Schwarzschild Black Holes 2

- Eddington - Finkelstein coordinates
- Kruskal - Szekeres coordinates

Schwarzschild metric in (t, r, θ, φ)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

- explicit timelike killing vector field ∂_t for $r > 2M$
  $(\partial_t g_{\mu\nu} = 0)$ (KVF)

- explicitly static: as $t \to \infty$ α) ∂_t timelike KVF
  β) $\partial_t \perp$ to $dt =$ const surface

- explicit Newtonian limit for $r \gg 2M$
  asymptotic flatness as $r \to \infty$

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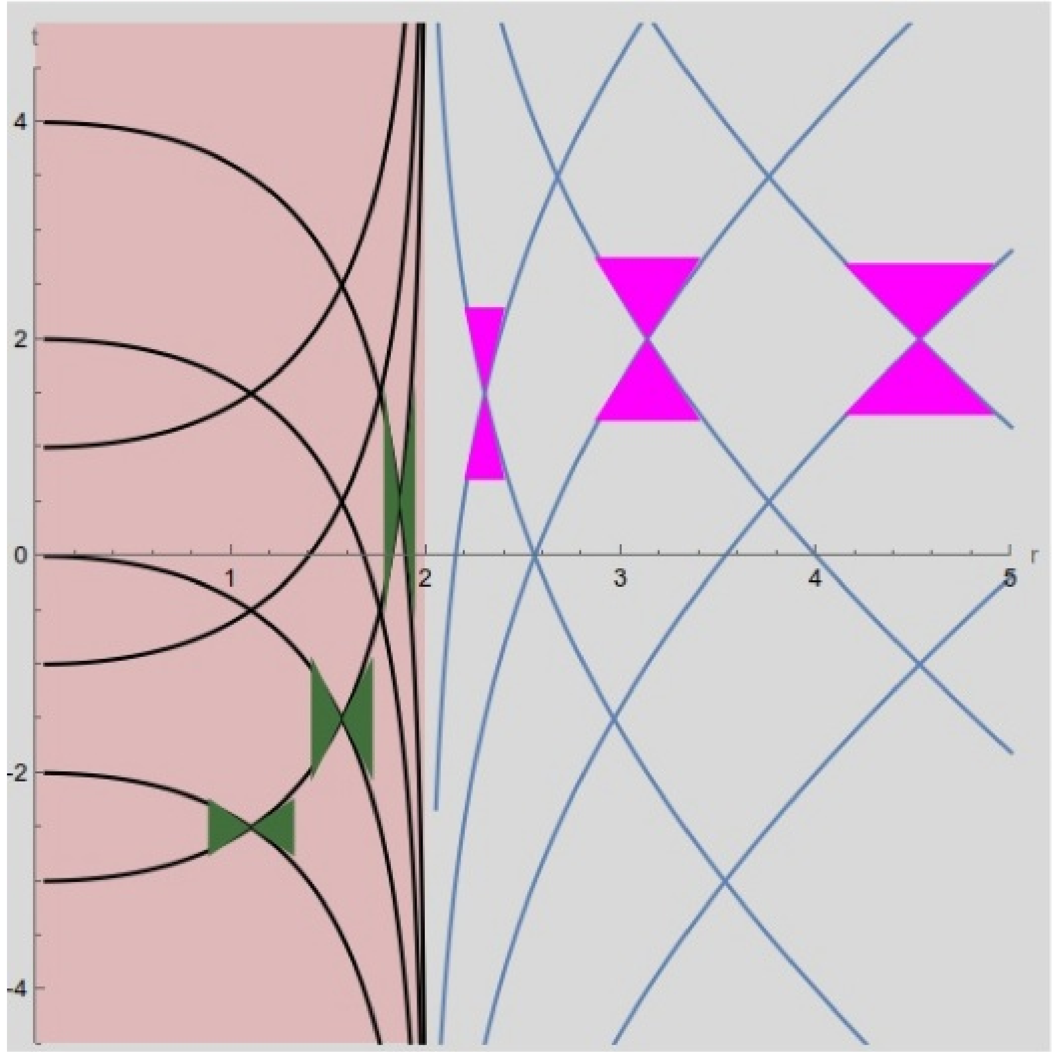

Not good for Black Hole physics $r \approx 2M$ and $r \leq 2M$

(α) coordinate singularity at $r = 2M$

(β) infinite t for particle to fall into BH

(γ) stationary observers not physically plausible for $r \approx 2M$ (infinite proper acceleration + energy needed to realize)

(δ) ∂_t becomes null on Horizon, then spacelike $-\partial_r$ in the direction of time for $r < 2M$ $\Big\}$ light cones tip at $r = 2M$ reflecting (α)

Eddington-Finkelstein coordinates

(E-F coordinates)

$t \to v$ $\qquad v = t + r + 2M \ln\left|\frac{r}{2M} - 1\right|$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2\, d\Omega^2$$

Eddington-Finkelstein coordinates

$t \to v$ $\qquad v = t + r + 2M \ln\left|\frac{r}{2M} - 1\right|$

coordinate singularity absorbed by logarithm

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2\, d\Omega^2$$

not singular at $r = 2M$

cross term: prevents singularity, but ∂_v not normal to ∂_r

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \overset{r=2M}{=} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & (2M)^2 & 0 \\ 0 & 0 & 0 & (2M)^2 \sin^2\theta \end{pmatrix}$$

not singular

Eddington-Finkelstein coordinates

$t \to v \qquad v = t + r + 2M \ln\left|\frac{r}{2M} - 1\right|$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2\, d\Omega^2$$

- explicitly non singular: $r > 2M$ and $r < 2M$ smootly connected

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- $r \to \infty$ $\Rightarrow$ ds^2 is flat with $t = v - r$

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- $r \to \infty \Rightarrow ds^2$ is flat with $t = v - r$
- $r \to 0$ true singularity $\left(R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} = \frac{48 M^2}{r^6} \right)$

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- $r \to 0$ true singularity $\left(R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} = \frac{48 M^2}{r^6} \right)$
- $g_{01} = g_{vr} = 1 \neq 0 \Rightarrow \partial_v \cdot \partial_r = 1 \neq 0$

Eddington-Finkelstein coordinates

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$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2\, d\Omega^2$$

- Light cone structure:

  consider radial null curves $\qquad d\theta = d\varphi = 0$

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$$\Rightarrow \begin{cases} dv = 0 \\ -\left(1 - \frac{2M}{r}\right) dv + 2\, dr = 0 \end{cases}$$

(also $r = 2M$!)

Eddington-Finkelstein coordinates

$t \to v \qquad v = t + r + 2M \ln\left|\frac{r}{2M} - 1\right|$

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$$\Rightarrow \left.\begin{cases} dv = 0 \\ -\left(1 - \frac{2M}{r}\right) dv + 2\, dr = 0 \end{cases}\right\} \Rightarrow \begin{matrix} v = \text{const} \\ \frac{dv}{dr} = 2\left(1 - \frac{2M}{r}\right)^{-1} \end{matrix}$$

Eddington-Finkelstein coordinates

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consider radial null curves $\quad d\theta = d\varphi = 0$

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$$\Rightarrow \left[-\left(1 - \frac{2M}{r}\right) dv + 2\, dr\right] dv = 0$$

$$\Rightarrow \left\{\begin{array}{ll} dv & = 0 \\ -\left(1 - \frac{2M}{r}\right) dv + 2\, dr & = 0 \end{array}\right\} \Rightarrow \begin{array}{l} v = \text{const} \\ \int dv = \int 2\left(1 - \frac{2M}{r}\right)^{-1} dr \end{array}$$

Eddington-Finkelstein coordinates

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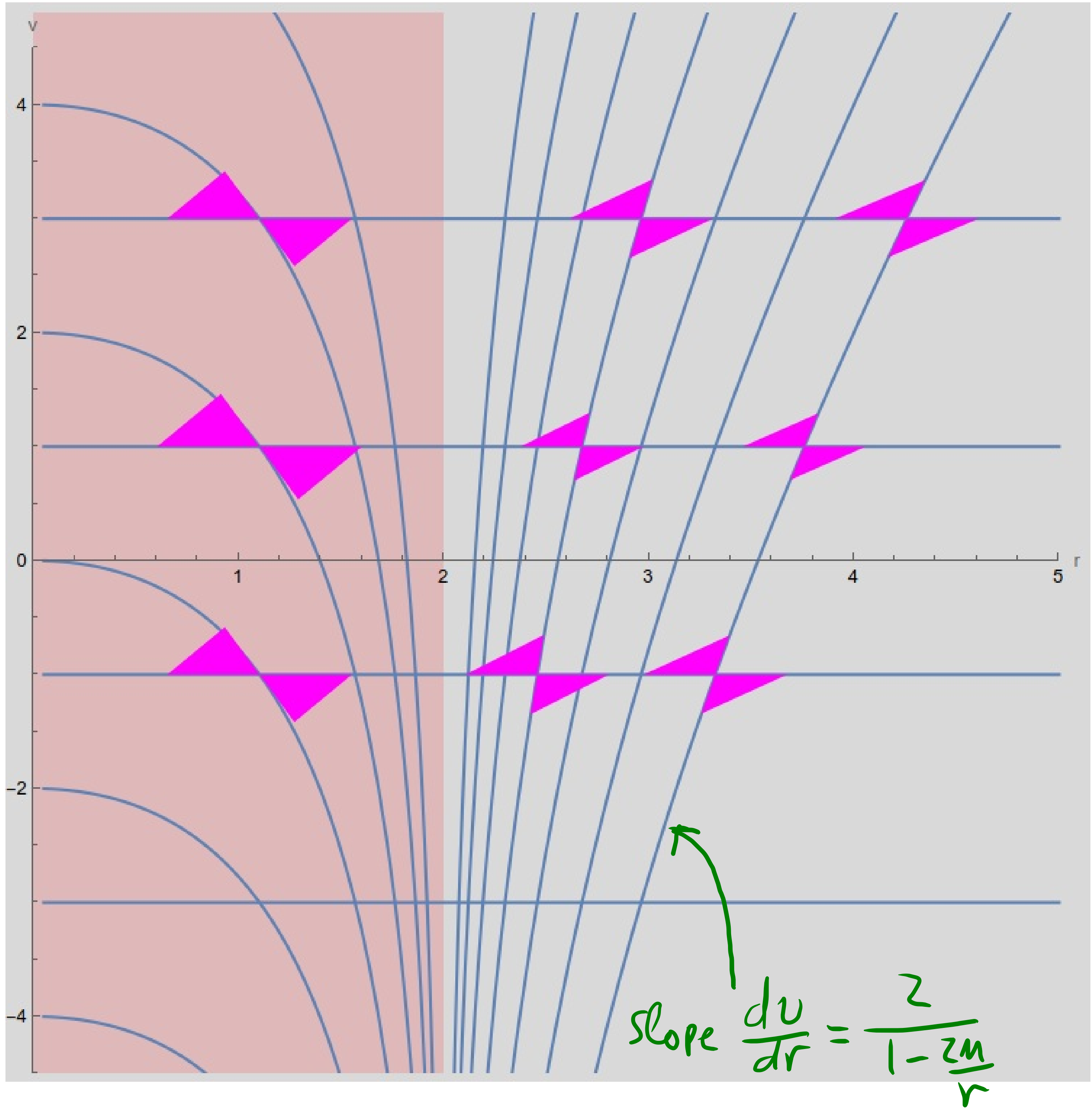

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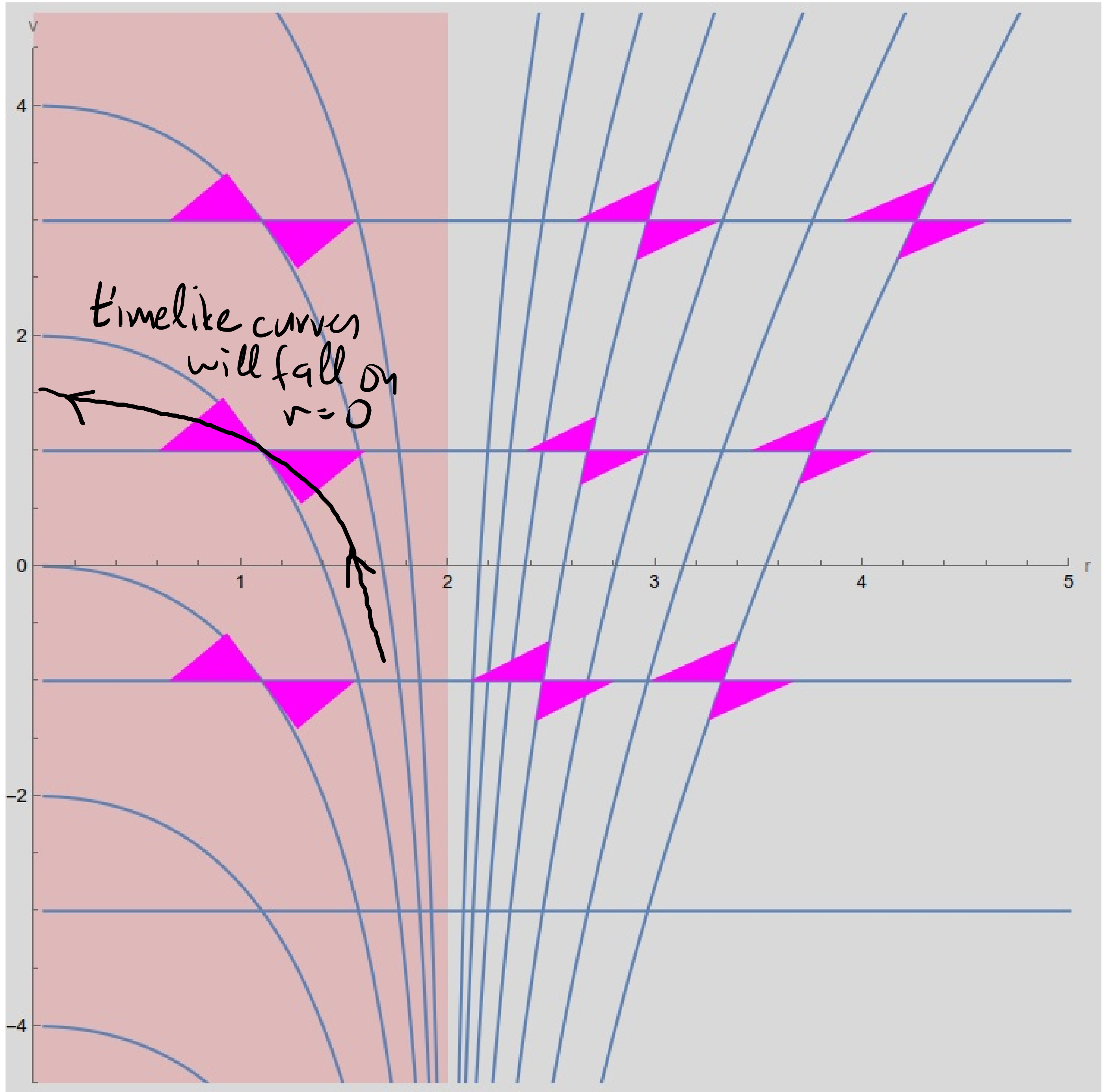

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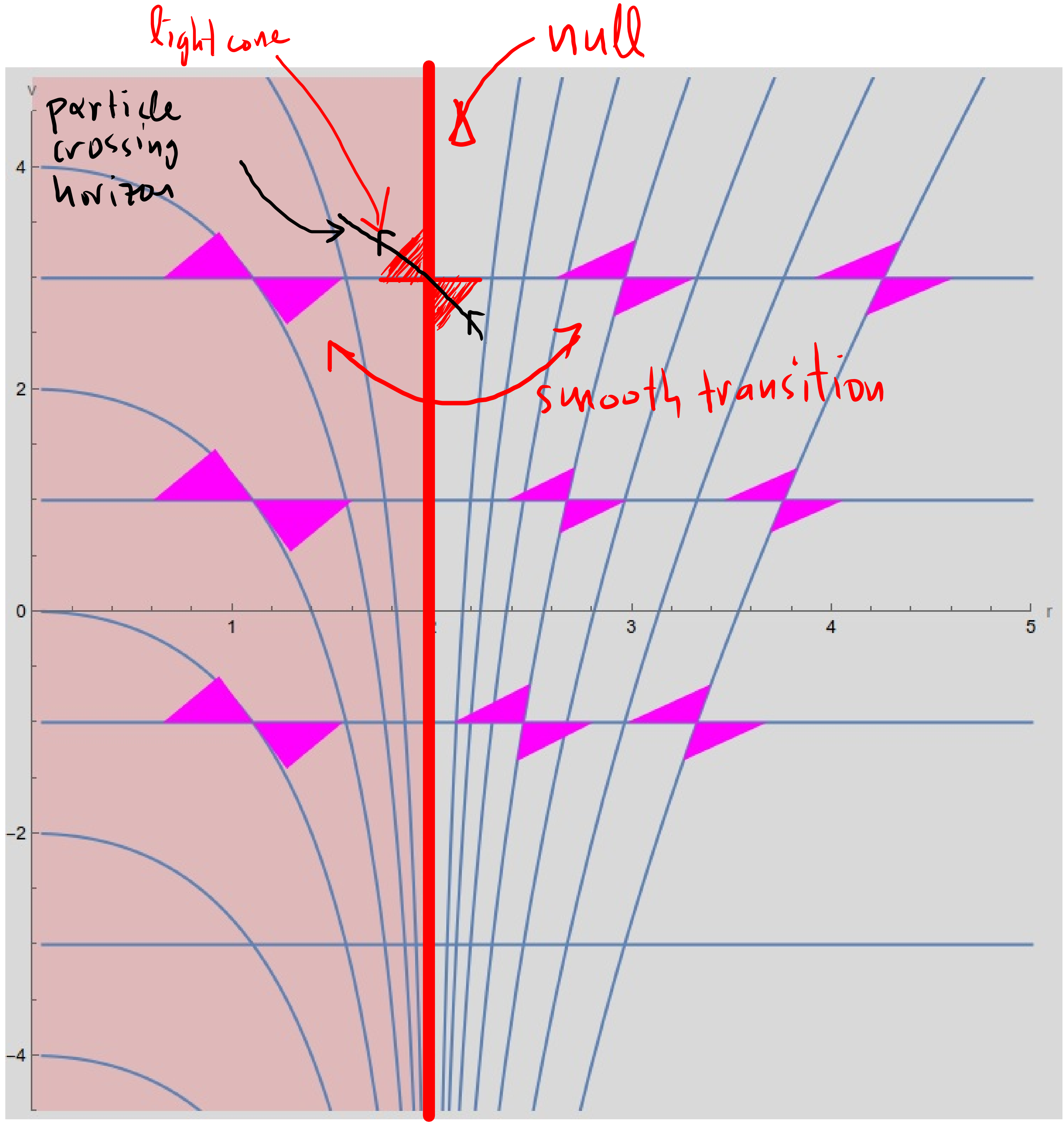

$r = 2M$ (null)

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- Define "time" $\tilde{t} = v - r = t + 2M \ln\left|\frac{r}{2M} - 1\right|$

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$\Rightarrow -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr = -\left(1 - \frac{2M}{r}\right)\left(d\tilde{t}^2 + dr^2 + 2\, d\tilde{t}\, dr\right) + 2\left(d\tilde{t} + dr\right) dr$

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$$= -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 - \left(\cancel{1} - \frac{2M}{r}\right) dr^2 - 2\left(\cancel{1} - \frac{2M}{r}\right) d\tilde{t}\, dr + \cancel{2\, d\tilde{t}\, dr} + \cancel{2}\, dr^2$$

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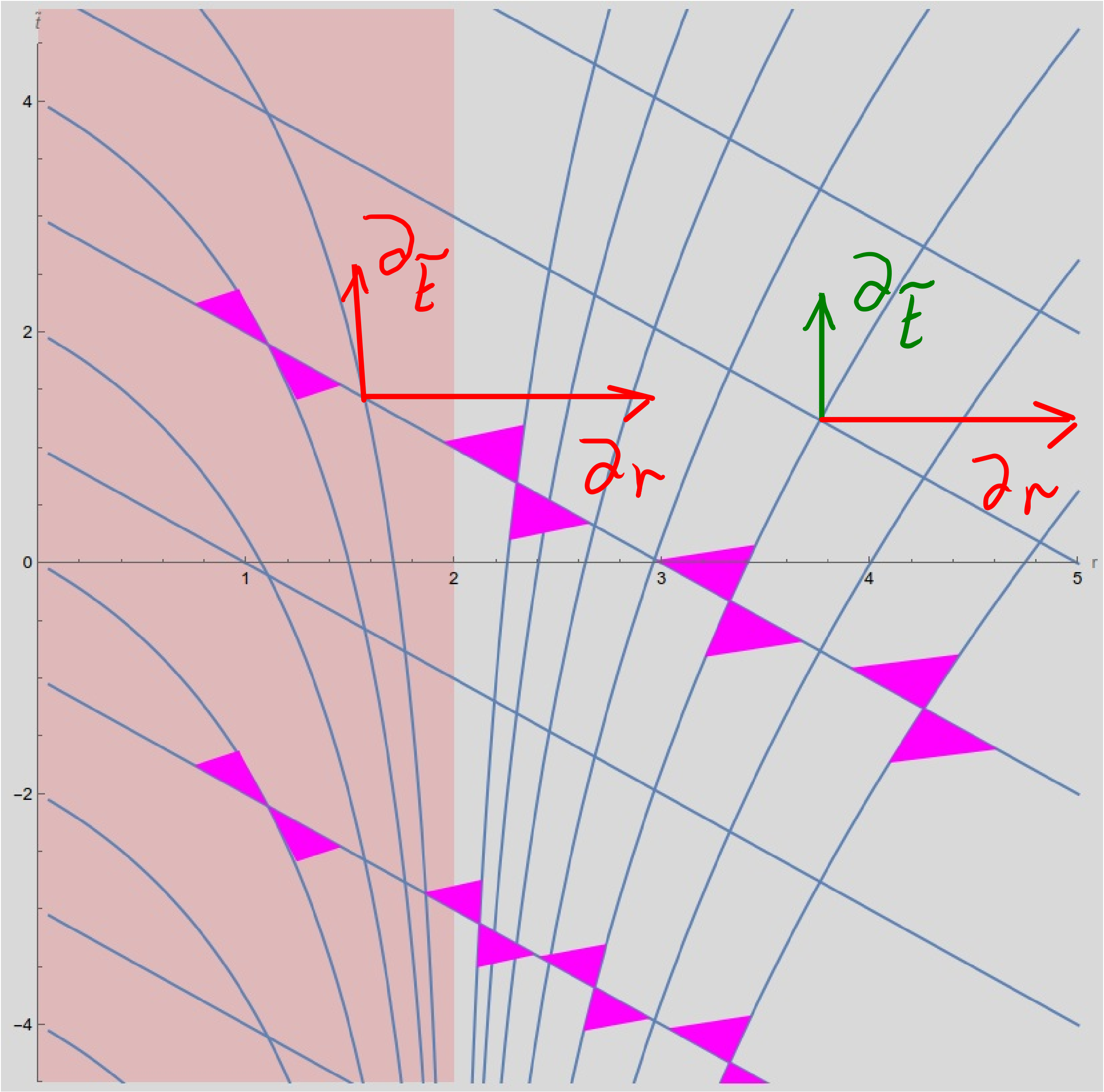
$\partial_{\tilde{t}} \cdot \partial_{\tilde{t}} = -\left(1 - \frac{2M}{r}\right)$ → timelike $r > 2M$; → spacelike $r < 2M$

$\partial_r \cdot \partial_r = 1 + \frac{2M}{r}$ → always spacelike



Eddington-Finkelstein coordinates

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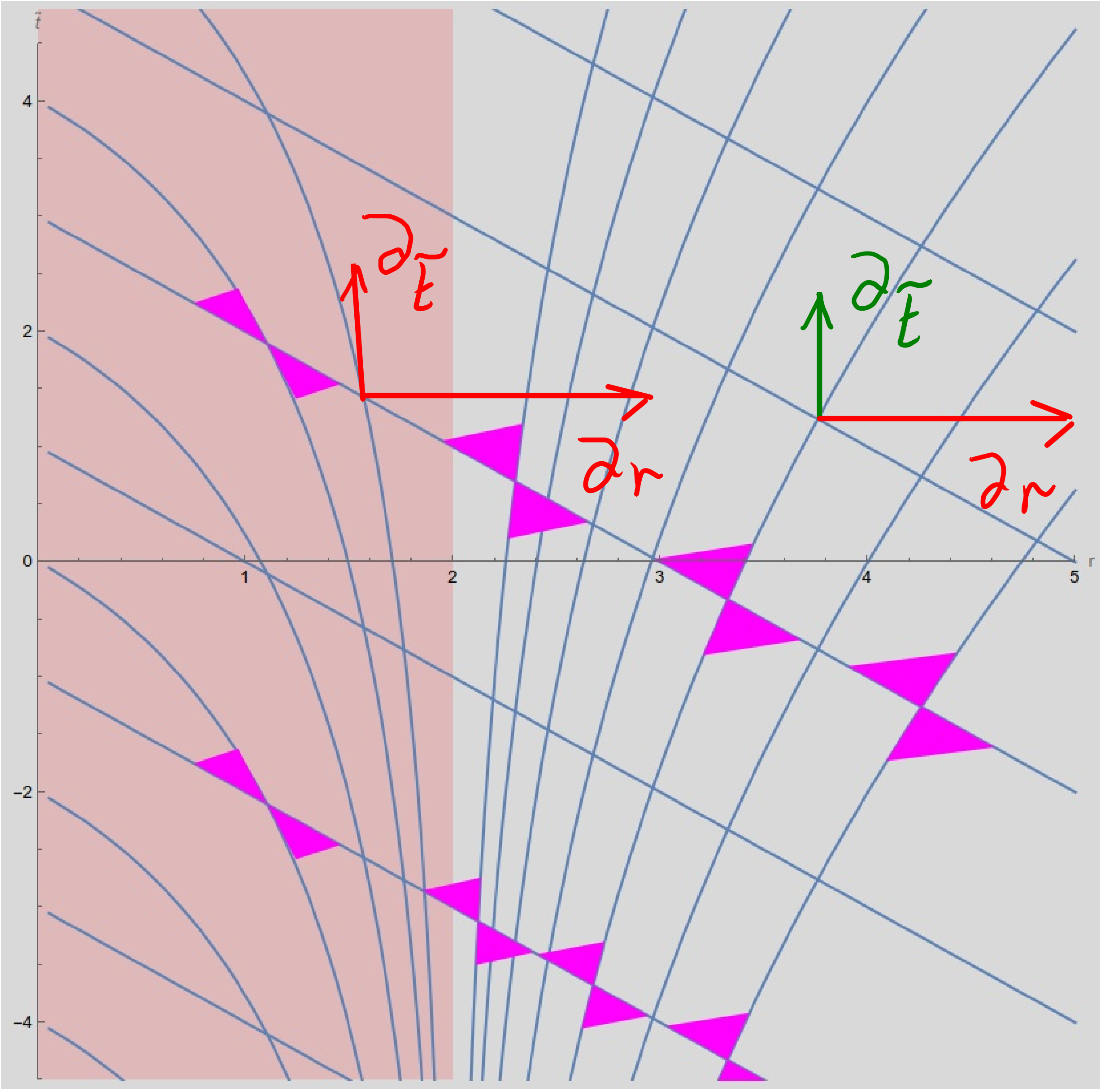

Light cone structure:

radial null curves: $d\theta = d\varphi = 0$

$ds^2 = 0$

Eddington-Finkelstein coordinates

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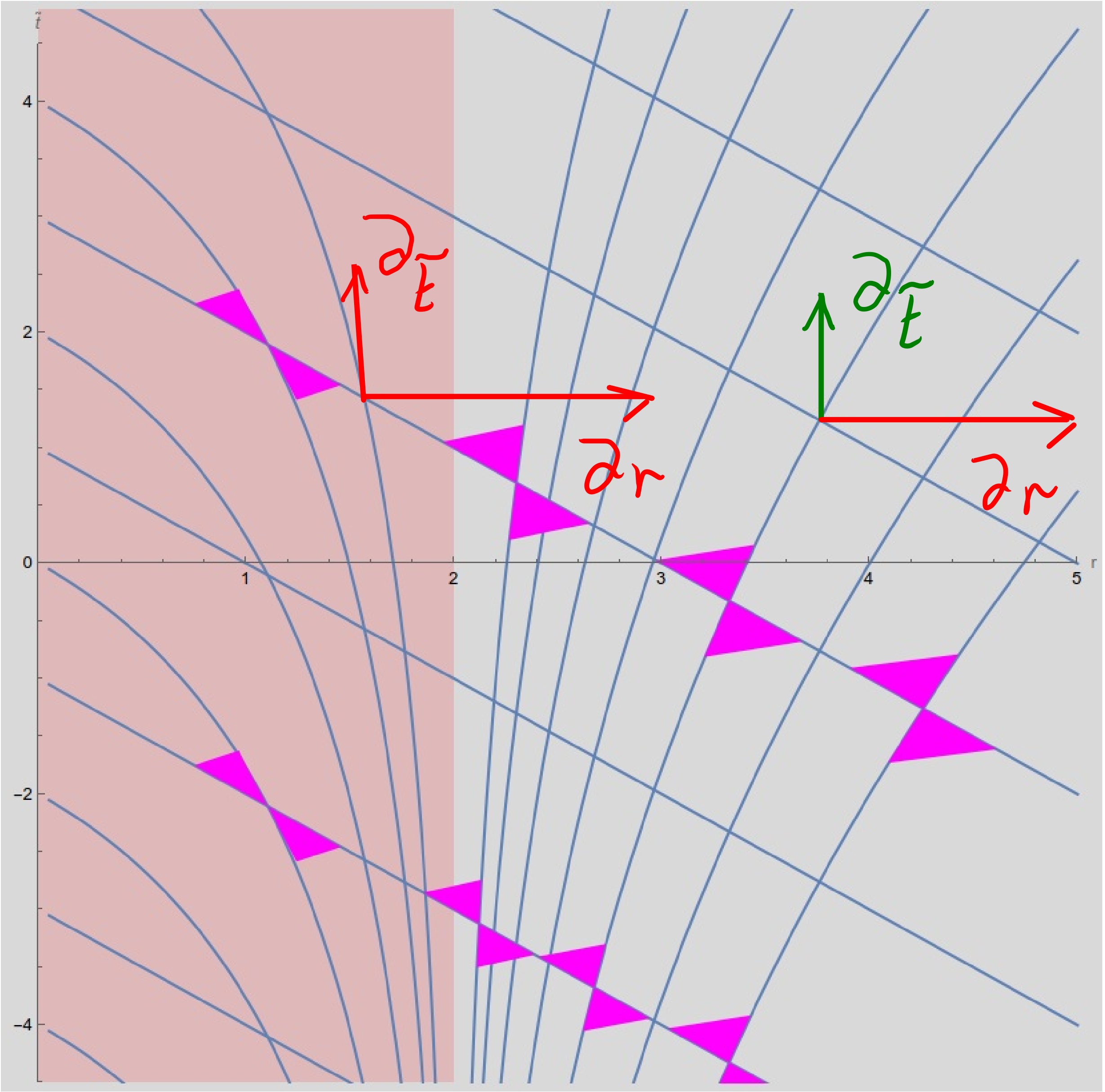

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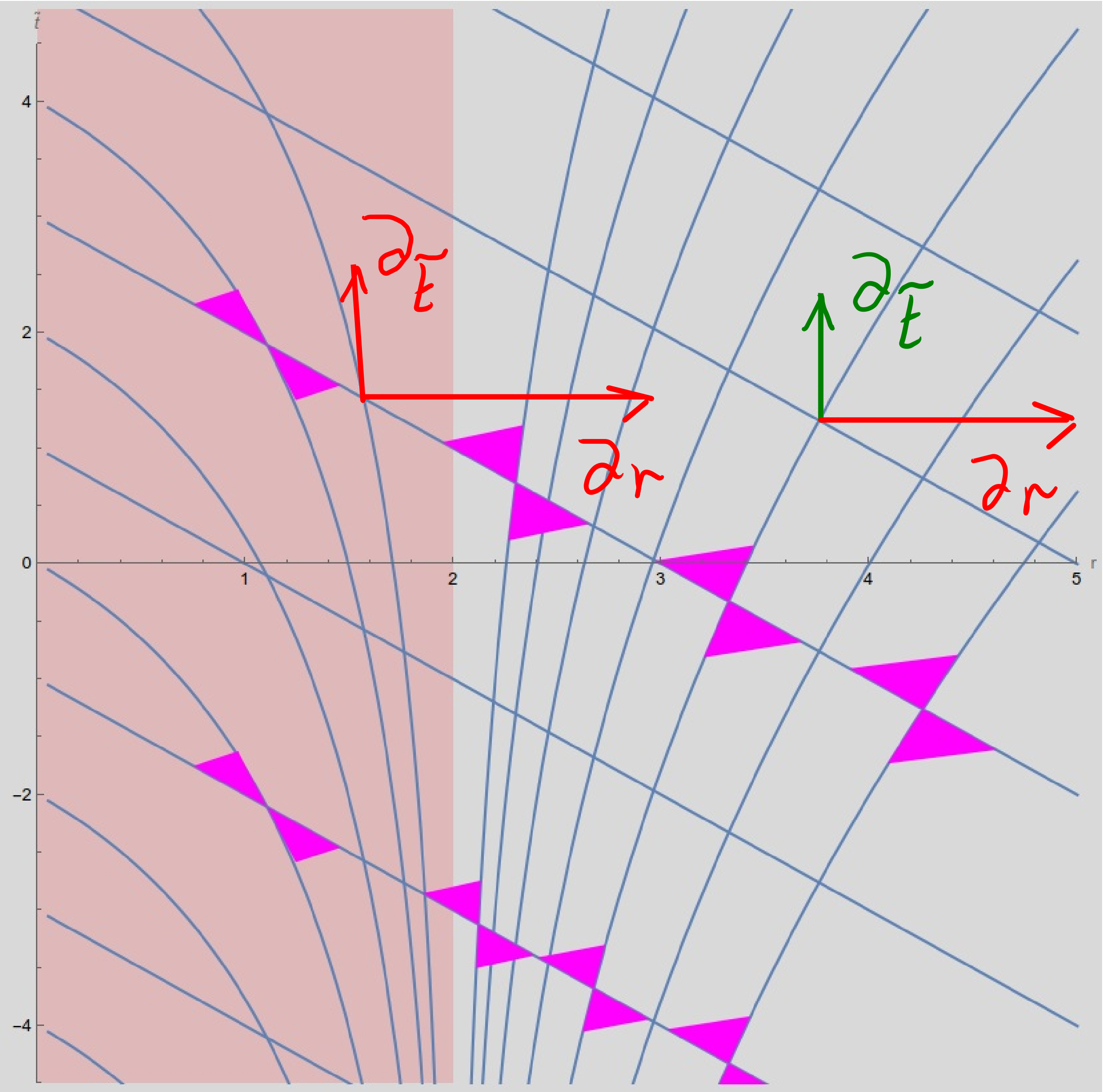
$$(d\tilde{t} + dr)\left[-\left(1 - \frac{2M}{r}\right) d\tilde{t} + \left(1 + \frac{2M}{r}\right) dr\right] = 0$$



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Light cone structure:

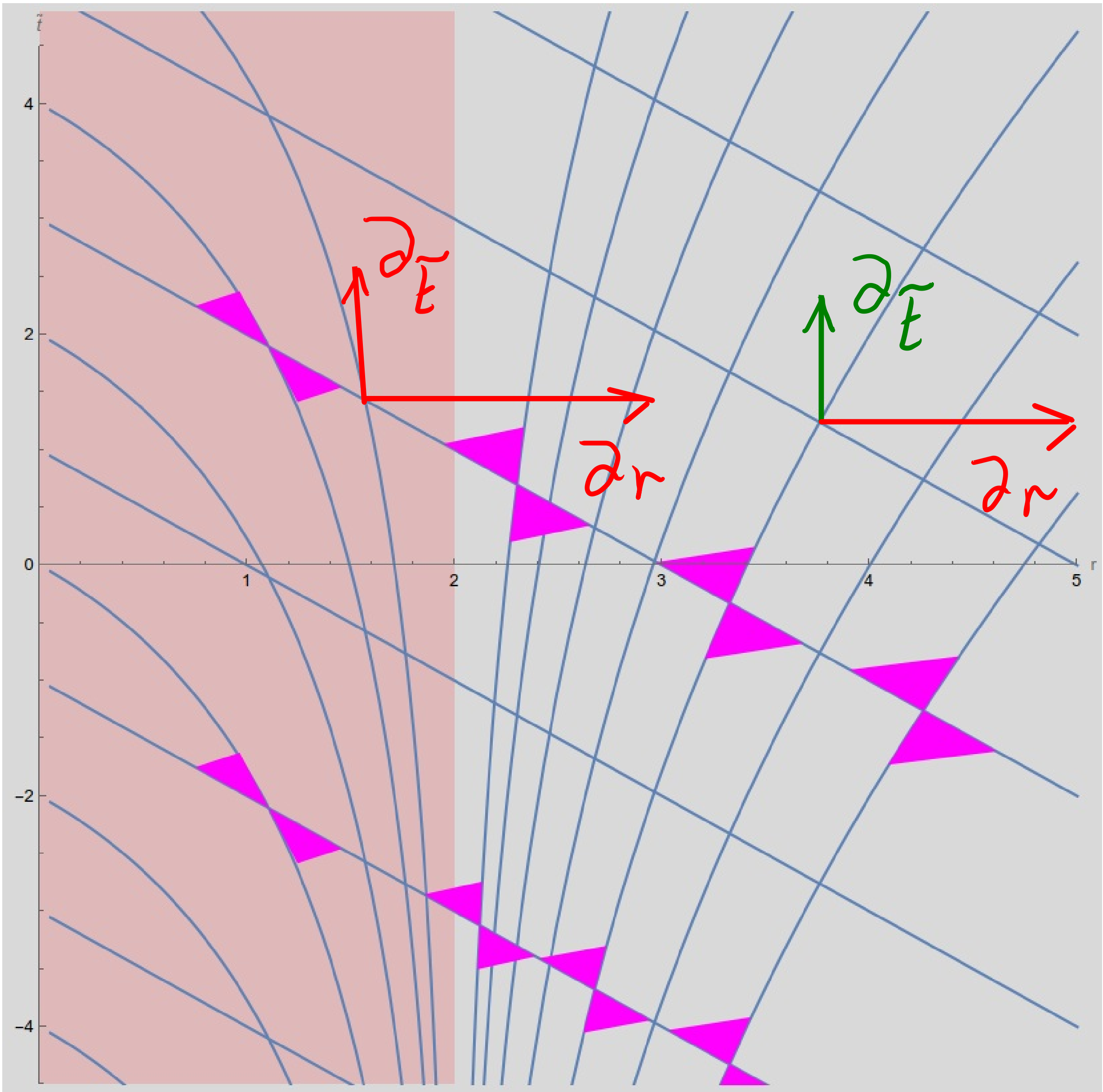
radial null curves: $d\theta = d\varphi = 0$

$$ds^2 = 0 \Rightarrow$$

$$-\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + \left(1 + \frac{2M}{r}\right) dr^2 + \frac{4M}{r} d\tilde{t}\, dr = 0 \Rightarrow$$

$$(d\tilde{t} + dr)\left[-\left(1 - \frac{2M}{r}\right) d\tilde{t} + \left(1 + \frac{2M}{r}\right) dr\right] = 0 \Rightarrow$$

$$\begin{cases} d\tilde{t} + dr = 0 \\ -\left(1 - \frac{2M}{r}\right) d\tilde{t} + \left(1 + \frac{2M}{r}\right) dr = 0 \end{cases}$$



