)$$

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timelike or null

$k = -1, 0$

$$\mathcal{E} = \frac{1}{2}\left(k_0^2 - k_3^2 + \frac{\sqrt{2}k_0 k_2}{a} + k\right)$$

$$A^2 = \frac{1}{2a^2}\left(k_0^2 + k_3^2 - k\right)$$

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$$V(r) = \frac{1}{2}A^2 r^2 + \frac{L^2}{2r^2}$$

$$\mathcal{E} = \frac{1}{2}(\dot{r})^2 + V(r)$$

$$\mathcal{E} = \frac{1}{2}\left(k_0^2 - k_3^2 + \frac{\sqrt{2}\,k_0 k_2}{a} + k\right)$$

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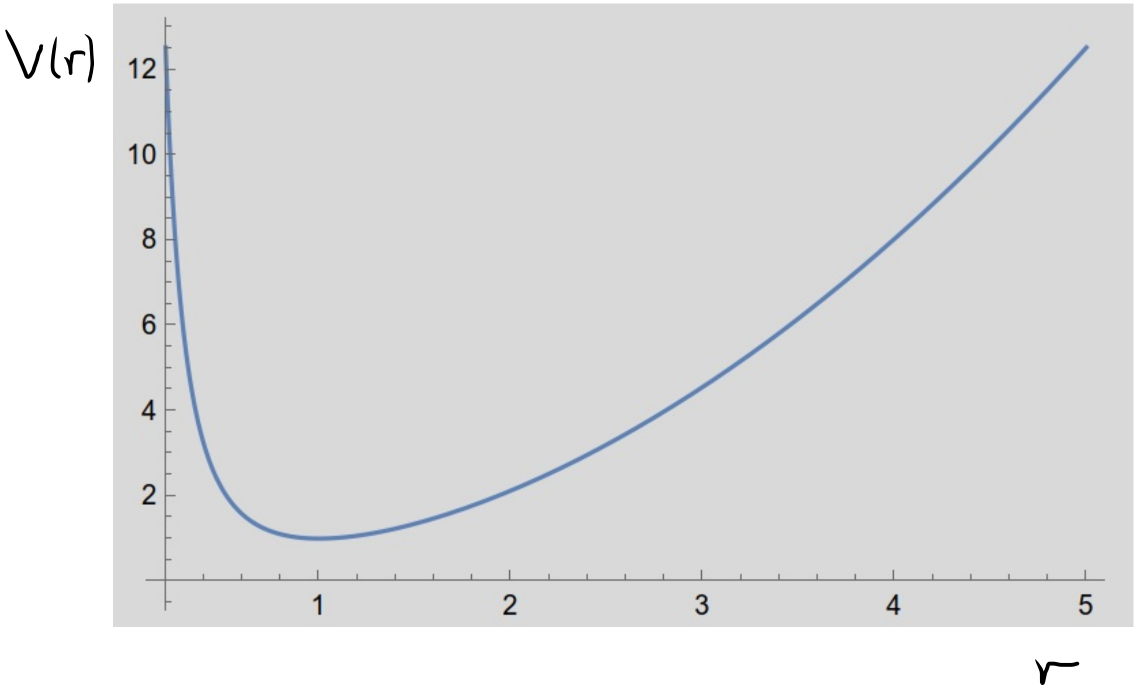

$$\mathcal{E} = \frac{1}{2} (\dot{r})^2 + V(r)$$

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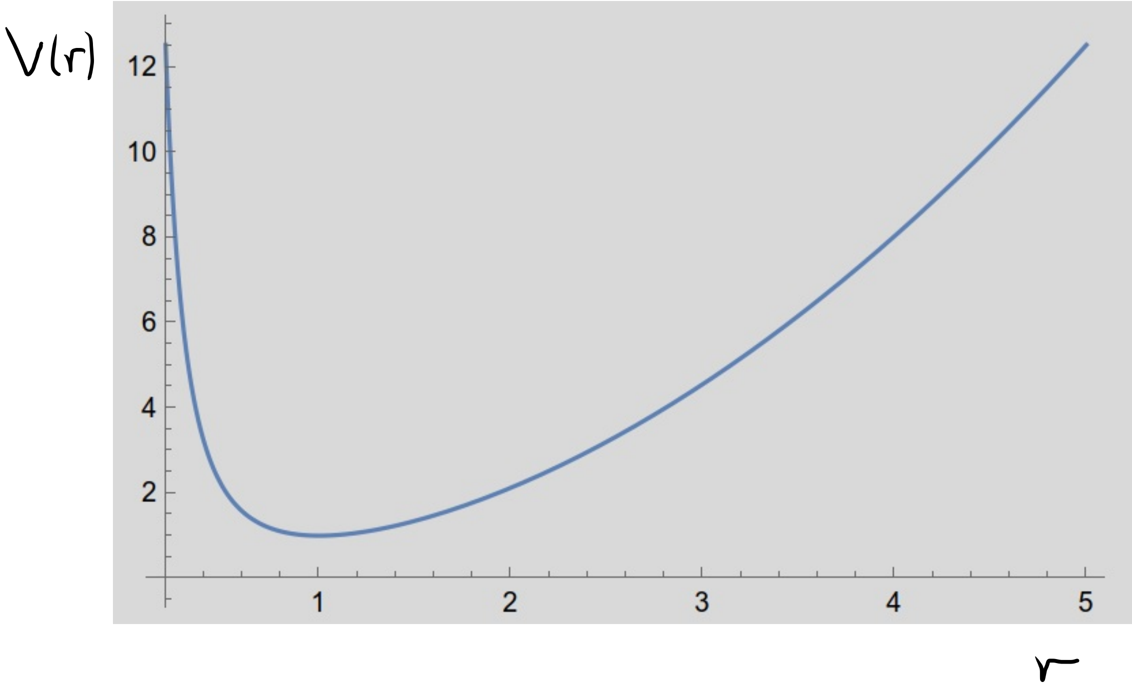

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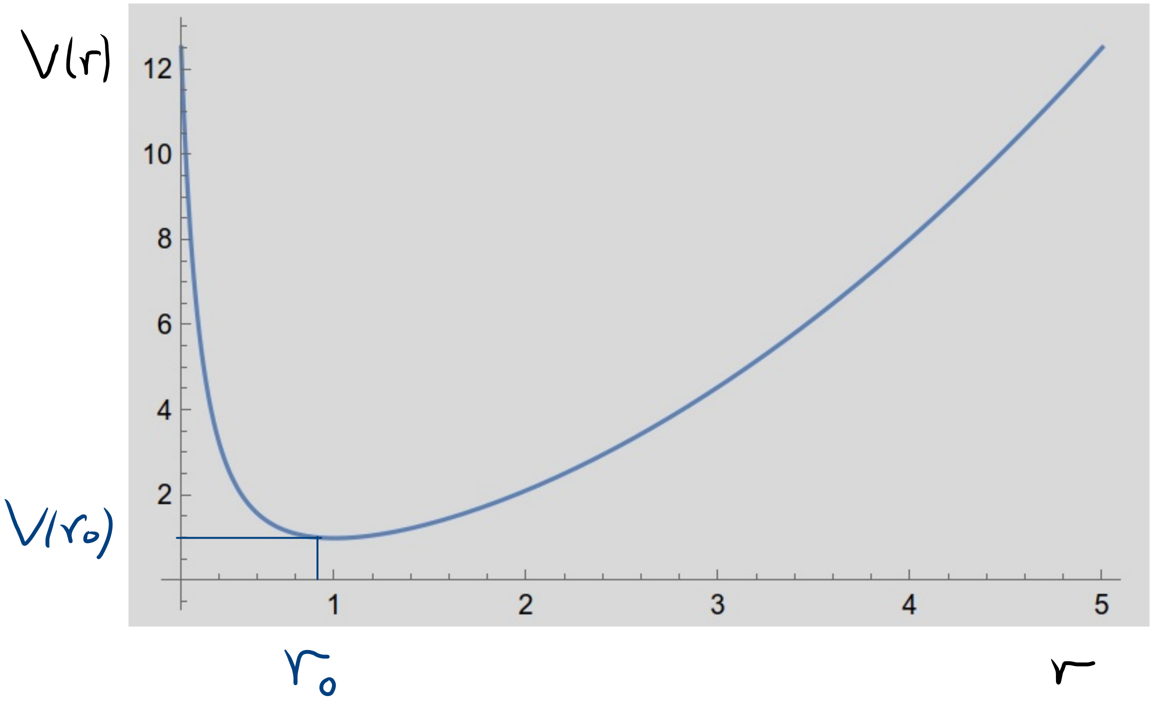

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$$V(r_0) = A \cdot L$$

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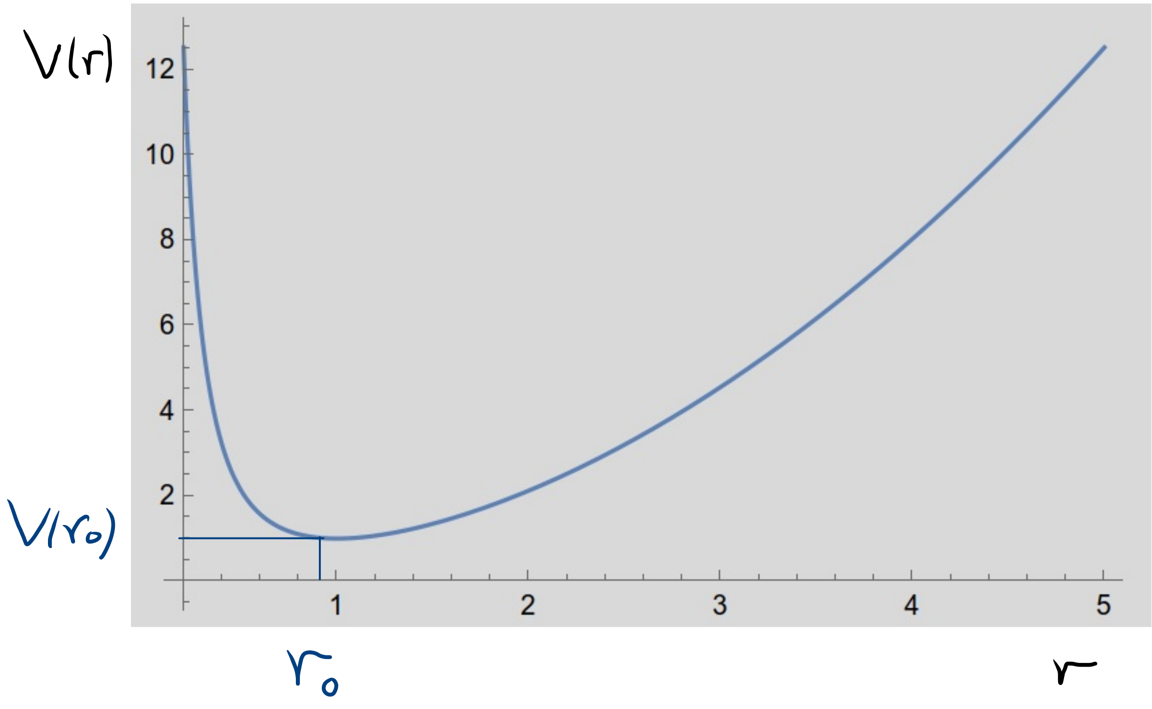

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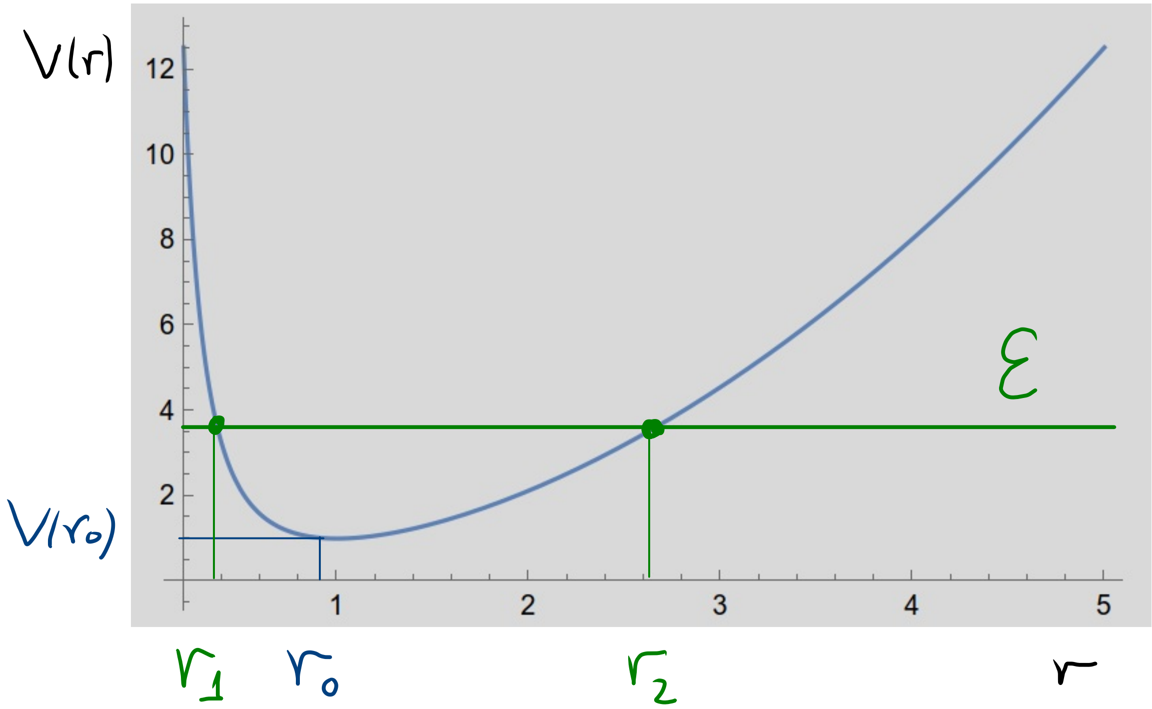

