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Problems in General Relativity

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The Gödel Universe

Consider the Gödel spacetime:

$$ds^2 = -dt^2 - 2\frac{r^2}{\sqrt{2}a}dtd\phi + \frac{dr^2}{1+\left(\frac{r}{2a}\right)^2} + r^2\left[1-\left(\frac{r}{2a}\right)^2\right]d\phi^2 + dz^2 \,. \tag{1}$$

1. Determine whether ∂_μ are timelike, null, or spacelike. From the kind of ∂_ϕ, discuss if it is possible to have (local future pointing) timelike geodesics moving in the negative t direction.
2. Show that $\xi_0 = \partial_t$, $\xi_2 = \partial_\phi$, $\xi_3 = \partial_z$, are Killing Vector Fields (KVF), and compute the corresponding conserved quantities k_0, k_2, and k_3 along a geodesic with tangent vector $u^\mu = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$.
3. Compute $\dot{t}$, $\dot{\phi}$ and $\dot{z}$ in terms of k_0, k_2, and k_3.
4. Show that $u^\mu u_\mu = \kappa$, $\kappa = 0, -1$ for null/timelike geodesics yield
$$\mathcal{E} = \frac{1}{2}(\dot{r})^2 + \frac{1}{2}A^2 r^2 + \frac{L^2}{2r^2}\,, \tag{2}$$
where $\mathcal{E}$, A, and L are constants, which you should calculate. Find conditions for motion $r_1 \le r \le r_2$, and compute $r_{1,2}$ in terms of $\mathcal{E}$, A, and L. (Notice that the problem of the radial motion is similar to the 3-dimensional harmonic oscillator)
5. Compute an orthonormal basis $\{e_a\}$. If $u = u^{(a)}e_a = u^\mu\partial_\mu$, compute $u^{(a)}$ in terms of u^μ, and vice-versa.
6. Compute k_0, k_2, and k_3 in terms of $u^{(a)}$, so that the $u^{(a)}$ can be used as initial conditions in the geodesic equations.
7. Free massless particle goes through the local inertial frame $\{e_a\}$ with 4-velocity $(u^{(0)}, u^{(1)}, 0, 0)$. Write down the geodesic equations for $(\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$ in terms of $u^{(0)}, u^{(1)}$.
8. Compute *all* the null vectors at a point with coordinate r (i.e. compute the lightcone). Give expressions for both $u^{(a)}$ and u^μ (Hint: You will need a 3-parameter family of vectors, start from $u^{(a)}$ which is easier).
9. Compute the Christoffel symbols of the Levi-Civita connection of the metric. The nonzero components of the inverse metric are
$$g^{tt} = -\frac{1-\left(\frac{r}{2a}\right)^2}{1+\left(\frac{r}{2a}\right)^2}\,,\quad g^{rr} = 1+\left(\frac{r}{2a}\right)^2\,,\quad g^{zz} = 1\,,\quad g^{t\phi} = -\frac{1}{\sqrt{2}a\left(1+\left(\frac{r}{2a}\right)^2\right)}\,, \tag{3}$$
$$g^{\phi\phi} = \frac{1}{r^2\left(1+\left(\frac{r}{2a}\right)^2\right)}\,,\quad g^{t\phi} = -\frac{1}{\sqrt{2}a\left(1+\left(\frac{r}{2a}\right)^2\right)}\,. \tag{4}$$
The result is:
$$\Gamma^t_{rt} = \frac{r}{2a^2}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,,\quad \Gamma^t_{\phi r} = \frac{r^3}{4\sqrt{2}a^3}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,,\quad \Gamma^r_{rr} = -\frac{r}{4a^2}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,, \tag{5}$$
$$\Gamma^r_{\phi t} = \frac{r}{\sqrt{2}a}\left(1+\left(\frac{r}{2a}\right)^2\right)\,,\quad \Gamma^r_{\phi\phi} = r\left(1+\left(\frac{r}{2a}\right)^2\right)\left(2\left(\frac{r}{2a}\right)^2-1\right)\,, \tag{6}$$
$$\Gamma^\phi_{\phi r} = \frac{1}{r}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,,\quad \Gamma^\phi_{rt} = -\frac{1}{\sqrt{2}ar}\frac{1}{1+\left(\frac{r}{2a}\right)^2}\,. \tag{7}$$

10. Consider the massive particle moving on the trajectory with $t = 0$, $r = R$, $\phi = \omega\tau$, $z = 0$, where R, ω are constants. Determine when the 4-velocity of the particle is timelike, in which case we have a closed timelike curve (CTC).

11. Compute the relation $\omega = \omega(R)$.

12. Compute the 4-acceleration of the particle $a^\mu = \ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho$, where $\dot{x}^\mu = dx^\mu/d\tau$. Conclude that the particle is not falling freely.

13. The vectors with components in the coordinate basis below are KVFs:

$$\xi_1 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}\left(\frac{r}{\sqrt{2}}\cos\phi, a\left(1+\left(\frac{r}{2a}\right)^2\right)\sin\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right)\cos\phi, 0\right) \tag{8}$$

$$\xi_4 = \frac{1}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}\left(\frac{r}{\sqrt{2}}\sin\phi, -a\left(1+\left(\frac{r}{2a}\right)^2\right)\cos\phi, \frac{a}{r}\left(1+2\left(\frac{r}{2a}\right)^2\right)\sin\phi, 0\right) \tag{9}$$

Compute the corresponding conserved quantities k_1 and k_4 along a geodesic with tangent vector $u^\mu = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$.

14. The KVF ξ_4 deforms isometrically the constant-t, circular, closed CTC, to a closed CTC on which the coordinate t varies. Show that $\mathcal{L}_\xi(u^\mu u_\mu) = 0$, so that the timelike kind of the curve does not change under this deformation.

15. Verify that $\nabla_t\xi_{1r} + \nabla_r\xi_{1t} = 0$ for the KVF ξ_1.

Reference: Frank Grave, Michael Buser, Thomas Müller, Günter Wunner, and Wolfgang P. Schleich, *"The Gödel universe: Exact geometrical optics and analytical investigations on motion"*, Phys. Rev. D 80, 103002 (2009)