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Light cone structure:

$$\frac{d\tilde{t}}{dr} = -1$$

$$\frac{d\tilde{t}}{dr} = \frac{1 + \frac{2M}{r}}{1 - \frac{2M}{r}}$$

$$r = 2M \quad \left(dr = 0,\ \frac{d\tilde{t}}{dr} = +\infty\right)$$

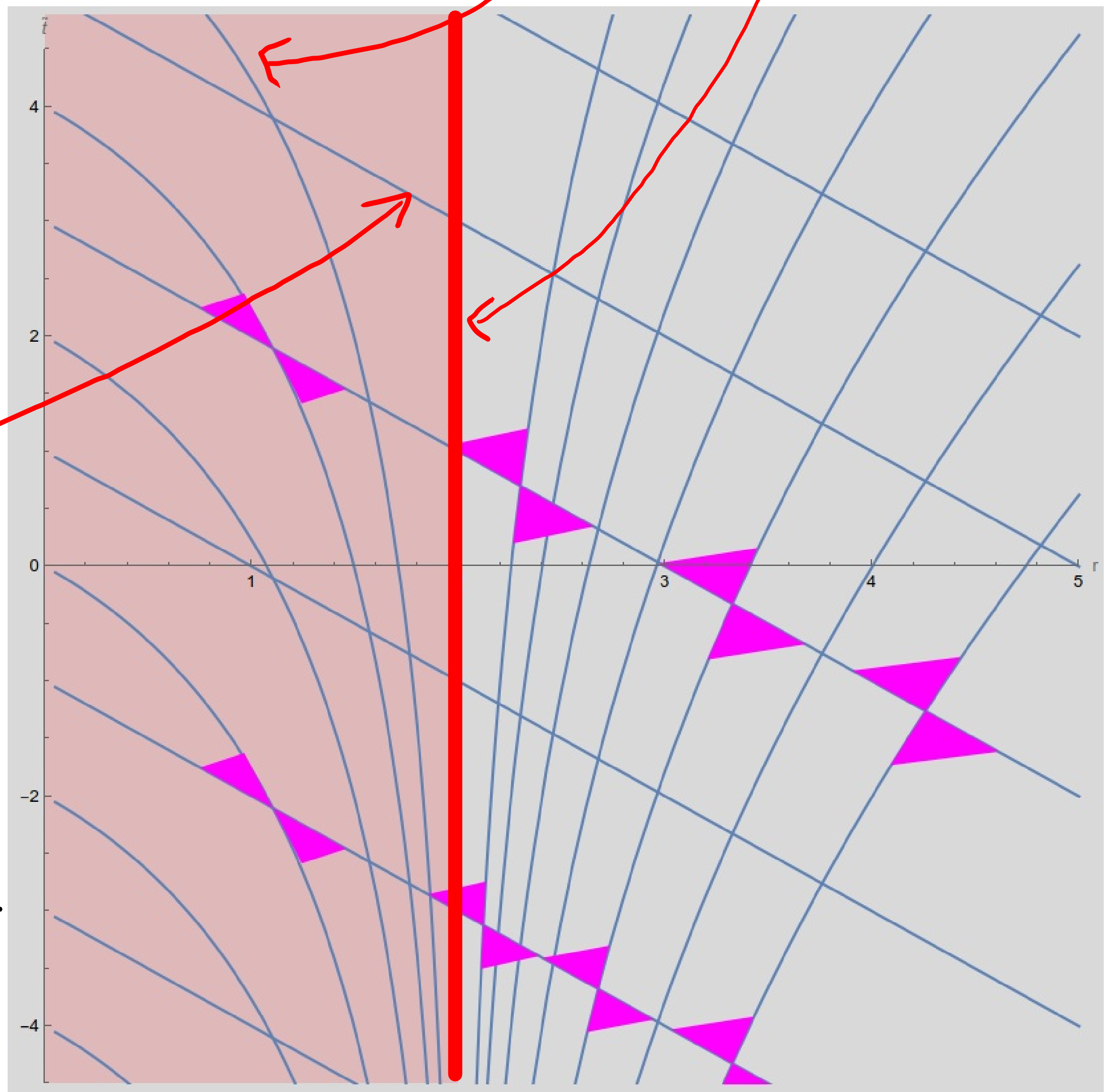
$$\begin{cases} d\tilde{t} + dr = 0 \\ -\left(1 - \frac{2M}{r}\right)d\tilde{t} + \left(1 + \frac{2M}{r}\right)dr = 0 \end{cases}$$

$\frac{d\tilde{t}}{dr} = \frac{1 + \frac{2M}{r}}{1 - \frac{2M}{r}}$

$r = 2M$ (null)

$\frac{d\tilde{t}}{dr} = -1$

$(-45°)$



Eddington-Finkelstein coordinates

$$t \to \tilde{t} \qquad \tilde{t} = t \qquad + 2M \ln\left|\frac{r}{2M} - 1\right|$$

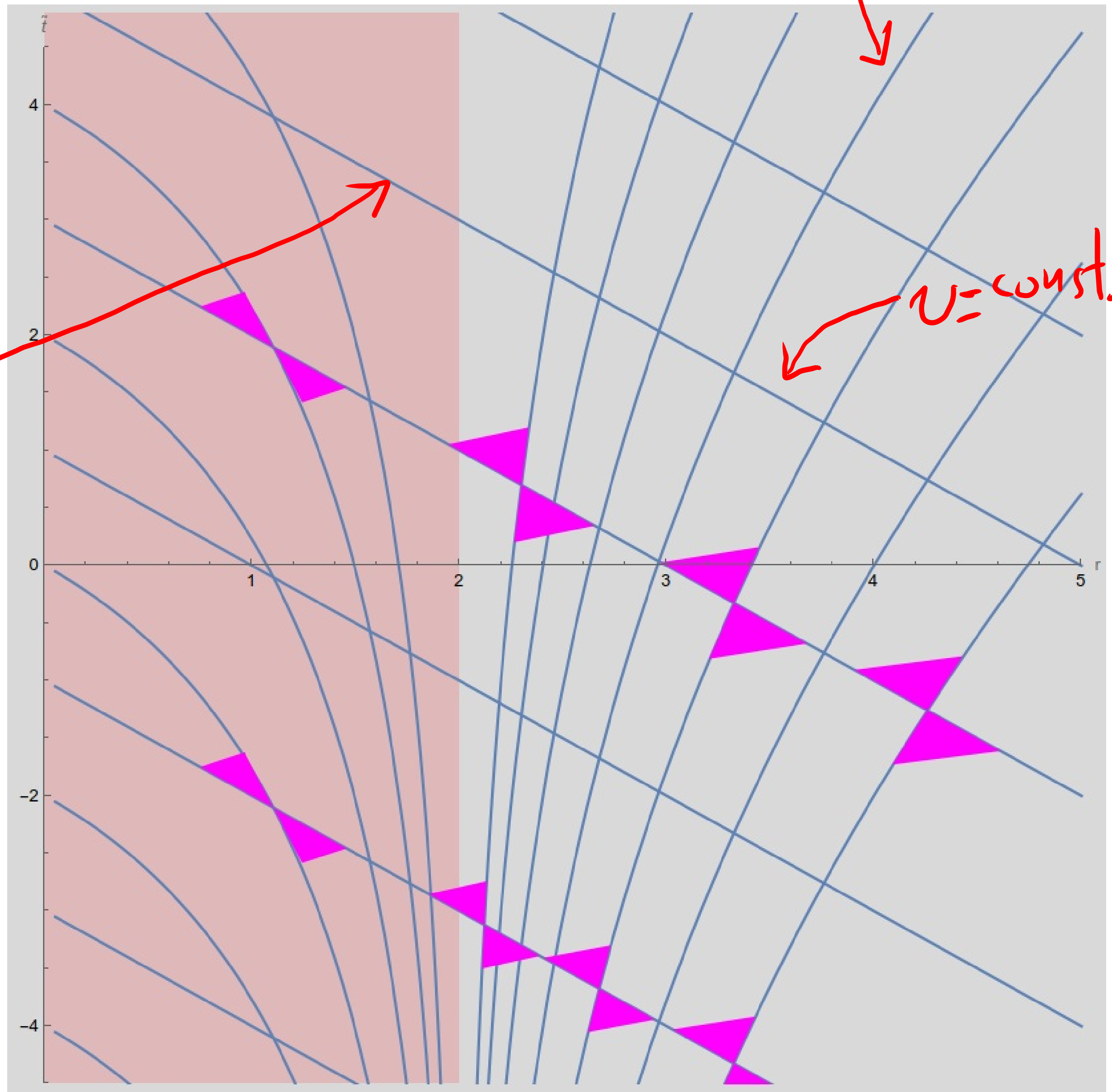
$$\frac{\tilde{t}}{4M} = \frac{r}{4M} + \ln\left|\frac{r}{2M} - 1\right| + \frac{\tilde{t}_0}{4M}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + \left(1 + \frac{2M}{r}\right) dr^2 + \frac{4M}{r}\, d\tilde{t}\, dr + r^2\, d\Omega^2$$

Light cone structure:

$$\left.\begin{array}{l} \dfrac{d\tilde{t}}{dr} = -1 \\[2ex] \dfrac{d\tilde{t}}{dr} = \dfrac{1 + \frac{2M}{r}}{1 - \frac{2M}{r}} \\[2ex] r = 2M \end{array}\right\} \Rightarrow \begin{array}{l} \tilde{t} = -r + \tilde{t}_0 \\[2ex] \dfrac{\tilde{t}}{4M} = \dfrac{r}{4M} + \ln\left|\dfrac{r}{2M} - 1\right| + \dfrac{\tilde{t}_0}{4M} \\[2ex] r = 2M \end{array}$$

$\tilde{t} = -r + \tilde{t}_0$



Eddington-Finkelstein coordinates

$t \to \tilde{t} \qquad \tilde{t} = t + 2M \ln\left|\frac{r}{2M} - 1\right|$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + \left(1 + \frac{2M}{r}\right) dr^2 + \frac{4M}{r}\, d\tilde{t}\, dr + r^2 d\Omega^2$$

Light cone structure:

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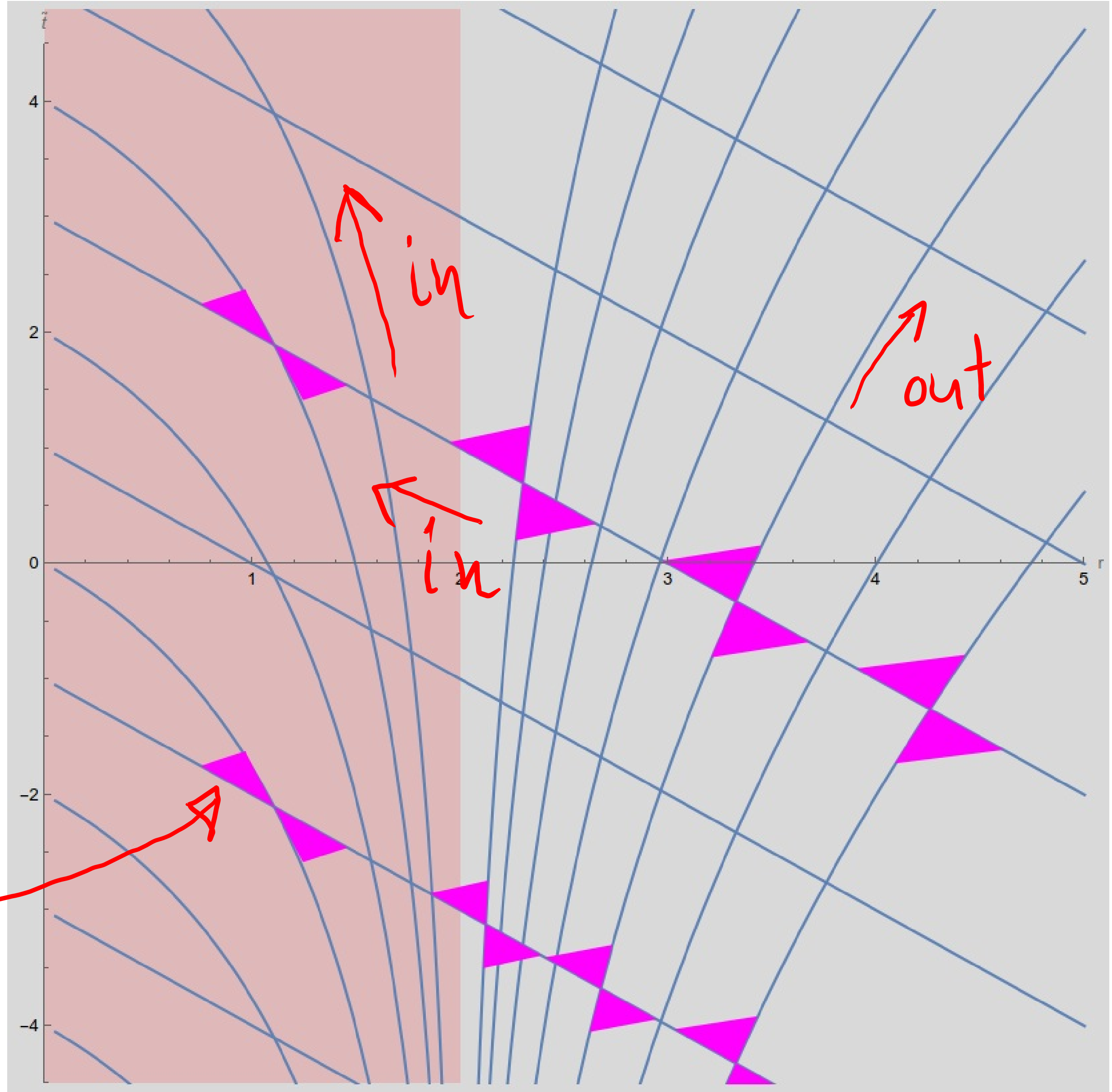

as $r \to 0$, cone tips towards $r=0$ axis and closes

The horizon

Back to E-F coordinates: $ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dv\,dr + r^2\,d\Omega^2$

$$(g_{\mu\nu}) = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad (g^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1-\frac{2M}{r} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}$$

The horizon

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- The horizon is a null surface: Defined by $r = f(v, r, \theta, \varphi) = 2M$

A 3-d surface!

The horizon

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  normal vector: $\eta^\mu = g^{\mu\nu}\nabla_\nu f(v,r,\theta,\varphi) = g^{\mu\nu}\partial_\nu r$

The horizon

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  $\eta^v = g^{vr}\partial_r r = 1\cdot 1 = 1$

The horizon

Back to E-F coordinates: $ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dv\,dr + r^2 d\Omega^2$

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  $\eta^r = g^{rr}\partial_r r = \left(1-\frac{2M}{r}\right)\cdot 1 = \left(1-\frac{2M}{r}\right)$

The horizon

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The horizon

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$$\eta^\mu \eta_\mu = g_{\mu\nu}\eta^\mu\eta^\nu$$

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- $v = \text{const} \Rightarrow ds^2 = (2M)^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$

The horizon

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- $v = \text{const} \Rightarrow ds^2 = (2M)^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$

  a 2-sphere of area: $A = 4\pi(2M)^2 = 16\pi M^2$ $\quad$ v-independent!

Kruskal - Szekeres coordinates

(V, U, θ, φ)

$r > 2M$

$$U = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

$$V = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$r < 2M$

$$U = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$$V = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

- smooth interpolation between $r > 2M$, $r < 2M$ regions
- simple light cone structure: $\pm 45°$ lines
- extendible $\rightarrow$ wormhole (non traversable)
- Penrose diagram makes clear spacetime structure

Kruskal - Szekeres coordinates

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$$\Rightarrow \quad U^2 - V^2 = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}$$

no absolute value

valid for $r > 2M$ & $r < 2M$

$r < 2M$

$$U = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

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- $\frac{V}{U} = \tanh\left(\frac{t}{4M}\right)$

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- $\frac{U}{V} = \tanh\left(\frac{t}{4M}\right)$

- fixed r: $U^2 - V^2 = \text{const}$
- fixed t: $V = \text{const} \cdot U$

Kruskal - Szekeres coordinates

(V, U, θ, φ)

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$$V = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

- $\frac{V}{U} = \tanh\left(\frac{t}{4M}\right)$

$$\Rightarrow \quad U^2 - V^2 = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}$$

$r < 2M$

$$U = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$$V = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

- $\frac{U}{V} = \tanh\left(\frac{t}{4M}\right)$

- fixed r: $U^2 - V^2 = \text{const}$

$r = 2M \quad V = \pm U$

$r > 2M \quad |V| < |U| \qquad r < 2M \quad |V| > |U|$

Kruskal - Szekeres coordinates

(V, U, θ, φ)

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$$V = \tanh\left(\frac{t}{4M}\right) U$$

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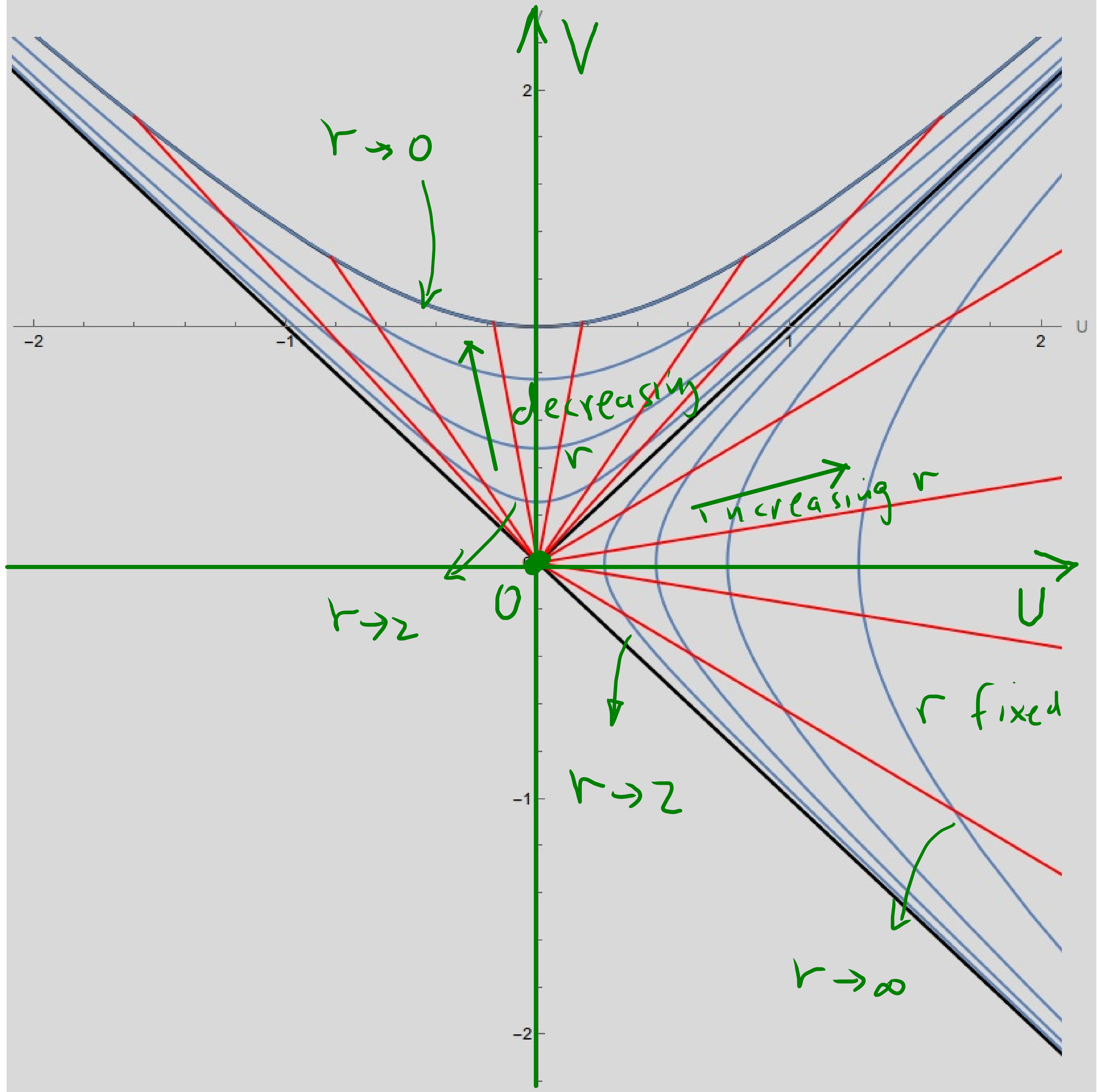
$$V = \coth\left(\frac{t}{4M}\right) U$$

$$U^2 - V^2 = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}$$



- fixed r: $U^2 - V^2 = \text{const}$

$r = 2M \quad V = \pm U$

$r > 2M \quad |V| < |U| \qquad r < 2M \quad |V| > |U|$

Kruskal - Szekeres coordinates

(V, U, θ, φ)

$r > 2M$

$$U = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

$$V = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$$\boxed{V = \tanh\left(\frac{t}{4M}\right) U}$$

$r < 2M$

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$$V = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

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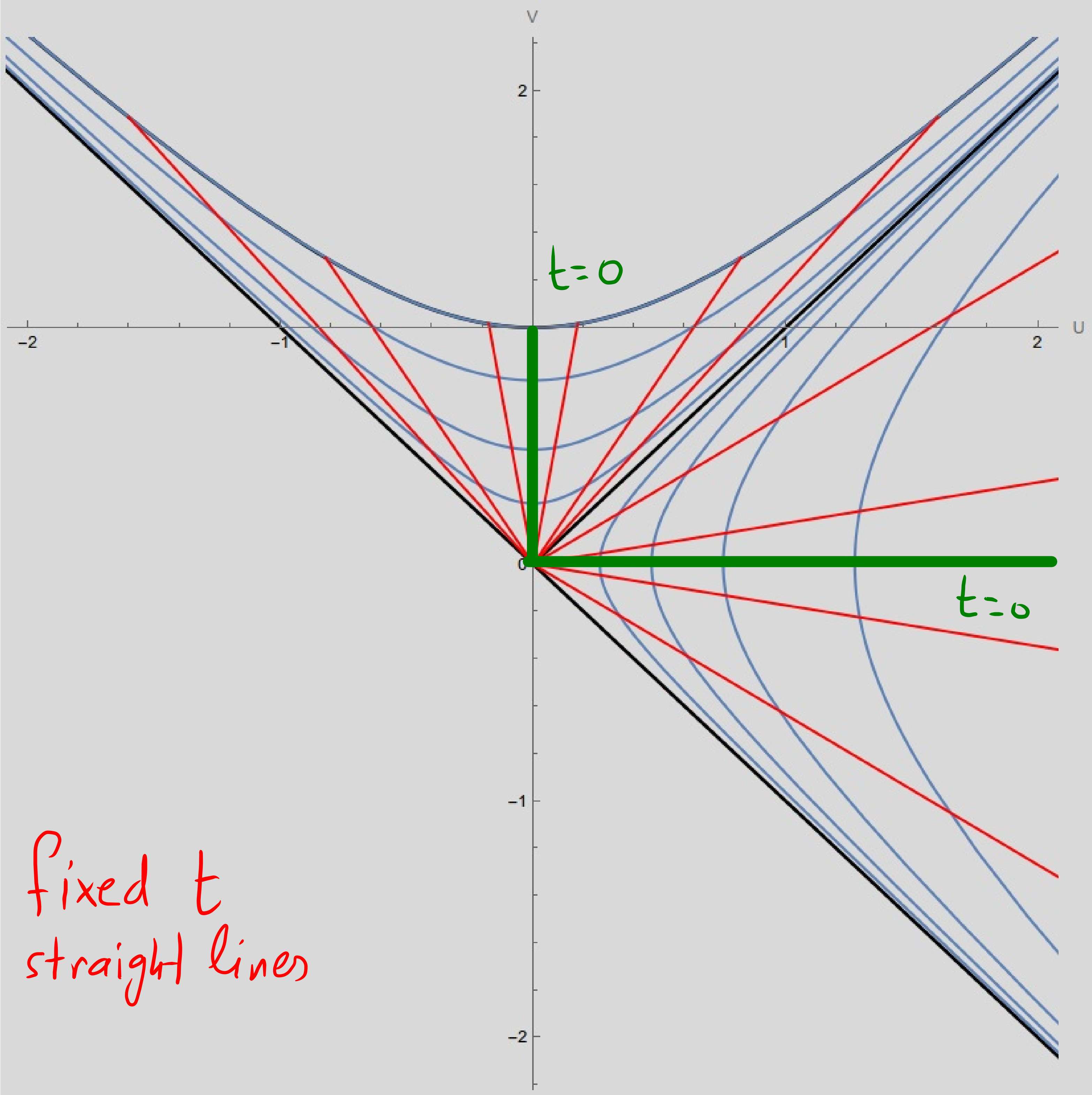
$$U^2 - V^2 = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}$$



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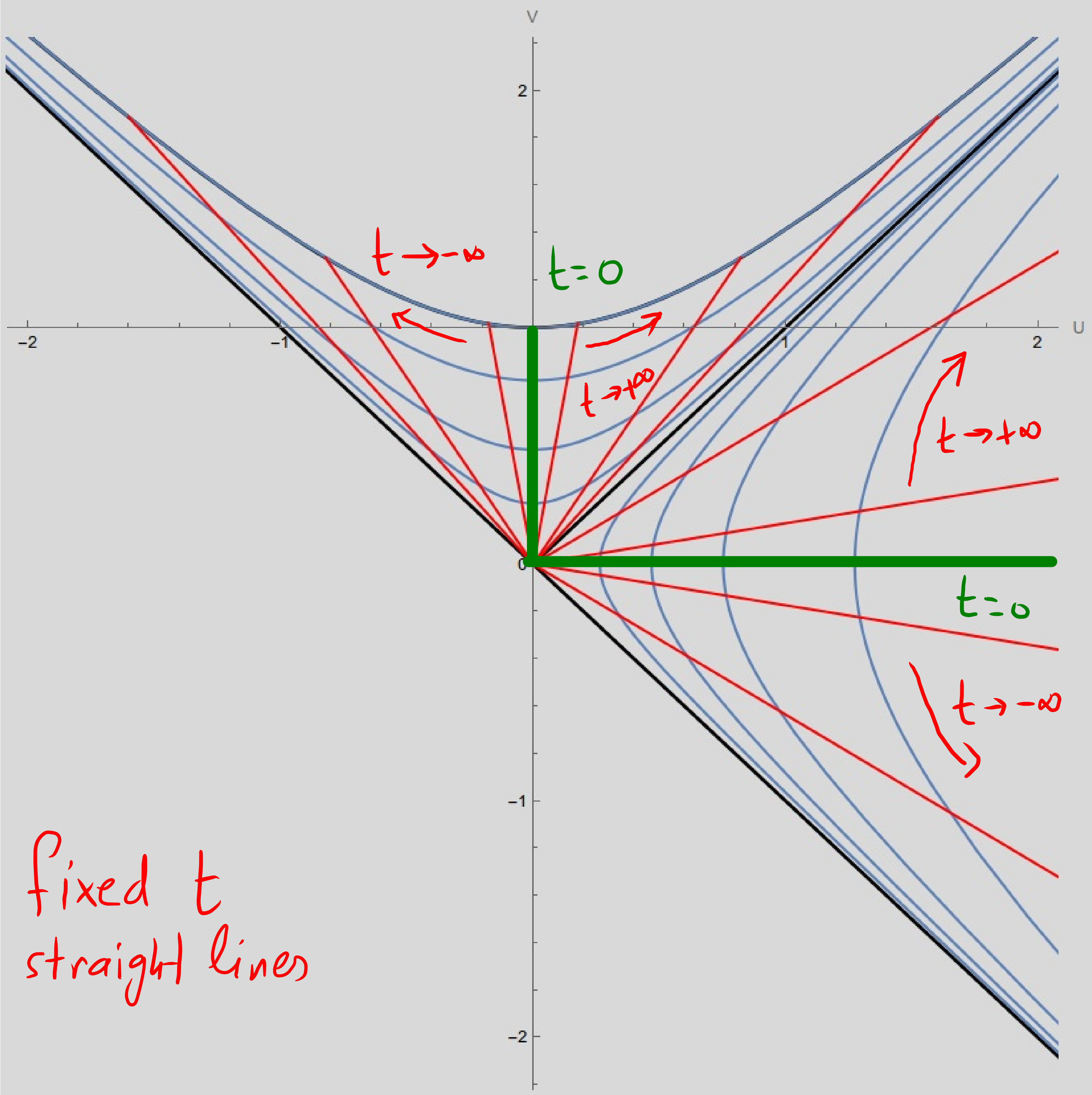

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$r < 2M$

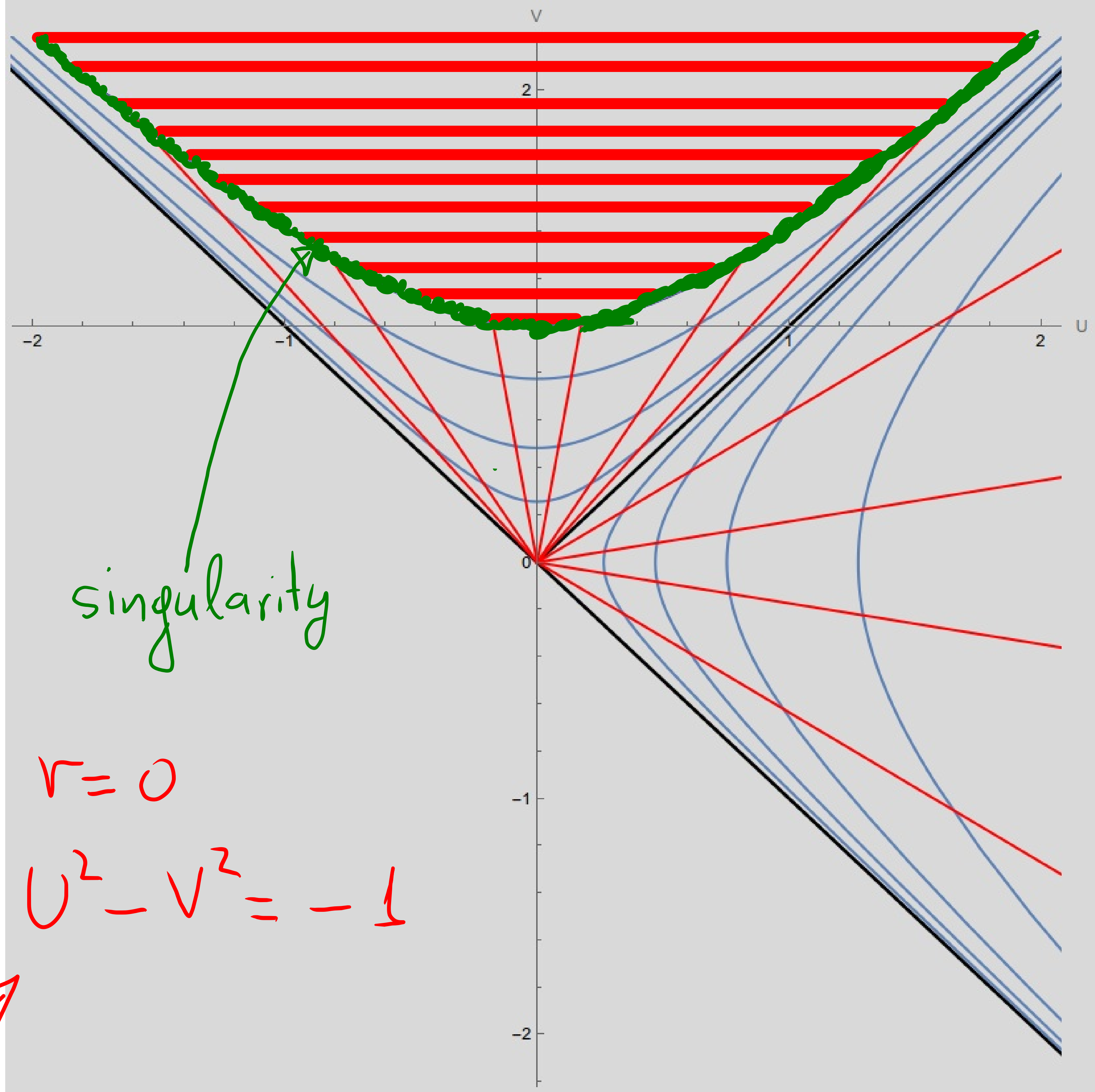
$$U = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$$V = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

$$V = \coth\left(\frac{t}{4M}\right) U$$

$$\boxed{U^2 - V^2 = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}}$$

$r = 0$



• fixed r: $U^2 - V^2 = \text{const}$

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$$V = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

$$V = \coth\left(\frac{t}{4M}\right) U$$

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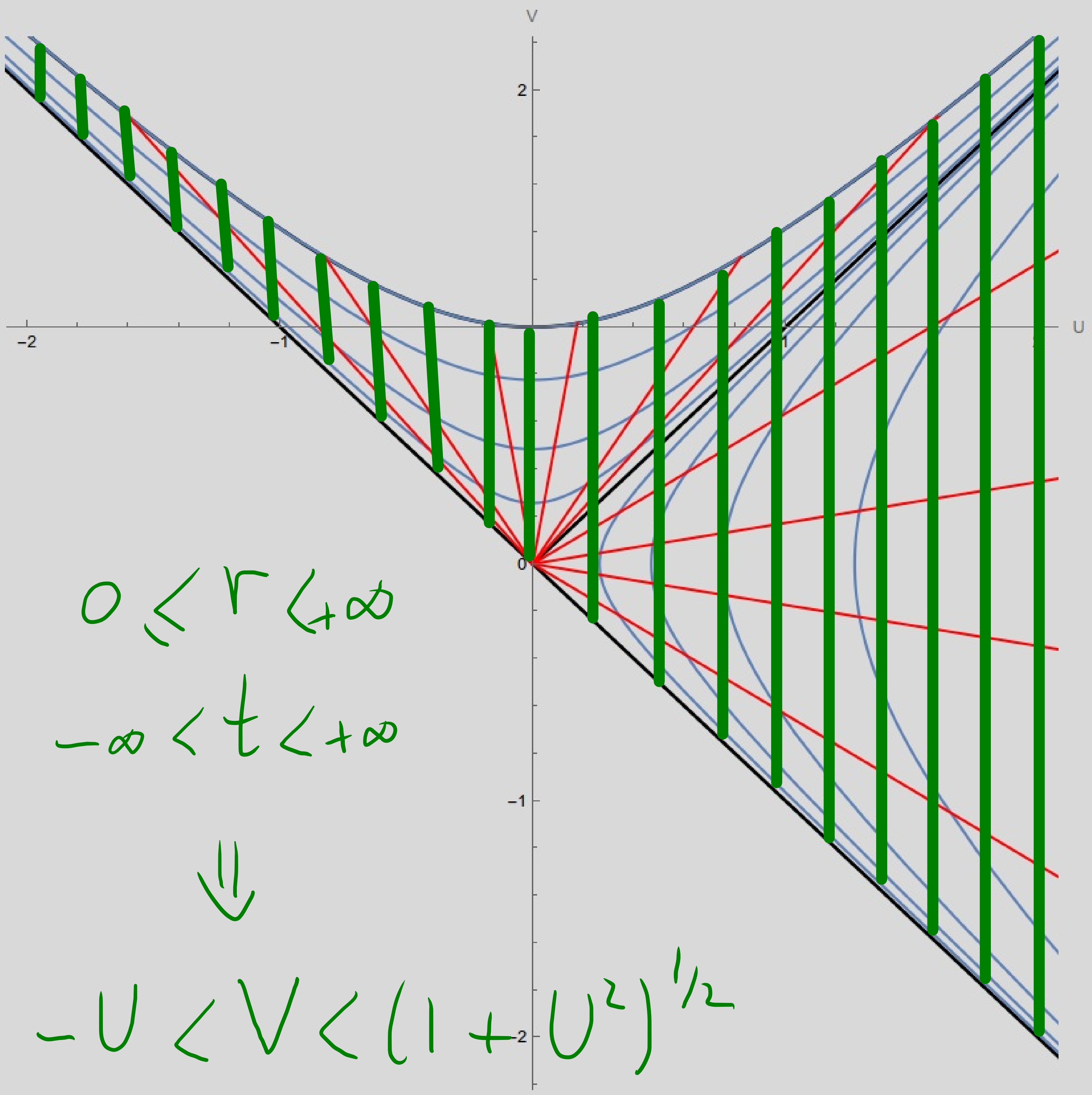

$0 \leq r < +\infty$

$-\infty < t < +\infty$

$\Downarrow$

$-U < V < (1 + U^2)^{1/2}$

- fixed r: $U^2 - V^2 = \text{const}$

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$$V = \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