For $\varepsilon > AL$, $\quad r_{1,2} = \frac{\varepsilon}{A^2} \pm \frac{1}{A^2}\sqrt{1 - \left(\frac{AL}{\varepsilon}\right)^2}$

$$V(r) = \frac{1}{2} A^2 r^2 + \frac{L^2}{2r^2}$$

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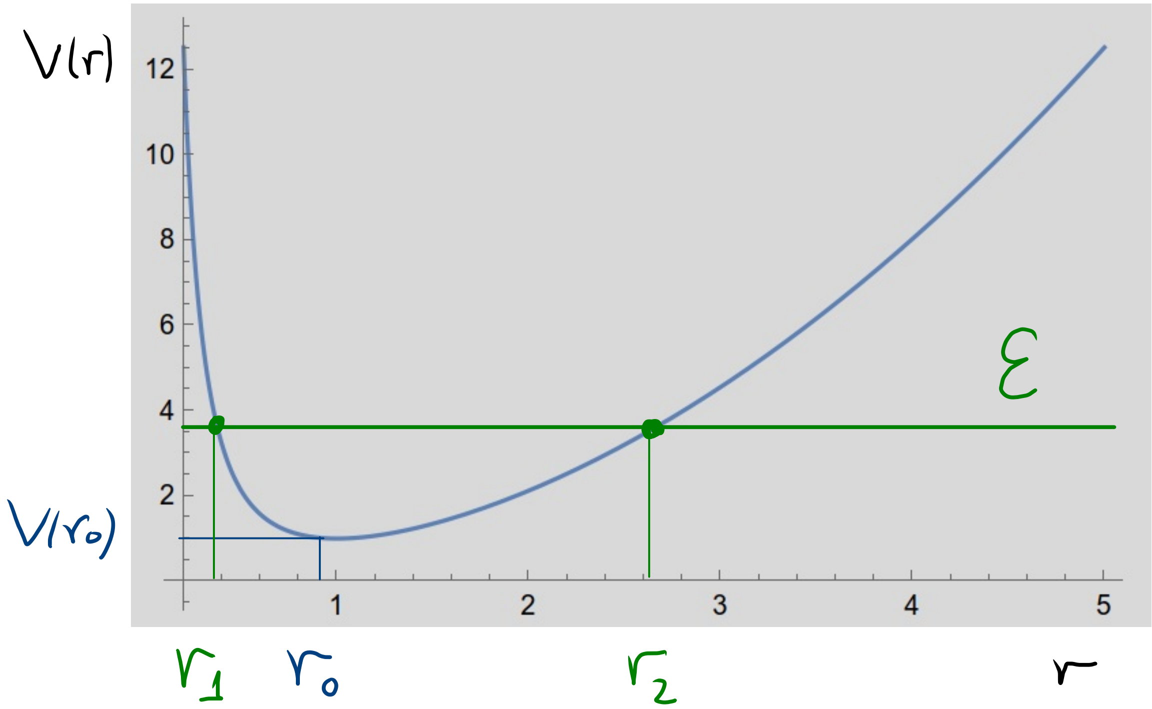

For $\mathcal{E} > AL$, $r_{1,2} = \frac{\mathcal{E}}{A^2} \pm \frac{1}{A^2}\sqrt{1 - \left(\frac{AL}{\mathcal{E}}\right)^2}$

For $\mathcal{E} = AL$, circular orbit with $R = \sqrt{\frac{L}{A}}$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

Brute force method explained in class:

1. Diagonalize the metric

$$g_d = O^T g O \quad , \quad O^{-1} = O^T \, , \quad g_d = \mathrm{diag}(g_0, g_1, g_2, g_3)$$

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3. Define $\Lambda = O D$, then $\eta = \Lambda^T g \Lambda$ $\qquad \eta = \mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1)$

the (-1) can be in any position

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

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1. Diagonalize the metric

$$g_d = O^T g O \quad , \quad O^{-1} = O^T \, , \quad g_d = \mathrm{diag}(g_0, g_1, g_2, g_3)$$

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3. Define $\Lambda = O D$, then $\eta = \Lambda^T g \Lambda \qquad \eta = \mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1)$

4. Rearrange the columns of Λ to bring (-1) at the first position, then $e_a = \Lambda^\mu{}_a \partial_\mu$ is an orthonormal basis

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

Brute force method simple if $g_{\mu\nu}$ a diagonal matrix

$e_\mu = \frac{1}{\sqrt{|g_{\mu\mu}|}} \partial_\mu$, but algebraically complicated otherwise.

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

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In our case only $\begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix}$ is a nondiagonal submatrix

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In our case only $\begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix}$ is a nondiagonal submatrix

We guess: $e_1 = \frac{1}{|g_{rr}|^{1/2}} \partial_r = \sqrt{1 + \left(\frac{r}{2a}\right)^2}\, \partial_r$ (diagonal part of matrix)

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

Brute force method simple if $g_{\mu\nu}$ a diagonal matrix

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In our case only $\begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix}$ is a nondiagonal submatrix

We guess: $e_1 = \frac{1}{|g_{rr}|^{1/2}} \partial_r = \sqrt{1 + \left(\frac{r}{2a}\right)^2}\ \partial_r$

$e_3 = \frac{1}{|g_{zz}|^{1/2}} \partial_z = \partial_z$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$\partial_t = \alpha \, e_0 + \beta \, e_2$$

$$\partial_\varphi = \gamma \, e_0 + \delta \, e_2$$

In our case only $\begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix}$ is a nondiagonal submatrix

We guess: $e_1 = \frac{1}{|g_{rr}|^{1/2}} \partial_r = \sqrt{1 + \left(\frac{r}{2a}\right)^2} \, \partial_r$

$$e_3 = \frac{1}{|g_{zz}|^{1/2}} \partial_z = \partial_z$$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$\left.\begin{aligned} \partial_t &= \alpha\, e_0 + \beta\, e_2 \\ \partial_\varphi &= \gamma\, e_0 + \delta\, e_2 \end{aligned}\right\} \Rightarrow \partial_t \cdot \partial_t = \alpha^2\, e_0 \cdot e_0 + \beta^2\, e_2 \cdot e_2 + 2\alpha\beta\, e_0 \cdot e_2$$

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$$\left.\begin{aligned} \partial_t &= \alpha\, e_0 + \beta\, e_2 \\ \partial_\varphi &= \gamma\, e_0 + \delta\, e_2 \end{aligned}\right\} \Rightarrow \partial_t \cdot \partial_t = \alpha^2 \underbrace{e_0 \cdot e_0}_{-1} + \beta^2 \underbrace{e_2 \cdot e_2}_{1} + 2\alpha\beta \underbrace{e_0 \cdot e_2}_{0}$$

$$e_\mu \cdot e_\nu = \eta_{\mu\nu}$$

In our case only $\begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix}$ is a nondiagonal submatrix

We guess: $e_1 = \frac{1}{|g_{rr}|^{1/2}} \partial_r = \sqrt{1 + \left(\frac{r}{2a}\right)^2}\, \partial_r$

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In our case only $\begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix}$ is a nondiagonal submatrix

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5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$\partial_t = \alpha e_0 + \beta e_2$$

$$\partial_\varphi = \gamma e_0 + \delta e_2$$

Choose $\alpha = 1$ (normalization) – in general $\alpha = |g_{tt}|^{1/2} \Rightarrow \beta = 0$

$$-\alpha^2 + \beta^2 = -1 \quad (1) \qquad -1 + \beta^2 = -1 \Rightarrow \beta = 0$$

$$-\gamma^2 + \delta^2 = r^2\left(1 - \left(\frac{r}{2a}\right)^2\right) \quad (2)$$

$$-\alpha\gamma + \beta\delta = -\frac{r^2}{\sqrt{2}\,a} \quad (3)$$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$\partial_t = e_0 + \beta e_2$$

$$\partial_\varphi = \gamma e_0 + \delta e_2$$

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$$-\gamma^2 + \delta^2 = r^2\left(1 - \left(\frac{r}{2a}\right)^2\right) \quad (2)$$

$$-\alpha\gamma + \beta\delta = -\frac{r^2}{\sqrt{2}a} \quad (3) \qquad -\gamma + 0 = -\frac{r^2}{\sqrt{2}a} \Rightarrow \gamma = \frac{r^2}{\sqrt{2}a}$$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$\partial_t = \quad e_0 + \beta e_2$$

$$\partial_\varphi = \gamma e_0 + \delta e_2$$

Choose $\alpha = 1$

$$-\alpha^2 + \beta^2 = -1 \qquad (1) \qquad -1 + \beta^2 = -1 \Rightarrow \beta = 0$$

$$-\gamma^2 + \delta^2 = r^2\left(1 - \left(\frac{r}{2a}\right)^2\right) \qquad (2) \qquad -\frac{r}{2a^2} + \delta^2 = r^2 - r^2\left(\frac{r}{2a}\right)^2$$

$$-\alpha\gamma + \beta\delta = -\frac{r^2}{\sqrt{2}\,a} \qquad (3) \qquad -\gamma + 0 = -\frac{r^2}{\sqrt{2}a} \Rightarrow \gamma = \frac{r^2}{\sqrt{2}\,a}$$

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$$-\gamma^2 + \delta^2 = r^2\left(1 - \left(\frac{r}{2a}\right)^2\right) \quad (2) \qquad \delta = r\sqrt{1 + \left(\frac{r}{2a}\right)^2}$$

$$-\alpha\gamma + \beta\delta = -\frac{r^2}{\sqrt{2}a} \quad (3) \qquad -\gamma + 0 = -\frac{r^2}{\sqrt{2}a} \Rightarrow \gamma = \frac{r^2}{\sqrt{2}a}$$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$\partial_t = e_0 + \beta e_2 \qquad \partial_t = e_0$$

$$\partial_\varphi = \gamma e_0 + \delta e_2 \qquad \partial_\varphi = \frac{r^2}{\sqrt{2}a} e_0 + r\sqrt{1+\left(\frac{r}{2a}\right)^2}\, e_2$$

Choose $\alpha = 1$

$$-\alpha^2 + \beta^2 = -1 \quad (1) \qquad -1 + \beta^2 = -1 \Rightarrow \beta = 0$$

$$-\gamma^2 + \delta^2 = r^2\left(1 - \left(\frac{r}{2a}\right)^2\right) \quad (2) \qquad \delta = r\sqrt{1 + \left(\frac{r}{2a}\right)^2}$$

$$-\alpha\gamma + \beta\delta = -\frac{r^2}{\sqrt{2}a} \quad (3) \qquad -\gamma + 0 = -\frac{r^2}{\sqrt{2}a} \Rightarrow \gamma = \frac{r^2}{\sqrt{2}a}$$

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$$\rightsquigarrow \partial_\varphi = \frac{r^2}{\sqrt{2}a} \partial_t + r\sqrt{1+\left(\frac{r}{2a}\right)^2}\, e_2$$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$\partial_t = e_0 + \beta e_2 \qquad \partial_t = e_0$$

$$\partial_\varphi = \gamma e_0 + \delta e_2 \qquad \partial_\varphi = \frac{r^2}{\sqrt{2}a} e_0 + r\sqrt{1+\left(\frac{r}{2a}\right)^2} e_2$$

$$\rightsquigarrow \partial_\varphi = \frac{r^2}{\sqrt{2}a} \partial_t + r\sqrt{1+\left(\frac{r}{2a}\right)^2} e_2$$

$$\rightsquigarrow e_2 = \frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(-\frac{r^2}{\sqrt{2}a} \partial_t + \partial_\varphi \right)$$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$e_0 = \partial_t$$

$$e_1 = \sqrt{1 + \left(\frac{r}{2a}\right)^2} \; \partial_r$$

$$e_2 = \frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(-\frac{r^2}{\sqrt{2}\, a} \partial_t + \partial_\varphi \right)$$

$$e_3 = \partial_z$$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$e_0 = \partial_t$$

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$$e_3 = \partial_z$$

$$\partial_t = e_0$$

$$\partial_r = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}\, e_1$$

$$\partial_\varphi = \frac{r^2}{\sqrt{2}\,a} e_0 + r\sqrt{1+\left(\frac{r}{2a}\right)^2}\, e_2$$

$$\partial_z = e_3$$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$u^{(a)} e_a = u^{(0)} e_0 + u^{(1)} e_1 + u^{(2)} e_2 + u^{(3)} e_3$$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$u^{(a)} e_a = u^{(0)} \partial_t + u^{(1)} \sqrt{1+\left(\frac{r}{2a}\right)^2}\, \partial_r + u^{(2)} \frac{1}{r} \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(-\frac{r^2}{\sqrt{2}\, a} \partial_t + \partial_\varphi \right) + u^{(3)} \partial_z$$

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$$= \left(u^{(0)} - \frac{r}{\sqrt{2}\, a \sqrt{1+\left(\frac{r}{2a}\right)^2}} u^{(2)} \right) \partial_t + \sqrt{1+\left(\frac{r}{2a}\right)^2}\, u^{(1)} \partial_r + \frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}} u^{(2)} \partial_\varphi + u^{(3)} \partial_z$$

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$$u^{(a)} e_a = u^{(0)} \partial_t + u^{(1)} \sqrt{1+\left(\frac{r}{2a}\right)^2}\, \partial_r + u^{(2)} \frac{1}{r} \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(-\frac{r^2}{\sqrt{2}\, a} \partial_t + \partial_\varphi \right) + u^{(3)} \partial_z$$

$$= \left(u^{(0)} - \frac{r}{\sqrt{2}\, a \sqrt{1+\left(\frac{r}{2a}\right)^2}} u^{(2)} \right) \partial_t + \sqrt{1+\left(\frac{r}{2a}\right)^2}\, u^{(1)} \partial_r + \frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}} u^{(2)} \partial_\varphi + u^{(3)} \partial_z$$

$$= u^t \partial_t + u^r \partial_r + u^\varphi \partial_\varphi + u^z \partial_z$$

5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)} e_a = u^\mu \partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.

$$u^t = \dot{t} = u^{(0)} - \frac{r}{\sqrt{2}\, a \sqrt{1+\left(\frac{r}{2a}\right)^2}} u^{(2)}$$

$$u^r = \dot{r} = \sqrt{1+\left(\frac{r}{2a}\right)^2}\; u^{(1)}$$

$$u^\phi = \dot{\phi} = \frac{1}{r\sqrt{1+(r/2a)^2}} u^{(2)} \qquad u^z = \dot{z} = u^{(3)}$$

$$= \left(u^{(0)} - \frac{r}{\sqrt{2}\, a \sqrt{1+\left(\frac{r}{2a}\right)^2}} u^{(2)} \right) \partial_t + \sqrt{1+\left(\frac{r}{2a}\right)^2}\; u^{(1)} \partial_r + \frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}} u^{(2)} \partial_\phi + u^{(3)} \partial_z$$

$$= u^t \partial_t + u^r \partial_r + u^\phi \partial_\phi + u^z \partial_z$$

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$$u^{(2)} = r\sqrt{1+\left(\frac{r}{2a}\right)^2}\; \dot{\phi}$$