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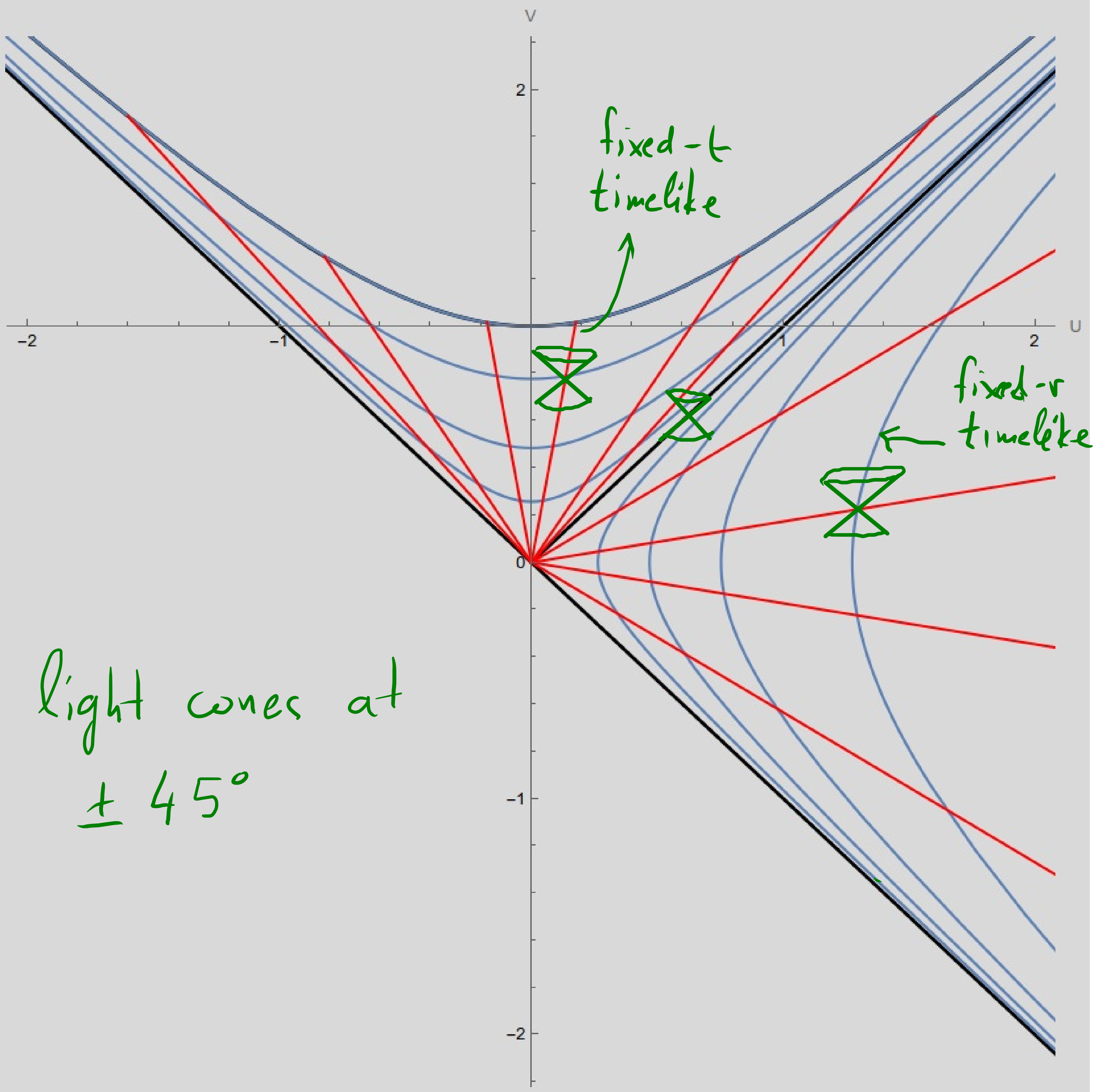

light cones at $\pm 45°$

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$r > 2M$ $\quad |V| < |U|$ $\quad r < 2M$ $\quad |V| > |U|$

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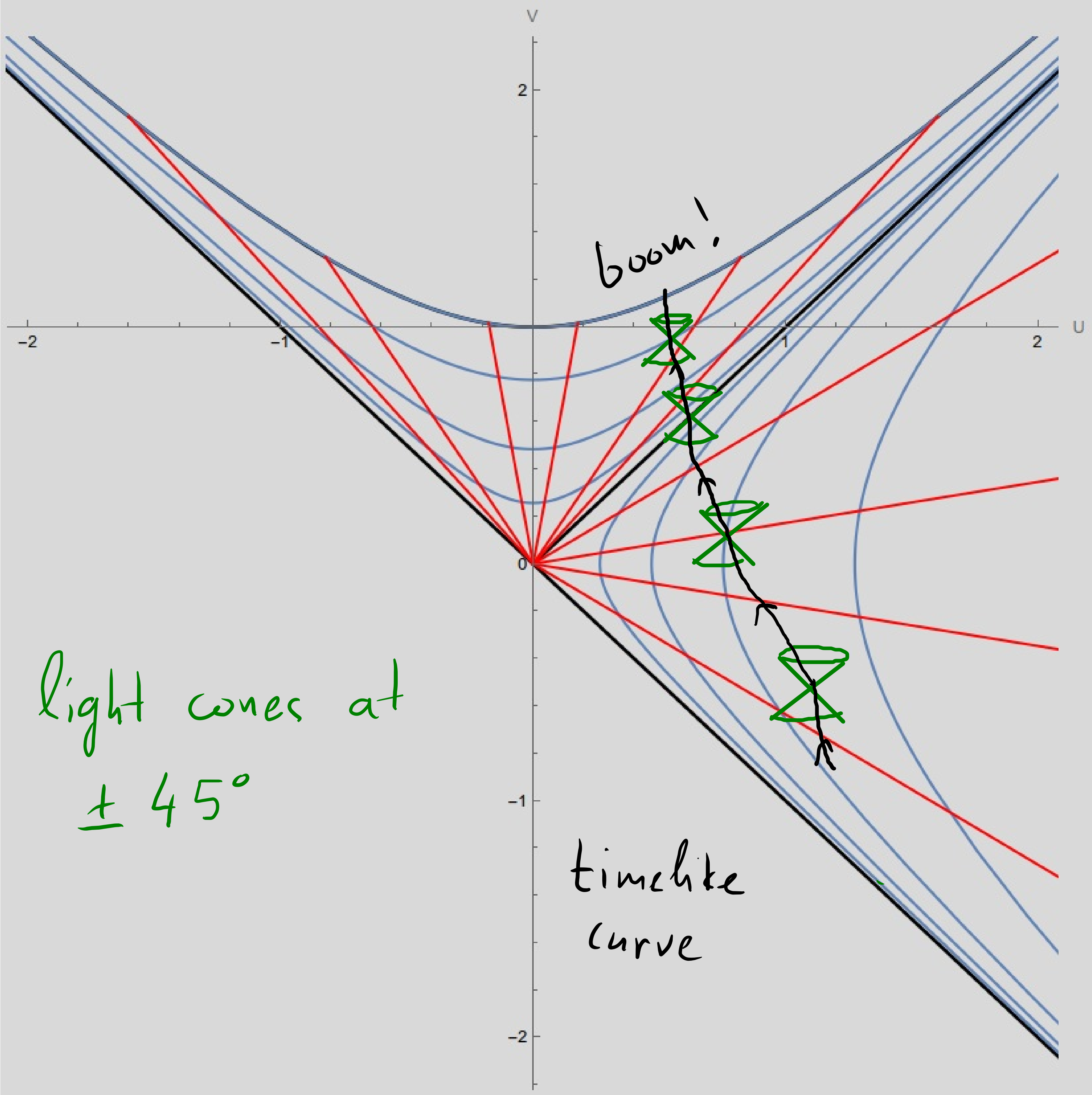
$$V = \coth\left(\frac{t}{4M}\right) U$$

$$U^2 - V^2 = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}$$



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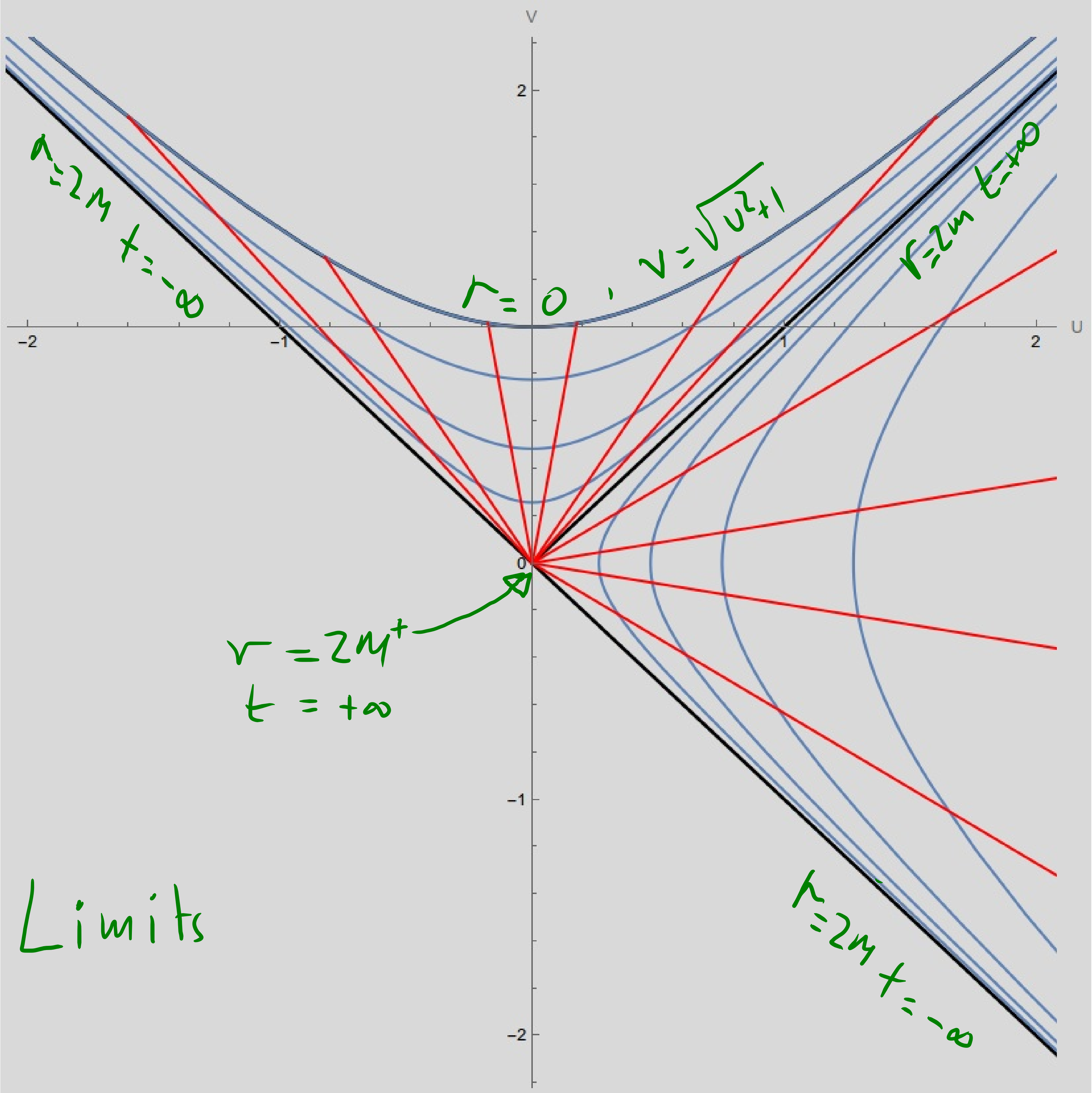
$$V = \coth\left(\frac{t}{4M}\right) U$$

$$U^2 - V^2 = \left(\frac{r}{2M}-1\right) e^{\frac{r}{2M}}$$



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Compute the metric

$r > 2M$

$$V = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/2M} \sinh\left(\frac{t}{4M}\right)$$

$$U = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/2M} \cosh\left(\frac{t}{4M}\right)$$

Compute the metric

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$\Rightarrow$

$$\partial_t V = \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

Compute the metric

$\underline{r > 2M}$

$$V = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/2M} \sinh\left(\frac{t}{4M}\right)$$

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$$\partial_t V = \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

$$\partial_r V = \frac{1}{4M}\,\frac{r}{2M}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

Compute the metric

$\underline{r > 2M}$

$$V = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/2M} \sinh\left(\frac{t}{4M}\right)$$

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$\Rightarrow$

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$$\partial_r V = \frac{1}{4M} \frac{r}{2M} \left(\frac{r}{2M} - 1\right)^{-1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$$\partial_t U = \frac{1}{4M} \left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

Compute the metric

$\underline{r > 2M}$

$$V = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/2M} \sinh\left(\frac{t}{4M}\right)$$

$$U = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/2M} \cosh\left(\frac{t}{4M}\right)$$

$$\Rightarrow$$

$$\partial_t V = \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

$$\partial_r V = \frac{1}{4M}\,\frac{r}{2M}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$$\partial_t U = \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$$\partial_r U = \frac{1}{4M}\,\frac{r}{2M}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

Compute the metric

$r > 2M$

$$V = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/2M} \sinh\left(\frac{t}{4M}\right)$$

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$$\partial_t U = \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$$\partial_r U = \frac{1}{4M}\,\frac{r}{2M}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

$$dV = \partial_t V\, dt + \partial_r V\, dr$$

$$dU = \partial_t U\, dt + \partial_r U\, dr$$

Compute the metric

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$$\partial_r U = \frac{1}{4M}\,\frac{r}{2M}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

$$dV = \partial_t V\, dt + \partial_r V\, dr = \frac{r}{8M^2}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{\frac{r}{4M}}\left[\left(1 - \frac{2M}{r}\right)\cosh\left(\frac{t}{4M}\right)dt + \sinh\left(\frac{t}{4M}\right) dr\right]$$

$$dU = \partial_t U\, dt + \partial_r U\, dr$$

Compute the metric

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$\Rightarrow$

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$$\partial_r V = \frac{1}{4M}\,\frac{r}{2M}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$$\partial_t U = \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$$\partial_r U = \frac{1}{4M}\,\frac{r}{2M}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

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$$dU = \partial_t U\, dt + \partial_r U\, dr = \frac{r}{8M^2}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{\frac{r}{4M}}\left[\left(1 - \frac{2M}{r}\right)\sinh\left(\frac{t}{4M}\right) dt + \cosh\left(\frac{t}{4M}\right) dr\right]$$

$$-dV^2 + dU^2 = \frac{r^2}{64M^4}\left(\frac{r}{2M}-1\right)^{-1} e^{r/2M} \Big\{$$

$$-\left(1-\frac{2M}{r}\right)^2 \cosh^2\left(\frac{t}{4M}\right) dt^2 - \sinh^2\left(\frac{t}{4M}\right) dr^2 - 2\left(1-\frac{2M}{r}\right)\cosh\left(\frac{t}{4M}\right)\sinh\left(\frac{t}{4M}\right) dt\, dr$$

$$+\left(1-\frac{2M}{r}\right)^2 \sinh^2\left(\frac{t}{4M}\right) dt^2 + \cosh^2\left(\frac{t}{4M}\right) dr^2 + 2\left(1-\frac{2M}{r}\right)\sinh\left(\frac{t}{4M}\right)\cosh\left(\frac{t}{4M}\right) dt\, dr \Big\}$$

$$dV = \partial_t V\, dt + \partial_r V\, dr = \frac{r}{8M^2}\left(\frac{r}{2M}-1\right)^{-1/2} e^{\frac{r}{4M}}\left[\left(1-\frac{2M}{r}\right)\cosh\left(\frac{t}{4M}\right)dt + \sinh\left(\frac{t}{4M}\right) dr\right]$$

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$$+\left(1-\frac{2M}{r}\right)^2 \underline{\sinh^2\left(\frac{t}{4M}\right)}\, dt^2 + \underline{\cosh^2\left(\frac{t}{4M}\right)}\, dr^2 + 2\left(1-\frac{2M}{r}\right)\cancel{\sinh\left(\frac{t}{4M}\right)\cosh\left(\frac{t}{4M}\right) dt\, dr} \Bigg\}$$

(-1) $\qquad$ $(+1)$

$$dV = \partial_t V\, dt + \partial_r V\, dr = \frac{r}{8M^2}\left(\frac{r}{2M}-1\right)^{-1/2} e^{\frac{r}{4M}}\left[\left(1-\frac{2M}{r}\right)\cosh\left(\frac{t}{4M}\right)dt + \sinh\left(\frac{t}{4M}\right)dr\right]$$

$$dU = \partial_t U\, dt + \partial_r U\, dr = \frac{r}{8M^2}\left(\frac{r}{2M}-1\right)^{-1/2} e^{\frac{r}{4M}}\left[\left(1-\frac{2M}{r}\right)\sinh\left(\frac{t}{4M}\right)dt + \cosh\left(\frac{t}{4M}\right)dr\right]$$

$$
-dV^2 + dU^2 = \frac{r^2}{64M^4}\left(\frac{r}{2M}-1\right)^{-1} e^{r/2M} \Big\{
$$

$$
-\left(1-\frac{2M}{r}\right)^2 \underline{\cosh^2\left(\frac{t}{4M}\right)}\, dt^2 - \underline{\sinh^2\left(\frac{t}{4M}\right)}\, dr^2 - 2\left(1-\frac{2M}{r}\right)\cosh\cancel{\left(\frac{t}{4M}\right)} \sinh\left(\frac{t}{4M}\right) dt\, dr
$$

$$
+\left(1-\frac{2M}{r}\right)^2 \underline{\sinh^2\left(\frac{t}{4M}\right)}\, dt^2 + \underline{\cosh^2\left(\frac{t}{4M}\right)}\, dr^2 + 2\left(1-\frac{2M}{r}\right)\sinh\cancel{\left(\frac{t}{4M}\right)} \cosh\left(\frac{t}{4M}\right) dt\, dr \Big\}
$$

$$
= \frac{r^2}{64M^4}\left(\frac{r}{2M}-1\right)^{-1} e^{r/2M} \left\{ -\left(1-\frac{2M}{r}\right)^2 dt^2 + dr^2 \right\}
$$

$$-dV^2 + dU^2 = \frac{r^2}{64M^4}\left(\frac{r}{2M}-1\right)^{-1} e^{r/2M}\Big\{$$

$$-\left(1-\frac{2M}{r}\right)^2 \cosh^2\left(\frac{t}{4M}\right) dt^2 - \sinh^2\left(\frac{t}{4M}\right) dr^2 - 2\left(1-\frac{2M}{r}\right)\cancel{\cosh\left(\frac{t}{4M}\right)} \sinh\left(\frac{t}{4M}\right) dt\,dr$$

$$+\left(1-\frac{2M}{r}\right)^2 \sinh^2\left(\frac{t}{4M}\right) dt^2 + \cosh^2\left(\frac{t}{4M}\right) dr^2 + 2\left(1-\frac{2M}{r}\right)\cancel{\sinh\left(\frac{t}{4M}\right)} \cosh\left(\frac{t}{4M}\right) dt\,dr \Big\}$$

$$= \frac{r^2}{64M^4}\left(\frac{r}{2M}-1\right)^{-1} e^{r/2M}\left\{-\left(1-\frac{2M}{r}\right)^2 dt^2 + dr^2\right\}$$

$$= \frac{r^2}{64M^4}\left(\frac{r}{2M}\right)^{-1}\left(1-\frac{2M}{r}\right)^{-1} e^{r/2M}\left\{-\left(1-\frac{2M}{r}\right)^2 dt^2 + dr^2\right\}$$

$$-dV^2 + dU^2 = \frac{r^2}{64M^4}\left(\frac{r}{2M}-1\right)^{-1} e^{r/2M}\Big\{$$

$$-\left(1-\frac{2M}{r}\right)^2 \underline{\cosh^2\left(\frac{t}{4M}\right)} dt^2 - \underline{\sinh^2\left(\frac{t}{4M}\right)} dr^2 - 2\left(1-\frac{2M}{r}\right)\cancel{\cosh\left(\frac{t}{4M}\right)\sinh\left(\frac{t}{4M}\right) dt\, dr}$$

$$+\left(1-\frac{2M}{r}\right)^2 \underline{\sinh^2\left(\frac{t}{4M}\right)} dt^2 + \underline{\cosh^2\left(\frac{t}{4M}\right)} dr^2 + 2\left(1-\frac{2M}{r}\right)\cancel{\sinh\left(\frac{t}{4M}\right)\cosh\left(\frac{t}{4M}\right) dt\, dr}\Big\}$$

$$= \frac{r^2}{64M^4}\left(\frac{r}{2M}-1\right)^{-1} e^{r/2M}\left\{-\left(1-\frac{2M}{r}\right)^2 dt^2 + dr^2\right\}$$

$$= \frac{r^2}{64M^4}\left(\frac{r}{2M}\right)^{-1}\left(1-\frac{2M}{r}\right)^{-1} e^{r/2M}\left\{-\left(1-\frac{2M}{r}\right)^2 dt^2 + dr^2\right\}$$

$$= \frac{r}{32M^3} e^{r/2M}\left\{-\left(1-\frac{2M}{r}\right) dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2\right\}$$

$$\Rightarrow -\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2=\frac{32M^3}{r}e^{-r/2M}\left[-dV^2+dU^2\right]$$

$$-dV^2+dU^2=$$

$$=\frac{r}{32M^3}e^{r/2M}\left\{-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2\right\}$$

$$\Rightarrow -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 = \frac{32M^3}{r}\, e^{-r/2M}\left[-dV^2+dU^2\right]$$

$$\Rightarrow ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)dr^2 + r^2\, d\Omega^2 =$$

$$= \frac{32M^3}{r}\, e^{-r/2M}\left[-dV^2+dU^2\right] + r^2\, d\Omega^2$$

where $r = r(U,V)$, solution of $U^2 - V^2 = \left(\frac{r}{2M}-1\right)e^{r/2M}$

$$-dV^2+dU^2 =$$

$$= \frac{r}{32M^3}\, e^{r/2M}\left\{-\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2\right\}$$

$$\Rightarrow -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 = \frac{32M^3}{r} e^{-r/2M}\left[-dV^2 + dU^2\right]$$

$$\Rightarrow ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)dr^2 + r^2 d\Omega^2 =$$

$$= \frac{32M^3}{r} e^{-r/2M}\left[-dV^2 + dU^2\right] + r^2 d\Omega^2$$

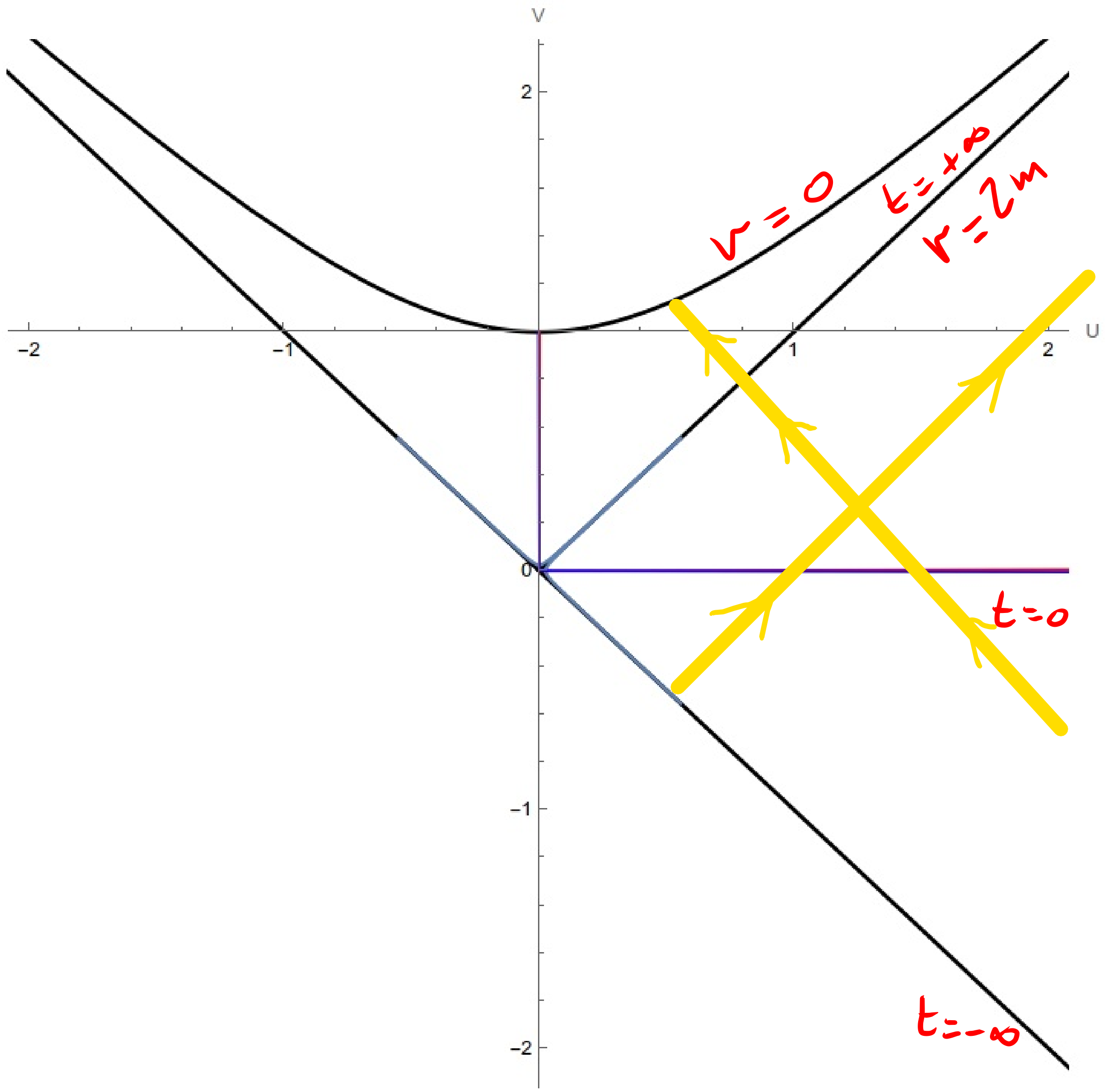
where $r = r(U,V)$, solution of $\quad U^2 - V^2 = \left(\frac{r}{2M} - 1\right) e^{r/2M}$

Exercise: Repeat calculation for $r < 2M$, show that the result is the same

Light Cone Structure

radial null curves: $d\theta = d\varphi = 0$

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \left[-dV^2 + dU^2 \right] = 0 \Rightarrow dV = \pm dU \Rightarrow V = \pm U + V_0$$



Light Cone Structure

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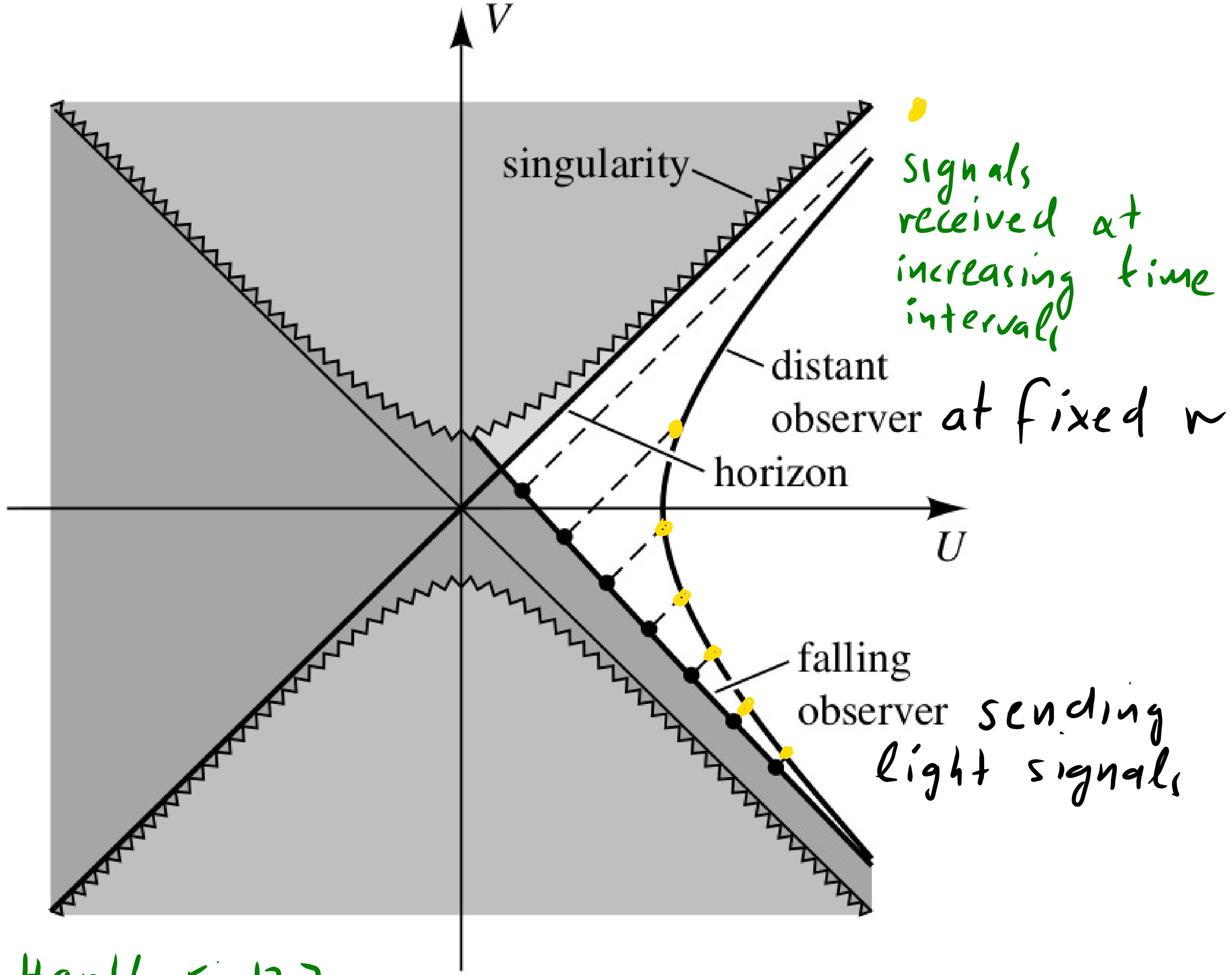
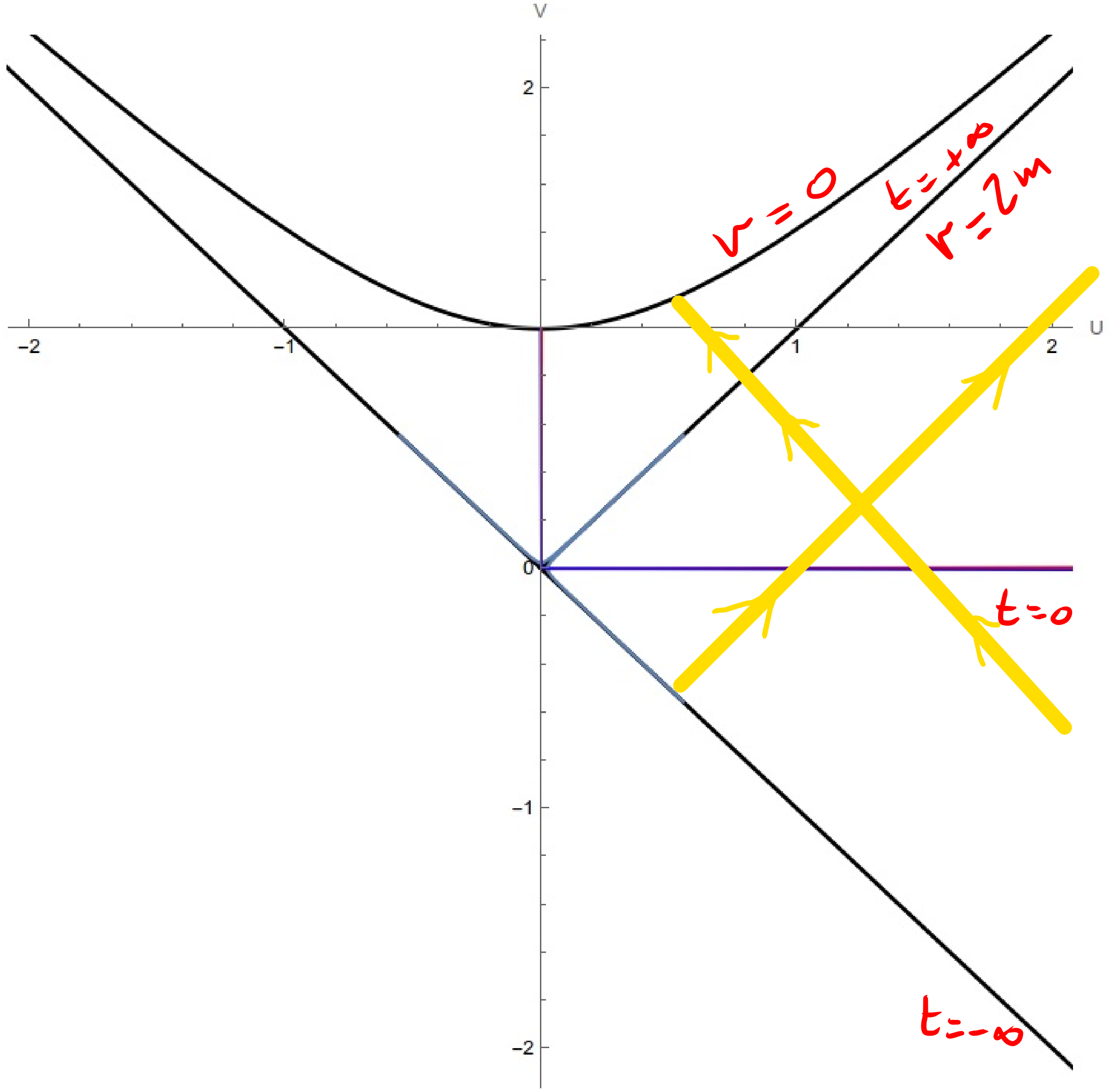
$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \left[-dV^2 + dU^2 \right] = 0 \Rightarrow dV = \pm dU \Rightarrow V = \pm U + V_0$$



Hartle, Fig 12.7

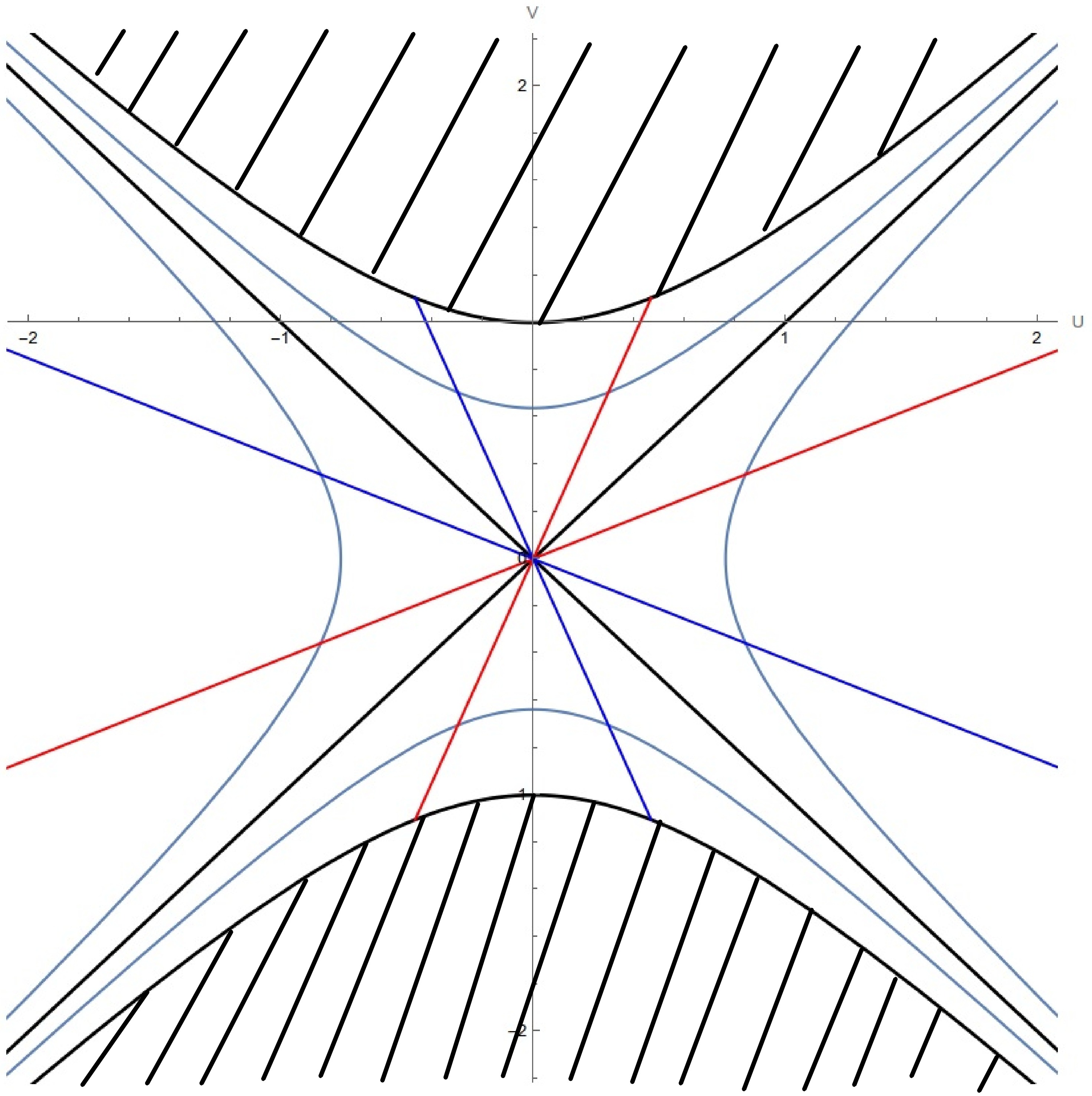


Kruskal Diagram

- extend (U,V) to all allowed values:

$$-\infty < U < +\infty$$

$$V^2 < U^2 + 1$$



Kruskal Diagram

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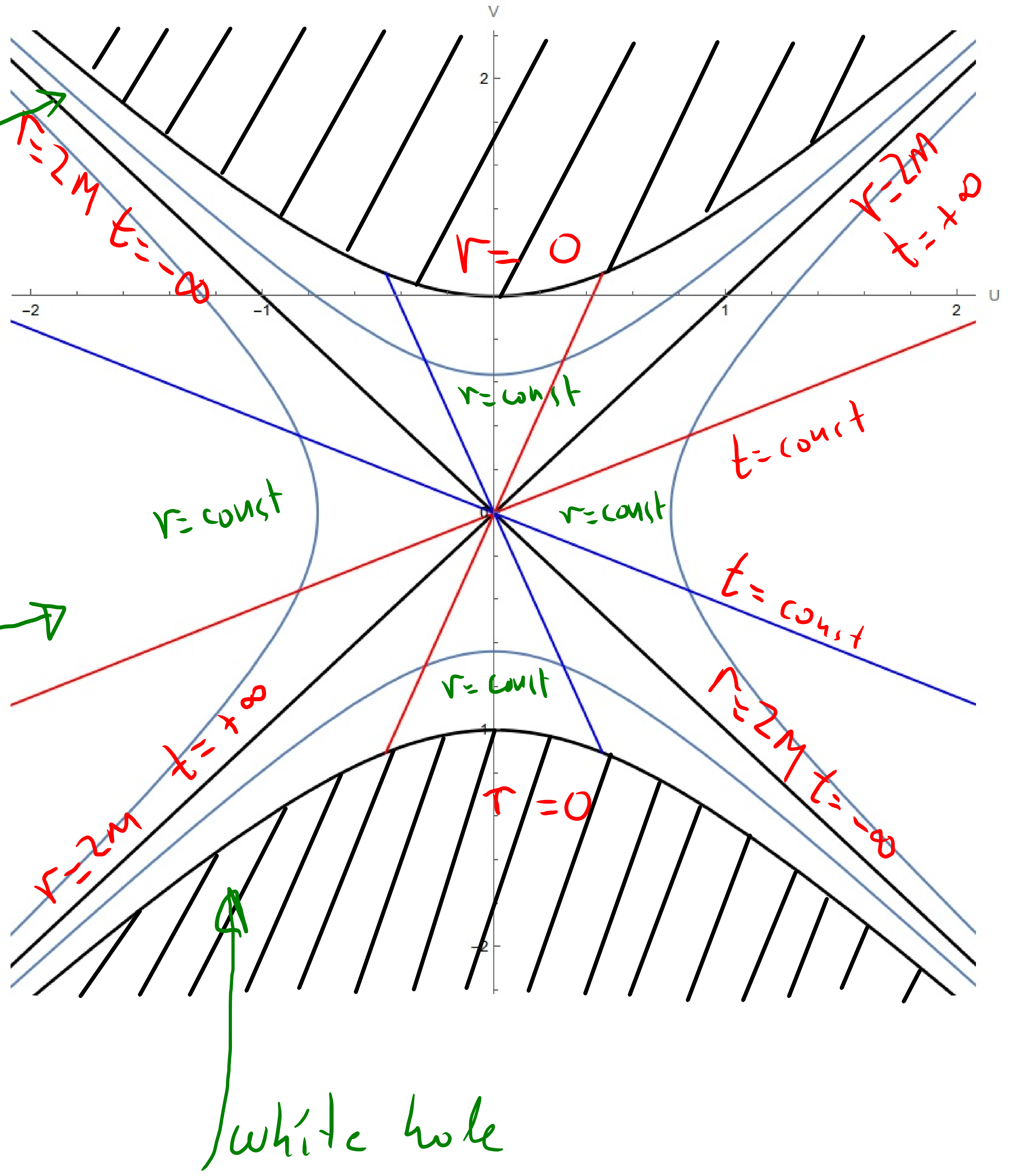


Careful: forward in time in this region is upwards, in the direction of decreasing t!

(t, r) not so good coordinates in extended diagram...

Kruskal Diagram

- extend (U,V) to all allowed values:

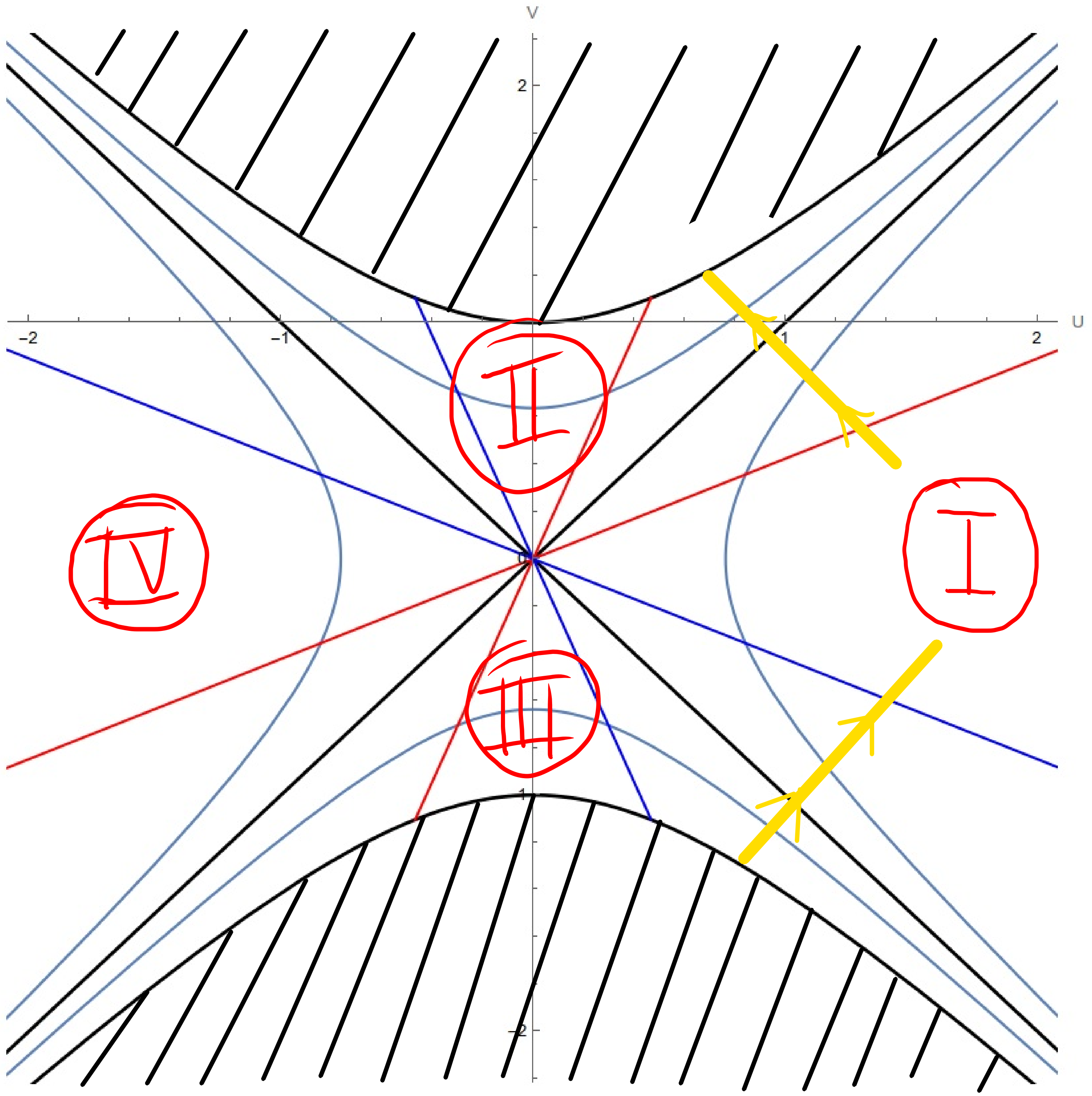
$$-\infty < U < +\infty$$

$$V^2 < U^2 + 1$$

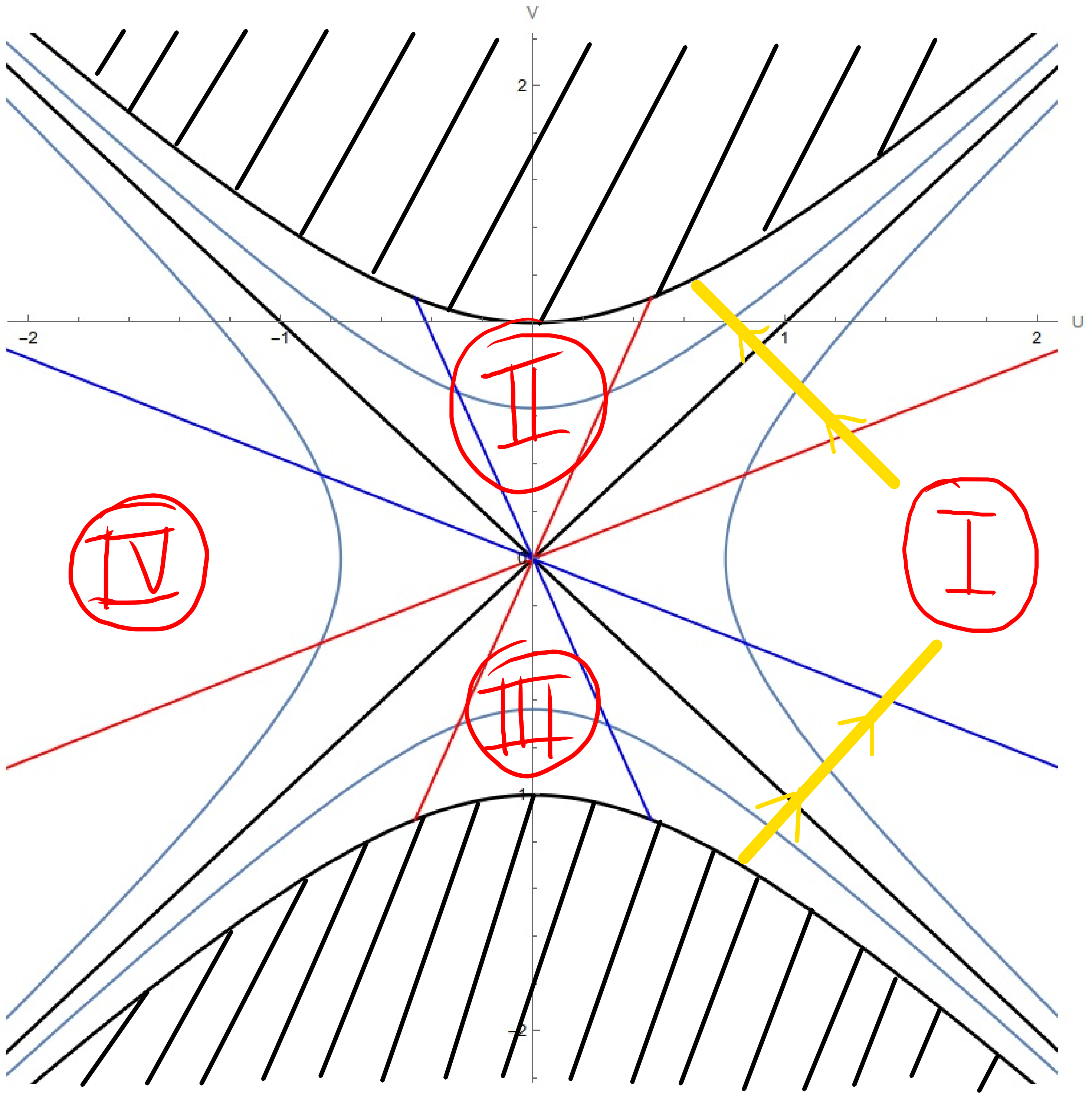
- future directed null lines

$$\mathrm{III} \to \mathrm{I} \qquad \mathrm{III} \to \mathrm{IV}$$

$$\mathrm{I} \to \mathrm{II} \qquad \mathrm{IV} \to \mathrm{II}$$



Kruskal Diagram



- extend (U,V) to all allowed values:

$$-\infty < U < +\infty$$

$$V^2 < U^2 + 1$$

- future directed null lines

$$\mathrm{III} \to \mathrm{I}$$

$$\mathrm{I} \to \mathrm{II}$$

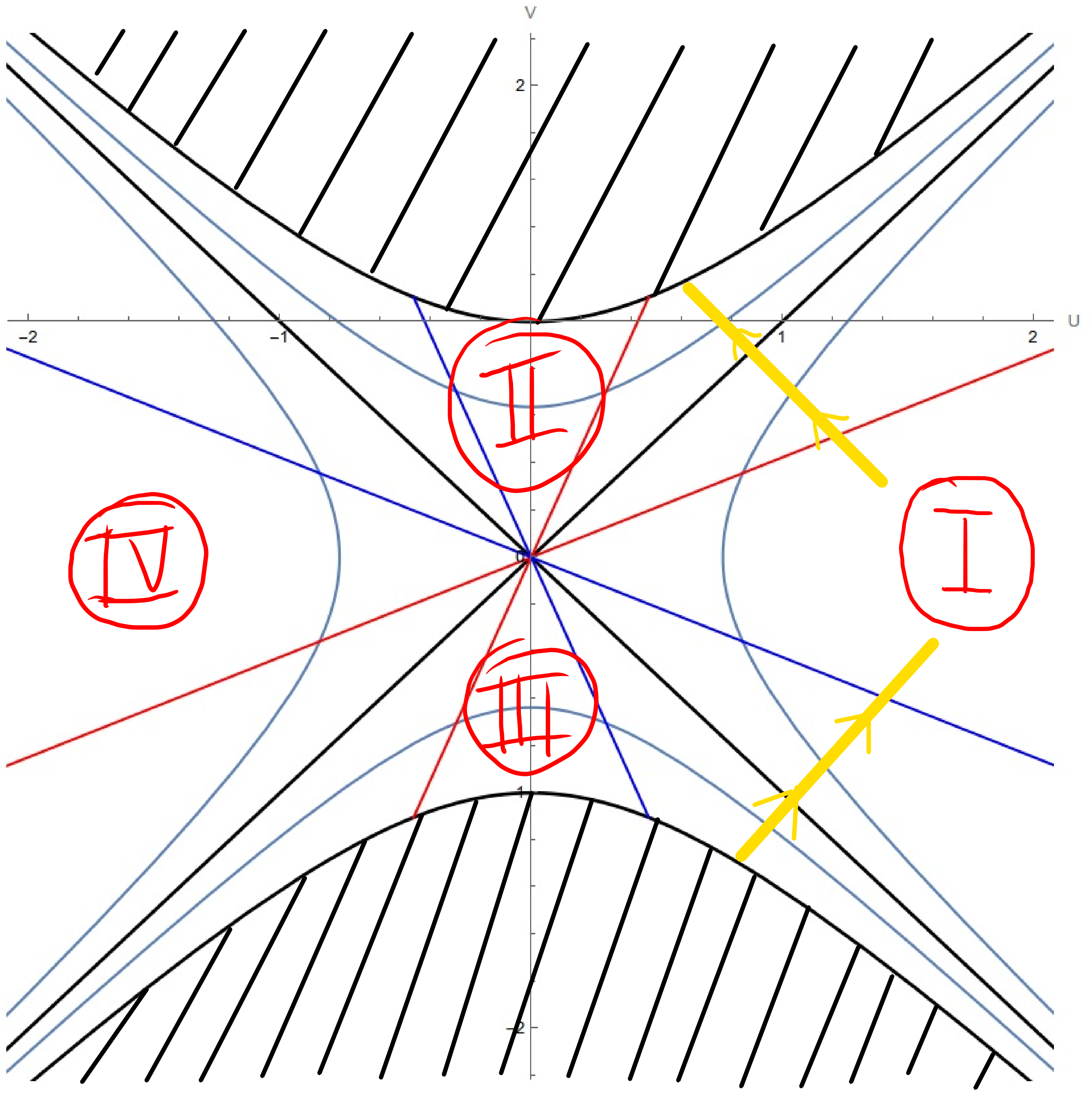
into black hole:
anything going in - never comes out
- hits singularity

Kruskal Diagram

- extend (U,V) to all allowed values:

$$-\infty < U < +\infty$$

$$V^2 < U^2 + 1$$



- future directed null lines

III → I out of a white hole:
- we cannot go there.

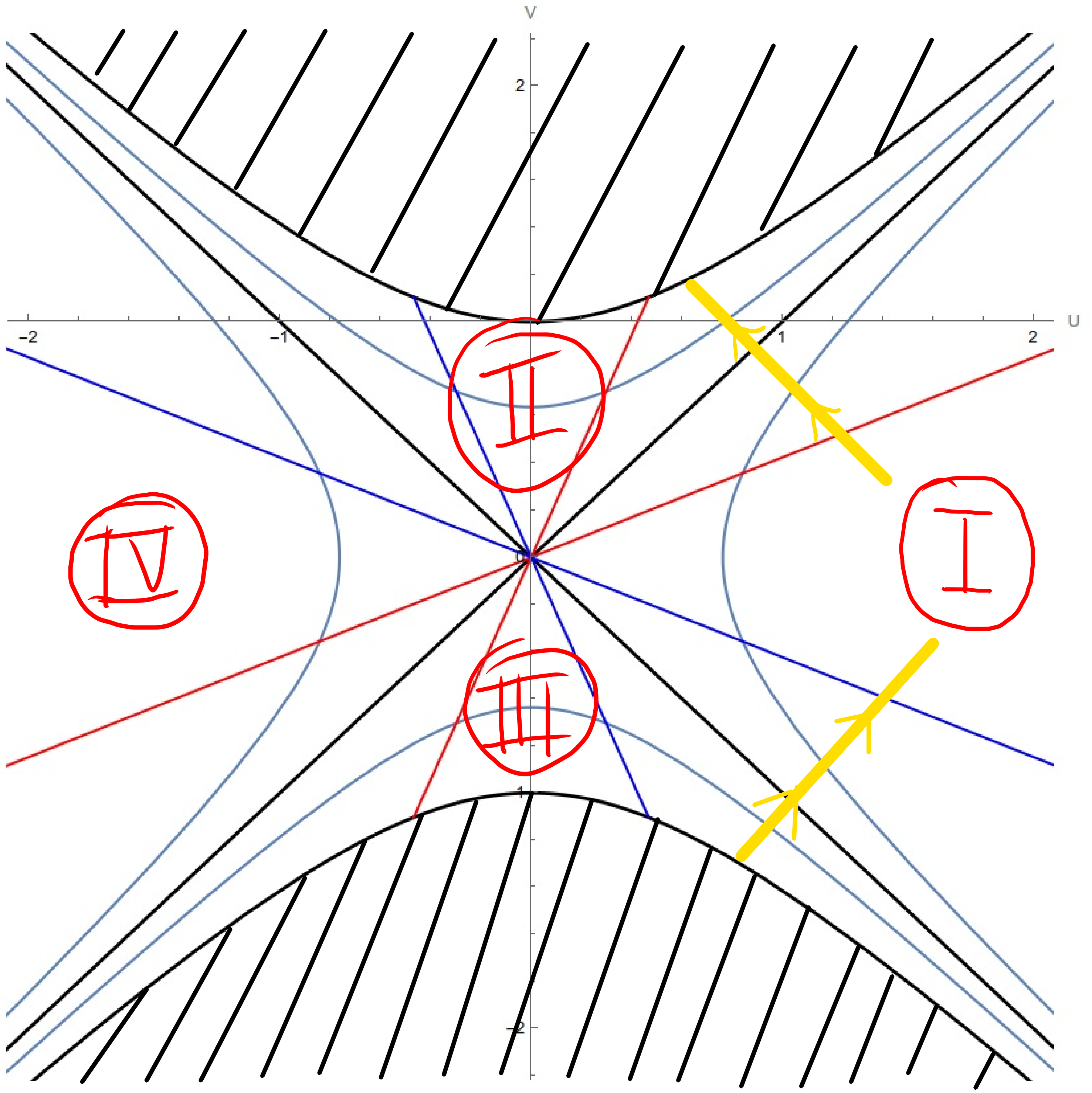
I → II
- things only emerge from white hole

Kruskal Diagram

- extend (U,V) to all allowed values:

$$-\infty < U < +\infty$$

$$V^2 < U^2 + 1$$



- future directed null lines

$\mathrm{III} \to \mathrm{I}$ $\quad r = 0$ a "naked" singularity

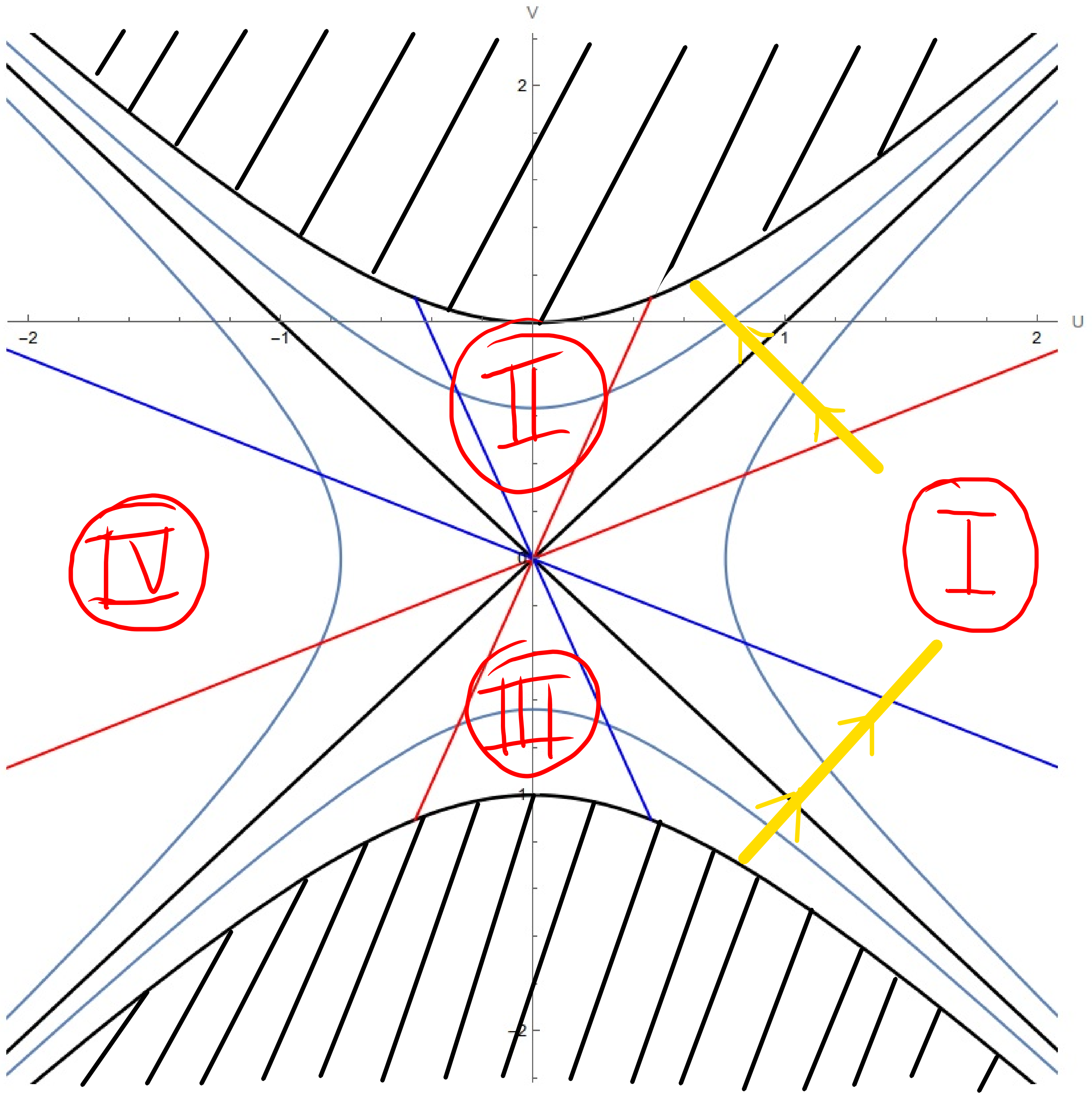
$\mathrm{I} \to \mathrm{II}$ $\quad$ " $\quad$ " hidden $\quad$ "

Kruskal Diagram

- extend (U,V) to all allowed values:

$$-\infty < U < +\infty$$

$$V^2 < U^2 + 1$$



- future directed null lines

$\mathrm{III} \to \mathrm{I}$ $r=0$ a "naked" singularity

$\mathrm{I} \to \mathrm{II}$ " " hidden "

↳ not nice: upredictability ...

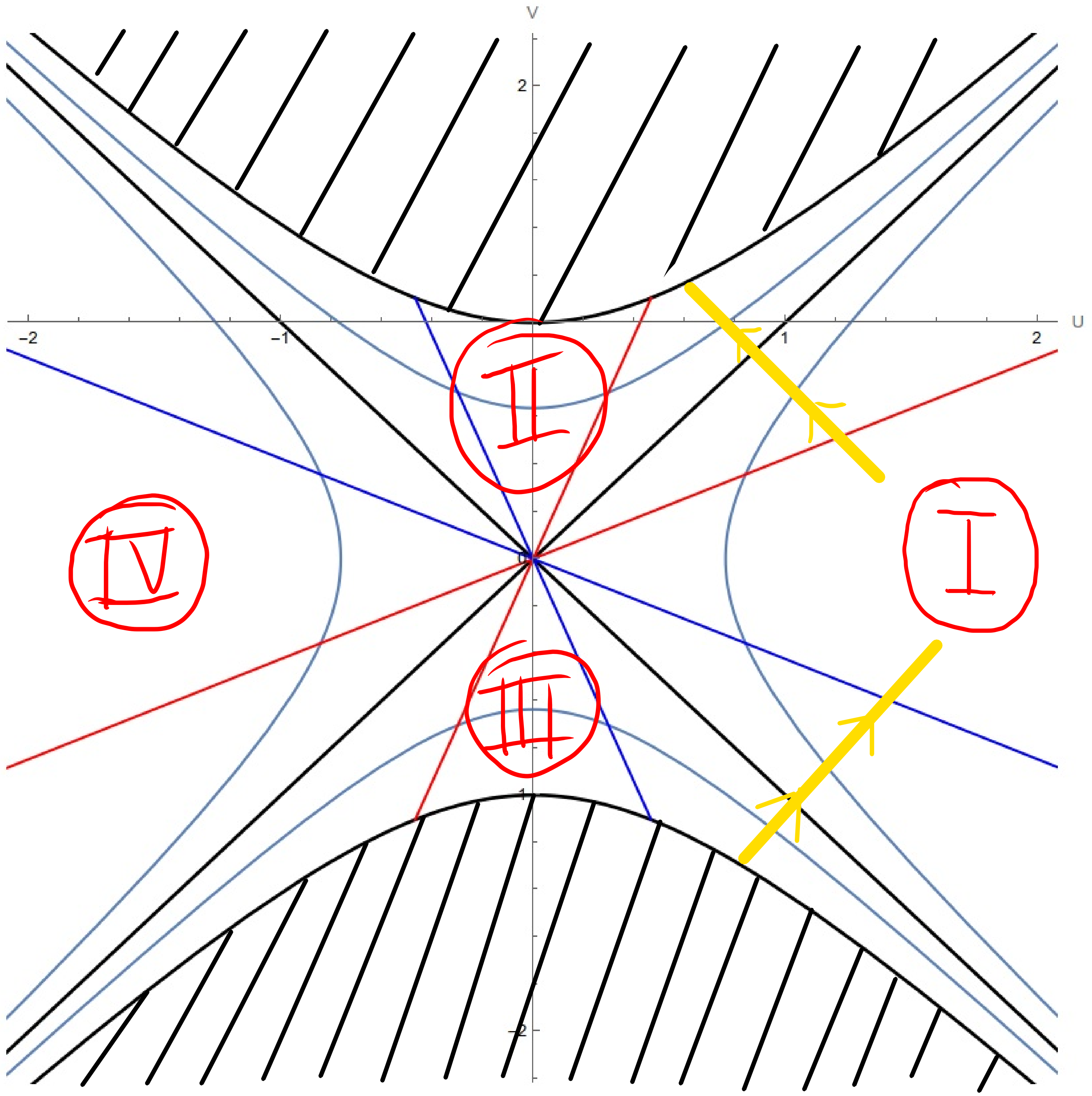
Cosmic censorship conjecture: there are no naked singularities

Kruskal Diagram

- extend (U, V) to all allowed values:

$$-\infty < U < +\infty$$

$$V^2 < U^2 + 1$$



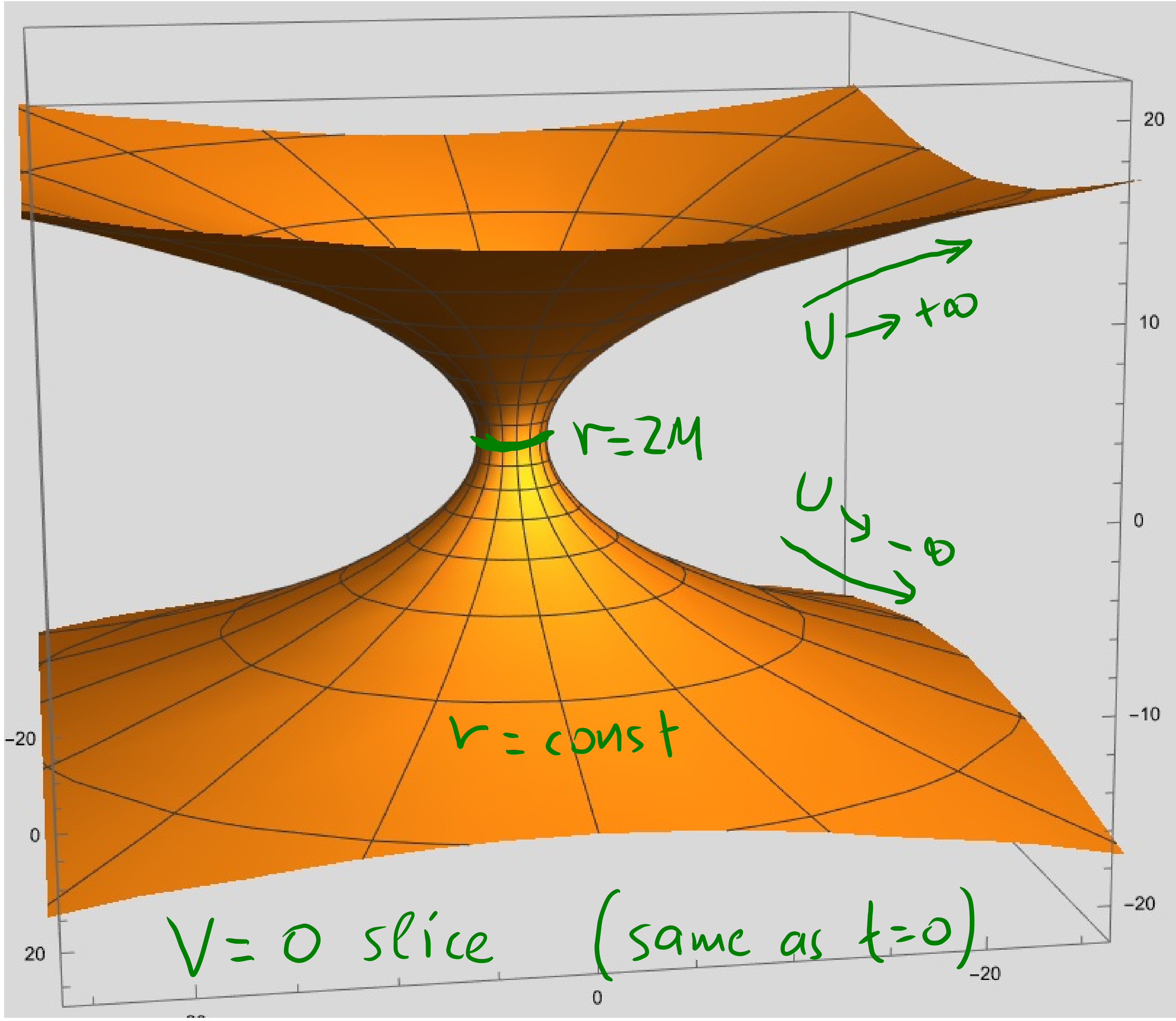
- future directed null lines

$\mathrm{III} \rightarrow \mathrm{I}$ $r = 0$ a "naked" singularity

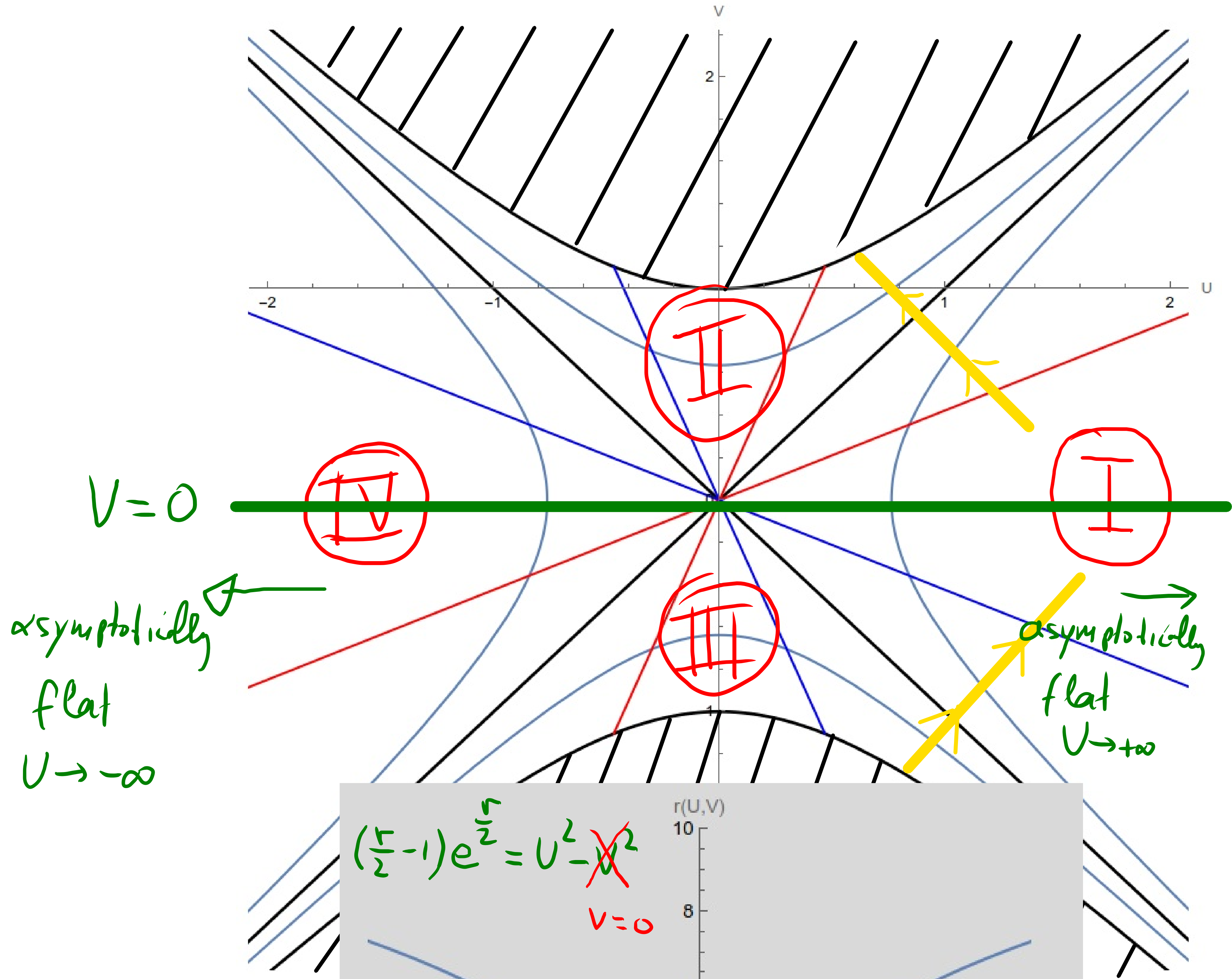
$\underbrace{\mathrm{I} \rightarrow \mathrm{II}}$ " " hidden "

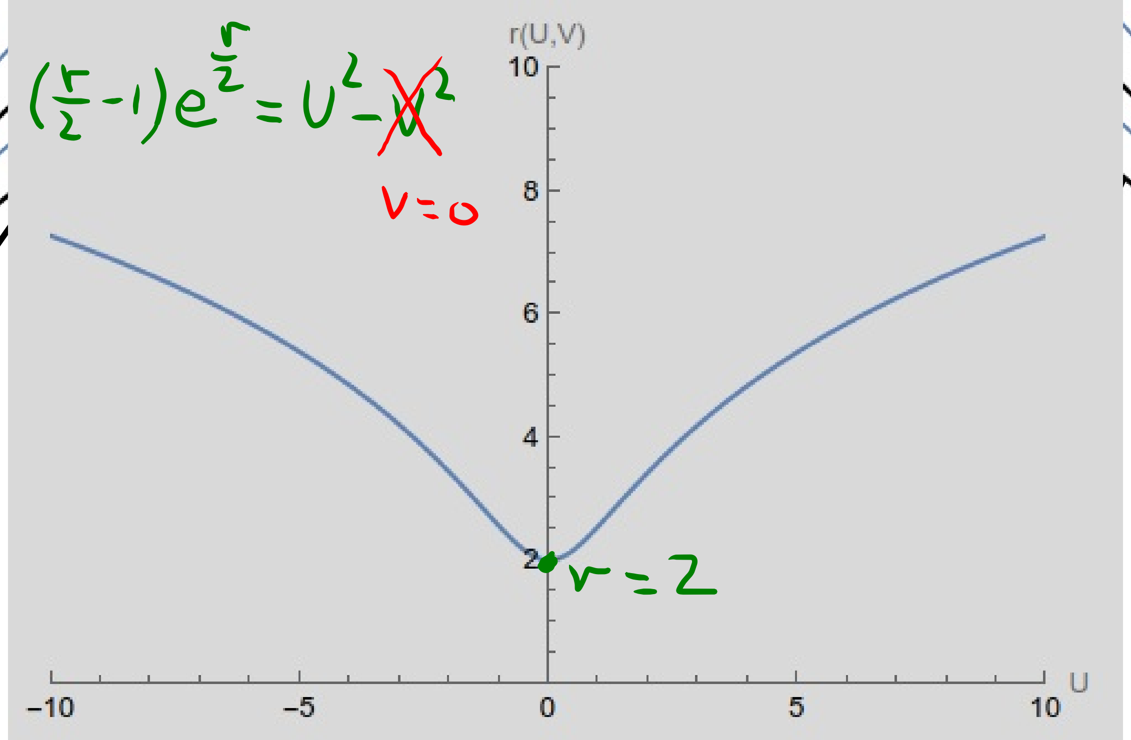
the only regions relevant to star collapse process