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$$u^\phi = \dot{\phi} = \frac{1}{r\sqrt{1+(r/2a)^2}}\, u^{(2)} \qquad u^z = \dot{z} = u^{(3)}$$

$$u^{(2)} = r\sqrt{1+\left(\frac{r}{2a}\right)^2}\;\dot{\phi}$$

$$u^{(0)} = \dot{t} + \frac{r}{\sqrt{2}\, a\sqrt{1+\left(\frac{r}{2a}\right)^2}}\, u^{(2)}$$

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$$u^\phi = \dot{\phi} = \frac{1}{r\sqrt{1 + (r/2a)^2}}\, u^{(2)} \qquad u^z = \dot{z} = u^{(3)}$$

$$u^{(2)} = r \sqrt{1 + \left(\frac{r}{2a}\right)^2}\; \dot{\phi}$$

$$u^{(0)} = \dot{t} + \frac{r}{\sqrt{2}\, a \sqrt{1 + \left(\frac{r}{2a}\right)^2}}\; r \sqrt{1 + \left(\frac{r}{2a}\right)^2}\; \dot{\phi}$$

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$$u^{(0)} = \dot{t} + \frac{r}{\sqrt{2}\, a\sqrt{1+\left(\frac{r}{2a}\right)^2}}\, r\sqrt{1+\left(\frac{r}{2a}\right)^2}\;\dot{\phi} = \dot{t} + \frac{r^2}{\sqrt{2}\, a}\dot{\phi}$$

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$$u^t = \dot{t} = u^{(0)} - \frac{r}{\sqrt{2}\, a \sqrt{1+\left(\frac{r}{2a}\right)^2}} u^{(2)}$$

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$$u^\phi = \dot{\phi} = \frac{1}{r\sqrt{1+(r/2a)^2}} u^{(2)} \qquad u^z = \dot{z} = u^{(3)}$$

$$u^{(2)} = r\sqrt{1+\left(\frac{r}{2a}\right)^2}\, \dot{\phi}$$

$$u^{(0)} = \dot{t} + \frac{r}{\sqrt{2}\, a\sqrt{1+\left(\frac{r}{2a}\right)^2}}\, r\sqrt{1+\left(\frac{r}{2a}\right)^2}\, \dot{\phi} = \dot{t} + \frac{r^2}{\sqrt{2}\, a}\dot{\phi}$$

$$u^{(1)} = \frac{1}{\sqrt{1+(r/2a)^2}}\dot{r} \qquad u^{(3)} = \dot{z}$$

6. Compute k_0, k_2, and k_3 in terms of $u^{(a)}$, so that the $u^{(a)}$ can be used as initial conditions in the geodesic equations.

$$k_0 = -\dot{t} - \frac{r^2}{\sqrt{2}a}\dot{\phi}$$

6. Compute k_0, k_2, and k_3 in terms of $u^{(a)}$, so that the $u^{(a)}$ can be used as initial conditions in the geodesic equations.

$$k_0 = -\dot{t} - \frac{r^2}{\sqrt{2}a}\dot{\phi} = -\left(u^{(0)} - \frac{r}{\sqrt{2}a}\frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(1)}\right) - \frac{r^2}{\sqrt{2}a}\frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)}$$

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$$= -u^{(0)}$$

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$$k_0 = -\overset{\circ}{\dot{t}} - \frac{r^2}{\sqrt{2}a}\overset{\circ}{\dot{\phi}} = -\left(u^{(0)} - \frac{r}{\sqrt{2}a}\frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)}\right) - \frac{r^2}{\sqrt{2}a}\frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)}$$

$$= -u^{(0)}$$

$$k_2 = r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\overset{\circ}{\dot{\phi}} - \frac{r^2}{\sqrt{2}a}\dot{t}$$

6. Compute k_0, k_2, and k_3 in terms of $u^{(a)}$, so that the $u^{(a)}$ can be used as initial conditions in the geodesic equations.

$$k_0 = -\dot{t} - \frac{r^2}{\sqrt{2}a}\dot{\phi} = -\left(u^{(0)} - \frac{r}{\sqrt{2}a}\frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)}\right) - \frac{r^2}{\sqrt{2}a}\frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)}$$

$$= -u^{(0)}$$

$$k_2 = r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\dot{\phi} - \frac{r^2}{\sqrt{2}a}\dot{t}$$

$$= r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)} - \frac{r^2}{\sqrt{2}a}u^{(0)} + \frac{r^2}{\sqrt{2}a}\frac{r}{\sqrt{2}a}\frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)}$$

6. Compute k_0, k_2, and k_3 in terms of $u^{(a)}$, so that the $u^{(a)}$ can be used as initial conditions in the geodesic equations.

$$k_0 = -\overset{0}{\dot t} - \frac{r^2}{\sqrt{2}a}\overset{0}{\dot\phi} = -\left(u^{(0)} - \frac{r}{\sqrt{2}a}\cancel{\frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}}u^{(2)}\right) - \frac{r^2}{\sqrt{2}a}\cancel{\frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}}u^{(2)}$$

$$= -u^{(0)}$$

$$k_2 = r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\overset{0}{\dot\phi} - \frac{r^2}{\sqrt{2}a}\dot t$$

$$= r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)} - \frac{r^2}{\sqrt{2}a}u^{(0)} + \frac{r^2}{\sqrt{2}a}\frac{r}{\sqrt{2}a}\frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)}$$

$$= r\sqrt{1+\left(\frac{r}{2a}\right)^2}\,u^{(2)} - \frac{r^2}{\sqrt{2}a}u^{(0)}$$

6. Compute k_0, k_2, and k_3 in terms of $u^{(a)}$, so that the $u^{(a)}$ can be used as initial conditions in the geodesic equations.

$$k_0 = -\overset{\circ}{t} - \frac{r^2}{\sqrt{2}a}\overset{\circ}{\phi} = -\left(u^{(0)} - \frac{r}{\sqrt{2}a}\frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)}\right) - \frac{r^2}{\sqrt{2}a}\frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)}$$

$$= -u^{(0)}$$

$$k_2 = r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\overset{\circ}{\phi} - \frac{r^2}{\sqrt{2}a}\dot{t}$$

$$= r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)} - \frac{r^2}{\sqrt{2}a}u^{(0)} + \frac{r^2}{\sqrt{2}a}\frac{r}{\sqrt{2}a}\frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)}$$

$$= r\sqrt{1+\left(\frac{r}{2a}\right)^2}\,u^{(2)} - \frac{r^2}{\sqrt{2}a}u^{(0)}$$

$$k_3 = u^{(3)}$$

6. Compute k_0, k_2, and k_3 in terms of $u^{(a)}$, so that the $u^{(a)}$ can be used as initial conditions in the geodesic equations.

$$k_0 = -u^{(0)}$$

$$k_2 = -\frac{r^2}{\sqrt{2}a} u^{(0)} + r\sqrt{1+\left(\frac{r}{2a}\right)^2}\; u^{(2)}$$

$$k_3 = u^{(3)}$$

Therefore, choose arbitrary $u^{(0)}, u^{(2)}, u^{(3)} \rightarrow k_0, k_2, k_3$

- determines $u^{(1)}$ from $u^{(a)} u^{(b)} \eta_{ab} = k \quad (=0, -1)$
- determines $\mathcal{E} = k_0^2 - k_3^2 + k + \frac{\sqrt{2}}{a} k_0 k_2$

  $A^2 = \frac{1}{(2a)^2}\left(k_0^2 + k_3^2 - k\right) \qquad L^2 = k_3^2$

6. Compute k_0, k_2, and k_3 in terms of $u^{(a)}$, so that the $u^{(a)}$ can be used as initial conditions in the geodesic equations.

- determines

$$\dot{t}(0) = u^{(0)} - \frac{r_0}{\sqrt{2}a\sqrt{1+\left(\frac{r_0}{2a}\right)^2}} u^{(2)}$$

$$\dot{r}(0) = \sqrt{1+\left(\frac{r_0}{2a}\right)^2}\, u^{(1)}$$

$$\dot{\phi}(0) = \frac{1}{r_0\sqrt{1+\left(\frac{r_0}{2a}\right)^2}} u^{(2)} \qquad \dot{z}(0) = u^{(3)}$$

Therefore, choose arbitrary $u^{(0)}, u^{(2)}, u^{(3)} \rightarrow k_0, k_2, k_3$

- determines $u^{(1)}$ from $u^{(a)} u^{(b)} \eta_{ab} = k \quad (=0, -1)$
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7. Free massless particle goes through the local inertial frame $\{e_a\}$ with 4-velocity $(u^{(0)}, u^{(1)}, 0, 0)$. Write down the geodesic equations for $(\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$ in terms of $u^{(0)}, u^{(1)}$.

We have: $k_0 = -u^{(0)}$ $\quad k_2 = -\frac{r_0^2}{\sqrt{2a}} u^{(0)}$ $\quad k_3 = 0$

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$$k_0 = -\dot{t}^2 - \frac{r^2}{\sqrt{2}a}\dot{\phi}$$

$$k_2 = -\frac{r^2}{\sqrt{2}a}\dot{t} + r^2\left[1 - \left(\frac{r}{2a}\right)^2\right]\dot{\phi}$$

7. Free massless particle goes through the local inertial frame $\{e_a\}$ with 4-velocity $(u^{(0)}, u^{(1)}, 0, 0)$. Write down the geodesic equations for $(\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$ in terms of $u^{(0)}, u^{(1)}$.

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$$\left.\begin{aligned} k_0 &= -\dot{t}^2 - \frac{r^2}{\sqrt{2}a}\dot{\phi} \\ k_2 &= -\frac{r^2}{\sqrt{2}a}\dot{t} + r^2\left[1 - \left(\frac{r}{2a}\right)^2\right]\dot{\phi} \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{t} &= -k_0 \frac{1-\left(\frac{r}{2a}\right)^2}{1+\left(\frac{r}{2a}\right)^2} - \frac{k_2}{\sqrt{2}a\left(1+\left(\frac{r}{2a}\right)^2\right)} \\ \dot{\phi} &= \frac{k_2 - \frac{r^2}{\sqrt{2}a}k_0}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \end{aligned}$$

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$$\Rightarrow \quad \dot{t} = -k_0 \frac{1-\left(\frac{r}{2a}\right)^2}{1+\left(\frac{r}{2a}\right)^2} - \frac{k_2}{\sqrt{2}a\left(1+\left(\frac{r}{2a}\right)^2\right)}$$

$$\dot{\phi} = \frac{k_2 - \frac{r^2}{\sqrt{2}a}k_0}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)}$$

$$\dot{t} = u^{(0)} \frac{1-\left(\frac{r}{2a}\right)^2}{1+\left(\frac{r}{2a}\right)^2} + \frac{r_0^2 u^{(0)}}{\sqrt{2}a\sqrt{2}a\left(1+\left(\frac{r}{2a}\right)^2\right)}$$

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$$\dot{t} = u^{(0)} \frac{1-\left(\frac{r}{2a}\right)^2}{1+\left(\frac{r}{2a}\right)^2} + \frac{r_0^2 u^{(0)}}{\sqrt{2}a\sqrt{2}a\left(1+\left(\frac{r}{2a}\right)^2\right)} = \frac{u^{(0)}}{1+\left(\frac{r}{2a}\right)^2}\left[1+\frac{r_0^2}{2a^2}-\left(\frac{r}{2a}\right)^2\right]$$

7. Free massless particle goes through the local inertial frame $\{e_a\}$ with 4-velocity $(u^{(0)}, u^{(1)}, 0, 0)$. Write down the geodesic equations for $(\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$ in terms of $u^{(0)}, u^{(1)}$.

We have: $k_0 = -u^{(0)}$ $\quad k_2 = -\frac{r_0^2}{\sqrt{2}a} u^{(0)}$ $\quad k_3 = 0$

$$\dot{t} = -k_0 \frac{1-\left(\frac{r}{2a}\right)^2}{1+\left(\frac{r}{2a}\right)^2} - \frac{k_2}{\sqrt{2}a\left(1+\left(\frac{r}{2a}\right)^2\right)}$$

$$\dot{\phi} = \frac{k_2 - \frac{r^2}{\sqrt{2}a} k_0}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)}$$

$$\dot{t} = \frac{u^{(0)}}{1+\left(\frac{r}{2a}\right)^2}\left[1 + \frac{r_0^2}{2a^2} - \left(\frac{r}{2a}\right)^2\right]$$

$$\dot{\phi} = \frac{1}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)}\left(-\frac{r_0^2}{\sqrt{2}a} u^{(0)} - \frac{r^2}{\sqrt{2}a}(-u^{(0)})\right)$$

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$$\dot{\phi} = \frac{k_2 - \frac{r^2}{\sqrt{2}a} k_0}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)}$$

$$\dot{t} = \frac{u^{(0)}}{1+\left(\frac{r}{2a}\right)^2}\left[1+\frac{r_0^2}{2a^2}-\left(\frac{r}{2a}\right)^2\right]$$

$$\dot{\phi} = \frac{1}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)}\left(-\frac{r_0^2}{\sqrt{2}a}u^{(0)} - \frac{r^2}{\sqrt{2}a}(-u^{(0)})\right) = \frac{u^{(0)}}{1+\left(\frac{r}{2a}\right)^2}\left(1-\left(\frac{r_0}{r}\right)^2\right)$$

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$$\dot{\phi} = \frac{u^{(0)}}{1+\left(\frac{r}{2a}\right)^2}\left(1 - \left(\frac{r_0}{r}\right)^2\right)$$

$$(\dot{r})^2 = (k - k_3^2)\left(1+\left(\frac{r}{2a}\right)^2\right) - \frac{k_2^2}{r^2} + \sqrt{2}\,\frac{k_0 k_2}{a} + k_0^2\left(1-\left(\frac{r}{2a}\right)^2\right)$$

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$$\dot{\phi} = \frac{u^{(0)}}{1+\left(\frac{r}{2a}\right)^2}\left(1 - \left(\frac{r_0}{r}\right)^2\right)$$

massless $\quad = 0$

$$(\dot{r})^2 = (\cancel{k} - \cancel{k_3^2})\left(1+\left(\frac{r}{2a}\right)^2\right) - \frac{k_2^2}{r^2} + \sqrt{2}\frac{k_0 k_2}{a} + k_0^2\left(1-\left(\frac{r}{2a}\right)^2\right)$$

$$= -\frac{r_0^4 (u^{(0)})^2}{2a^2}\frac{1}{r^2} + \frac{\sqrt{2}}{a} u^{(0)} \frac{r_0^2 u^{(0)}}{\sqrt{2}\, a} + (u_0^{(0)})^2\left(1-\left(\frac{r}{2a}\right)^2\right)$$

$$= (u^{(0)})^2\left[-\frac{r_0^4}{2a^2}\frac{1}{r^2} - \left(\frac{r}{2a}\right)^2 + \frac{r_0^2}{a^2} + 1\right]$$

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$$\dot{\phi} = \frac{u^{(0)}}{1+\left(\frac{r}{2a}\right)^2}\left(1 - \left(\frac{r_0}{r}\right)^2\right)$$

$$\dot{z} = 0$$

$$(\dot{r})^2 = (u^{(0)})^2\left[-\frac{r_0^4}{2a^2}\frac{1}{r^2} - \left(\frac{r}{2a}\right)^2 + \frac{r_0^2}{a^2} + 1\right]$$

8. Compute *all* the null vectors at a point with coordinate r (i.e. compute the lightcone). Give expressions for both $u^{(a)}$ and u^{μ} (Hint: You will need a 3-parameter family of vectors, start from $u^{(a)}$ which is easier).

We will work in the local inertial frame $\{e_a\}$

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Null vectors: $u = u^{(a)} e_a$

$$-(u^{(0)})^2 + (u^{(1)})^2 + (u^{(2)})^2 + (u^{(3)})^2 = 0$$

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Null vectors: $u = u^{(a)} e_a$

$$-(u^{(0)})^2 + (u^{(1)})^2 + (u^{(2)})^2 + (u^{(3)})^2 = 0$$

Solution:

$$u^{(0)} = \beta \qquad \beta \in \mathbb{R}$$
$$u^{(1)} = \beta \sin\alpha \cos\gamma \qquad 0 \leq \alpha \leq \pi$$
$$u^{(2)} = \beta \sin\alpha \sin\gamma \qquad 0 \leq \gamma < 2\pi$$
$$u^{(3)} = \beta \cos\alpha$$

remember from spherical coordinates!

8. Compute *all* the null vectors at a point with coordinate r (i.e. compute the lightcone). Give expressions for both $u^{(a)}$ and u^{μ} (Hint: You will need a 3-parameter family of vectors, start from $u^{(a)}$ which is easier).

$$u^t = u^{(0)} - \frac{r}{\sqrt{2}a} \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \quad u^{(2)}$$

$$u^{(0)} = \beta$$

$$u^{(1)} = \beta \sin\alpha \cos\gamma$$

$$u^{(2)} = \beta \sin\alpha \sin\gamma$$

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$$u^t = u^{(0)} - \frac{r}{\sqrt{2}a} \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \qquad u^{(2)} = \beta\left\{1 - \frac{r}{\sqrt{2}a}\frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \sin\alpha \sin\gamma\right\}$$

$$u^{(0)} = \beta$$
$$u^{(1)} = \beta \sin\alpha \cos\gamma$$
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$$u^t = u^{(0)} - \frac{r}{\sqrt{2}a} \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} u^{(2)} = \beta\left\{1 - \frac{r}{\sqrt{2}a} \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \sin\alpha \sin\gamma\right\}$$

$$u^r = \sqrt{1+\left(\frac{r}{2a}\right)^2}\, u^{(1)}$$

$$u^{(0)} = \beta$$

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$$u^r = \sqrt{1+\left(\frac{r}{2a}\right)^2}\, u^{(1)} = \beta\sqrt{1+\left(\frac{r}{2a}\right)}\sin\alpha\cos\gamma$$

$$u^{\varphi} = \frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}} u^{(2)} = \beta\frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}\sin\alpha\sin\gamma$$

$$u^{(0)} = \beta$$

$$u^{(1)} = \beta\sin\alpha\cos\gamma$$

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$$u^r = \sqrt{1+\left(\frac{r}{2a}\right)^2}\, u^{(1)} = \beta\sqrt{1+\left(\frac{r}{2a}\right)}\,\sin\alpha\cos\gamma$$

$$u^\varphi = \frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}\, u^{(2)} = \beta\,\frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}\sin\alpha\sin\gamma$$

$$u^z = \beta\cos\alpha$$

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$$u^z = \beta\cos\alpha$$

- $\beta = 0$ the tip of the lightcone
- fixed $\beta \neq 0$: a 2-sphere

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$$u^r = \sqrt{1+\left(\frac{r}{2a}\right)^2}\,u^{(1)} = \beta\sqrt{1+\left(\frac{r}{2a}\right)}\,\sin\alpha\cos\gamma$$

$$u^\varphi = \frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}u^{(2)} = \beta\frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}\sin\alpha\sin\gamma$$

$$u^z = \beta\cos\alpha$$

- $\beta = 0$ the tip of the lightcone
- fixed $\beta \neq 0$: a 2-sphere
- (α, γ) fixed : a ray of the lightcone

Easy to calculate numerically:

$u^{(a)}$ components

```
ue[0] = β; ue[1] = β Sin[α] Cos[γ] ; ue[2] = β Sin[α] Sin[γ]; ue[3] = β Cos[α];
```

$$u[\beta\_, \gamma\_, \alpha\_] = \left\{ ue[0] - \frac{r}{\sqrt{2}\, a \sqrt{1 + \left(\frac{r}{2a}\right)^2}}\, ue[2],\ \sqrt{1 + \left(\frac{r}{2a}\right)^2}\, ue[1],\ \frac{1}{r\sqrt{1 + \left(\frac{r}{2a}\right)^2}}\, ue[2],\ ue[3] \right\};$$

```
(*Cartesian components: {u^x,u^y,u^t} for 3D plots: the tip of the lightcone is placed at (x,y,0) *)
ux[β_, γ_, α_] = {
    r Cos[ϕ] + r Cos[ϕ] u[β, γ, α][[2]] - r Sin[ϕ] u[β, γ, α][[3]],
    r Sin[ϕ] + r Sin[ϕ] u[β, γ, α][[2]] + r Cos[ϕ] u[β, γ, α][[3]],
    u[β, γ, α][[1]]
    };
```

u^μ components as function of (α, β, γ) span all lightcone

Easy to calculate numerically:

place tip of cone at $x = r\cos\phi$, $y = r\sin\phi$

```
ue[0] = β; ue[1] = β Sin[α] Cos[γ] ; ue[2] = β Sin[α] Sin[γ]; ue[3] = β Cos[α];
```

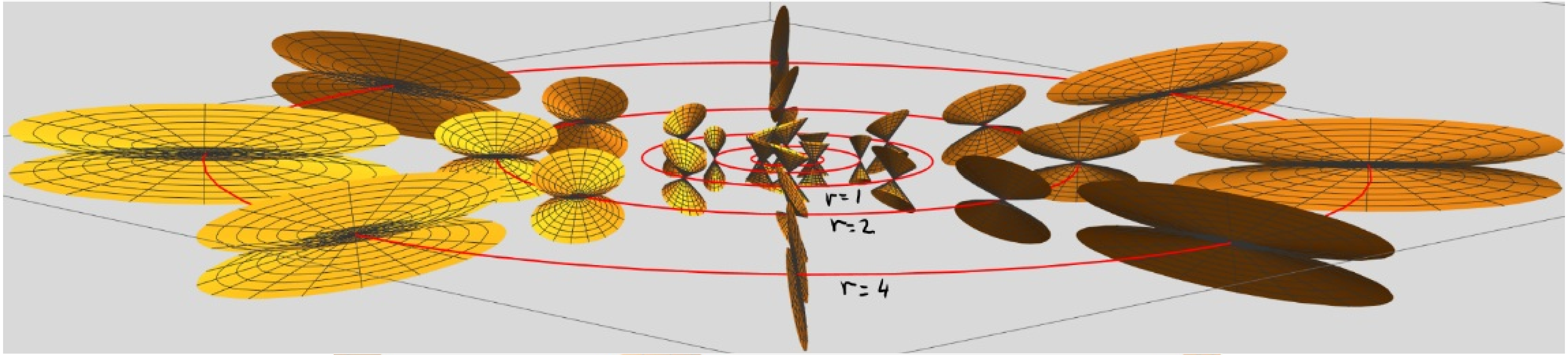
$$u[\beta\_, \gamma\_, \alpha\_] = \left\{ ue[0] - \frac{r}{\sqrt{2}\, a \sqrt{1+\left(\frac{r}{2a}\right)^2}}\, ue[2],\ \sqrt{1+\left(\frac{r}{2a}\right)^2}\, ue[1],\ \frac{1}{r\sqrt{1+\left(\frac{r}{2a}\right)^2}}\, ue[2],\ ue[3] \right\};$$

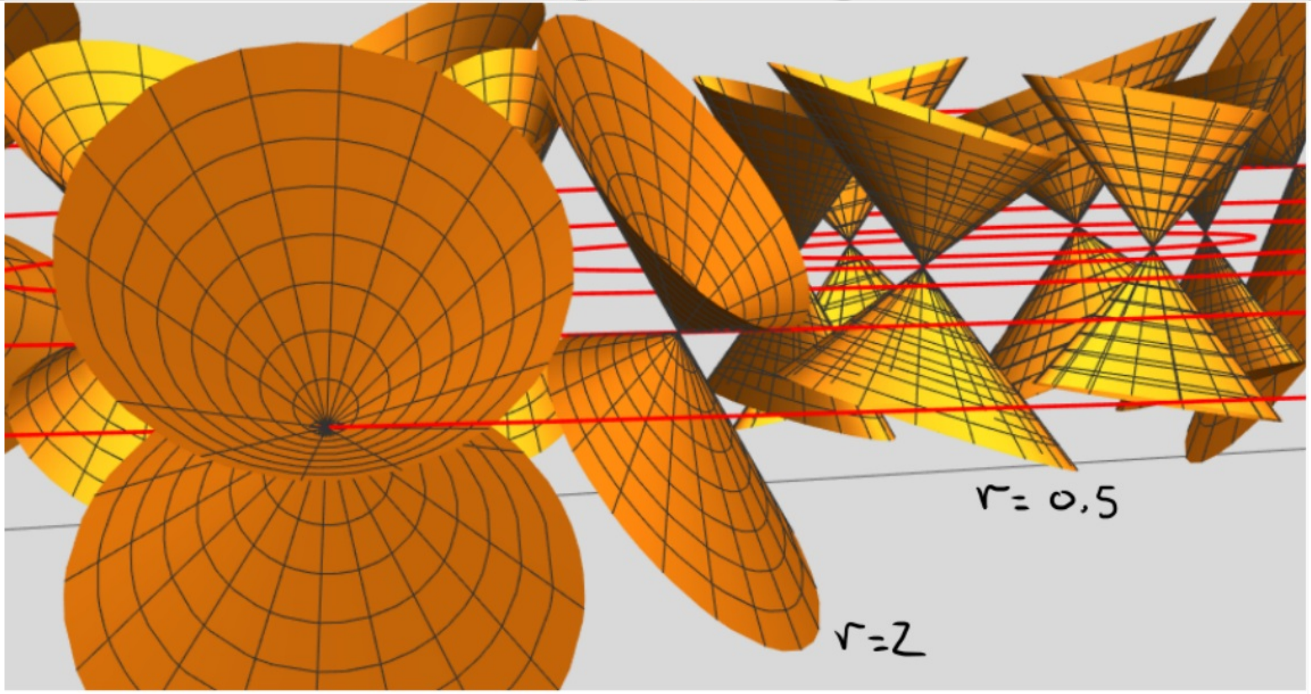
```
(*Cartesian components: {u^x,u^y,u^t} for 3D plots: the tip of the lightcone is placed at (x,y,0) *)
ux[β_, γ_, α_] = {
   r Cos[ϕ] + r Cos[ϕ] u[β, γ, α][[2]] - r Sin[ϕ] u[β, γ, α][[3]],
   r Sin[ϕ] + r Sin[ϕ] u[β, γ, α][[2]] + r Cos[ϕ] u[β, γ, α][[3]],
   u[β, γ, α][[1]]
  };
```

plot in Cartesian axes projecting out z

$$u^x = \cos\phi\, u^r - r\sin\phi\, u^\phi$$

$$u^y = \sin\phi\, u^r + r\cos\phi\, u^\phi$$





Lightcones tip at r ≳ 2

9. Compute the Christoffel symbols of the Levi-Civita connection of the metric. The nonzero components of the inverse metric are

$$g^{tt} = -\frac{1-\left(\frac{r}{2a}\right)^2}{1+\left(\frac{r}{2a}\right)^2}, \quad g^{rr} = 1+\left(\frac{r}{2a}\right)^2, \quad g^{zz} = 1, \quad g^{t\phi} = -\frac{1}{\sqrt{2}a\left(1+\left(\frac{r}{2a}\right)^2\right)}, \tag{3}$$

$$g^{\phi\phi} = \frac{1}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)}, \quad g^{t\phi} = -\frac{1}{\sqrt{2}a\left(1+\left(\frac{r}{2a}\right)^2\right)}. \tag{4}$$

Calculate

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right)$$

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Calculate

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right)$$

Start from:

$$\Gamma_{\sigma\nu\rho} = \frac{1}{2} \left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g\right)$$

9. Compute the Christoffel symbols of the Levi-Civita connection of the metric. The nonzero components of the inverse metric are

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Calculate

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right)$$

Notice that only ∂_r - derivatives can be non zero

Start from:

$$\Gamma_{\sigma\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right)$$

$$\Gamma = \frac{1}{2}(\partial g + \partial g - \partial g) =$$
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$$\Gamma = \frac{1}{2}(\partial g + \partial g - \partial g) =$$
$$\Gamma = \frac{1}{2}(\partial g + \partial g - \partial g) =$$

$$\Gamma_{t} = \frac{1}{2}\left(\partial g_{t} + \partial g_{t} - \partial_{t} g \right) =$$

$$\Gamma_{t} = \frac{1}{2}\left(\partial g_{t} + \partial g_{t} - \partial_{t} g \right) =$$