Kruskal Diagram



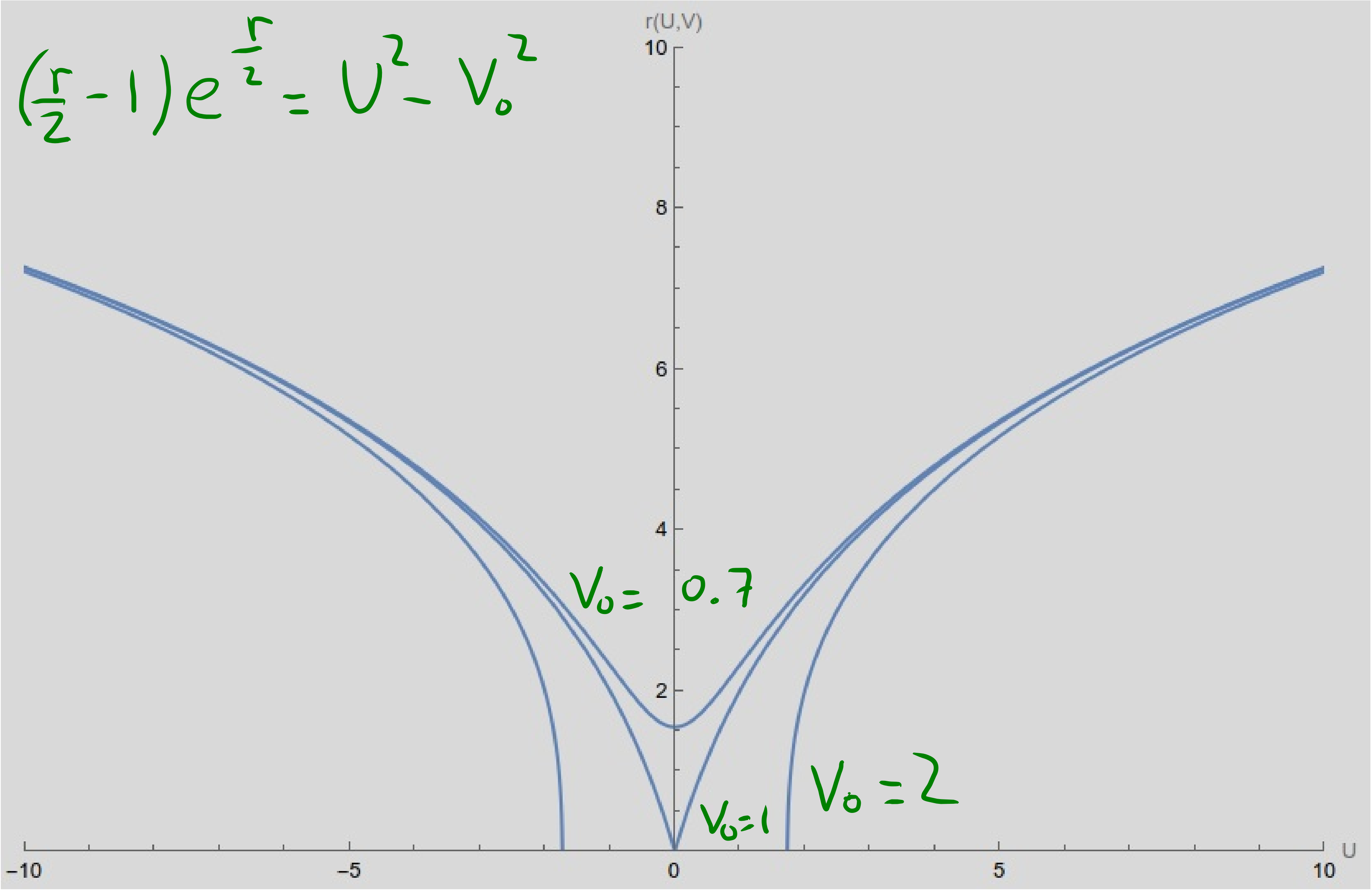
$V = 0$ slice (same as $t = 0$)

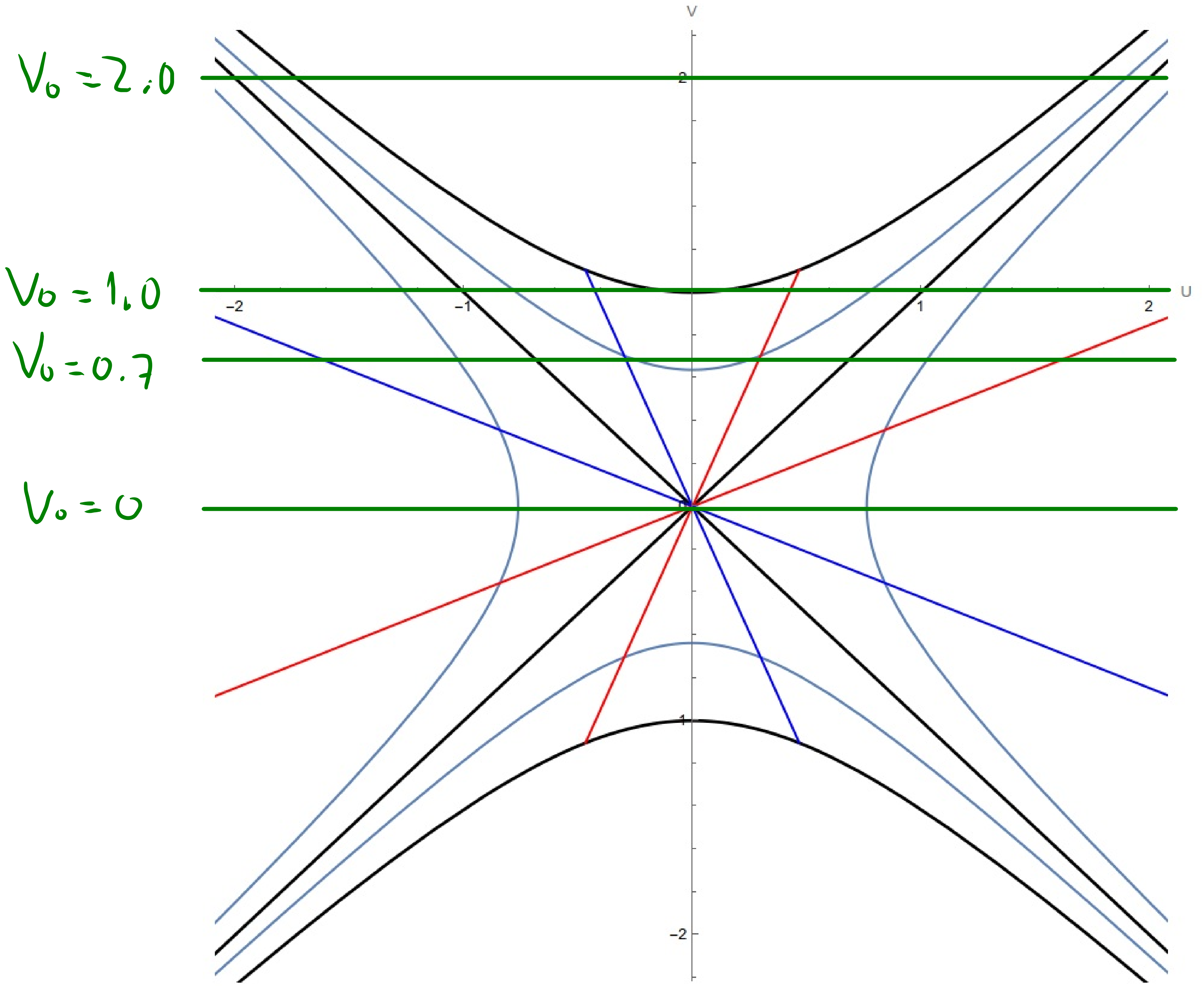




- a wormhole
- not static: as value of V changes, throat $r = 2M \to$ vanish
- no causal curve connecting I + IV: throat closes

Kruskal Diagram

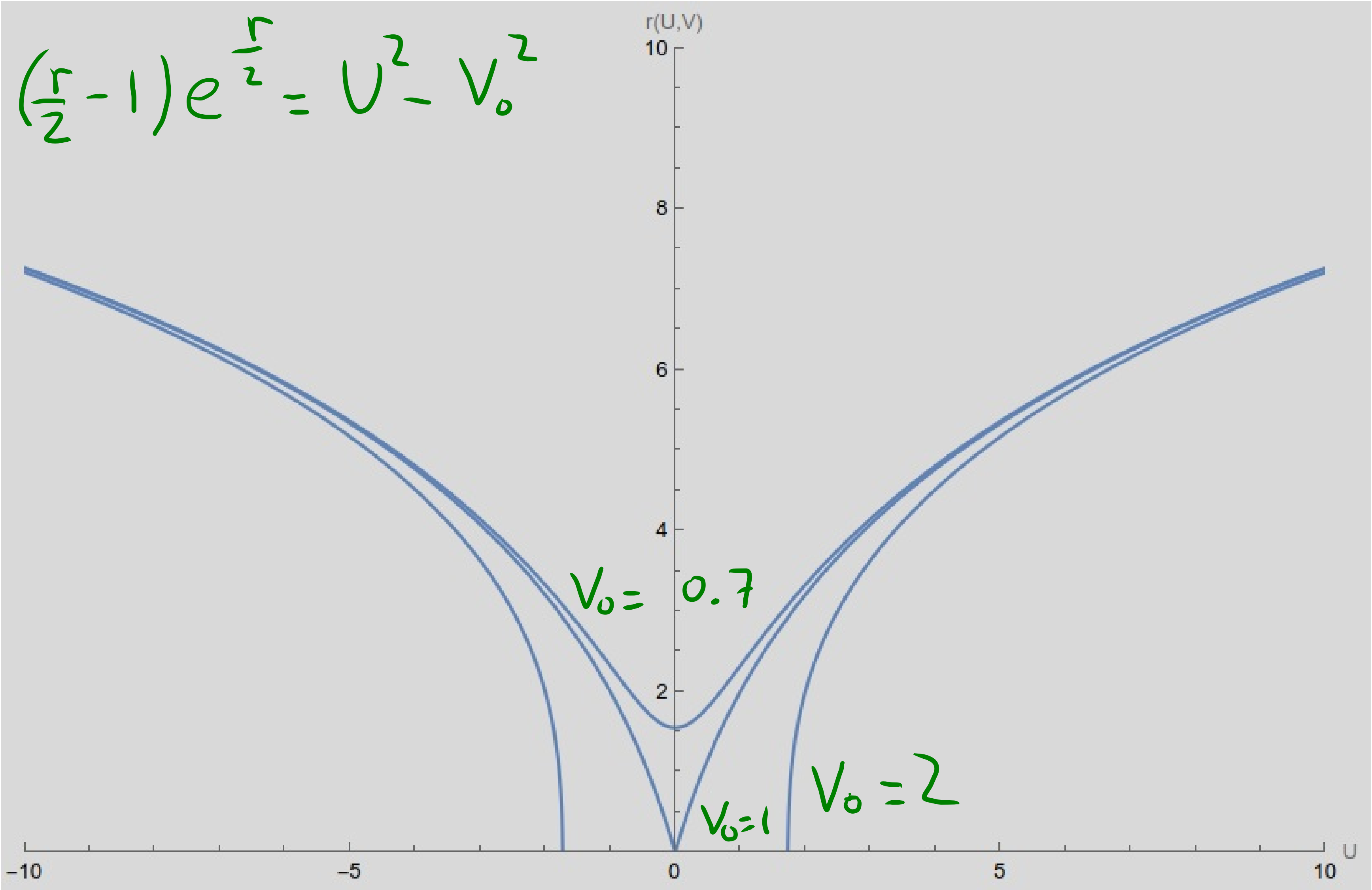


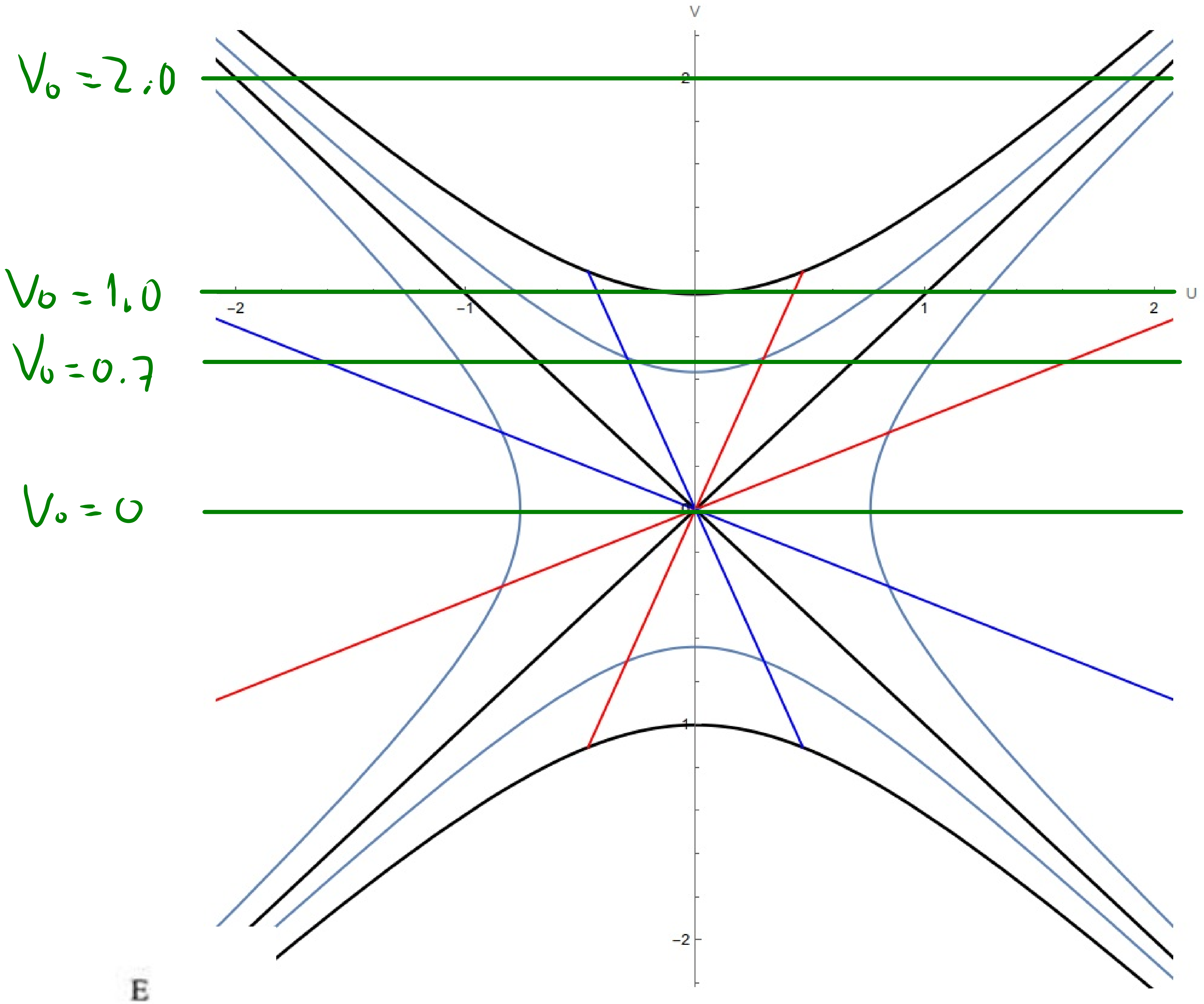


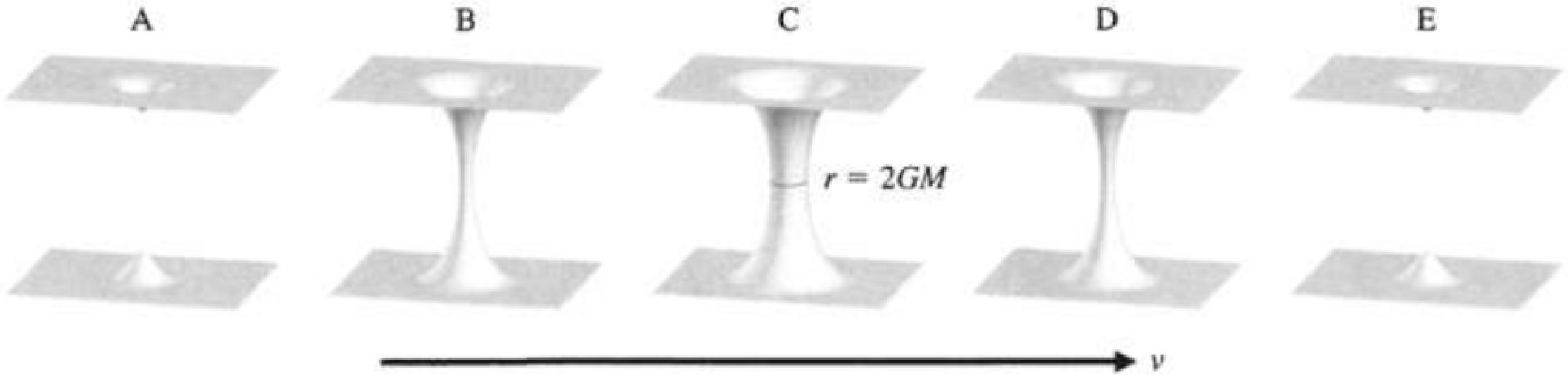
- a wormhole
- not static: as value of V changes, throat $r = 2M \rightarrow$ vanish
- no causal curve connecting I + IV: throat closes

Kruskal Diagram

$$\left(\frac{r}{2}-1\right)e^{\frac{r}{2}} = U^2 - V_0^2$$



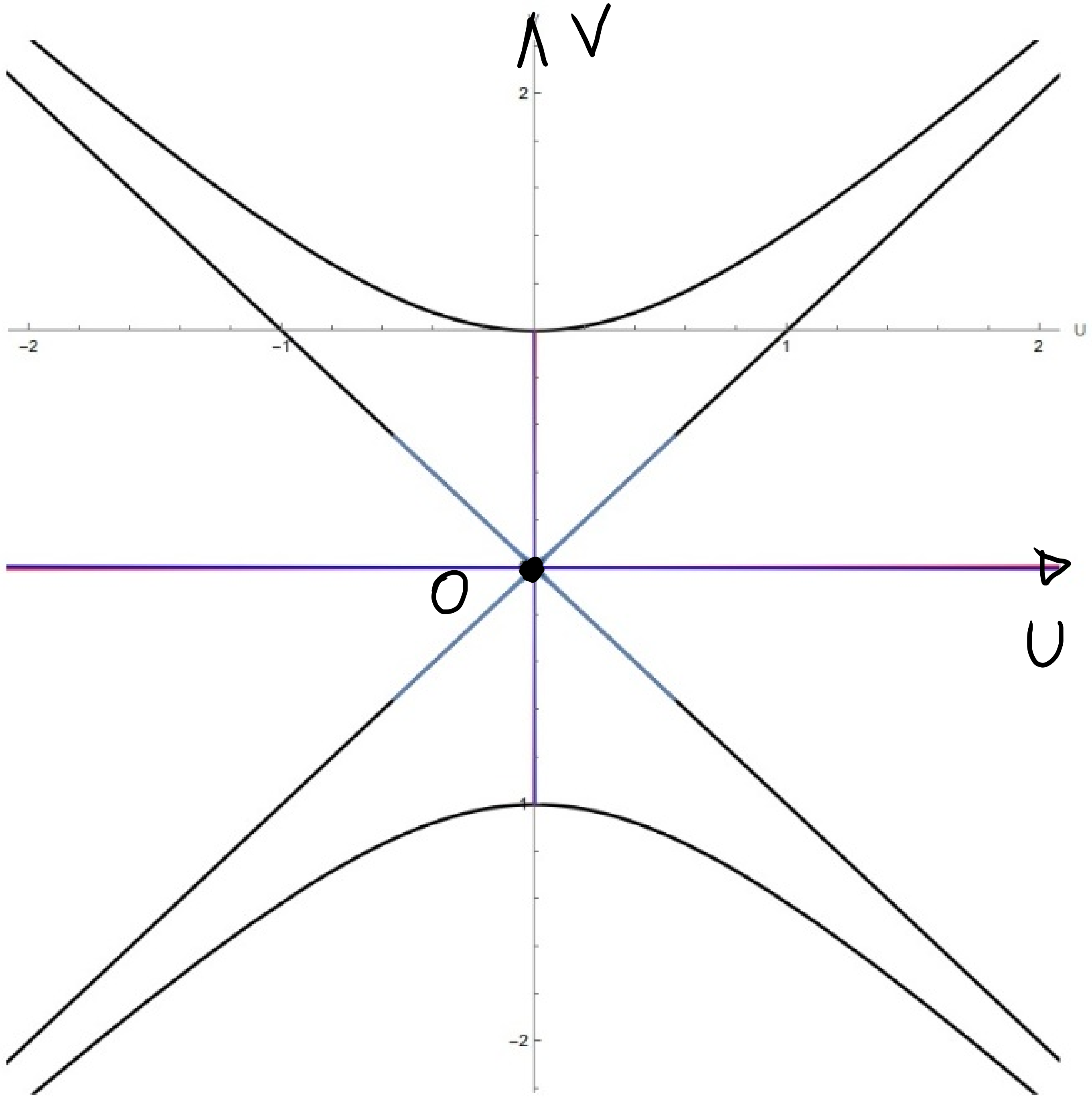




Carroll, Fig 5.15

Penrose Diagram of Schwarzschild Geometry

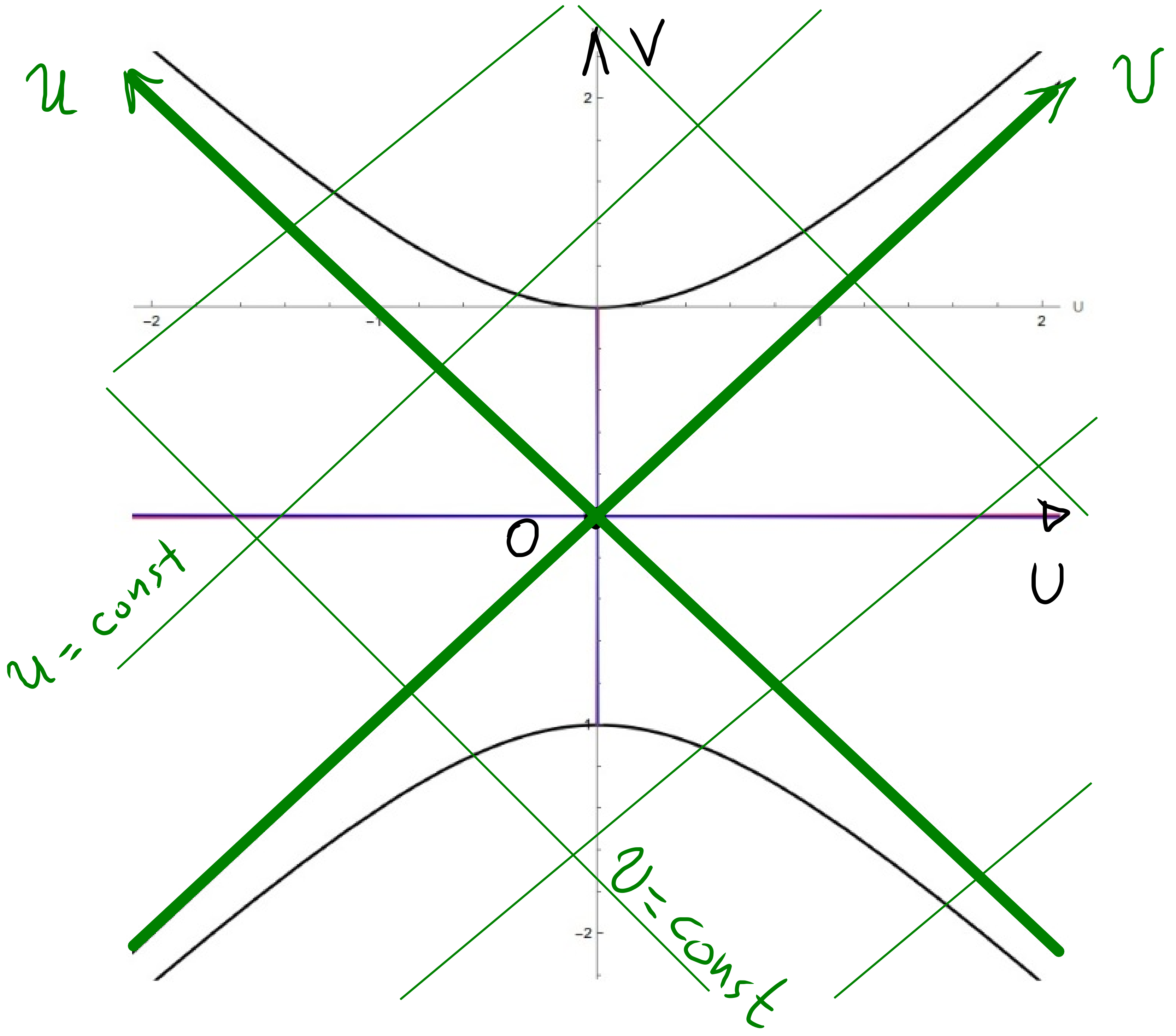
- As we did for Minkowski space, so that we:
  - bring infinities to finite distance
  - light rays at $\pm 45°$
  - visualize asymptotic behavior of light like curves in a simple way
  - visualise global spacetime structure, esp. causal structure



Penrose Diagram of Schwarzschild Geometry

$u = V - U \qquad v = V + U$

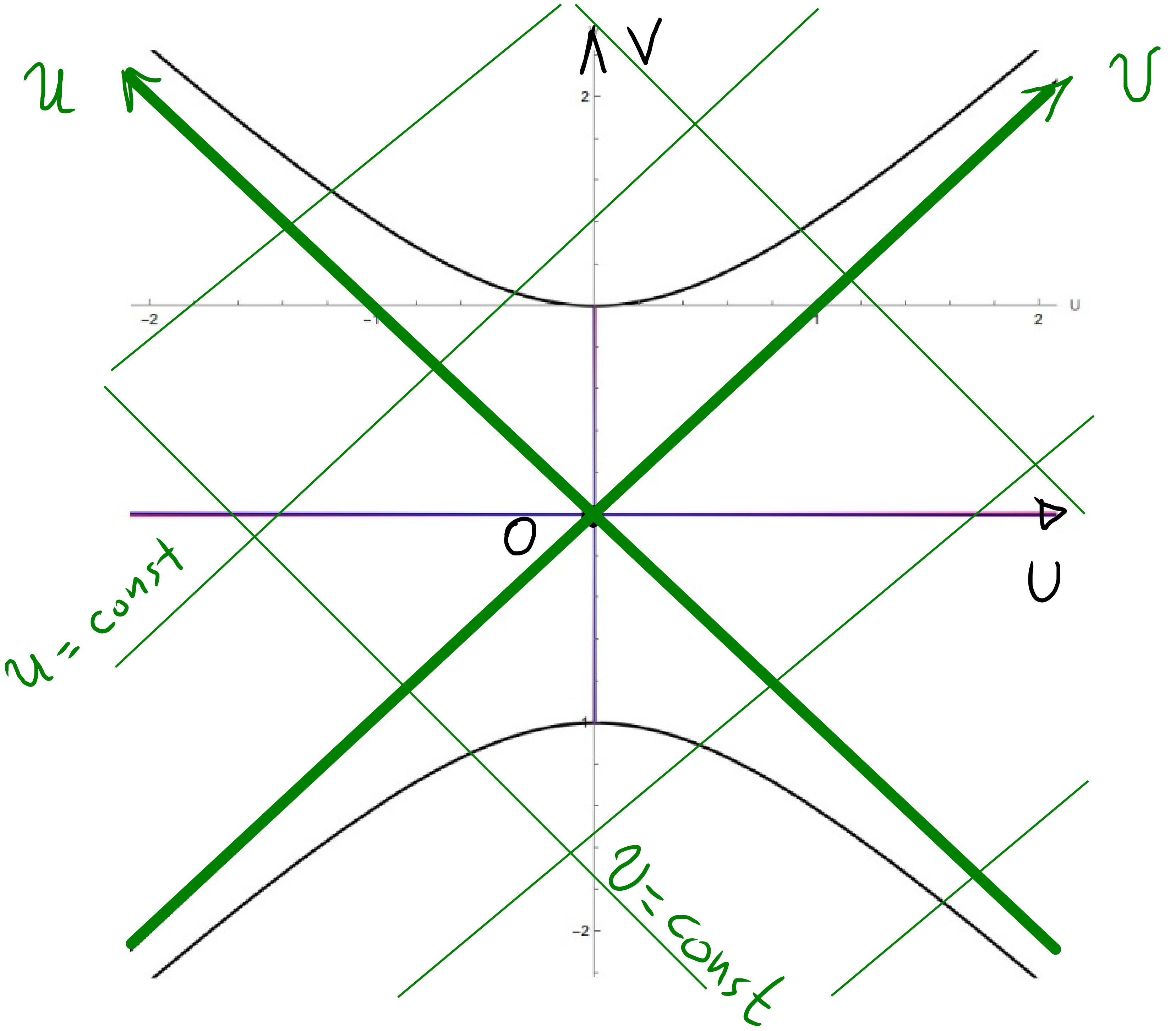
$U = \dfrac{v-u}{2} \qquad V = \dfrac{v+u}{2}$



Penrose Diagram of Schwarzschild Geometry

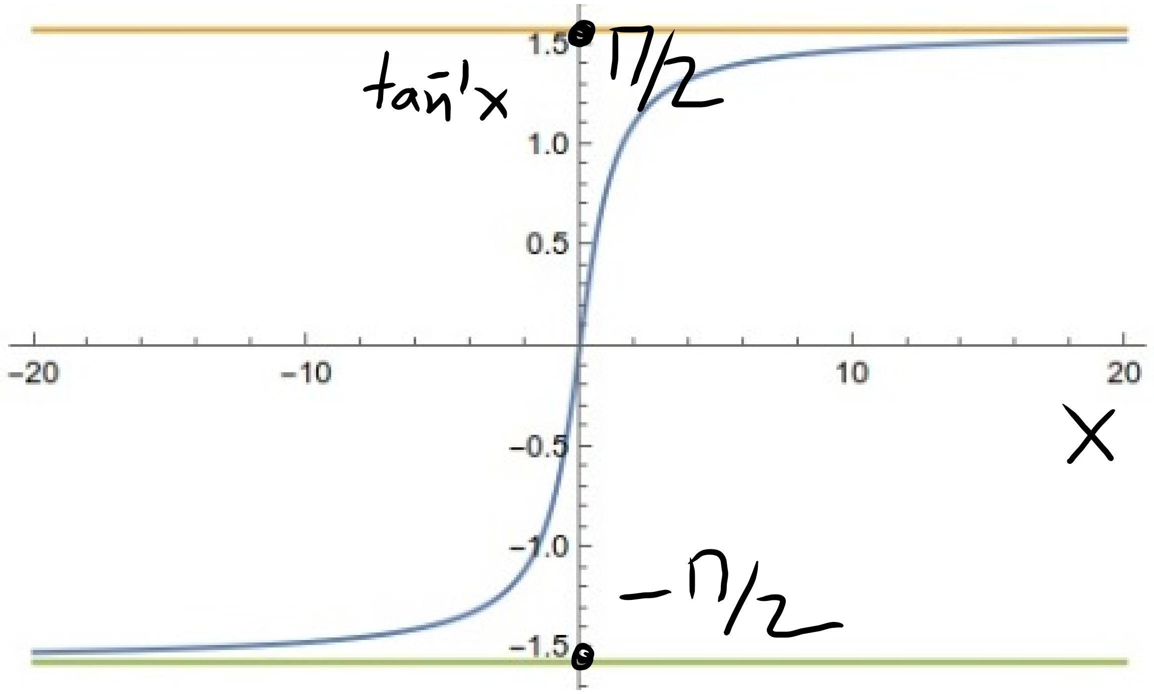
$u = V - U \qquad v = V + U$

$U = \frac{v - u}{2} \qquad V = \frac{v + u}{2}$



$u' = \tan^{-1} u$

$v' = \tan^{-1} v$



$\frac{\pi}{2} < u' < \frac{\pi}{2}$

$\frac{\pi}{2} < v' < \frac{\pi}{2}$

Penrose Diagram of Schwarzschild Geometry

$$u = V - U \qquad v = V + U$$

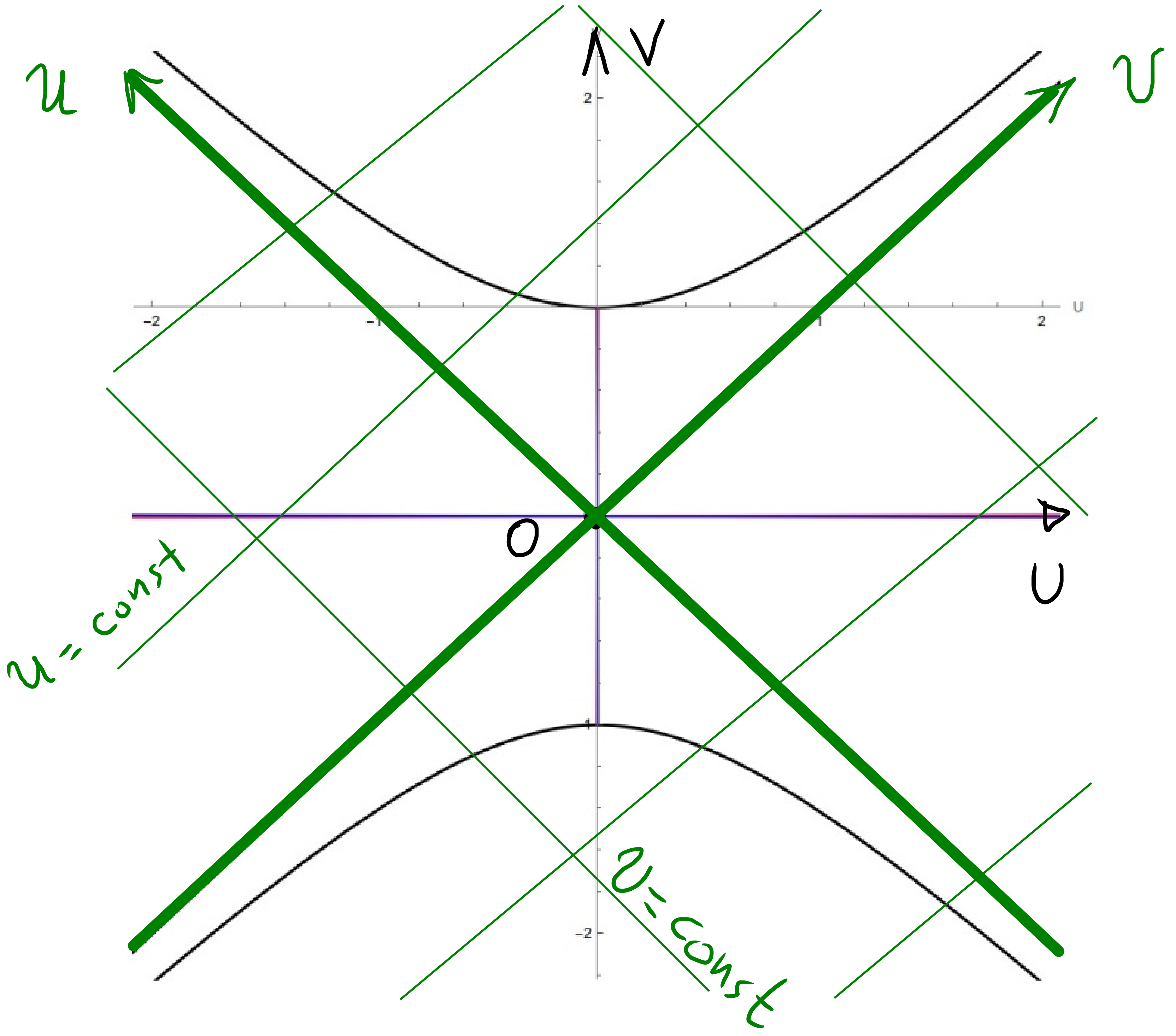
$$U = \frac{v - u}{2} \qquad V = \frac{v + u}{2}$$

$$u' = \tan^{-1} u$$

$$v' = \tan^{-1} v$$

$$u' = V' - U' \qquad \Longleftrightarrow \qquad U' = \frac{v' - u'}{2}$$

$$v' = V' + U' \qquad\qquad V' = \frac{v' + u'}{2}$$



Penrose Diagram of Schwarzschild Geometry

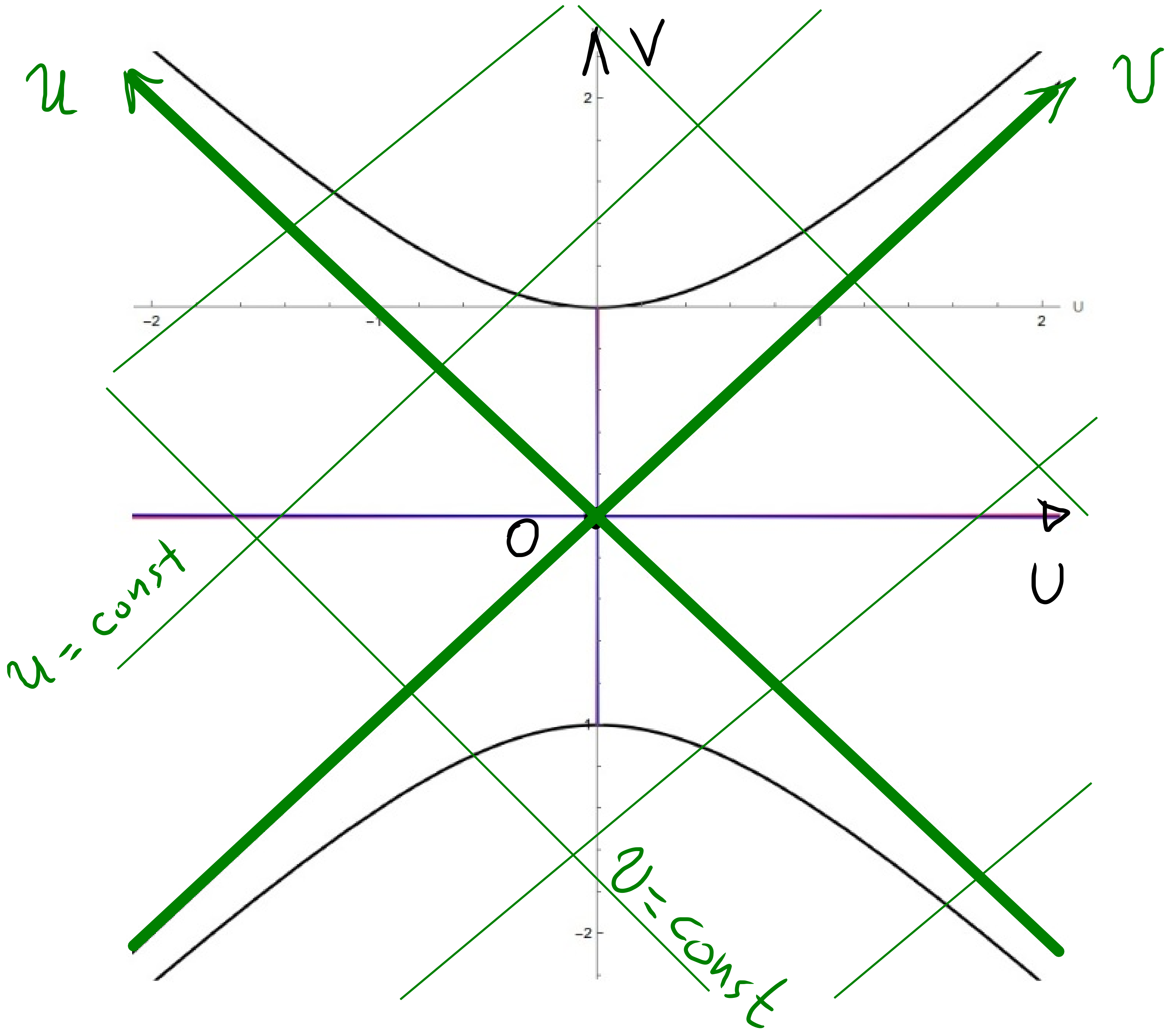
$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$u' = \tan^{-1} u$$

$$v' = \tan^{-1} v$$

$$u' = V' - U' \qquad v' = V' + U' \iff U' = \frac{v'-u'}{2} \qquad V' = \frac{v'+u'}{2}$$



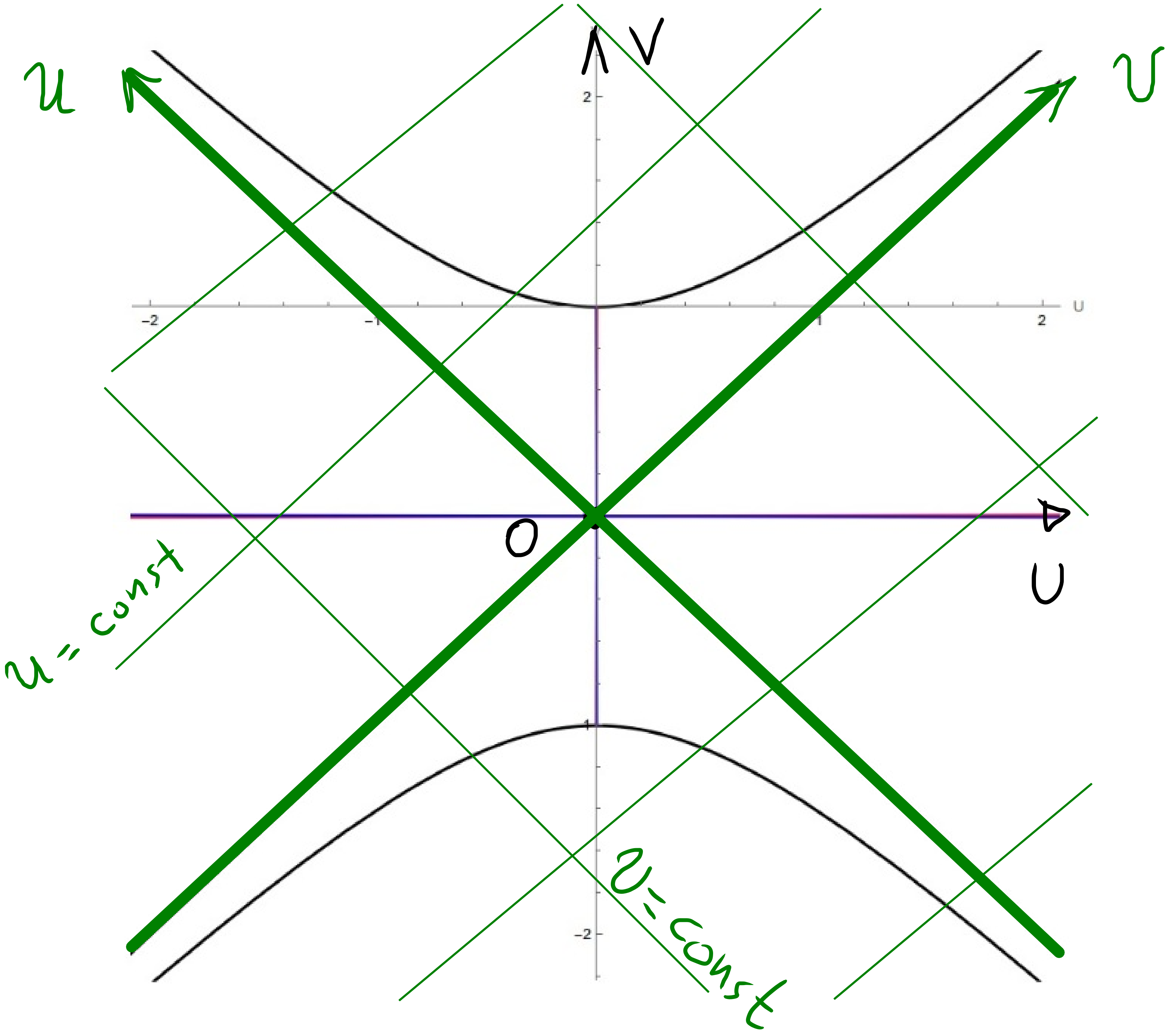
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$\tan^{-1}(V-U) = V' - U'$$

$$\tan^{-1}(V+U) = V' + U'$$



$$u' = V' - U'$$

$$v' = V' + U'$$

$$\Longleftrightarrow$$

$$U' = \frac{v' - u'}{2}$$

$$V' = \frac{v' + u'}{2}$$

Penrose Diagram of Schwarzschild Geometry

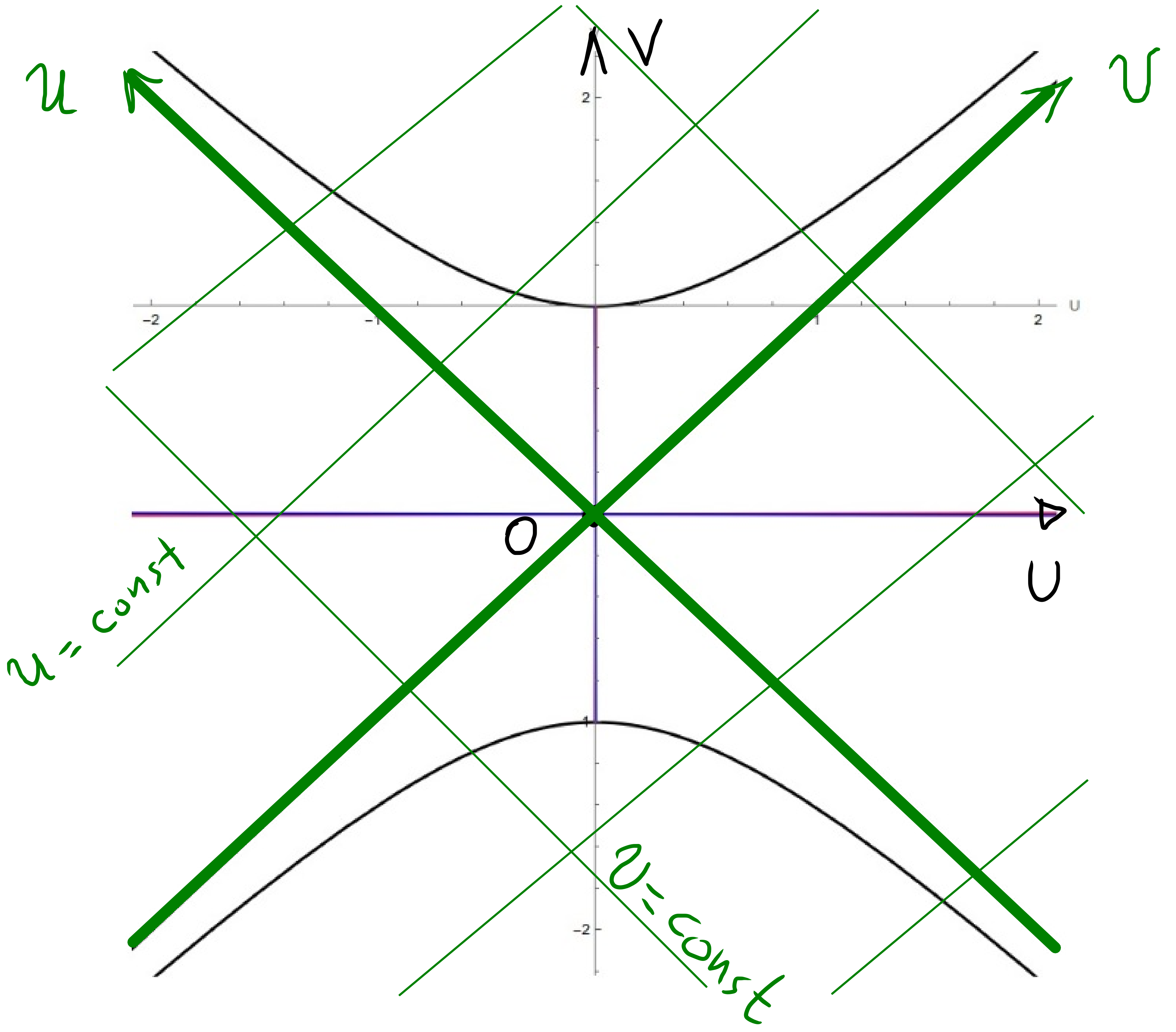
$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$\tan^{-1}(V-U) = V'-U'$$
$$\tan^{-1}(V+U) = V'+U' \quad \Rightarrow$$

$$V-U = \tan(V'-U')$$

$$V+U = \tan(V'+U')$$



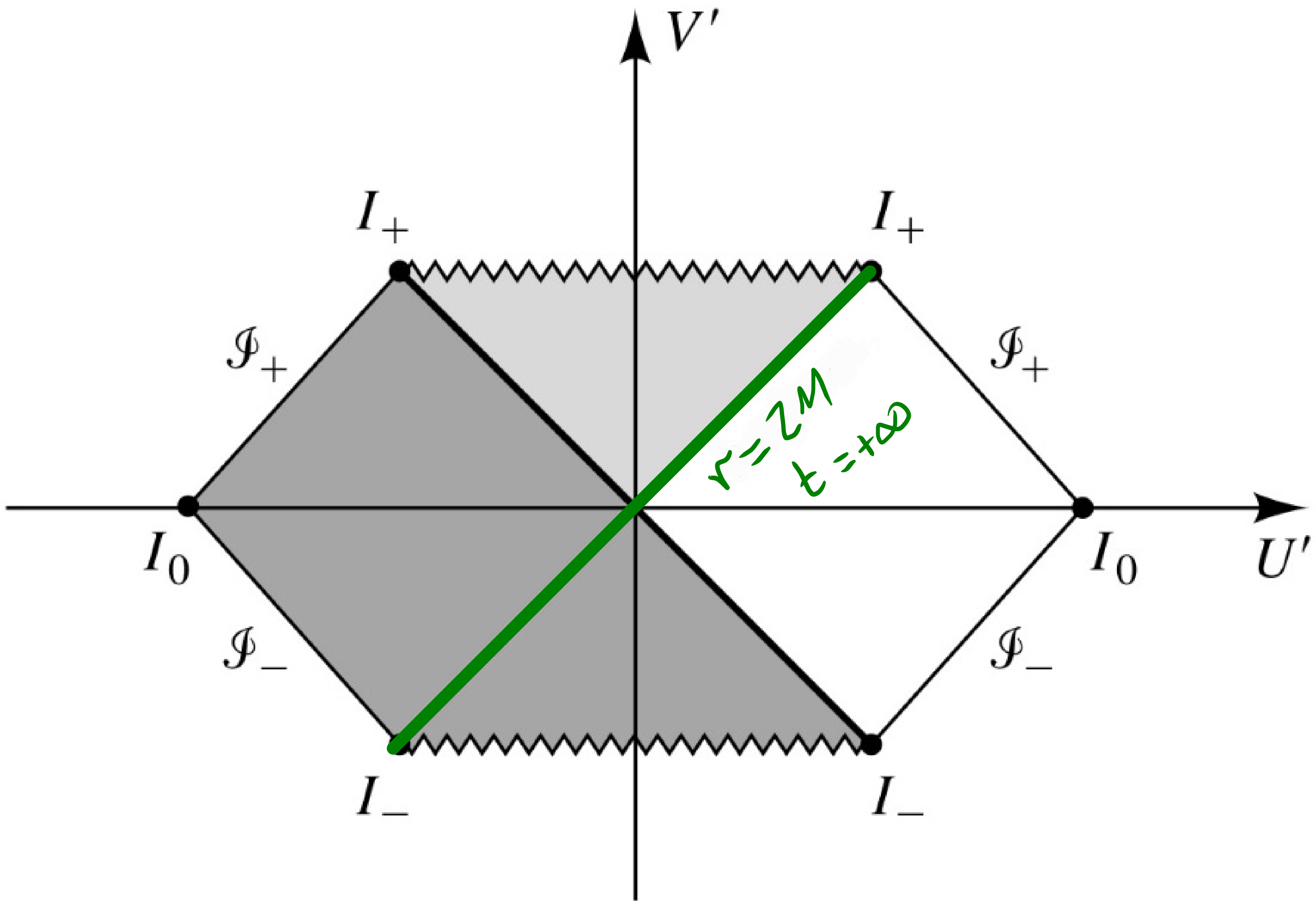
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V' - U')$$

$$V + U = \tan(V' + U')$$



Horizon $r=2M$, $t=+\infty$ (future Horizon)

$$V = U$$

Hartle Box 12.5

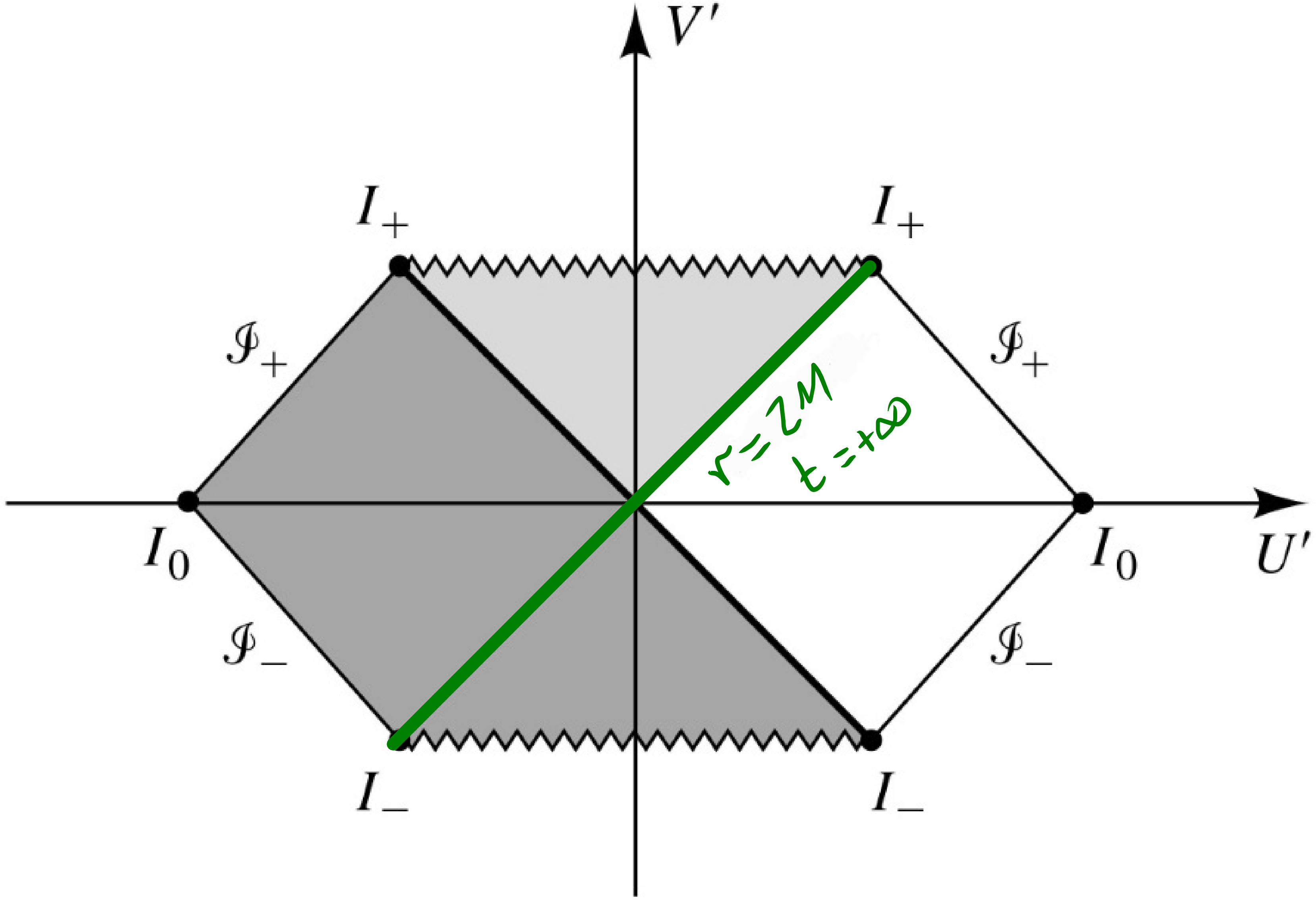
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V' - U')$$

$$V + U = \tan(V' + U')$$



Horizon $r=2M$, $t=+\infty$ (future Horizon)

$$V = U \Rightarrow \begin{cases} U' = \frac{1}{2}\tan^{-1}(V+U) \\ V' = \frac{1}{2}\tan^{-1}(V+U) \end{cases}$$

Hartle Box 12.5

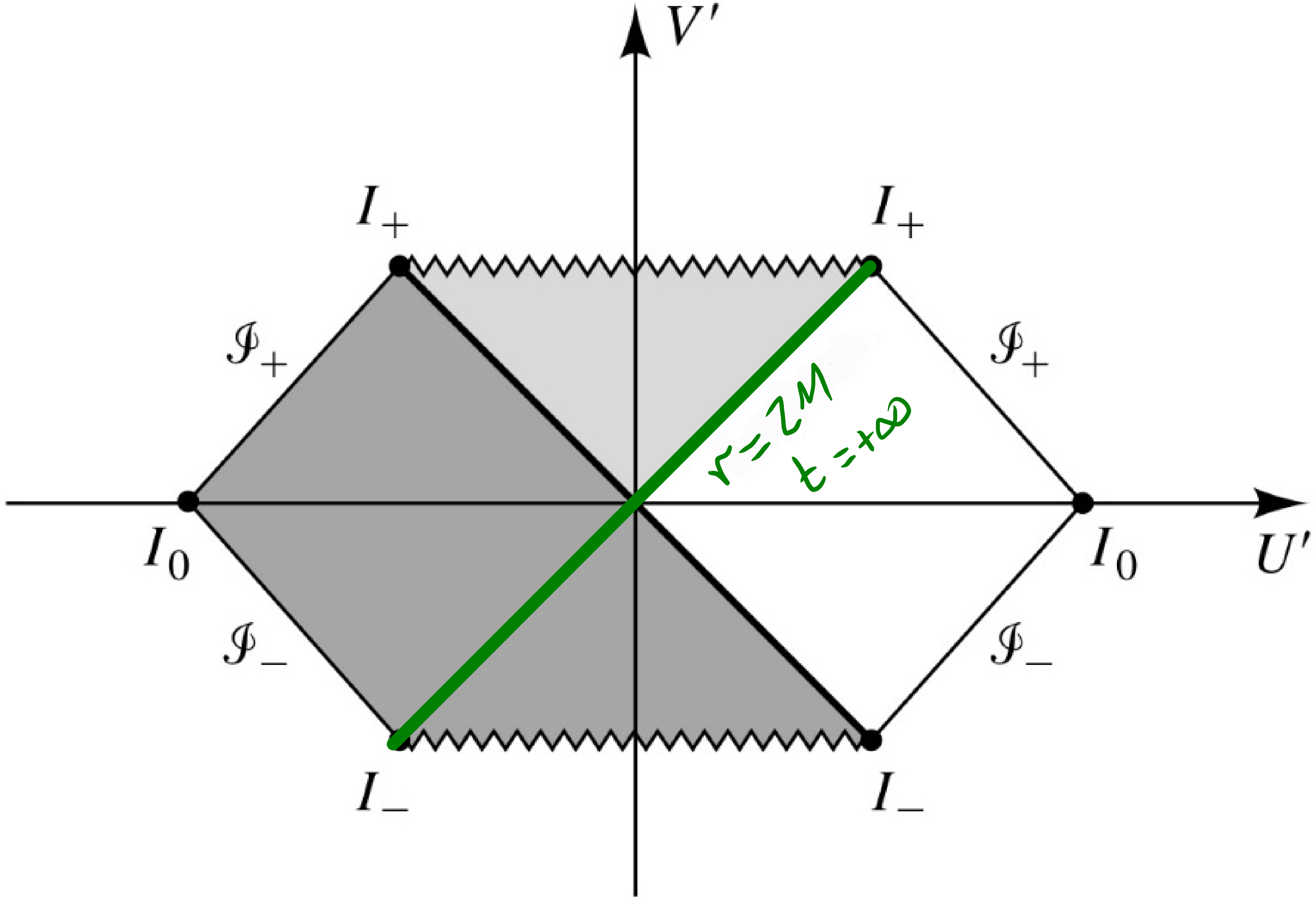
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V-U = \tan(V'-U')$$

$$V+U = \tan(V'+U')$$



Horizon $r=2M$, $t=+\infty$ (future Horizon)

$$V=U \Rightarrow \begin{Bmatrix} U' = \frac{1}{2}\tan^{-1}(V+U) \\ V' = \frac{1}{2}\tan^{-1}(V+U) \end{Bmatrix} \Rightarrow V'=U'$$

Hartle Box 12.5

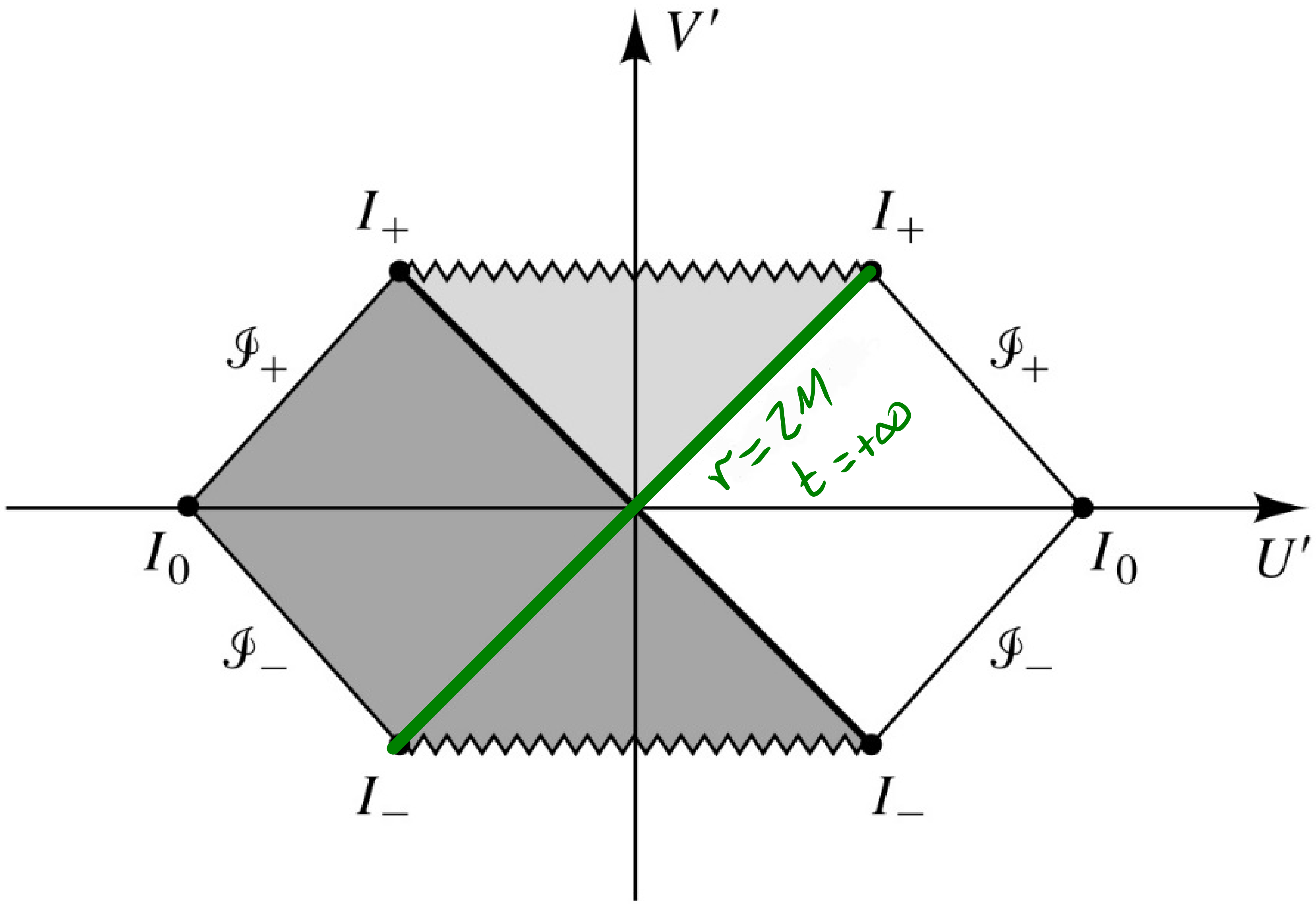
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V'-U')$$

$$V + U = \tan(V'+U')$$



Horizon $r=2M$, $t=+\infty$ (future Horizon)

$$V = U \Rightarrow \left\{ \begin{array}{l} U' = \frac{1}{2}\tan^{-1}(V+U) \\ V' = \frac{1}{2}\tan^{-1}(V+U) \end{array} \right\} \Rightarrow V' = U'$$

Hartle Box 12.5

$$\frac{1}{2}\left(-\frac{\pi}{2}\right) \le U' = V' = \frac{1}{2}\tan^{-1}(V+U) \le \frac{1}{2}\,\frac{\pi}{2}$$

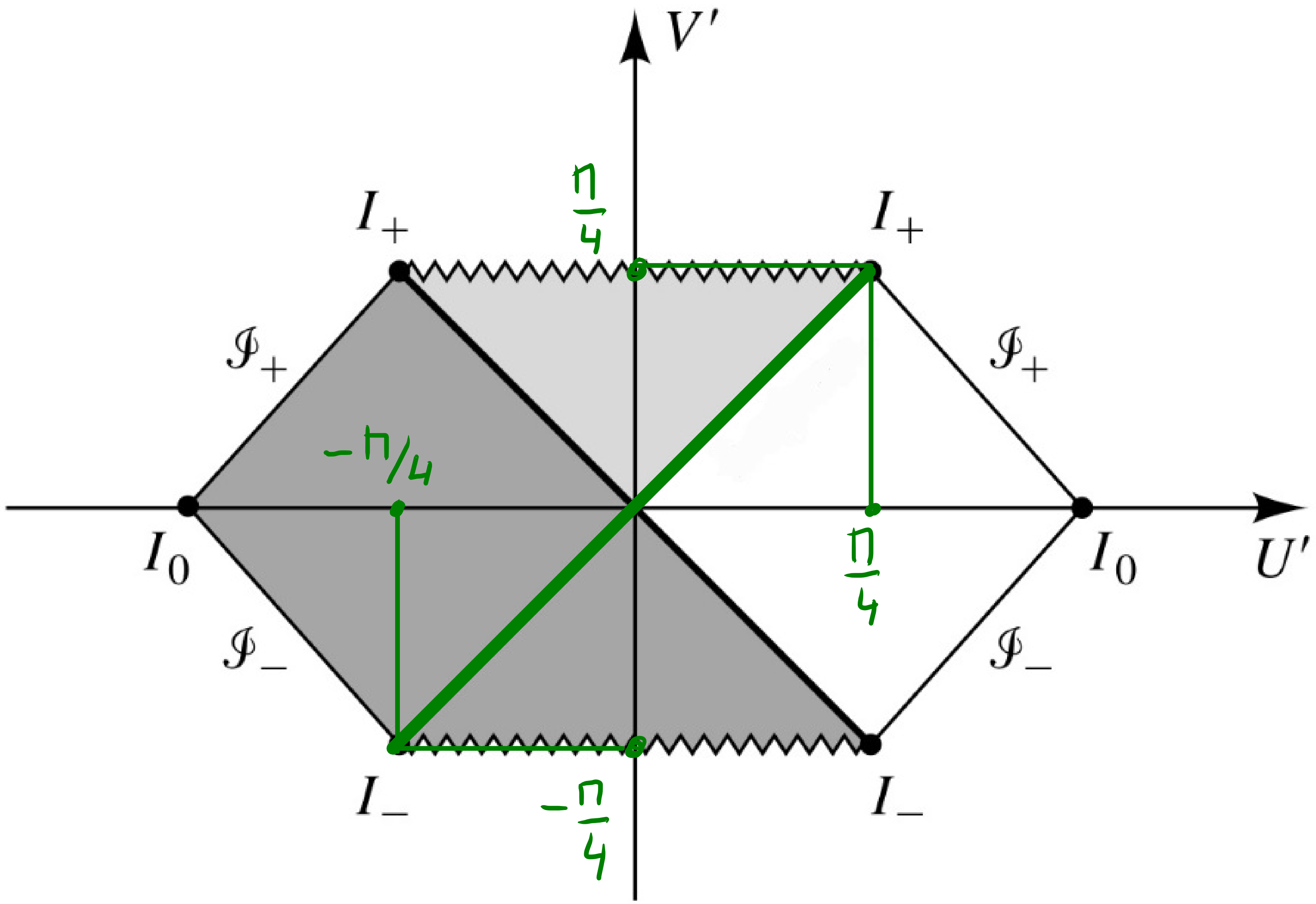
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V' - U')$$

$$V + U = \tan(V' + U')$$



Horizon $r = 2M$, $t = +\infty$ (future Horizon)

$$V = U \Rightarrow \left\{ \begin{array}{l} U' = \frac{1}{2}\tan^{-1}(V+U) \\ V' = \frac{1}{2}\tan^{-1}(V+U) \end{array} \right\} \Rightarrow V' = U'$$

$$-\frac{\pi}{4} \le U' = V' = \frac{1}{2}\tan^{-1}(V+U) \le \frac{\pi}{4}$$

Hartle Box 12.5

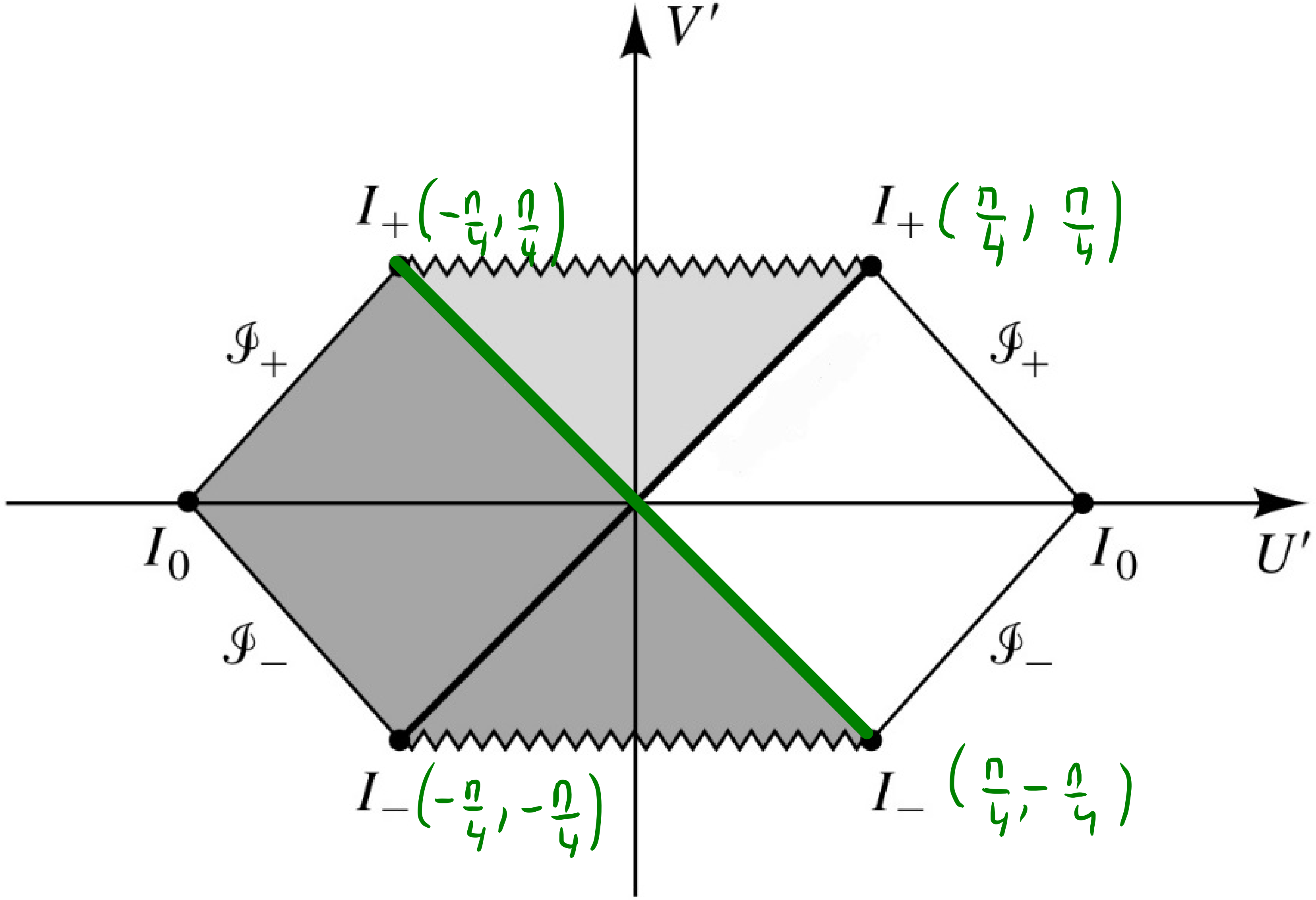
Penrose Diagram of Schwarzschild Geometry

$$U' = \tfrac{1}{2}\tan^{-1}\cancel{(V+U)} - \tfrac{1}{2}\tan^{-1}(V-U)$$

$$V' = \tfrac{1}{2}\tan^{-1}\cancel{(V+U)} + \tfrac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V' - U')$$

$$V + U = \tan(V' + U')$$



Hartle Box 12.5

Horizon $r = 2M$, $t = -\infty$ (future Horizon)

$$V = -U \Rightarrow \left\{ \begin{aligned} U' &= -\tfrac{1}{2}\tan^{-1}(V-U) \\ V' &= \tfrac{1}{2}\tan^{-1}(V-U) \end{aligned} \right\} \Rightarrow V' = -U'$$

$$-\frac{\pi}{4} \le U' = V' = \tfrac{1}{2}\tan^{-1}(V-U) \le \frac{\pi}{4}$$

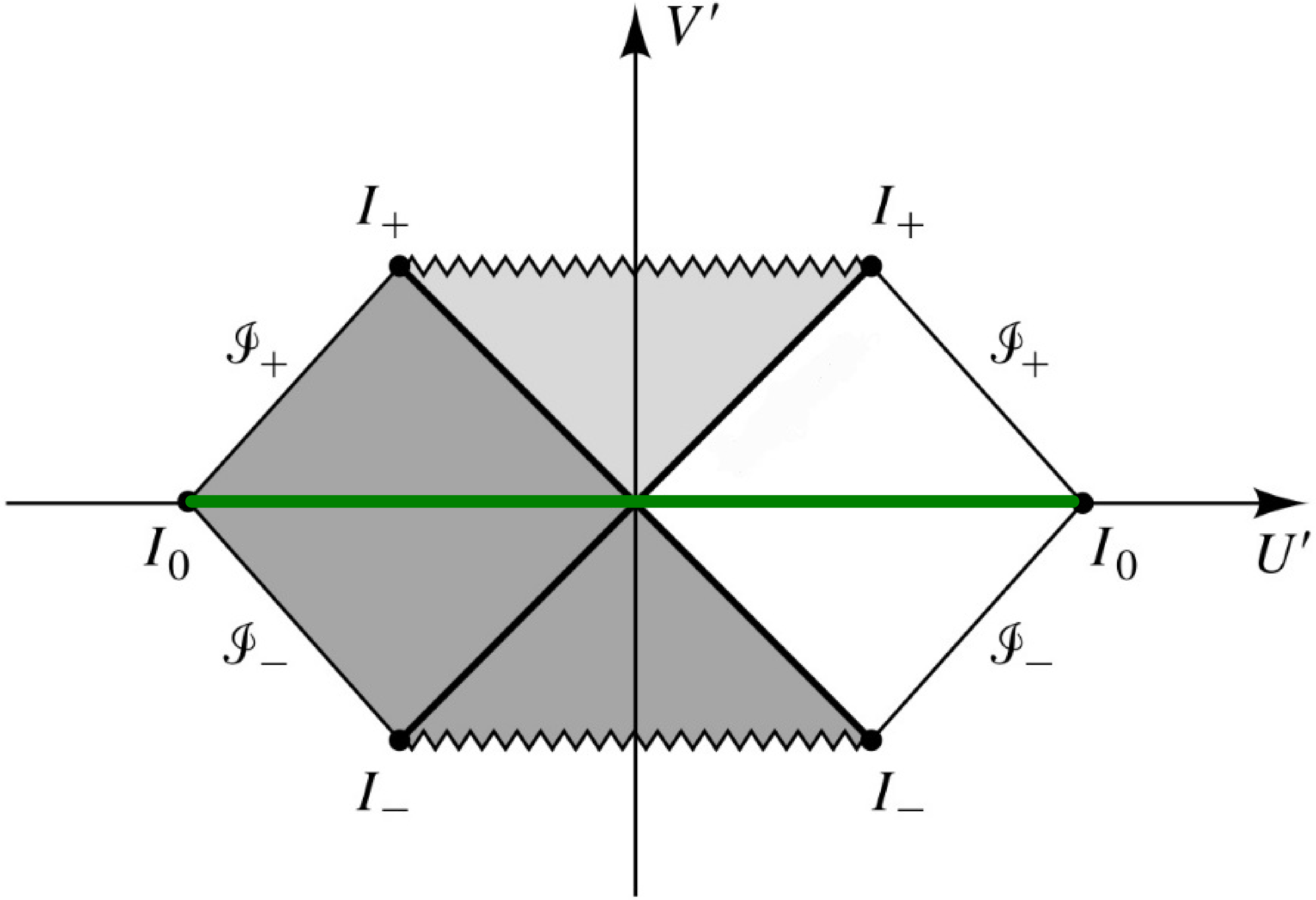
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(\cancel{V}+U) - \frac{1}{2}\tan^{-1}(\cancel{V}-U)$$

$$V' = \frac{1}{2}\tan^{-1}(\cancel{V}+U) + \frac{1}{2}\tan^{-1}(\cancel{V}-U)$$

$$V - U = \tan(V'-U')$$

$$V + U = \tan(V'+U')$$



Horizontal axis

$$V = 0 \Rightarrow U' = \frac{1}{2}\tan^{-1}(U) - \frac{1}{2}\tan^{-1}(-U)$$

$$V' = \frac{1}{2}\tan^{-1}(U) + \frac{1}{2}\tan^{-1}(-U)$$

Hartle Box 12.5

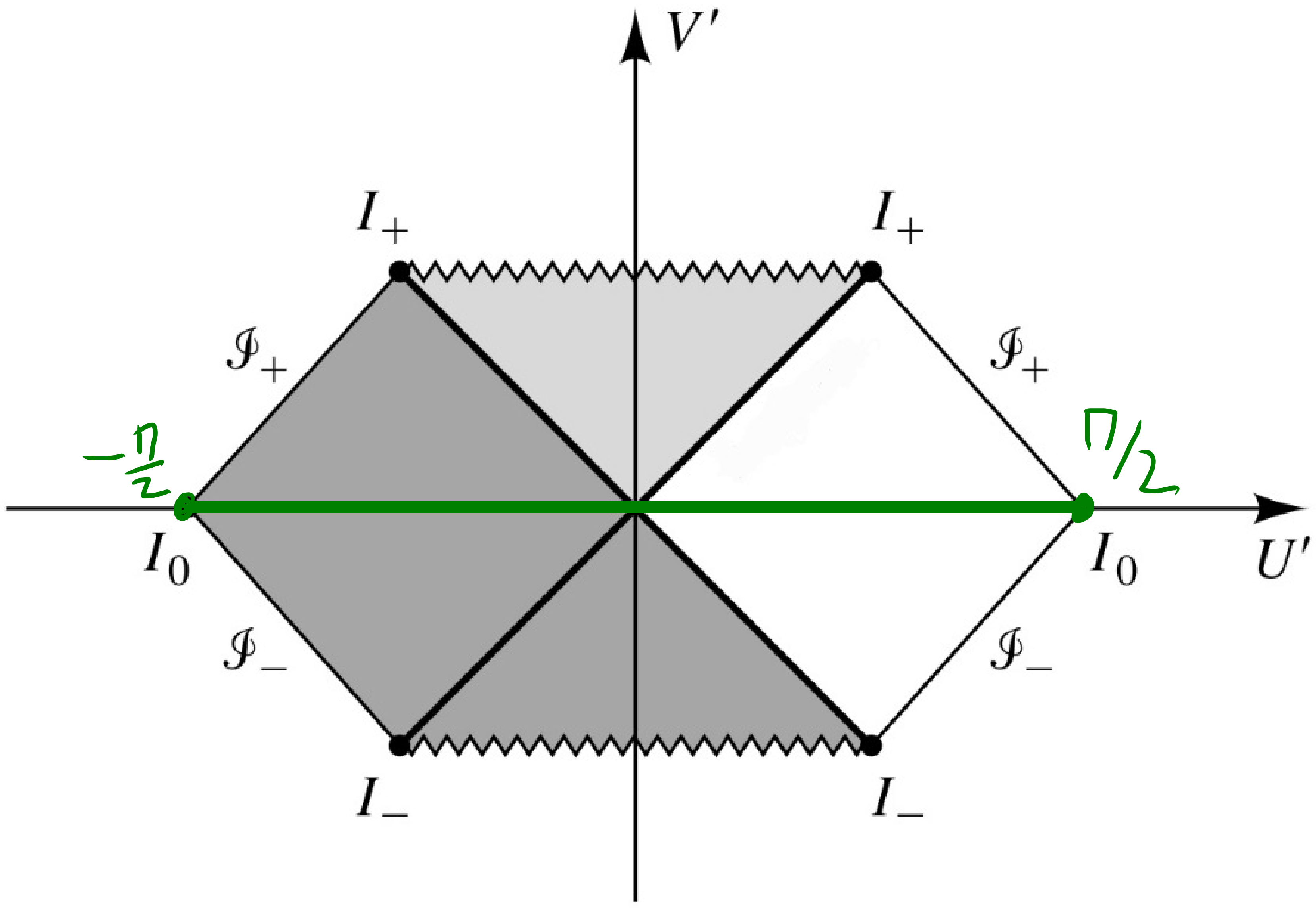
Penrose Diagram of Schwarzschild Geometry