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$$\Gamma_{t} = \frac{1}{2}\left(\partial g_{t} + \partial g_{t} - \partial_{t} g \right) =$$

$$\Gamma_{t\,tt} = \frac{1}{2}\left(\partial_t g_{tt} + \partial_t g_{tt} - \partial_t g_{tt} \right) =$$

$$\Gamma_{t\,tr} = \frac{1}{2}\left(\partial_t g_{rt} + \partial_r g_{tt} - \partial_t g_{tr} \right) =$$

$$\Gamma_{t\,t\phi} = \frac{1}{2}\left(\partial_t g_{\phi t} + \partial_\phi g_{tt} - \partial_t g_{t\phi} \right) =$$

$$\Gamma_{t\,tz} = \frac{1}{2}\left(\partial_t g_{zt} + \partial_z g_{tt} - \partial_t g_{tz} \right) =$$

$$\Gamma_{t\,rr} = \frac{1}{2}\left(\partial_r g_{rt} + \partial_r g_{rt} - \partial_t g_{rr} \right) =$$

$$\Gamma_{t\,r\phi} = \frac{1}{2}\left(\partial_r g_{\phi t} + \partial_\phi g_{rt} - \partial_t g_{r\phi} \right) =$$

$$\Gamma_{t\,rz} = \frac{1}{2}\left(\partial_r g_{zt} + \partial_z g_{rt} - \partial_t g_{rz} \right) =$$

$$\Gamma_{t\,\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi t} + \partial_\phi g_{\phi t} - \partial_t g_{\phi\phi} \right) =$$

$$\Gamma_{t\,\phi z} = \frac{1}{2}\left(\partial_\phi g_{zt} + \partial_z g_{\phi t} - \partial_t g_{\phi z} \right) =$$

$$\Gamma_{t\,zz} = \frac{1}{2}\left(\partial_z g_{zt} + \partial_z g_{zt} - \partial_t g_{zz} \right) =$$

$$\Gamma_{t\,tt} = \frac{1}{2}\left(\partial_t g_{tt} + \partial_t g_{tt} - \partial_t g_{tt}\right) =$$

$$\Gamma_{t\,tr} = \frac{1}{2}\left(\partial_t g_{rt} + \partial_r g_{tt} - \partial_t g_{tr}\right) =$$

$$\Gamma_{t\,t\phi} = \frac{1}{2}\left(\partial_t g_{\phi t} + \partial_\phi g_{tt} - \partial_t g_{t\phi}\right) =$$

$$\Gamma_{t\,tz} = \frac{1}{2}\left(\partial_t g_{zt} + \partial_z g_{tt} - \partial_t g_{tz}\right) =$$

$$\Gamma_{t\,rr} = \frac{1}{2}\left(\partial_r g_{rt} + \partial_r g_{rt} - \partial_t g_{rr}\right) =$$

$$\Gamma_{t\,r\phi} = \frac{1}{2}\left(\partial_r g_{\phi t} + \partial_\phi g_{rt} - \partial_t g_{r\phi}\right) =$$

$$\Gamma_{t\,rz} = \frac{1}{2}\left(\partial_r g_{zt} + \partial_z g_{rt} - \partial_t g_{rz}\right) =$$

$$\Gamma_{t\,\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi t} + \partial_\phi g_{\phi t} - \partial_t g_{\phi\phi}\right) =$$

$$\Gamma_{t\,\phi z} = \frac{1}{2}\left(\partial_\phi g_{zt} + \partial_z g_{\phi t} - \partial_t g_{\phi z}\right) =$$

$$\Gamma_{t\,zz} = \frac{1}{2}\left(\partial_z g_{zt} + \partial_z g_{zt} - \partial_t g_{zz}\right) =$$

Look only at ∂_r

$$\Gamma^t_{tt} = \frac{1}{2}\left(\partial_t g_{tt} + \partial_t g_{tt} - \partial_t g_{tt} \right) = 0$$

$$\Gamma^t_{tr} = \frac{1}{2}\left(\partial_t g_{rt} + \partial_r g_{tt} - \partial_t g_{tr} \right) = 0$$

$$\Gamma^t_{t\phi} = \frac{1}{2}\left(\partial_t g_{\phi t} + \partial_\phi g_{tt} - \partial_t g_{t\phi} \right) = 0$$

$$\Gamma^t_{tz} = \frac{1}{2}\left(\partial_t g_{zt} + \partial_z g_{tt} - \partial_t g_{tz} \right) = 0$$

$$\Gamma^t_{rr} = \frac{1}{2}\left(\partial_r g_{rt} + \partial_r g_{rt} - \partial_t g_{rr} \right) = 0$$

$$\Gamma^t_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi t} + \partial_\phi g_{rt} - \partial_t g_{r\phi} \right) = \frac{1}{2} \partial_r \left(- \frac{r^2}{\sqrt{2}a} \right) = - \frac{r}{\sqrt{2}a}$$

$$\Gamma^t_{rz} = \frac{1}{2}\left(\partial_r g_{zt} + \partial_z g_{rt} - \partial_t g_{rz} \right) = 0$$

$$\Gamma^t_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi t} + \partial_\phi g_{\phi t} - \partial_t g_{\phi\phi} \right) = 0$$

$$\Gamma^t_{\phi z} = \frac{1}{2}\left(\partial_\phi g_{zt} + \partial_z g_{\phi t} - \partial_t g_{\phi z} \right) = 0$$

$$\Gamma^t_{zz} = \frac{1}{2}\left(\partial_z g_{zt} + \partial_z g_{zt} - \partial_t g_{zz} \right) = 0$$

Look only at ∂_r

$$\Gamma_{r} = \frac{1}{2}\left(\partial g_{r} + \partial g_{r} - \partial_{r} g \right) =$$

$$\Gamma_{r} = \frac{1}{2}\left(\partial g_{r} + \partial g_{r} - \partial_{r} g \right) =$$

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$$\Gamma_{r} = \frac{1}{2}\left(\partial g_{r} + \partial g_{r} - \partial_{r} g \right) =$$

$$\Gamma_{r\,tt} = \frac{1}{2}\left(\partial_t g_{t\,r} + \partial_t g_{t\,r} - \partial_r g_{tt}\right) =$$

$$\Gamma_{r\,tr} = \frac{1}{2}\left(\partial_t g_{r\,r} + \partial_r g_{t\,r} - \partial_r g_{tr}\right) =$$

$$\Gamma_{r\,t\phi} = \frac{1}{2}\left(\partial_t g_{\phi\,r} + \partial_\phi g_{t\,r} - \partial_r g_{t\phi}\right) =$$

$$\Gamma_{r\,tz} = \frac{1}{2}\left(\partial_t g_{z\,r} + \partial_z g_{t\,r} - \partial_r g_{tz}\right) =$$

$$\Gamma_{r\,rr} = \frac{1}{2}\left(\partial_r g_{r\,r} + \partial_r g_{r\,r} - \partial_r g_{rr}\right) =$$

$$\Gamma_{r\,r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\,r} + \partial_\phi g_{r\,r} - \partial_r g_{r\phi}\right) =$$

$$\Gamma_{r\,rz} = \frac{1}{2}\left(\partial_r g_{z\,r} + \partial_z g_{r\,r} - \partial_r g_{rz}\right) =$$

$$\Gamma_{r\,\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\,r} + \partial_\phi g_{\phi\,r} - \partial_r g_{\phi\phi}\right) =$$

$$\Gamma_{r\,\phi z} = \frac{1}{2}\left(\partial_\phi g_{z\,r} + \partial_z g_{\phi\,r} - \partial_r g_{\phi z}\right) =$$

$$\Gamma_{r\,zz} = \frac{1}{2}\left(\partial_z g_{z\,r} + \partial_z g_{z\,r} - \partial_r g_{zz}\right) =$$

$$\Gamma_{rtt} = \frac{1}{2}\left(\partial_t g_{tr} + \partial_t g_{tr} - \partial_r g_{tt} \right) =$$

$$\Gamma_{rtr} = \frac{1}{2}\left(\partial_t g_{rr} + \partial_r g_{tr} - \partial_r g_{tr} \right) =$$

$$\Gamma_{rt\phi} = \frac{1}{2}\left(\partial_t g_{\phi r} + \partial_\phi g_{tr} - \partial_r g_{t\phi} \right) =$$

$$\Gamma_{rtz} = \frac{1}{2}\left(\partial_t g_{zr} + \partial_z g_{tr} - \partial_r g_{tz} \right) =$$

$$\Gamma_{rrr} = \frac{1}{2}\left(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr} \right) =$$

$$\Gamma_{rr\phi} = \frac{1}{2}\left(\partial_r g_{\phi r} + \partial_\phi g_{rr} - \partial_r g_{r\phi} \right) =$$

$$\Gamma_{rrz} = \frac{1}{2}\left(\partial_r g_{zr} + \partial_z g_{rr} - \partial_r g_{rz} \right) =$$

$$\Gamma_{r\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{\phi r} - \partial_r g_{\phi\phi} \right) =$$

$$\Gamma_{r\phi z} = \frac{1}{2}\left(\partial_\phi g_{zr} + \partial_z g_{\phi r} - \partial_r g_{\phi z} \right) =$$

$$\Gamma_{rzz} = \frac{1}{2}\left(\partial_z g_{zr} + \partial_z g_{zr} - \partial_r g_{zz} \right) =$$

$$\Gamma_{rtt} = \frac{1}{2}\left(\partial_t g_{tr} + \partial_t g_{tr} - \partial_r g_{tt}\right) = 0$$

$$\Gamma_{rtr} = \frac{1}{2}\left(\partial_t g_{rr} + \partial_r g_{tr} - \partial_r g_{tr}\right) = 0$$

$$\Gamma_{rt\phi} = \frac{1}{2}\left(\partial_t g_{\phi r} + \partial_\phi g_{tr} - \partial_r g_{t\phi}\right) = -\frac{1}{2}\partial_r\left(-\frac{r^2}{\sqrt{2}a}\right) = \frac{r}{\sqrt{2}a}$$

$$\Gamma_{rtz} = \frac{1}{2}\left(\partial_t g_{zr} + \partial_z g_{tr} - \partial_r g_{tz}\right) = 0$$

$$\Gamma_{rrr} = \frac{1}{2}\left(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}\right) = \frac{1}{2}\partial_r\left(\frac{1}{1+\left(\frac{r}{2a}\right)^2}\right) = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)^2}$$

$$\Gamma_{rr\phi} = \frac{1}{2}\left(\partial_r g_{\phi r} + \partial_\phi g_{rr} - \partial_r g_{r\phi}\right) = 0$$

$$\Gamma_{rrz} = \frac{1}{2}\left(\partial_r g_{zr} + \partial_z g_{rr} - \partial_r g_{rz}\right) = 0$$

$$\Gamma_{r\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{\phi r} - \partial_r g_{\phi\phi}\right) = -\frac{1}{2}\partial_r\left[r^2\left(1-\left(\frac{r}{2a}\right)^2\right)\right] = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{r\phi z} = \frac{1}{2}\left(\partial_\phi g_{zr} + \partial_z g_{\phi r} - \partial_r g_{\phi z}\right) = 0$$

$$\Gamma_{rzz} = \frac{1}{2}\left(\partial_z g_{zr} + \partial_z g_{zr} - \partial_r g_{zz}\right) = 0$$

$$\Gamma_{\phi} = \frac{1}{2}\left(\partial g_{\phi} + \partial g_{\phi} - \partial_{\phi} g\right) =$$

$$\Gamma_{\phi} = \frac{1}{2}\left(\partial g_{\phi} + \partial g_{\phi} - \partial g_{\phi}\right) =$$

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$$\Gamma_{\phi} = \frac{1}{2}\left(\partial g_{\phi} + \partial g_{\phi} - \partial g_{\phi}\right) =$$

$$\Gamma_{\phi tt} = \frac{1}{2}\left(\partial_t g_{t\phi} + \partial_t g_{t\phi} - \partial_\phi g_{tt}\right) =$$

$$\Gamma_{\phi tr} = \frac{1}{2}\left(\partial_t g_{r\phi} + \partial_r g_{t\phi} - \partial_\phi g_{tr}\right) =$$

$$\Gamma_{\phi t\phi} = \frac{1}{2}\left(\partial_t g_{\phi\phi} + \partial_\phi g_{t\phi} - \partial_\phi g_{t\phi}\right) =$$

$$\Gamma_{\phi tz} = \frac{1}{2}\left(\partial_t g_{z\phi} + \partial_z g_{t\phi} - \partial_\phi g_{tz}\right) =$$

$$\Gamma_{\phi rr} = \frac{1}{2}\left(\partial_r g_{r\phi} + \partial_r g_{r\phi} - \partial_\phi g_{rr}\right) =$$

$$\Gamma_{\phi r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{r\phi} - \partial_\phi g_{r\phi}\right) =$$

$$\Gamma_{\phi rz} = \frac{1}{2}\left(\partial_r g_{z\phi} + \partial_z g_{r\phi} - \partial_\phi g_{rz}\right) =$$

$$\Gamma_{\phi\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\phi} + \partial_\phi g_{\phi\phi} - \partial_\phi g_{\phi\phi}\right) =$$

$$\Gamma_{\phi\phi z} = \frac{1}{2}\left(\partial_\phi g_{z\phi} + \partial_z g_{\phi\phi} - \partial_\phi g_{\phi z}\right) =$$

$$\Gamma_{\phi zz} = \frac{1}{2}\left(\partial_z g_{z\phi} + \partial_z g_{z\phi} - \partial_\phi g_{zz}\right) =$$

$$\Gamma_{\phi tt} = \frac{1}{2}\left(\partial_t g_{t\phi} + \partial_t g_{t\phi} - \partial_\phi g_{tt}\right) =$$

$$\Gamma_{\phi tr} = \frac{1}{2}\left(\partial_t g_{r\phi} + \partial_r g_{t\phi} - \partial_\phi g_{tr}\right) =$$

$$\Gamma_{\phi t\phi} = \frac{1}{2}\left(\partial_t g_{\phi\phi} + \partial_\phi g_{t\phi} - \partial_\phi g_{t\phi}\right) =$$

$$\Gamma_{\phi tz} = \frac{1}{2}\left(\partial_t g_{z\phi} + \partial_z g_{t\phi} - \partial_\phi g_{tz}\right) =$$

$$\Gamma_{\phi rr} = \frac{1}{2}\left(\partial_r g_{r\phi} + \partial_r g_{r\phi} - \partial_\phi g_{rr}\right) =$$

$$\Gamma_{\phi r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{r\phi} - \partial_\phi g_{r\phi}\right) =$$

$$\Gamma_{\phi rz} = \frac{1}{2}\left(\partial_r g_{z\phi} + \partial_z g_{r\phi} - \partial_\phi g_{rz}\right) =$$

$$\Gamma_{\phi\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\phi} + \partial_\phi g_{\phi\phi} - \partial_\phi g_{\phi\phi}\right) =$$

$$\Gamma_{\phi\phi z} = \frac{1}{2}\left(\partial_\phi g_{z\phi} + \partial_z g_{\phi\phi} - \partial_\phi g_{\phi z}\right) =$$

$$\Gamma_{\phi zz} = \frac{1}{2}\left(\partial_z g_{z\phi} + \partial_z g_{z\phi} - \partial_\phi g_{zz}\right) =$$

$$\Gamma_{\phi tt} = \frac{1}{2}\left(\partial_t g_{t\phi} + \partial_t g_{t\phi} - \partial_\phi g_{tt} \right) = 0$$

$$\Gamma_{\phi tr} = \frac{1}{2}\left(\partial_t g_{r\phi} + \partial_r g_{t\phi} - \partial_\phi g_{tr} \right) = \frac{1}{2}\partial_r\left(-\frac{r^2}{\sqrt{2}a}\right) = -\frac{r}{\sqrt{2}a}$$

$$\Gamma_{\phi t\phi} = \frac{1}{2}\left(\partial_t g_{\phi\phi} + \partial_\phi g_{t\phi} - \partial_\phi g_{t\phi} \right) = 0$$

$$\Gamma_{\phi tz} = \frac{1}{2}\left(\partial_t g_{z\phi} + \partial_z g_{t\phi} - \partial_\phi g_{tz} \right) = 0$$

$$\Gamma_{\phi rr} = \frac{1}{2}\left(\partial_r g_{r\phi} + \partial_r g_{r\phi} - \partial_\phi g_{rr} \right) = 0$$

$$\Gamma_{\phi r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{r\phi} - \partial_\phi g_{r\phi} \right) = \frac{1}{2}\partial_r\left[r^2\left(1-\left(\frac{r}{2a}\right)^2\right)\right] = -r\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

$$\Gamma_{\phi rz} = \frac{1}{2}\left(\partial_r g_{z\phi} + \partial_z g_{r\phi} - \partial_\phi g_{rz} \right) = 0$$

$$\Gamma_{\phi\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\phi} + \partial_\phi g_{\phi\phi} - \partial_\phi g_{\phi\phi} \right) = 0$$

$$\Gamma_{\phi\phi z} = \frac{1}{2}\left(\partial_\phi g_{z\phi} + \partial_z g_{\phi\phi} - \partial_\phi g_{\phi z} \right) = 0$$

$$\Gamma_{\phi zz} = \frac{1}{2}\left(\partial_z g_{z\phi} + \partial_z g_{z\phi} - \partial_\phi g_{zz} \right) = 0$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a}$$

$$\Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a}$$

$$\Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)}$$

$$\Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a}$$

$$\Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma^{r}{}_{rr} = g^{rr}\,\Gamma_{rrr} = \left[1+\left(\frac{r}{2a}\right)^2\right)\frac{-r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)^2} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)}$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\Gamma_{\sigma\nu\rho}$

$$\Gamma^{r}{}_{\phi\phi} = g^{rr}\Gamma_{r\phi\phi} = \left[1+\left(\frac{r}{2a}\right)^2\right] r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma^{t}{}_{r\phi} = g^{tt}\,\Gamma_{tr\phi} + g^{\phi\phi}\,\Gamma_{\phi r\phi}$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma^{t}{}_{r\phi} = g^{tt}\,\Gamma_{tr\phi} + g^{\phi\phi}\,\Gamma_{\phi r\phi}$$

$$= \frac{1-\left(\frac{r}{2a}\right)^2}{1+\left(\frac{r}{2a}\right)^2}\,\frac{r}{\sqrt{2}\,a} + \frac{1}{\sqrt{2}\,a\left(1+\left(\frac{r}{2a}\right)^2\right)}\,r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma^{t}{}_{r\phi} = g^{tt}\,\Gamma_{tr\phi} + g^{\phi\phi}\,\Gamma_{\phi r\phi}$$

$$= \frac{1-\left(\frac{r}{2a}\right)^2}{1+\left(\frac{r}{2a}\right)^2}\,\frac{r}{\sqrt{2}\,a} + \frac{1}{\sqrt{2}\,a\left(1+\left(\frac{r}{2a}\right)^2\right)}\,r\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

$$= \frac{r}{\sqrt{2}\,a}\left(\frac{r}{2a}\right)^2\frac{1}{1+\left(\frac{r}{2a}\right)^2}$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma^{\phi}{}_{r\phi} = g^{\phi t}\,\Gamma_{tr\phi} + g^{\phi\phi}\,\Gamma_{\phi r\phi}$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma^{\phi}{}_{r\phi} = g^{\phi t}\,\Gamma_{tr\phi} + g^{\phi\phi}\,\Gamma_{\phi r\phi}$$

$$= -\frac{1}{\sqrt{2}\,a\left(1+\left(\frac{r}{2a}\right)^2\right)}\left(-\frac{r}{\sqrt{2}\,a}\right) + \frac{1}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)}\left[-r\left(2\left(\frac{r}{2a}\right)^2 - 1\right)\right]$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

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Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma^{\phi}{}_{r\phi} = g^{\phi t}\,\Gamma_{tr\phi} + g^{\phi\phi}\,\Gamma_{\phi r\phi}$$

$$= -\frac{1}{\sqrt{2}\,a\left(1+\left(\frac{r}{2a}\right)^2\right)}\left(-\frac{r}{\sqrt{2}\,a}\right) + \frac{1}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)}\left[-r\left(2\left(\frac{r}{2a}\right)^2 - 1\right)\right]$$

$$= \frac{1}{1+\left(\frac{r}{2a}\right)^2}\left[\frac{r}{2a^2} - \frac{2}{r}\left(\frac{r}{2a}\right)^2 + \frac{1}{r}\right] = \frac{1}{r\left(1+\left(\frac{r}{2a}\right)^2\right)}$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma^{t}{}_{tr} = g^{t\phi}\,\Gamma_{\phi tr} = -\frac{1}{\sqrt{2}\,a\left(1+\left(\frac{r}{2a}\right)^2\right)}\left(-\frac{r}{\sqrt{2}\,a}\right)$$

$$= \frac{r}{2a^2\left(1+\left(\frac{r}{2a}\right)^2\right)}$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma^{\phi}{}_{tr} = g^{\phi\phi}\,\Gamma_{\phi tr}$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

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Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma^{\phi}{}_{tr} = g^{\phi\phi}\,\Gamma_{\phi tr} = \frac{1}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)}\left(-\frac{r}{\sqrt{2}\,a}\right)$$

$$= -\frac{1}{\sqrt{2}\,ar\left(1+\left(\frac{r}{2a}\right)^2\right)}$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

Compute $\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\sigma}\,\Gamma_{\sigma\nu\rho}$

$$\Gamma^{r}{}_{t\phi} = g^{rr}\,\Gamma_{rt\phi} = \left(1 + \left(\frac{r}{2a}\right)^2\right)\frac{r}{\sqrt{2}\,a}$$

$$\Gamma_{tr\phi} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rt\phi} = \frac{r}{\sqrt{2}\,a} \qquad \Gamma_{rrr} = -\frac{r}{4a^2\left(1+\left(\frac{r}{2a}\right)^2\right)} \qquad \Gamma_{r\phi\phi} = r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma_{\phi tr} = -\frac{r}{\sqrt{2}\,a} \qquad \Gamma_{\phi r\phi} = -r\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\Gamma^t_{rt} = \frac{r}{2a^2}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,, \quad \Gamma^t_{\phi r} = \frac{r^3}{4\sqrt{2}a^3}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,, \quad \Gamma^r_{rr} = -\frac{r}{4a^2}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,,$$

$$\Gamma^r_{\phi t} = \frac{r}{\sqrt{2}a}\left(1+\left(\frac{r}{2a}\right)^2\right)\,, \quad \Gamma^r_{\phi\phi} = r\left(1+\left(\frac{r}{2a}\right)^2\right)\left(2\left(\frac{r}{2a}\right)^2 - 1\right)\,,$$

$$\Gamma^\phi_{\phi r} = \frac{1}{r}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,, \quad \Gamma^\phi_{rt} = -\frac{1}{\sqrt{2}ar}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,.$$

10. Consider the massive particle moving on the trajectory with $t = 0$, $r = R$, $\phi = \omega\tau$, $z = 0$, where R, ω are constants. Determine when the 4-velocity of the particle is timelike, in which case we have a closed timelike curve (CTC).

$t = 0$

$r = R$

$\phi = \omega t$

$z = 0$

10. Consider the massive particle moving on the trajectory with $t = 0$, $r = R$, $\phi = \omega\tau$, $z = 0$, where R, ω are constants. Determine when the 4-velocity of the particle is timelike, in which case we have a closed timelike curve (CTC).

$$t = 0 \qquad u^t = \dot{t} = 0$$

$$r = R \qquad u^r = \dot{r} = 0$$

$$\phi = \omega t \quad \Rightarrow \quad u^\phi = \dot{\phi} = \omega$$

$$z = 0 \qquad u^z = \dot{z} = 0$$

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$$r = R \quad \Rightarrow \quad u^r = \dot{r} = 0$$

$$\phi = \omega t \qquad u^\phi = \dot{\phi} = \omega$$

$$z = 0 \qquad u^z = \dot{z} = 0$$

$$\Rightarrow \quad u^{(0)} - \frac{R}{\sqrt{2}a} \frac{1}{\sqrt{1+\left(\frac{R}{2a}\right)^2}} u^{(2)} = 0$$

$$\sqrt{1 + \left(\frac{R}{2a}\right)^2}\, u^{(1)} = 0$$

$$\frac{1}{R\sqrt{1+\left(\frac{R}{2a}\right)^2}}\, u^{(2)} = \omega$$

$$u^{(3)} = 0$$

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$$u^{(2)} = \omega R \sqrt{1 + \left(\frac{R}{2a}\right)^2}$$

$$u^{(0)} - \frac{R}{\sqrt{2}a} \frac{1}{\sqrt{1 + \left(\frac{R}{2a}\right)^2}} u^{(2)} = 0$$

$$\sqrt{1 + \left(\frac{R}{2a}\right)^2} \, u^{(1)} = 0$$

$$\frac{1}{R\sqrt{1 + \left(\frac{R}{2a}\right)^2}} u^{(2)} = \omega$$

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$$u^{(0)} = \frac{R}{\sqrt{2}a} \frac{1}{\sqrt{1+\left(\frac{R}{2a}\right)^2}} \omega R \sqrt{1+\left(\frac{R}{2a}\right)^2}$$

$$u^{(0)} - \frac{R}{\sqrt{2}a} \frac{1}{\sqrt{1+\left(\frac{R}{2a}\right)^2}} u^{(2)} = 0$$

$$\sqrt{1 + \left(\frac{R}{2a}\right)^2} u^{(1)} = 0$$

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$$u^{(2)} = \omega R \sqrt{1 + \left(\frac{R}{2a}\right)^2}$$

$$u^{(0)} = \frac{R}{\sqrt{2}a} \frac{1}{\sqrt{1+\left(\frac{R}{2a}\right)^2}} \omega R \sqrt{1+\left(\frac{R}{2a}\right)^2}$$

$$= \frac{R^2 \omega}{\sqrt{2}\, a}$$

$$u^{(0)} - \frac{R}{\sqrt{2}a} \frac{1}{\sqrt{1+\left(\frac{R}{2a}\right)^2}} u^{(2)} = 0$$

$$\sqrt{1 + \left(\frac{R}{2a}\right)^2}\, u^{(1)} = 0$$

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$$u^{(2)} = \omega R \sqrt{1 + \left(\frac{R}{2a}\right)^2}$$

$$u^{(0)} = \frac{\omega R^2}{\sqrt{2}\, a}$$

$$u^{(1)} = 0 \qquad u^{(3)} = 0$$

$$u^{(0)} - \frac{R}{\sqrt{2}\, a} \frac{1}{\sqrt{1 + \left(\frac{R}{2a}\right)^2}} u^{(2)} = 0$$

$$\sqrt{1 + \left(\frac{R}{2a}\right)^2}\, u^{(1)} = 0$$

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$$u^{(2)} = \omega R \sqrt{1 + \left(\frac{R}{2a}\right)^2}$$

$$u^{(0)} = \frac{\omega R^2}{\sqrt{2}\, a}$$

$$u^{(1)} = 0 \qquad u^{(3)} = 0$$

$$-(u^{(0)})^2 + (u^{(2)})^2 = -1$$

10. Consider the massive particle moving on the trajectory with $t = 0$, $r = R$, $\phi = \omega\tau$, $z = 0$, where R, ω are constants. Determine when the 4-velocity of the particle is timelike, in which case we have a closed timelike curve (CTC).