$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$

$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$

$V-U = \tan(V'-U')$

$V+U = \tan(V'+U')$



Hartle Box 12.5

Horizontal axis

$$V=0 \Rightarrow U' = \frac{1}{2}\tan^{-1}(U) - \frac{1}{2}\tan^{-1}(-U)$$

$$V' = \frac{1}{2}\tan^{-1}(U) + \frac{1}{2}\tan^{-1}(-U)$$

$$\Rightarrow U' = \tan^{-1}(U) \quad , \quad -\frac{\pi}{2} < U' < \frac{\pi}{2}$$

$$V' = 0$$

Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+\cancel{\psi}) - \frac{1}{2}\tan^{-1}(V-\cancel{\psi})$$

$$V' = \frac{1}{2}\tan^{-1}(V+\cancel{\psi}) + \frac{1}{2}\tan^{-1}(V-\cancel{\psi})$$

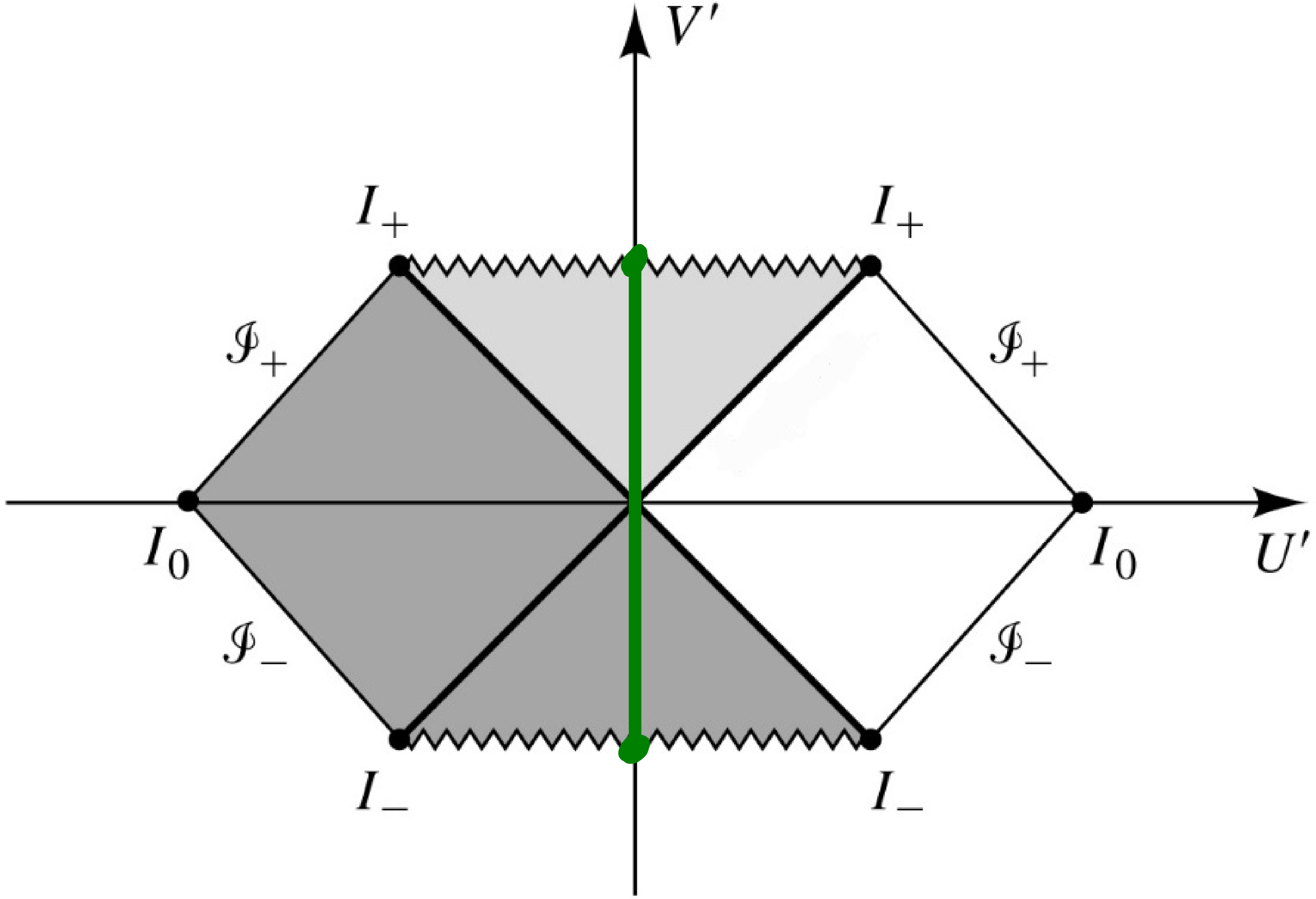
$$V - U = \tan(V'-U')$$

$$V+U = \tan(V'+U')$$

Vertical Axis:

$$U=0 \Rightarrow U' = \frac{1}{2}\tan^{-1}V - \frac{1}{2}\tan^{-1}V$$

$$V' = \frac{1}{2}\tan^{-1}V + \frac{1}{2}\tan^{-1}V$$



Hartle Box 12.5

Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V'-U')$$

$$V + U = \tan(V'+U')$$

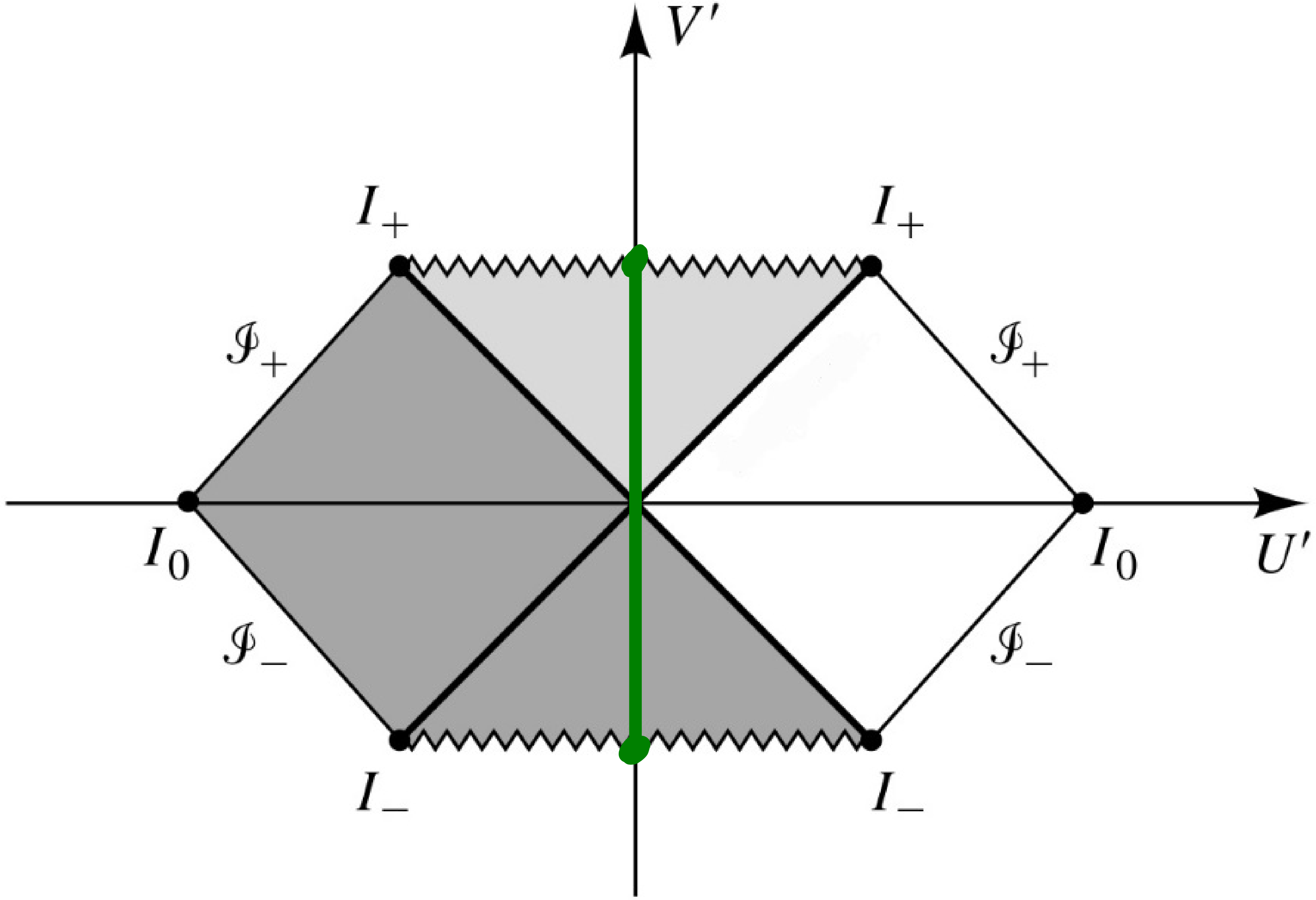
Vertical Axis:

$$U = 0 \Rightarrow U' = \frac{1}{2}\tan^{-1}V - \frac{1}{2}\tan^{-1}V$$

$$V' = \frac{1}{2}\tan^{-1}V + \frac{1}{2}\tan^{-1}V$$

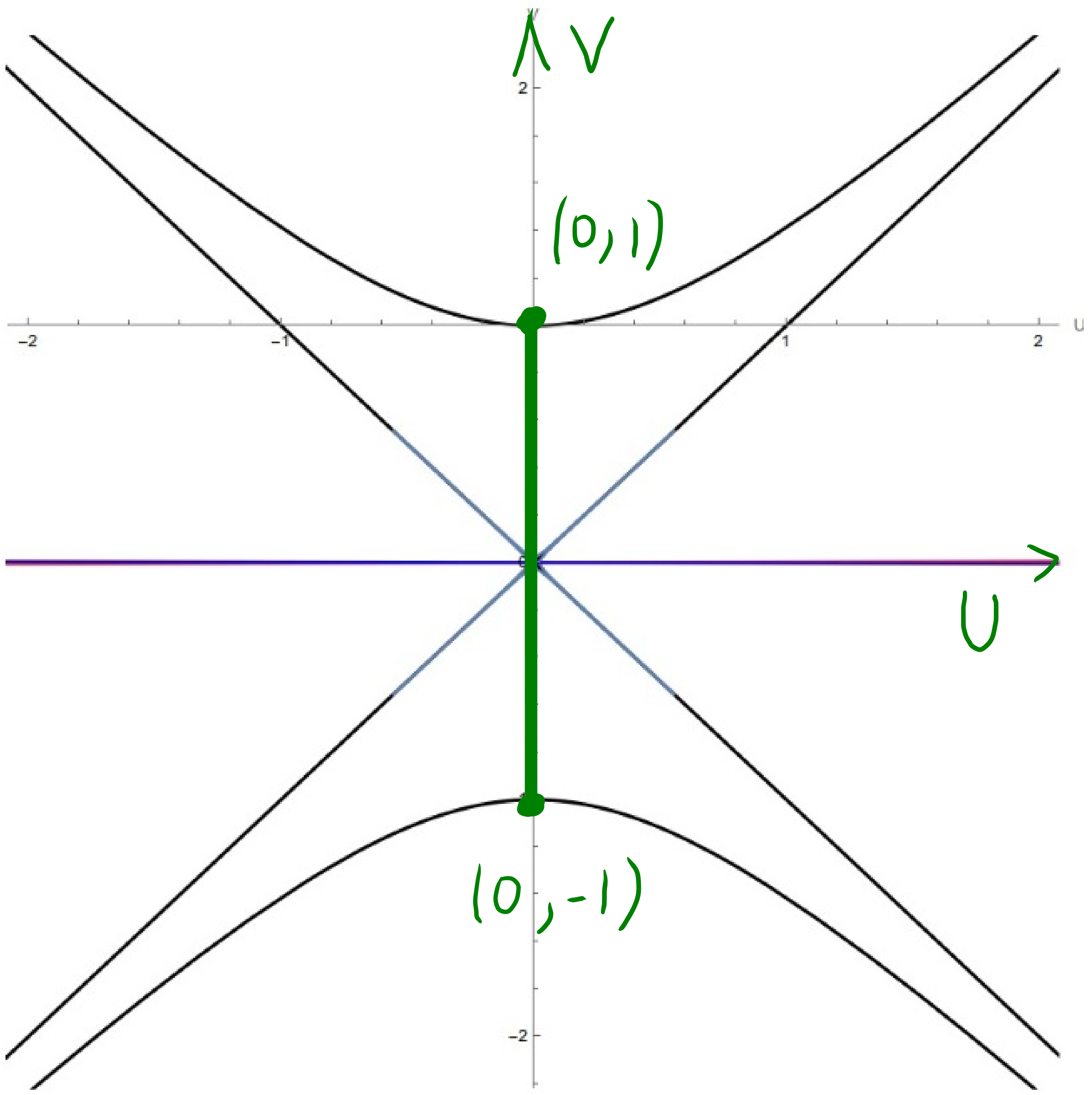
$$\Rightarrow U' = 0$$

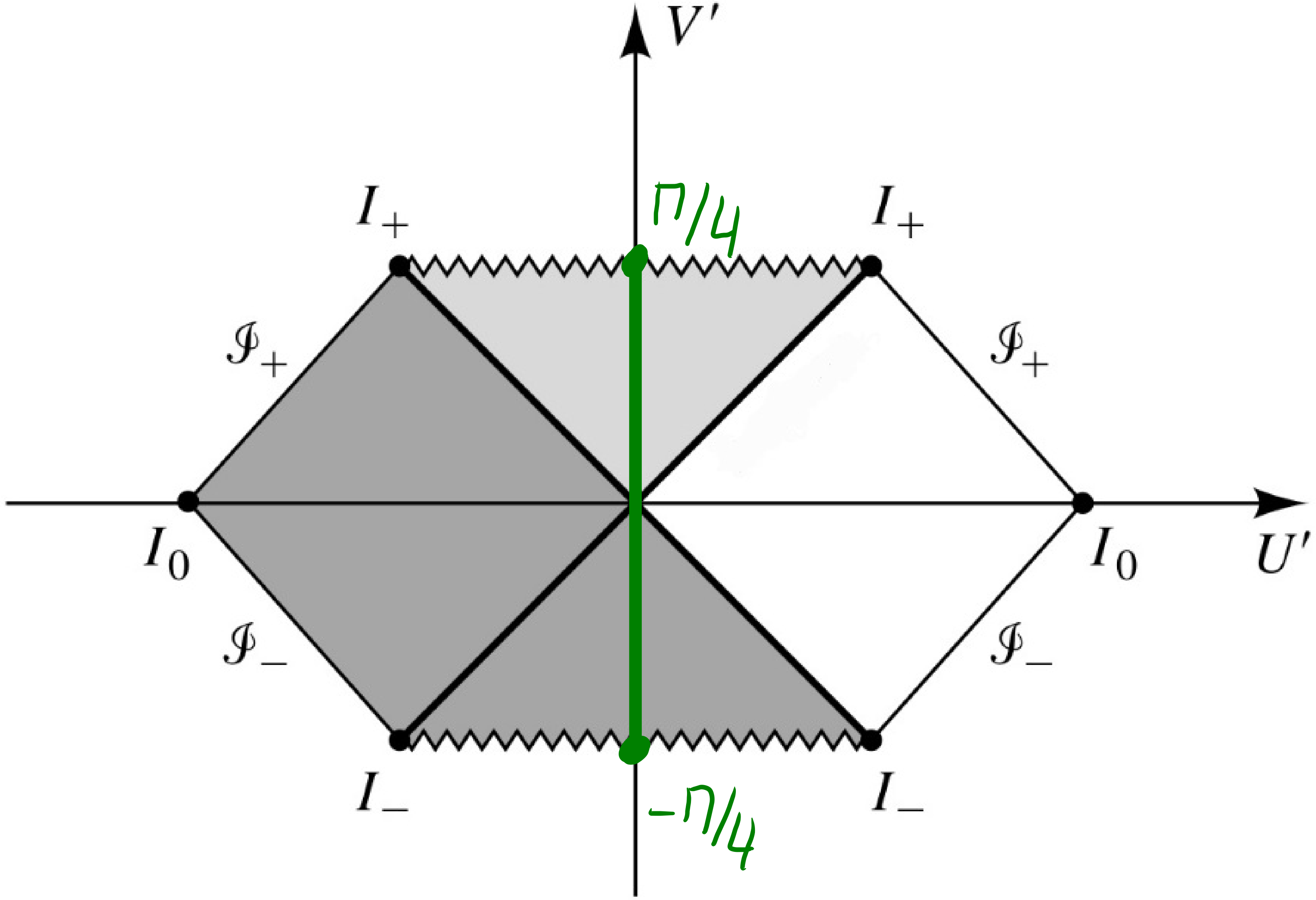
$$V' = \tan^{-1}V$$



Hartle Box 12.5

Penrose Diagram of Schwarzschild Geometry





Hartle Box 12.5

But $-1 < V < 1 \Rightarrow \tan^{-1}(-1) \leq V' \leq \tan^{-1}(1)$

$$\Rightarrow -\frac{\pi}{4} \leq V' \leq \frac{\pi}{4}$$

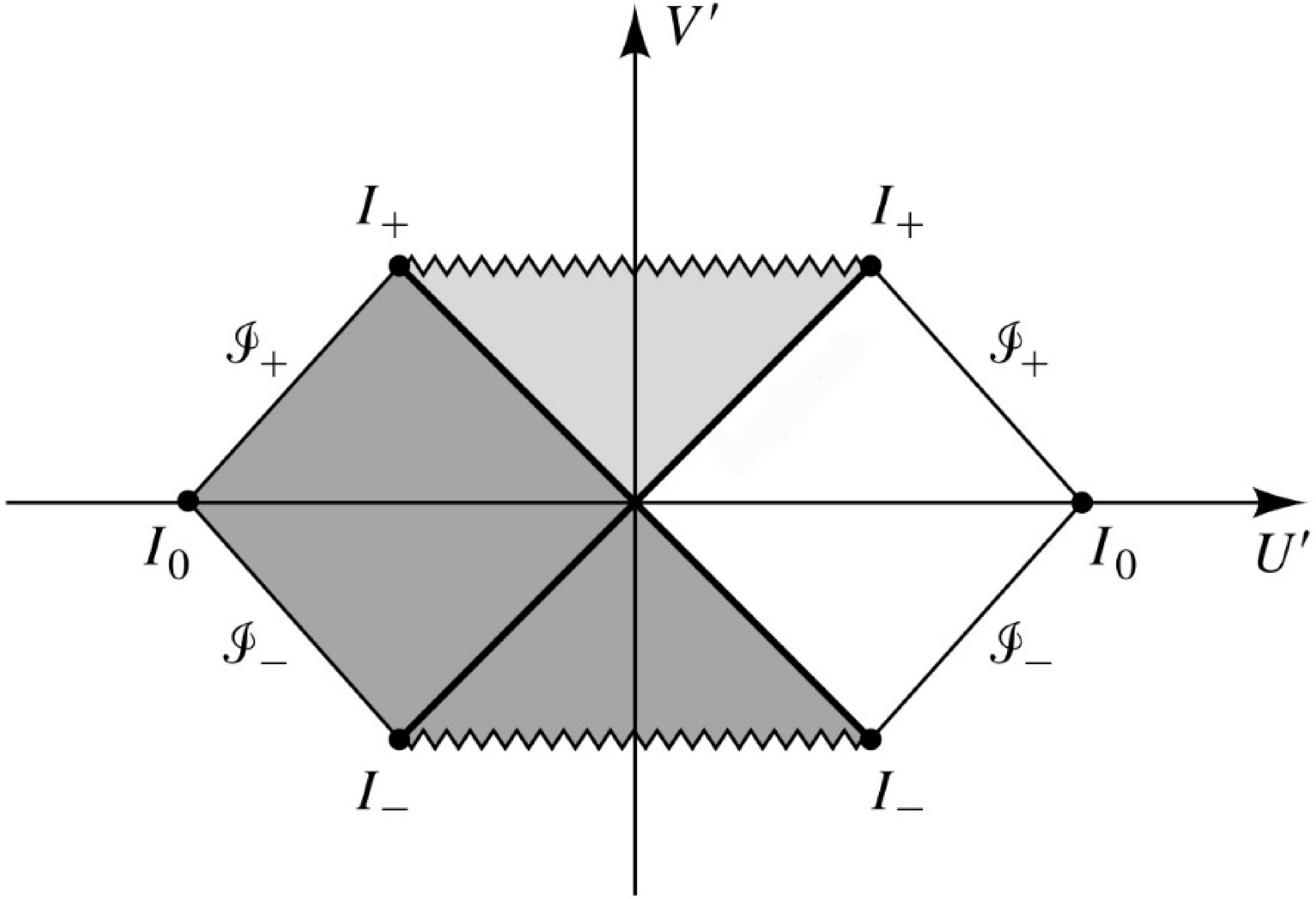
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V'-U')$$

$$V' + U' = \tan(V'+U')$$



Singularity

$r = 0$

$$V^2 - U^2 = 1 \Rightarrow (V-U)(V+U) = 1 \Rightarrow$$

$$\tan(V'-U')\tan(V'+U') = 1$$

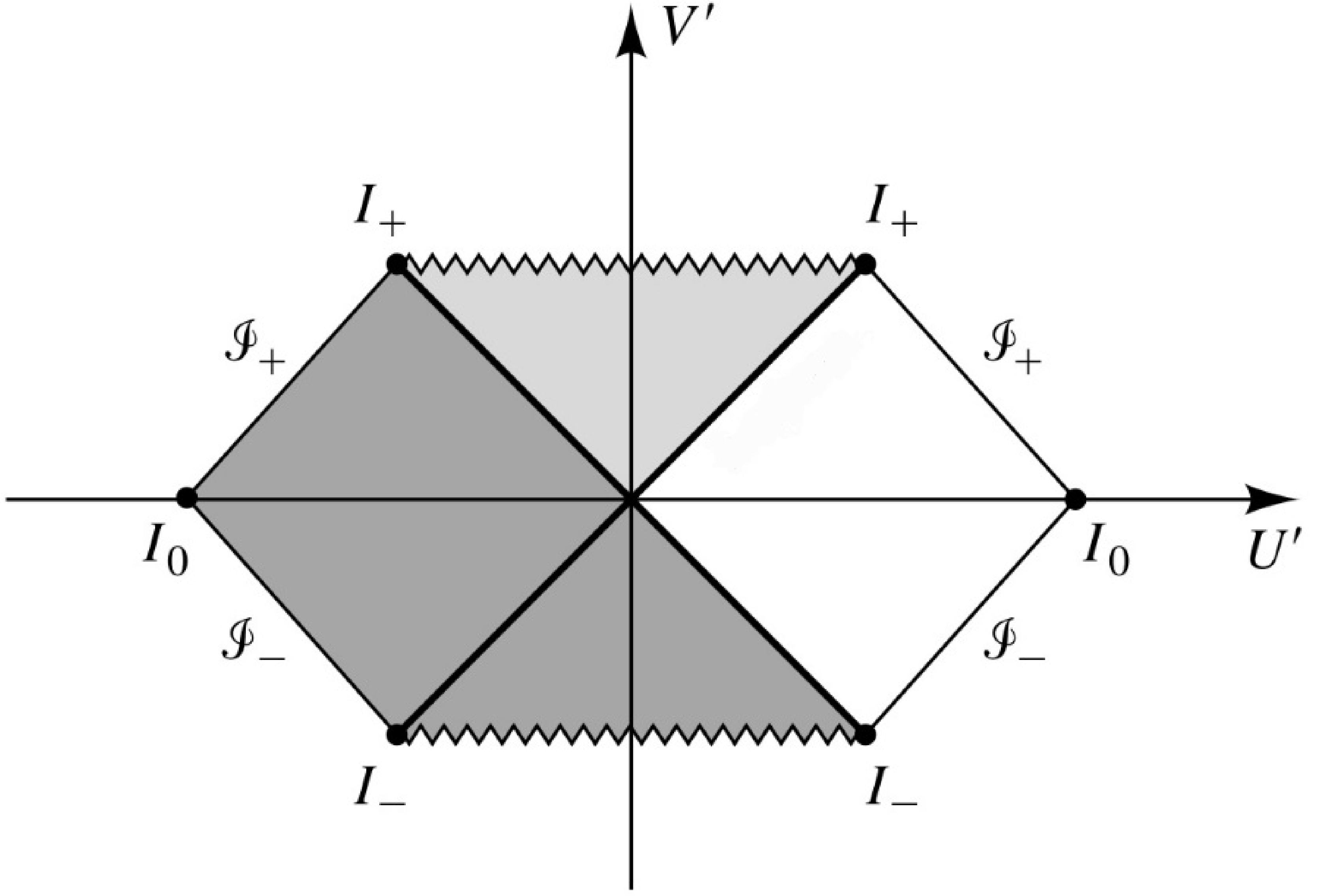
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V'-U')$$

$$V'+U' = \tan(V'+U')$$



Singularity

$r = 0$

$$V^2 - U^2 = 1 \Rightarrow (V-U)(V+U) = 1 \Rightarrow$$

$$\tan(V'-U')\,\tan(V'+U') = 1 \Rightarrow$$

$$\tan(V'+U') = \cot(V'-U') \Rightarrow \begin{cases} V'+U' = \frac{\pi}{2} - (V'-U') \\ V'+U' = -\frac{\pi}{2} - (V'-U') \end{cases}$$

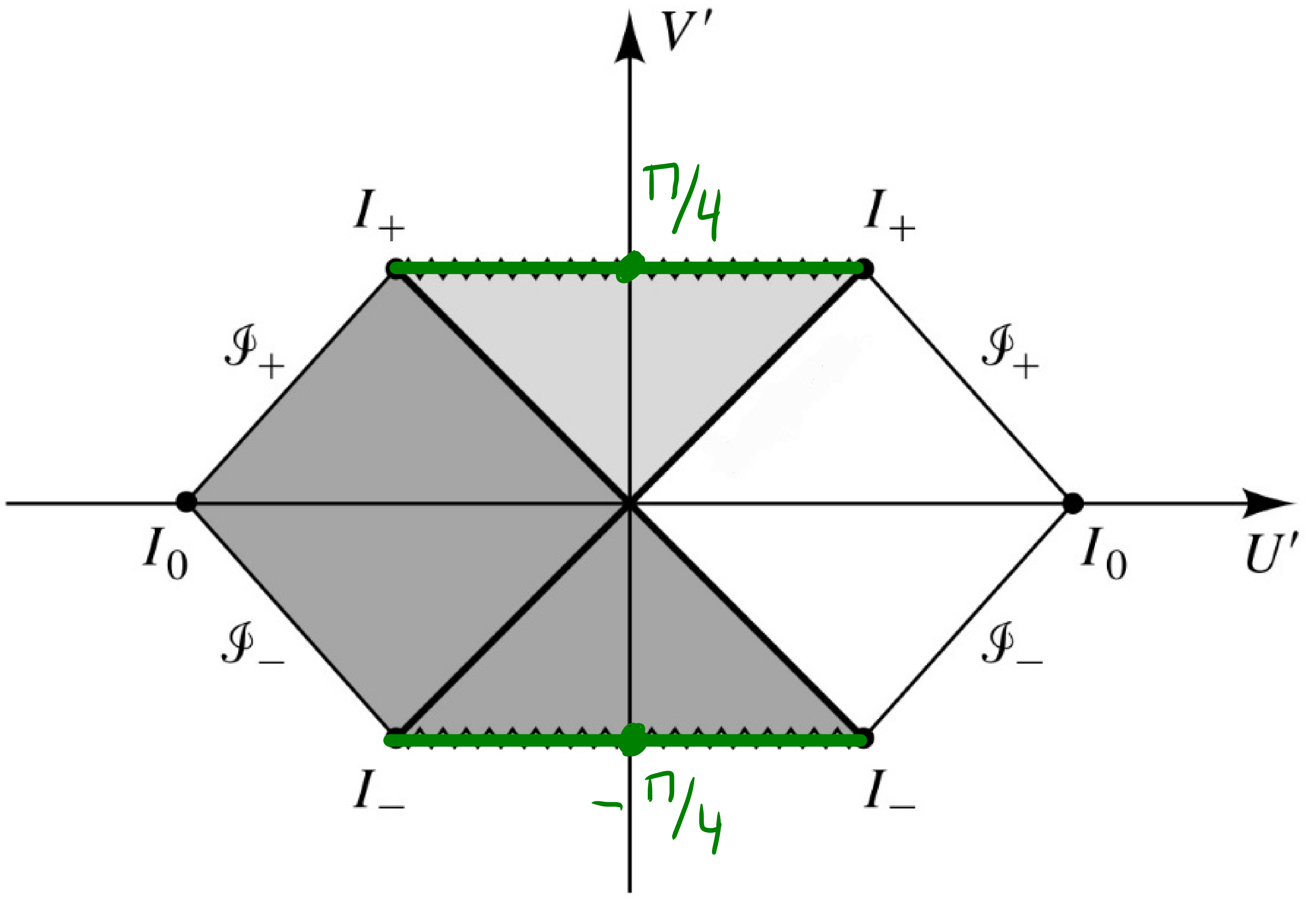
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V-U = \tan(V'-U')$$

$$V'+U' = \tan(V'+U')$$



Singularity $r=0$

$$V^2 - U^2 = 1 \Rightarrow (V-U)(V+U) = 1 \Rightarrow$$

$$\tan(V'-U')\tan(V'+U') = 1 \Rightarrow$$

$$\tan(V'+U') = \cot(V'-U') \Rightarrow \left\{ \begin{array}{l} V'+U' = \frac{\pi}{2} - (V'-U') \\ V'+U' = -\frac{\pi}{2} - (V'-U') \end{array} \right\} \Rightarrow \begin{array}{l} V' = \frac{\pi}{4} \\ V' = -\frac{\pi}{4} \end{array}$$

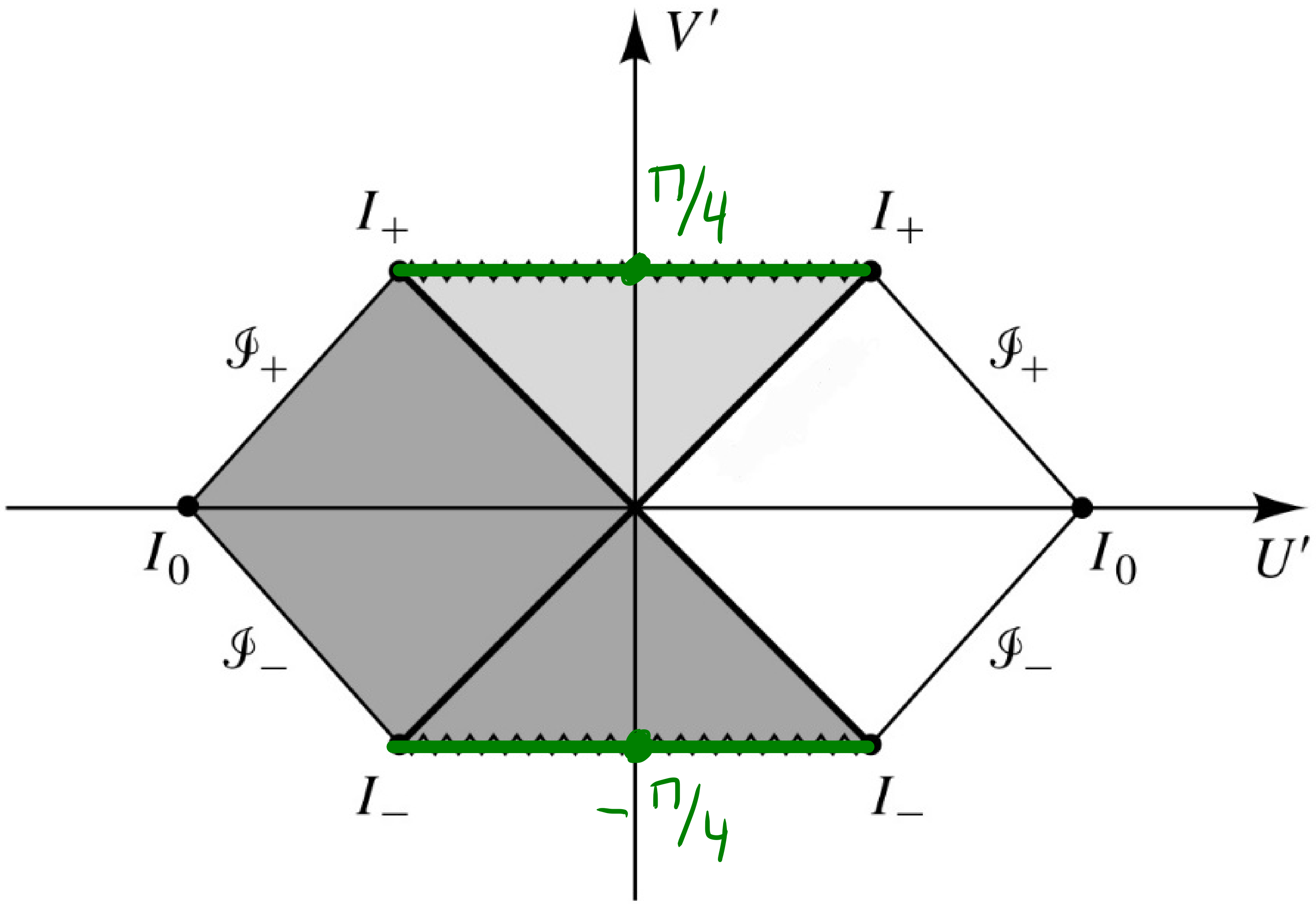
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V-U = \tan(V'-U')$$

$$V'+U' = \tan(V'+U')$$



Singularity

$r = 0$

$$V' = \frac{\pi}{4} \quad \text{or} \quad V' = -\frac{\pi}{4}$$

$$V' = \frac{\pi}{4} \Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}+U\right) + \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}-U\right) = \frac{\pi}{4}$$