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$$u^{(0)} = \frac{\omega R^2}{\sqrt{2}\, a}$$

$$u^{(1)} = 0 \qquad u^{(3)} = 0$$

$$-(u^{(0)})^2 + (u^{(2)})^2 = -1 \Rightarrow -\frac{\omega^2 R^4}{2a^2} + \omega^2 R^2 \left(1 + \left(\frac{R}{2a}\right)^2\right) = -1$$

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$$u^{(0)} = \frac{\omega R^2}{\sqrt{2}\, a}$$

$$u^{(1)} = 0 \qquad u^{(3)} = 0$$

$$-(u^{(0)})^2 + (u^{(2)})^2 = -1 \Rightarrow -\frac{\omega^2 R^4}{2a^2} + \omega^2 R^2 \left(1 + \left(\frac{R}{2a}\right)^2\right) = -1 \Rightarrow$$

$$\omega^2 R^2 \left[1 - \left(\frac{R}{2a}\right)^2\right] = -1$$

10. Consider the massive particle moving on the trajectory with $t = 0$, $r = R$, $\phi = \omega\tau$, $z = 0$, where R, ω are constants. Determine when the 4-velocity of the particle is timelike, in which case we have a closed timelike curve (CTC).

$$u^{(2)} = \omega R \sqrt{1 + \left(\frac{R}{2a}\right)^2} \qquad \omega^2 = \frac{1}{R^2\left[\left(\frac{R}{2a}\right)^2 - 1\right]}$$

$$u^{(0)} = \frac{\omega R^2}{\sqrt{2}\, a}$$

$$u^{(1)} = 0 \qquad u^{(3)} = 0$$

$$-\left(u^{(0)}\right)^2 + \left(u^{(2)}\right)^2 = -1 \Rightarrow -\frac{\omega^2 R^4}{2a^2} + \omega^2 R^2\left(1 + \left(\frac{R}{2a}\right)^2\right) = -1 \Rightarrow$$

$$\omega^2 R^2\left[1 - \left(\frac{R}{2a}\right)^2\right] = -1$$

10. Consider the massive particle moving on the trajectory with $t = 0$, $r = R$, $\phi = \omega\tau$, $z = 0$, where R, ω are constants. Determine when the 4-velocity of the particle is timelike, in which case we have a closed timelike curve (CTC).

$$u^{(2)} = \omega R \sqrt{1 + \left(\frac{R}{2a}\right)^2}$$

$$u^{(0)} = \frac{\omega R^2}{\sqrt{2}\, a}$$

$$u^{(1)} = 0 \qquad u^{(3)} = 0$$

$$\omega^2 = \frac{1}{R^2\left[\left(\frac{R}{2a}\right)^2 - 1\right]} \qquad R > 2a$$

$\omega^2 > 0 \Leftrightarrow R > 2a$

must be outside Gödel's horizon

Then u is timelike and the curve a CTC!

10. Consider the massive particle moving on the trajectory with $t = 0$, $r = R$, $\phi = \omega\tau$, $z = 0$, where R, ω are constants. Determine when the 4-velocity of the particle is timelike, in which case we have a closed timelike curve (CTC).

$$u^{(2)} = \omega R \sqrt{1 + \left(\frac{R}{2a}\right)^2} \qquad \omega = \frac{1}{R\left[\left(\frac{R}{2a}\right)^2 - 1\right]^{1/2}} \qquad R > 2a$$

$$u^{(0)} = \frac{\omega R^2}{\sqrt{2}\, a}$$

$$u^{(1)} = 0 \qquad u^{(3)} = 0$$

Then

$$u^{(2)} = \frac{1}{R\left[\left(\frac{R}{2a}\right)^2 - 1\right]} R\left[1 + \left(\frac{R}{2a}\right)^2\right]^{1/2} = \sqrt{\frac{\left(\frac{R}{2a}\right)^2 + 1}{\left(\frac{R}{2a}\right)^2 - 1}}$$

10. Consider the massive particle moving on the trajectory with $t = 0$, $r = R$, $\phi = \omega\tau$, $z = 0$, where R, ω are constants. Determine when the 4-velocity of the particle is timelike, in which case we have a closed timelike curve (CTC).

$$u^{(2)} = \omega R \sqrt{1 + \left(\frac{R}{2a}\right)^2} \qquad \omega = \frac{1}{R\left[\left(\frac{R}{2a}\right)^2 - 1\right]^{1/2}} \qquad R > 2a$$

$$u^{(0)} = \frac{\omega R^2}{\sqrt{2}\, a}$$

$$u^{(1)} = 0 \qquad u^{(3)} = 0$$

Then

$$u^{(2)} = \frac{1}{R\left[\left(\frac{R}{2a}\right)^2 - 1\right]} R\left[1 + \left(\frac{R}{2a}\right)^2\right]^{1/2} = \sqrt{\frac{\left(\frac{R}{2a}\right)^2 + 1}{\left(\frac{R}{2a}\right)^2 - 1}}$$

$$u^{(0)} = \frac{R}{\sqrt{2}\, a}\left[\left(\frac{R}{2a}\right)^2 - 1\right]^{-1/2}$$

11. Compute the relation $\omega = \omega(R)$.

$$\omega = \frac{1}{R\left[\left(\frac{R}{2a}\right)^2 - 1\right]^{1/2}} \qquad R > 2a$$

12. Compute the 4-acceleration of the particle $a^\mu = \ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho$, where $\dot{x}^\mu = dx^\mu/d\tau$. Conclude that the particle is not falling freely.

We have $\ddot{x}^\mu = 0$ for all μ

only $\dot{\phi} = \omega$ noy zero

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$$a^\mu = \ddot{x}^\mu + \Gamma^\mu{}_{\rho\sigma}\,\dot{x}^\rho\,\dot{x}^\sigma$$

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$$a^\mu = \ddot{x}^\mu + \Gamma^\mu{}_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma$$

$$= 0 + \Gamma^\mu{}_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma$$

$=$

12. Compute the 4-acceleration of the particle $a^\mu = \ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho$, where $\dot{x}^\mu = dx^\mu/d\tau$. Conclude that the particle is not falling freely.

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$$a^\mu = \ddot{x}^\mu + \Gamma^\mu{}_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma$$

$$= 0 + \Gamma^\mu{}_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma$$

$$= \Gamma^\mu{}_{\phi\phi} \dot{\phi} \dot{\phi}$$

$=$

12. Compute the 4-acceleration of the particle $a^\mu = \ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho$, where $\dot{x}^\mu = dx^\mu/d\tau$. Conclude that the particle is not falling freely.

We have $\ddot{x}^\mu = 0$ for all μ

only $\dot{\phi} = \omega$ non zero

$$a^\mu = \ddot{x}^\mu + \Gamma^\mu{}_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma$$

$$= 0 + \Gamma^\mu{}_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma$$

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Only nonzero:

$$\Gamma^r{}_{\phi\phi} = r\left[1 + \left(\frac{r}{2a}\right)^2\right]\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

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$$= R\left[1+\left(\frac{R}{2a}\right)^2\right]\left[2\left(\frac{R}{2a}\right)^2-1\right]\frac{1}{R^2\left[\left(\frac{R}{2a}\right)^2-1\right]}$$

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$$= \left[1+\left(\frac{R}{2a}\right)^2\right]\left[2\left(\frac{R}{2a}\right)^2-1\right]\frac{1}{R\left[\left(\frac{R}{2a}\right)^2-1\right]}$$

$R \to 2a$, $a^r \to \infty$

13. The vectors with components in the coordinate basis below are KVFs:

$$\xi_1 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(\frac{r}{\sqrt{2}}\cos\phi, a\left(1+\left(\frac{r}{2a}\right)^2\right)\sin\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right)\cos\phi, 0\right) \tag{8}$$

$$\xi_4 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(\frac{r}{\sqrt{2}}\sin\phi, -a\left(1+\left(\frac{r}{2a}\right)^2\right)\cos\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right)\sin\phi, 0\right) \tag{9}$$

Compute the corresponding conserved quantities k_1 and k_4 along a geodesic with tangent vector $u^\mu = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$.

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Compute the corresponding conserved quantities k_1 and k_4 along a geodesic with tangent vector $u^\mu = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$.

$$K_1 = \xi_1 \cdot u$$

$$= g_{tt}\xi_1^{\,t}\dot{t} + g_{t\phi}\xi_1^{\,t}\dot{\phi} + g_{\phi t}\xi_1^{\,\phi}\dot{t} + g_{\phi\phi}\xi_1^{\,\phi}\dot{\phi} + g_{rr}\xi_1^{\,r}\dot{r} + g_{zz}\xi_1^{\,z}\dot{z}$$

$=0$

$$\sqrt{1+\left(\frac{r}{2a}\right)^2}\, k_1 = -1\,\frac{r}{\sqrt{2}}\cos\phi\,\dot{t} - \frac{r^2}{\sqrt{2}a}\frac{r}{\sqrt{2}}\cos\phi\,\dot{\phi} - \frac{r^2}{\sqrt{2}a}\frac{a}{r}\left(1+\left(\frac{r}{2a}\right)^2\right)\cos\phi\,\dot{t}$$

$$+ r^2\left(1-\left(\frac{r}{2a}\right)^2\right)\frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right)\cos\phi\,\dot{\phi}$$

$$+ \frac{1}{1+\left(\frac{r}{2a}\right)^2}\, a\left(1+\left(\frac{r}{2a}\right)^2\right)\sin\phi\,\dot{r}$$

$$k_1 = \xi_1 \cdot u$$

$$= g_{tt}\,\xi_1^{\,t}\,\dot{t} + g_{t\phi}\,\xi_1^{\,t}\,\dot{\phi} + g_{\phi t}\,\xi_1^{\,\phi}\,\dot{t} + g_{\phi\phi}\,\xi_1^{\,\phi}\,\dot{\phi} + g_{rr}\,\xi_1^{\,r}\,\dot{r} + g_{zz}\,\xi_1^{\,t}\,\dot{z}$$

$= 0$

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$$\sqrt{1+\left(\frac{r}{2a}\right)^2}\, k_1 = -1\, \frac{r}{\sqrt{2}} \cos\phi\, \dot{t} - \frac{r^2}{\sqrt{2}\, a} \frac{r}{\sqrt{2}} \cos\phi\, \dot{\phi} - \frac{r^2}{\sqrt{2}\, a} \frac{a}{r} \left(1+\left(\frac{r}{2a}\right)^2\right) \cos\phi\, \dot{t}$$

$$+ r^2 \left(1-\left(\frac{r}{2a}\right)^2\right) \frac{a}{r} \left(1+2\left(\frac{r}{2a}\right)^2\right) \cos\phi\, \dot{\phi}$$

$$+ \frac{1}{1+\left(\frac{r}{2a}\right)^2}\, a \left(1+\left(\frac{r}{2a}\right)^2\right) \sin\phi\, \dot{r}$$

$$= -\frac{r}{\sqrt{2}} \cos\phi \left[1+\left(1+\left(\frac{r}{2a}\right)^2\right)\right] \dot{t} + a r \cos\phi \left[-\frac{r^2}{2a^2} + \left(1-\left(\frac{r}{2a}\right)^2\right)\left(1+2\left(\frac{r}{2a}\right)^2\right)\right] \dot{\phi} + a \sin\phi\, \dot{r}$$

$$\xi_1 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(\frac{r}{\sqrt{2}} \cos\phi,\, a\left(1+\left(\frac{r}{2a}\right)^2\right)\sin\phi,\, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right)\cos\phi,\, 0\right) \qquad (8)$$

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$$+\,\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,a\left(1+\left(\frac{r}{2a}\right)^2\right)\sin\phi\,\dot{r}$$

$$= -\frac{r}{\sqrt{2}}\cos\phi\left[1+\left(1+\left(\frac{r}{2a}\right)^2\right)\right]\dot{t} + ar\cos\phi\left[-\frac{r^2}{2a^2}+\left(1-\left(\frac{r}{2a}\right)^2\right)\left(1+2\left(\frac{r}{2a}\right)^2\right)\right]\dot{\phi} + a\sin\phi\,\dot{r}$$

$$= -\sqrt{2}\cos\phi\left[1+\left(\frac{r}{2a}\right)^2\right]\dot{t} - ar\cos\phi\left(1+\left(\frac{r}{2a}\right)^2\right)\left(2\left(\frac{r}{2a}\right)^2-1\right)\dot{\phi} + a\sin\phi\,\dot{r}$$

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$$+\, r^2 \left(1-\left(\frac{r}{2a}\right)^2\right) \frac{a}{r} \left(1+2\left(\frac{r}{2a}\right)^2\right) \cos\phi\, \dot{\phi}$$

$$+\, \frac{1}{1+\left(\frac{r}{2a}\right)^2}\, a \left(1+\left(\frac{r}{2a}\right)^2\right) \sin\phi\, \dot{r}$$

$$= -\frac{r}{\sqrt{2}} \cos\phi \left[1+\left(1+\left(\frac{r}{2a}\right)^2\right)\right] \dot{t} + a r \cos\phi \left[-\frac{r^2}{2a^2} + \left(1-\left(\frac{r}{2a}\right)^2\right)\left(1+2\left(\frac{r}{2a}\right)^2\right)\right] \dot{\phi} + a \sin\phi\, \dot{r}$$

$$= -\sqrt{2} \cos\phi \left[1+\left(\frac{r}{2a}\right)^2\right] \dot{t} - a r \cos\phi \left(1+\left(\frac{r}{2a}\right)^2\right)\left(2\left(\frac{r}{2a}\right)^2-1\right) \dot{\phi} + a \sin\phi\, \dot{r} \Rightarrow$$

$$k_1 = -\frac{\sqrt{2}\cos\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[1+\left(\frac{r}{2a}\right)^2\right] \dot{t} - \frac{a r \cos\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[1+\left(\frac{r}{2a}\right)^2\right]\left[2\left(\frac{r}{2a}\right)^2-1\right] \dot{\phi} + \frac{a \sin\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \dot{r}$$

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Substitute:

$$\dot{t} = k_0 - \frac{1}{2a}\frac{4k_0a - \sqrt{2}k_2}{1+\left(\frac{r}{2a}\right)^2} \qquad \dot{\phi} = \frac{1}{2ar^2}\frac{2ak_2 - \sqrt{2}k_0r^2}{1+\left(\frac{r}{2a}\right)^2}$$

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$$= \frac{k_2 a}{r}\left[2\left(\frac{r}{2a}\right)^2+1\right]\frac{\cos\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} + \frac{k_0 r\cos\phi}{\sqrt{2}\sqrt{1+\left(\frac{r}{2a}\right)^2}} + \frac{a\sin\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}\,\dot{r}$$

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$$u^{(1)} = \frac{\dot{r}}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}$$

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$$u^{(1)} = \frac{\dot{r}}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}$$

$$k_1 = a\sin\phi\, u^{(1)} + \frac{\cos\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}\left[\frac{k_2 a}{r}\left(2\left(\frac{r}{2a}\right)^2+1\right) + \frac{k_0 r}{\sqrt{2}}\right]$$

$$k_1 = -\sqrt{2}\cos\phi\sqrt{1+\left(\frac{r}{2a}\right)^2}\,\dot{t} - ar\cos\phi\sqrt{1+\left(\frac{r}{2a}\right)^2}\left[2\left(\frac{r}{2a}\right)^2-1\right]\dot{\phi} + \frac{a\sin\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}\dot{r}$$

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$u^{(1)}$

$$u^{(1)} = \frac{\dot{r}}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}$$

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Derive similar expression for k_4

14. The KVF ξ_4 deforms isometrically the constant-t, circular, closed CTC, to a closed CTC on which the coordinate t varies. Show that $\mathcal{L}_\xi(u^\mu u_\mu) = 0$, so that the timelike kind of the curve does not change under this deformation.

$$\mathcal{L}_\xi (u^\mu u_\mu) = \mathcal{L}_\xi (g_{\mu\nu} u^\mu u^\nu)$$

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$$= (\mathcal{L}_\xi g_{\mu\nu}) u^\mu u^\nu + g_{\mu\nu}(\mathcal{L}_\xi u^\mu) u^\nu + g_{\mu\nu} u^\mu (\mathcal{L}_\xi u^\nu)$$

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$\mathcal{L}_\xi g_{\mu\nu} = 0$ since ξ a KVF

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$\mathcal{L}_\xi u^\mu = 0$ since u is Lie dragged by ξ

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$$\mathcal{L}_\xi (u^\mu u_\mu) = \frac{d}{d\eta}(u^\mu u_\mu) = 0$$

(η the parameter on integral curve of ξ)

$\mathcal{L}_\xi g_{\mu\nu} = 0$ since ξ a KVF

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$$\mathcal{L}_\xi (u^\mu u_\mu) = \frac{d}{d\eta}(u^\mu u_\mu) = 0$$ (η the parameter on integral curve of ξ)

$u^\mu u_\mu$ = const along the integral curve $\gamma_\xi(\eta)$

In particular, it does not change sign!

u^μ timelike on original curve $\leadsto$ remains timelike

15. Verify that $\nabla_t \xi_{1r} + \nabla_r \xi_{1t} = 0$ for the KVF ξ_1.

$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho{}_{\mu\nu} \xi_\rho$$

15. Verify that $\nabla_t \xi_{1r} + \nabla_r \xi_{1t} = 0$ for the KVF ξ_1.

$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho{}_{\mu\nu} \xi_\rho$$

$$\nabla_t \xi_r = \partial_t \xi_r - \Gamma^t{}_{tr} \xi_t - \Gamma^\phi{}_{rt} \xi_\phi$$

$$\Gamma^t{}_{rt} = \frac{r}{2a^2} \frac{1}{1+\left(\frac{r}{2a}\right)^2}, \quad \Gamma^t{}_{\phi r} = \frac{r^3}{4\sqrt{2}a^3} \frac{1}{1+\left(\frac{r}{2a}\right)^2}, \quad \Gamma^r{}_{rr} = -\frac{r}{4a^2} \frac{1}{1+\left(\frac{r}{2a}\right)^2},$$

$$\Gamma^r{}_{\phi t} = \frac{r}{\sqrt{2}a}\left(1+\left(\frac{r}{2a}\right)^2\right), \quad \Gamma^r{}_{\phi\phi} = r\left(1+\left(\frac{r}{2a}\right)^2\right)\left(2\left(\frac{r}{2a}\right)^2 - 1\right),$$

$$\Gamma^\phi{}_{\phi r} = \frac{1}{r} \frac{1}{1+\left(\frac{r}{2a}\right)^2}, \quad \Gamma^\phi{}_{rt} = -\frac{1}{\sqrt{2}ar} \frac{1}{1+\left(\frac{r}{2a}\right)^2}.$$

No other $\Gamma^\mu{}_{rt} \neq 0$

15. Verify that $\nabla_t \xi_{1r} + \nabla_r \xi_{1t} = 0$ for the KVF ξ_1.

$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho{}_{\mu\nu} \xi_\rho$$

$$\nabla_t \xi_r = \partial_t \xi_r - \Gamma^t{}_{tr} \xi_t - \Gamma^\phi{}_{rt} \xi_\phi$$

$$\xi_r = g_{r\mu} \xi^\mu$$

$$\xi_1 = \frac{1}{\sqrt{1 + \left(\frac{r}{2a}\right)^2}} \left(\frac{r}{\sqrt{2}} \cos\phi, a\left(1 + \left(\frac{r}{2a}\right)^2\right) \sin\phi, \frac{a}{r}\left(1 + 2\left(\frac{r}{2a}\right)^2\right) \cos\phi, 0 \right)$$

15. Verify that $\nabla_t \xi_{1r} + \nabla_r \xi_{1t} = 0$ for the KVF ξ_1.

$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho{}_{\mu\nu} \xi_\rho$$

$$\nabla_t \xi_r = \partial_t \xi_r - \Gamma^t{}_{tr} \xi_t - \Gamma^\phi{}_{rt} \xi_\phi$$

$$\xi_r = g_{r\mu} \xi^\mu = g_{rr} \xi^r$$

$$\xi_1 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(\frac{r}{\sqrt{2}} \cos\phi, a\left(1+\left(\frac{r}{2a}\right)^2\right) \sin\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right) \cos\phi, 0 \right)$$

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$$= \frac{1}{1+\left(\frac{r}{2a}\right)^2} \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} a\left(1+\left(\frac{r}{2a}\right)^2\right) \sin\phi$$

$$\xi_1 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(\frac{r}{\sqrt{2}} \cos\phi, a\left(1+\left(\frac{r}{2a}\right)^2\right) \sin\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right) \cos\phi, 0\right)$$

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$$= \frac{1}{1+\left(\frac{r}{2a}\right)^2} \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} a\left(1+\left(\frac{r}{2a}\right)^2\right) \sin\phi$$

$$= \frac{a}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \sin\phi$$

$$\xi_1 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(\frac{r}{\sqrt{2}}\cos\phi, a\left(1+\left(\frac{r}{2a}\right)^2\right)\sin\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right)\cos\phi, 0\right)$$

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$$\xi_r = \frac{a \sin\phi}{\sqrt{1 + \left(\frac{r}{2a}\right)^2}}$$

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$$\xi_r = \frac{a \sin\phi}{\sqrt{1 + \left(\frac{r}{2a}\right)^2}}$$

$$\xi_t = g_{tt} \xi^t + g_{t\phi} \xi^\phi$$

$$\xi_1 = \frac{1}{\sqrt{1 + \left(\frac{r}{2a}\right)^2}} \left(\frac{r}{\sqrt{2}} \cos\phi, a \left(1 + \left(\frac{r}{2a}\right)^2\right) \sin\phi, \frac{a}{r} \left(1 + 2\left(\frac{r}{2a}\right)^2\right) \cos\phi, 0 \right)$$

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$$\xi_t = g_{tt} \xi^t + g_{t\phi} \xi^\phi = -\frac{r}{\sqrt{2}} \frac{\cos\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} - \frac{r^2}{\sqrt{2}\, a} \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(\frac{a}{r}\right)\left(1+2\left(\frac{r}{2a}\right)^2\right) \cos\phi$$

$$\xi_1 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(\frac{r}{\sqrt{2}} \cos\phi, a\left(1+\left(\frac{r}{2a}\right)^2\right) \sin\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right) \cos\phi, 0\right)$$

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$$= -\sqrt{2}\, r \cos\phi \sqrt{1+\left(\frac{r}{2a}\right)^2}$$

$$\xi_1 = \frac{1}{\sqrt{1 + \left(\frac{r}{2a}\right)^2}} \left(\frac{r}{\sqrt{2}} \cos\phi, a\left(1 + \left(\frac{r}{2a}\right)^2\right) \sin\phi, \frac{a}{r}\left(1 + 2\left(\frac{r}{2a}\right)^2\right) \cos\phi, 0\right)$$

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$$\xi_\phi = g_{\phi t} \xi^t + g_{\phi\phi} \xi^\phi$$

$$\xi_1 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left(\frac{r}{\sqrt{2}} \cos\phi, a\left(1+\left(\frac{r}{2a}\right)^2\right) \sin\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right) \cos\phi, 0 \right)$$

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$$\xi_\phi = g_{\phi t}\xi^t + g_{\phi\phi}\xi^\phi = -\frac{r^2}{\sqrt{2}a}\frac{r}{\sqrt{2}}\frac{\cos\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} + r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\frac{a}{r}\frac{1+2\left(\frac{r}{2a}\right)^2}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}\cos\phi$$

$$\xi_1 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}\left(\frac{r}{\sqrt{2}}\cos\phi, a\left(1+\left(\frac{r}{2a}\right)^2\right)\sin\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right)\cos\phi, 0\right)$$

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$$= -ar\cos\phi\sqrt{1+\left(\frac{r}{2a}\right)^2}\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

$$\xi_1 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}\left(\frac{r}{\sqrt{2}}\cos\phi, a\left(1+\left(\frac{r}{2a}\right)^2\right)\sin\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right)\cos\phi, 0\right)$$

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$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho{}_{\mu\nu}\xi_\rho$$

$$\nabla_t \xi_r = - \Gamma^t{}_{tr}\xi_t - \Gamma^\phi{}_{rt}\xi_\phi$$

$$\xi_r = \frac{a\sin\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \qquad \xi_t = -\sqrt{2}\, r\cos\phi\sqrt{1+\left(\frac{r}{2a}\right)^2} \qquad \xi_\phi = -ar\cos\phi\sqrt{1+\left(\frac{r}{2a}\right)^2}\left[2\left(\frac{r}{2a}\right)^2-1\right]$$

$$\Gamma^t{}_{tr}\xi_t = -\frac{r}{2a^2}\,\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,\sqrt{2}\, r\cos\phi\sqrt{1+\left(\frac{r}{2a}\right)^2}$$

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$$\Gamma^t{}_{tr} \xi_t = -\frac{r}{2a^2} \frac{1}{1+\left(\frac{r}{2a}\right)^2} \sqrt{2}\, r \cos\phi \sqrt{1+\left(\frac{r}{2a}\right)^2} = -\frac{r^2}{\sqrt{2}a^2} \frac{\cos\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}$$

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$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho{}_{\mu\nu} \xi_\rho$$

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$$\Gamma^t{}_{tr} \xi_t = - \frac{r^2}{\sqrt{2}\, a^2} \frac{\cos\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}$$

$$\Gamma^\phi{}_{rt} \xi_\phi = - \frac{1}{\sqrt{2}\, ar} \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[-ar\cos\phi\sqrt{1+\left(\frac{r}{2a}\right)^2}\left(2\left(\frac{r}{2a}\right)^2 - 1\right)\right]$$

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$$\Gamma^t{}_{tr}\xi_t = -\frac{r^2}{\sqrt{2}\, a^2} \frac{\cos\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}$$

$$\Gamma^\phi{}_{rt}\xi_\phi = -\frac{1}{\sqrt{2}\, ar} \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[-ar\cos\phi\sqrt{1+\left(\frac{r}{2a}\right)^2}\left(2\left(\frac{r}{2a}\right)^2 - 1\right)\right]$$

$$= \frac{\cos\phi}{\sqrt{2}\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

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$$\Gamma^t{}_{tr} \xi_t = -\frac{r^2}{\sqrt{2} a^2} \frac{\cos\phi}{\sqrt{1+\left(\frac{r}{2a}\right)^2}} \qquad \Gamma^\phi{}_{rt} \xi_\phi = \frac{\cos\phi}{\sqrt{2}\sqrt{1+\left(\frac{r}{2a}\right)^2}}\left[2\left(\frac{r}{2a}\right)^2 - 1\right]$$

$$\nabla_t \xi_r = \frac{\cos\phi}{\sqrt{2}\sqrt{1+\left(\frac{r}{2a}\right)^2}}\left[2\left(\frac{r}{2a}\right)^2 + 1\right]$$

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$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho{}_{\mu\nu} \xi_\rho$$

$$\nabla_t \xi_r = \frac{\cos\phi}{\sqrt{2}\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[2\left(\frac{r}{2a}\right)^2 + 1\right]$$

$$\nabla_r \xi_t = \partial_r \xi_t \underbrace{- \Gamma^t{}_{rt} \xi_t - \Gamma^\phi{}_{rt} \xi_\phi}_{= \nabla_t \xi_r}$$

15. Verify that $\nabla_t \xi_{1r} + \nabla_r \xi_{1t} = 0$ for the KVF ξ_1.

$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho{}_{\mu\nu} \xi_\rho$$

$$\nabla_t \xi_r = \frac{\cos\phi}{\sqrt{2}\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[2\left(\frac{r}{2a}\right)^2 + 1\right]$$

$$\nabla_r \xi_t = \partial_r \xi_t - \Gamma^t{}_{rt} \xi_t - \Gamma^\phi{}_{rt} \xi_\phi$$

$$= \partial_r \xi_t + \frac{\cos\phi}{\sqrt{2}\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[2\left(\frac{r}{2a}\right)^2 + 1\right]$$

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$$\nabla_r \xi_t = \partial_r \xi_t + \frac{\cos\phi}{\sqrt{2}\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[2\left(\frac{r}{2a}\right)^2 + 1\right]$$

$$\partial_r \xi_t = \partial_r \left[-\sqrt{2}\, r \cos\phi \sqrt{1+\left(\frac{r}{2a}\right)^2}\right]$$

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$$\partial_r \xi_t = \partial_r \left[-\sqrt{2}\, r \cos\phi \sqrt{1+\left(\frac{r}{2a}\right)^2}\right] = -\frac{2\cos\phi}{\sqrt{2}\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[2\left(\frac{r}{2a}\right)^2 + 1\right]$$

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$$\nabla_r \xi_t = -\frac{\cos\phi}{\sqrt{2}\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[2\left(\frac{r}{2a}\right)^2 + 1\right] = -\nabla_t \xi_r$$

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$$\nabla_r \xi_t = -\frac{\cos\phi}{\sqrt{2}\sqrt{1+\left(\frac{r}{2a}\right)^2}} \left[2\left(\frac{r}{2a}\right)^2 + 1\right] = -\nabla_t \xi_r \quad \Rightarrow \nabla_r \xi_t + \nabla_t \xi_r = 0$$