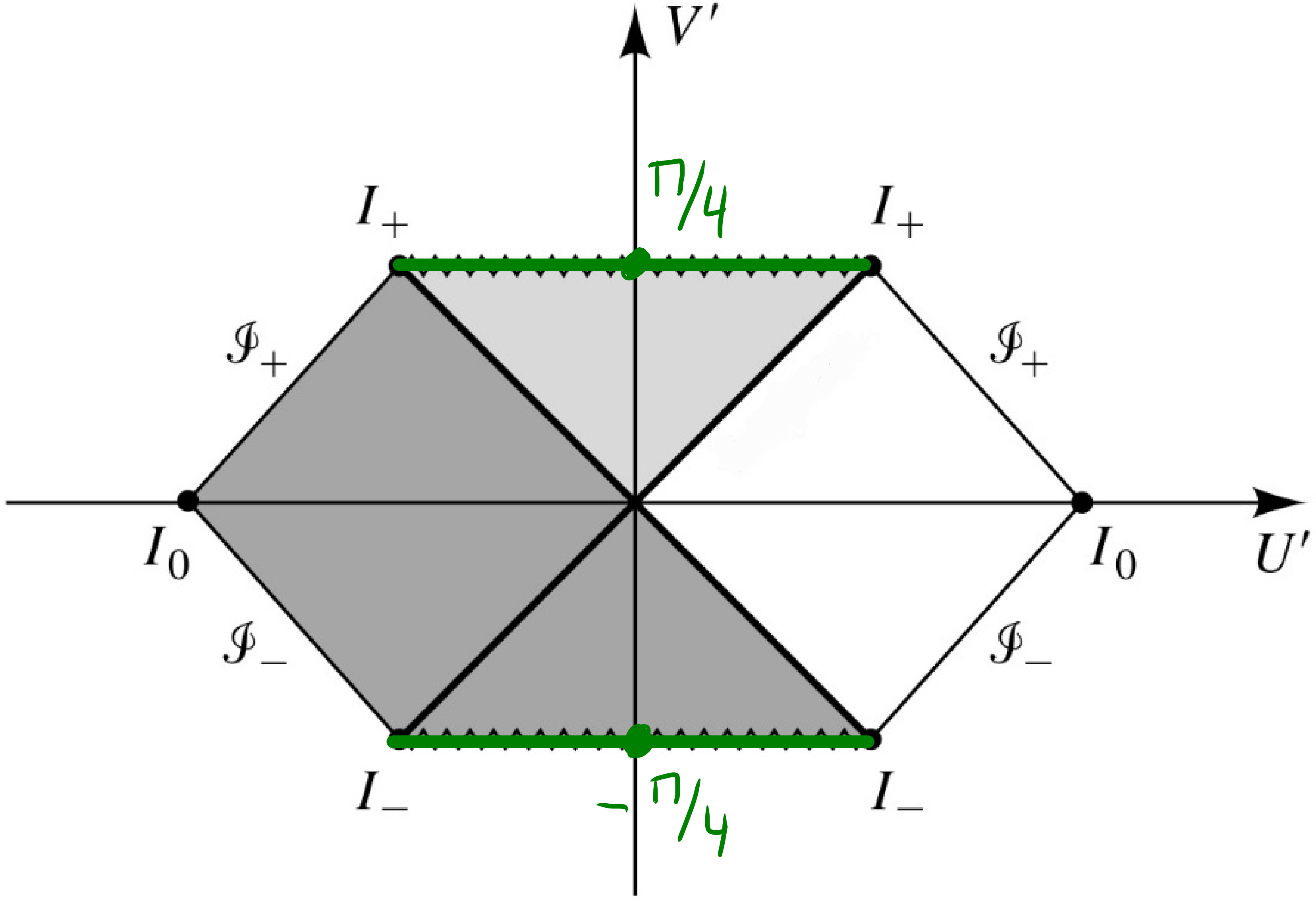
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V'-U')$$

$$V'+U' = \tan(V'+U')$$



Singularity

$r=0$

$$V' = \frac{\pi}{4} \quad \text{or} \quad V' = -\frac{\pi}{4}$$

$$V' = \frac{\pi}{4} \Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}+U\right) + \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}-U\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}-U\right) = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}+U\right)$$

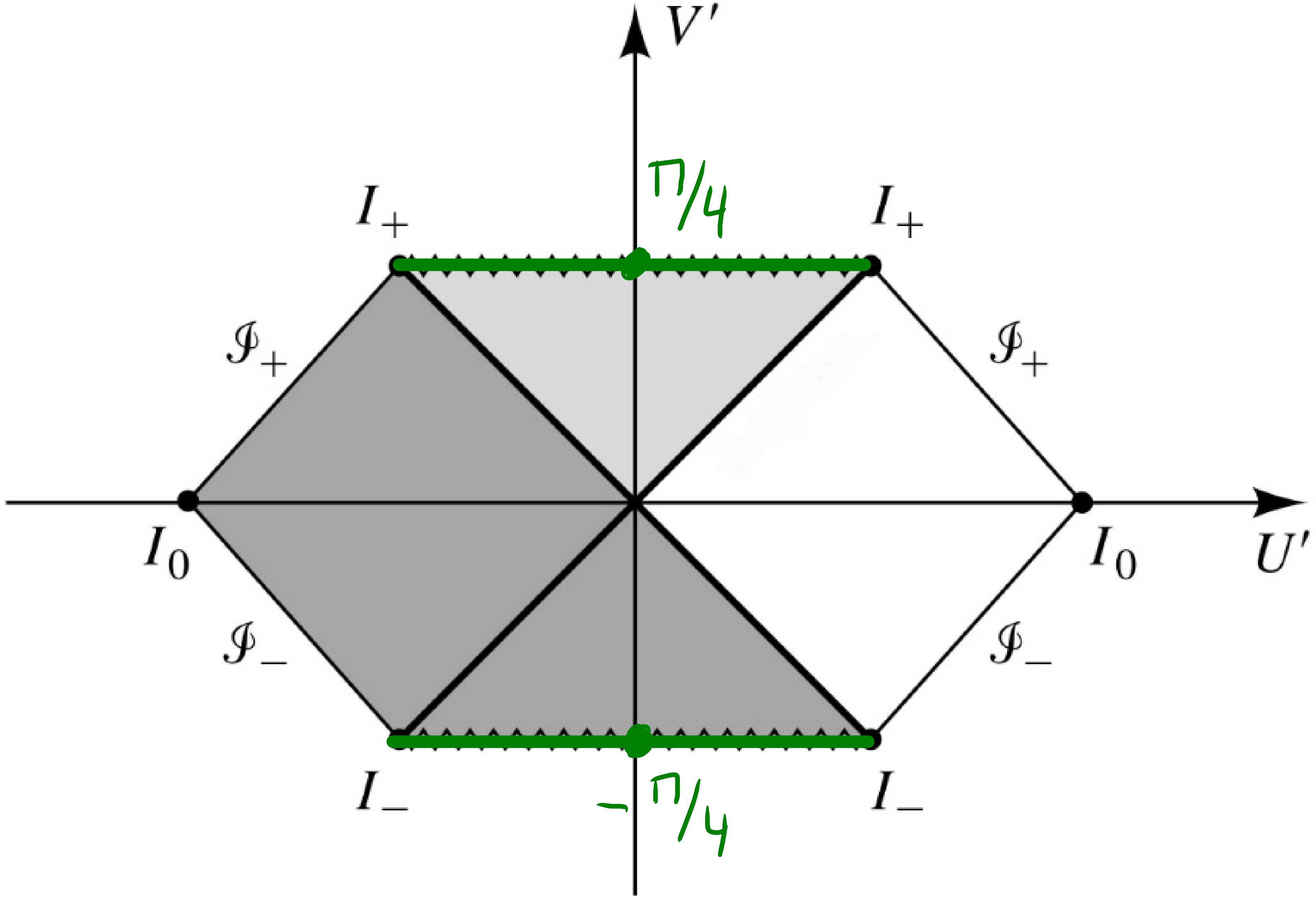
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V'-U')$$

$$V' + U' = \tan(V'+U')$$



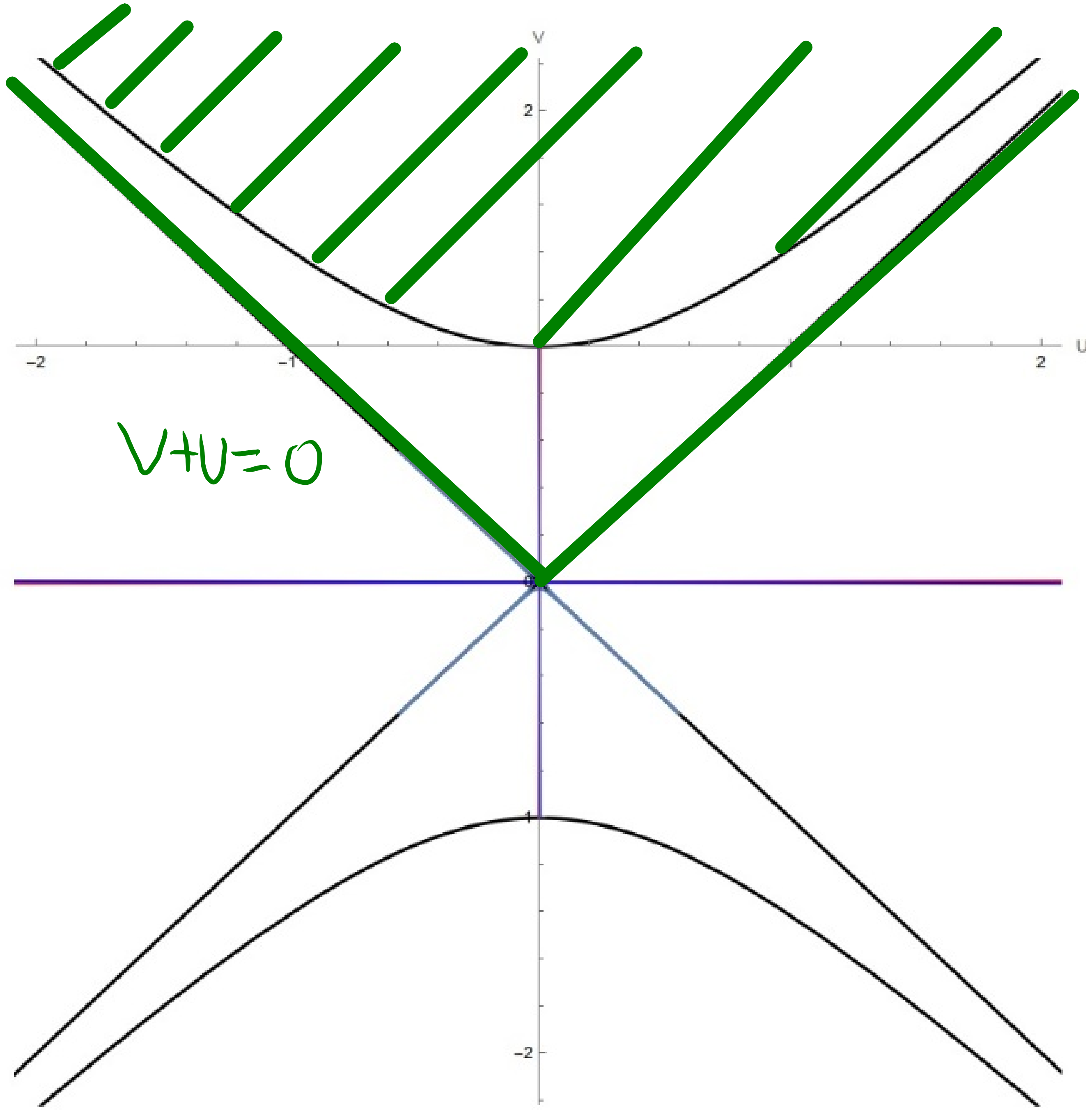
Singularity $r = 0$

$$V' = \frac{\pi}{4} \quad \text{or} \quad V' = -\frac{\pi}{4}$$

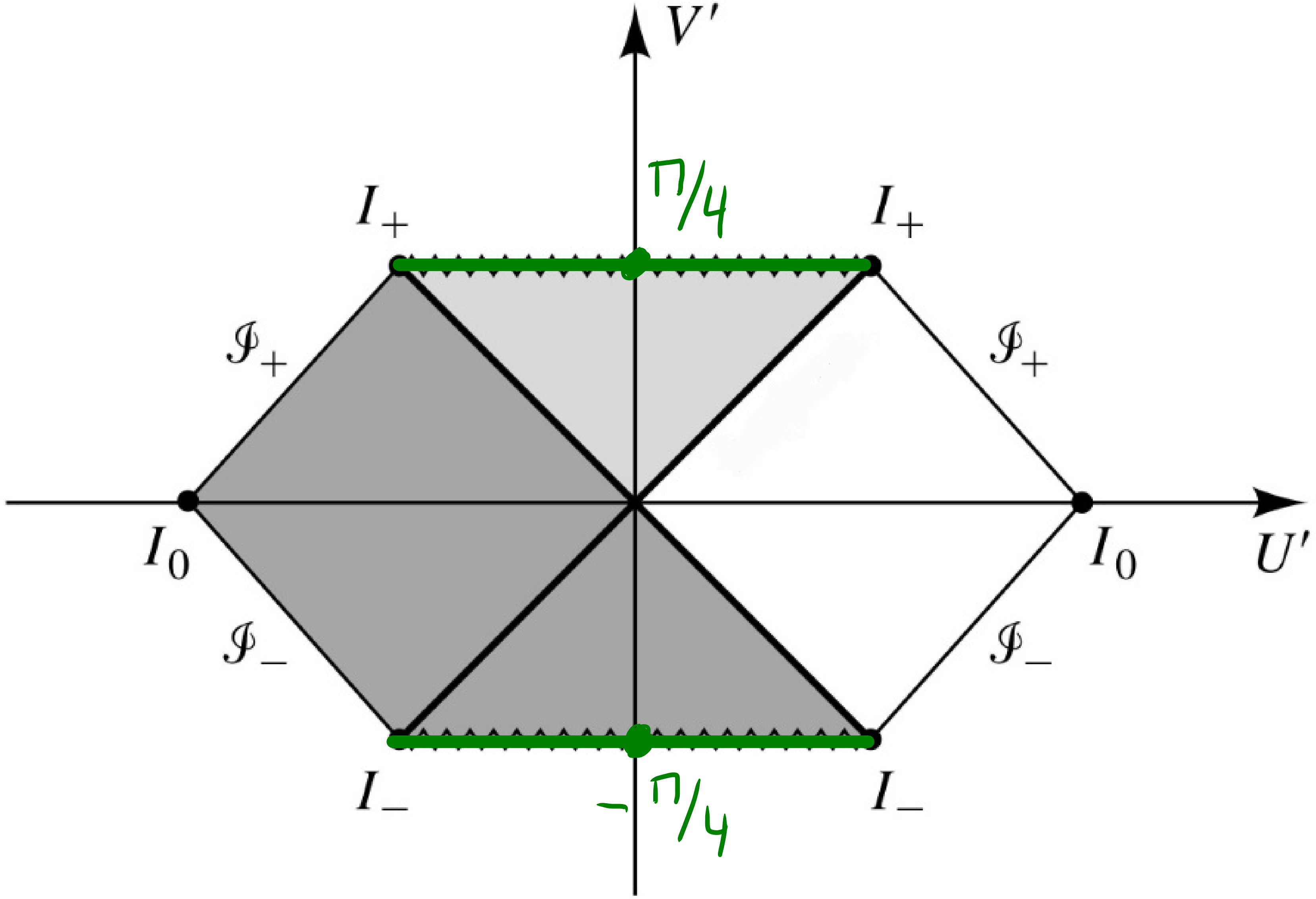
$$V' = \frac{\pi}{4} \Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}+U\right) + \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}-U\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}-U\right) = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}+U\right), \quad \text{then}$$

$$U' = \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}+U\right) - \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}-U\right) = \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}+U\right) - \frac{\pi}{4} + \frac{1}{2}\tan^{-1}\left(\frac{\pi}{4}+U\right)$$



V
U
2
-2
-1
1
V+U=0



V'
U'
I_+
I_+
$\mathscr{I}_+$
$\mathscr{I}_+$
I_0
I_0
$\mathscr{I}_-$
$\mathscr{I}_-$
I_-
I_-
π/4
-π/4

$$0 < V + U < +\infty$$

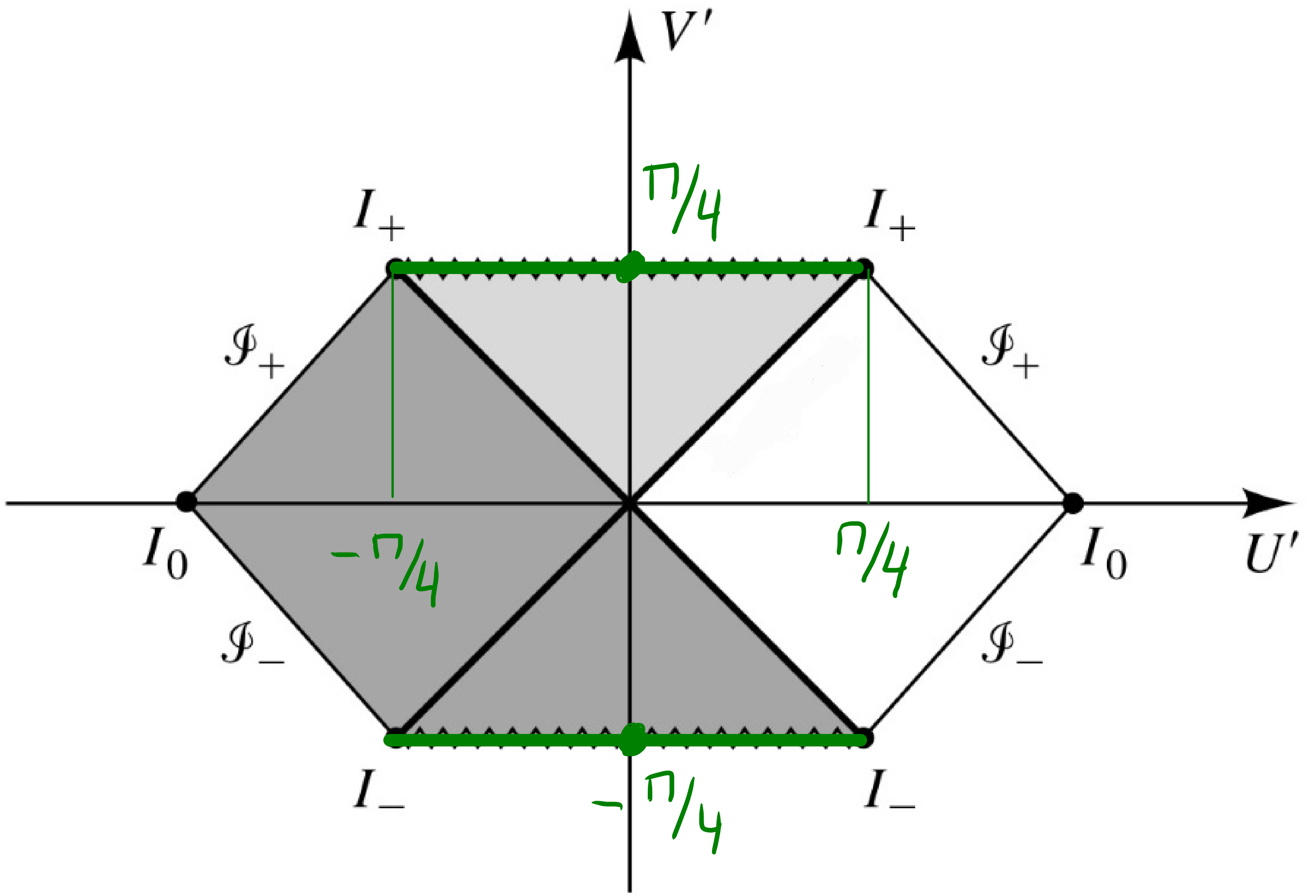
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V'-U')$$

$$V'+U' = \tan(V'+U')$$



Singularity $\quad r=0$

$$V' = \frac{\pi}{4} \quad \text{or} \quad V' = -\frac{\pi}{4}$$

$$V' = \frac{\pi}{4} \Rightarrow U' = \tan^{-1}\left(\frac{\pi}{4}+U\right) - \frac{\pi}{4}$$

$$0 < V+U < +\infty \Rightarrow U'_{max} = \tan^{-1}(+\infty) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$U'_{min} = \tan^{-1}(0) - \frac{\pi}{4} = 0 - \frac{\pi}{4} = -\frac{\pi}{4}$$

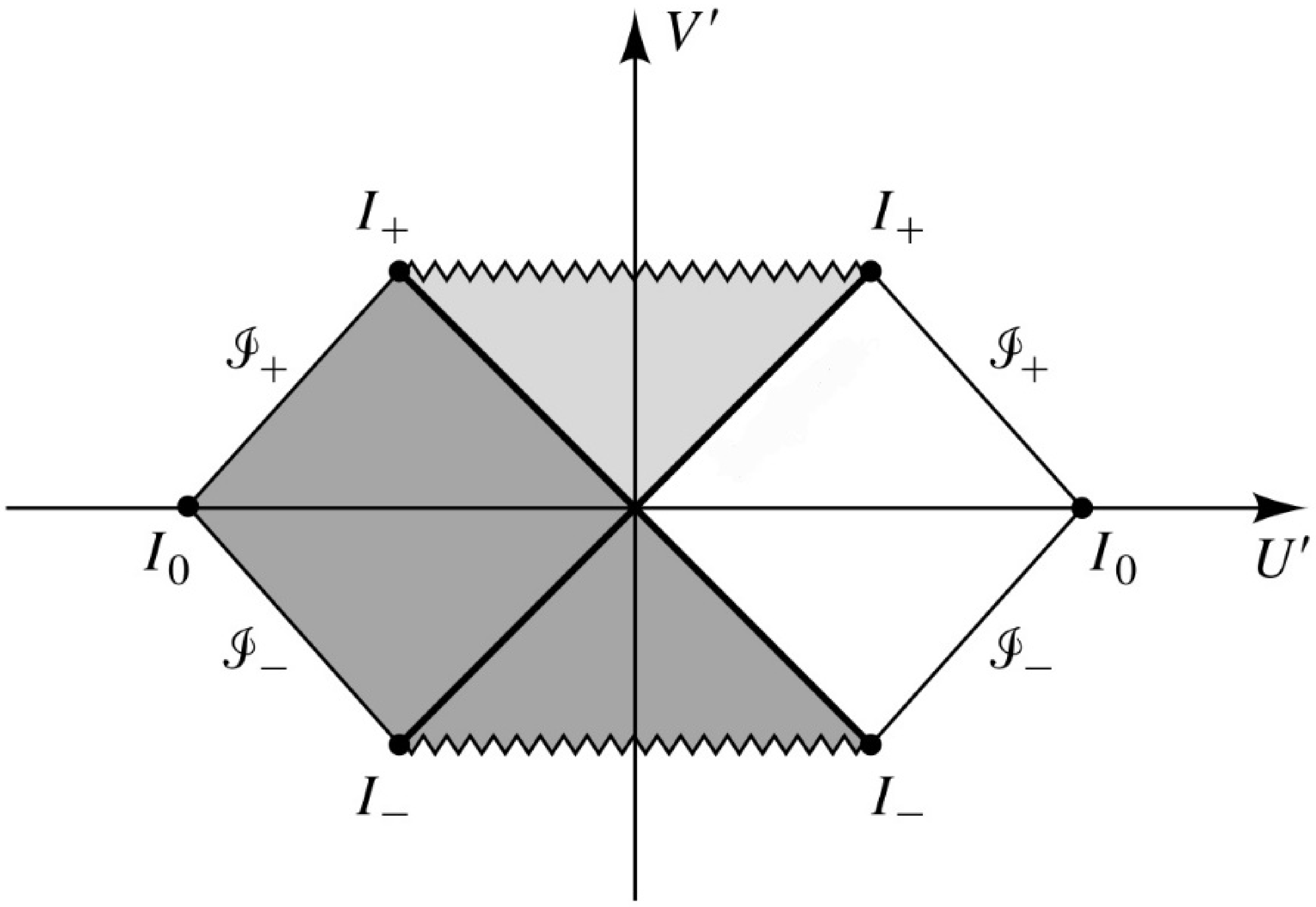
Penrose Diagram of Schwarzschild Geometry

$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$

$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$

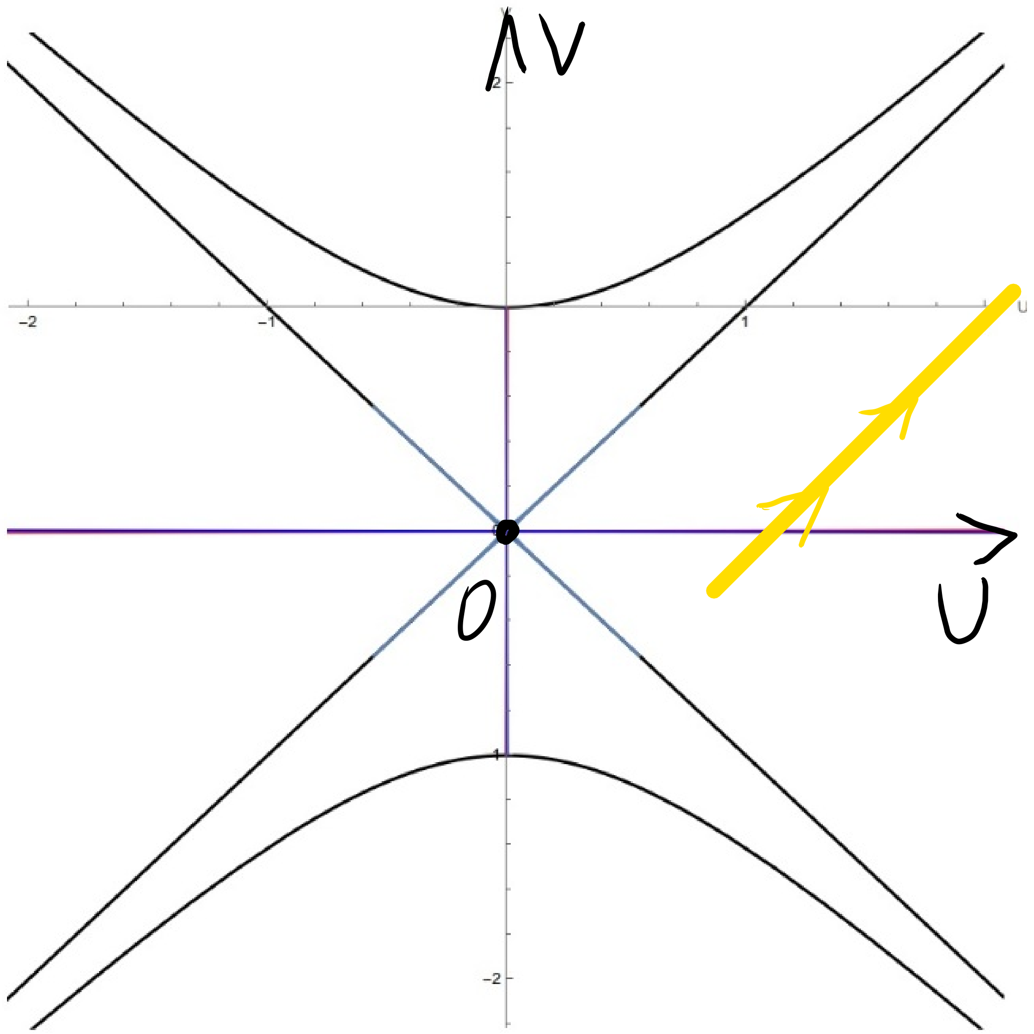
$V - U = \tan(V'-U')$

$V' + U' = \tan(V'+U')$



Light rays

$V = U + V_0$



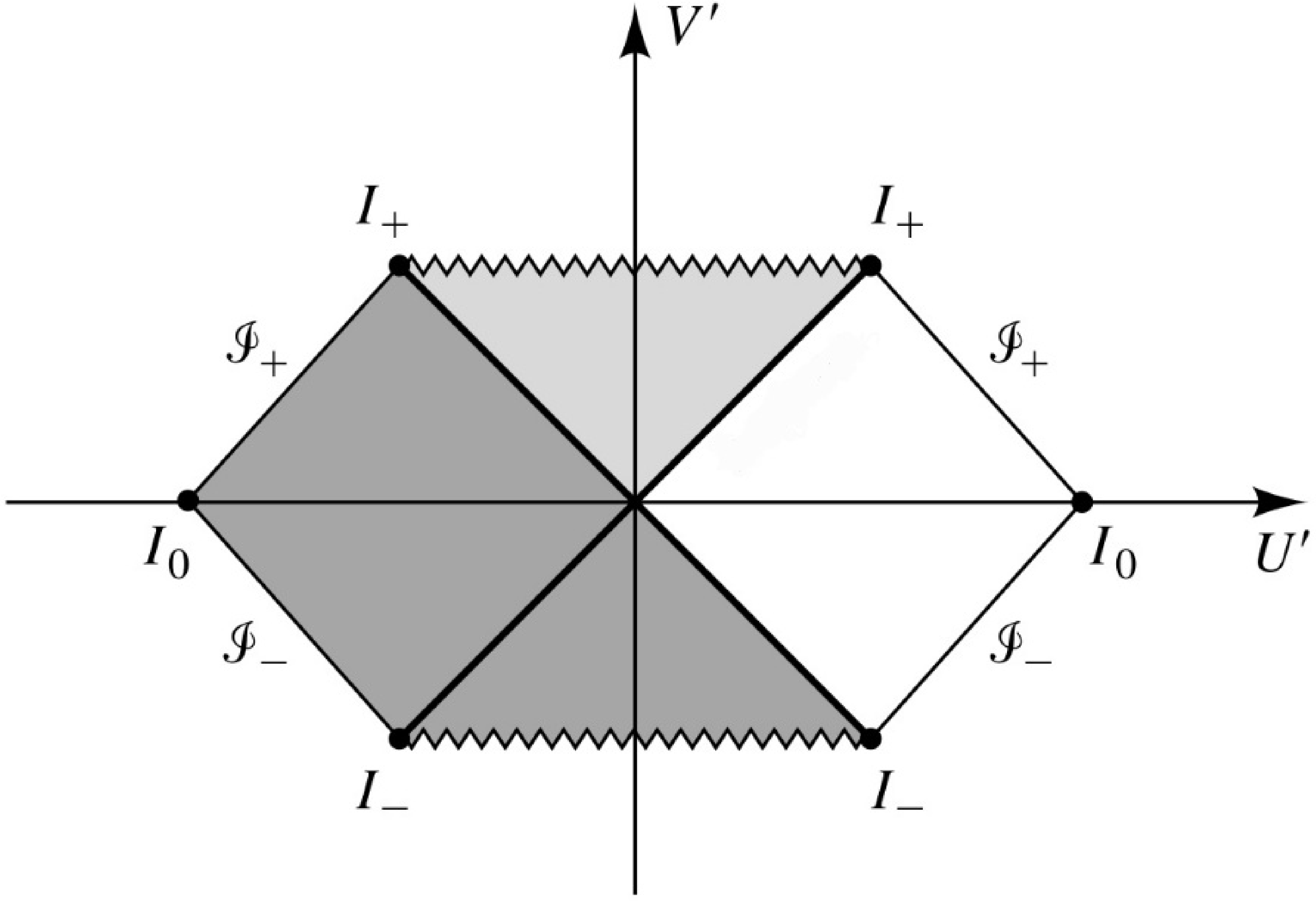
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V-U = \tan(V'-U')$$

$$V'+U' = \tan(V'+U')$$



Light rays

$$V = U + U_0 \Rightarrow \qquad V - U = U_0 = \text{const}$$

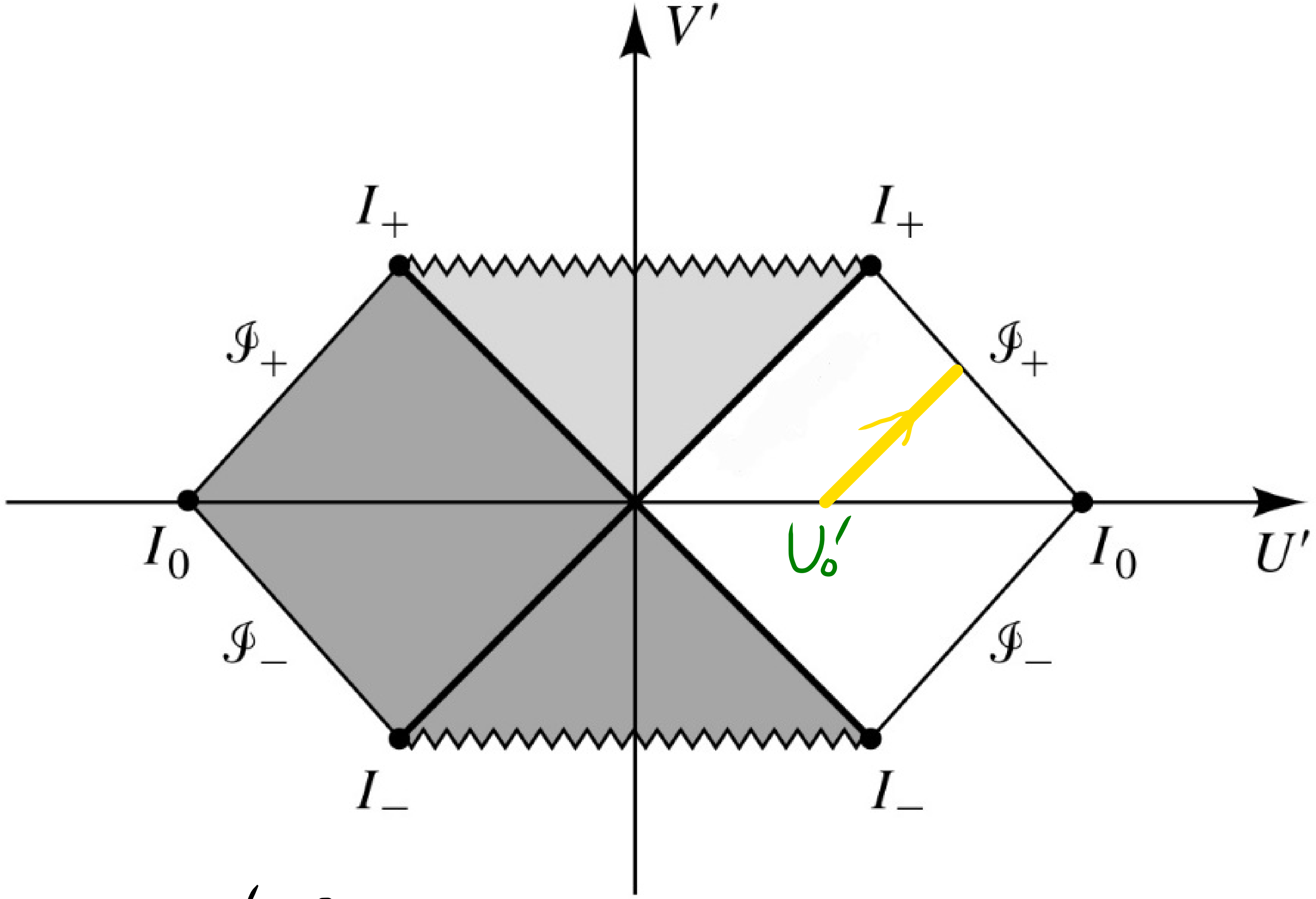
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V-U = \tan(V'-U')$$

$$V'+U' = \tan(V'+U')$$



Light rays

$$V = U + U_0 \Rightarrow \quad V - U = U_0 = \text{const}$$

$$\Rightarrow V' - U' = \tan^{-1}(V-U) = \tan^{-1} U_0 \equiv U_0', \quad 0 < U_0' < \frac{\pi}{2}$$

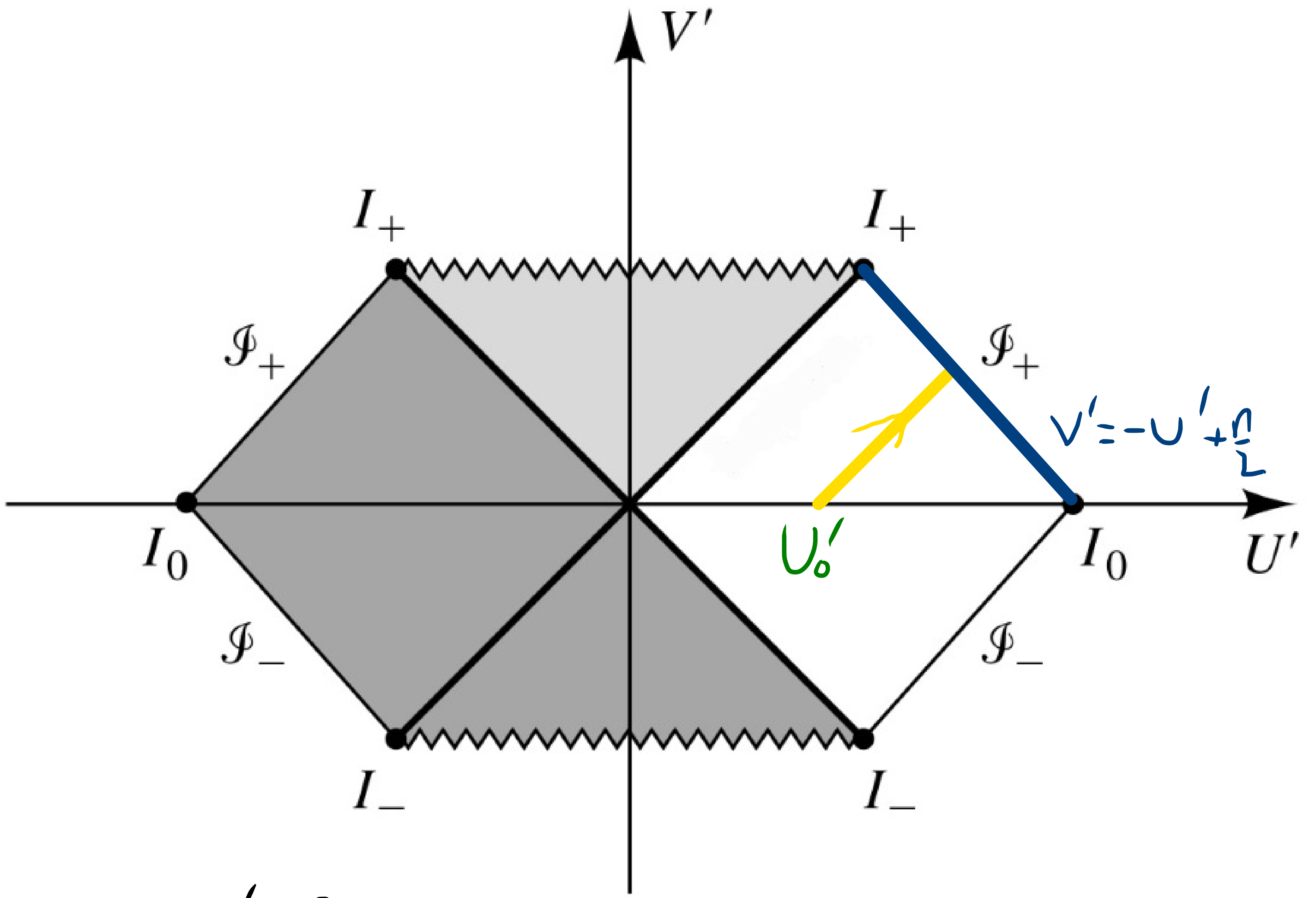
Penrose Diagram of Schwarzschild Geometry

$$U' = \tfrac{1}{2}\tan^{-1}(V+U) - \tfrac{1}{2}\tan^{-1}(V-U)$$

$$V' = \tfrac{1}{2}\tan^{-1}(V+U) + \tfrac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V'-U')$$

$$V'+U' = \tan(V'+U')$$



Light rays

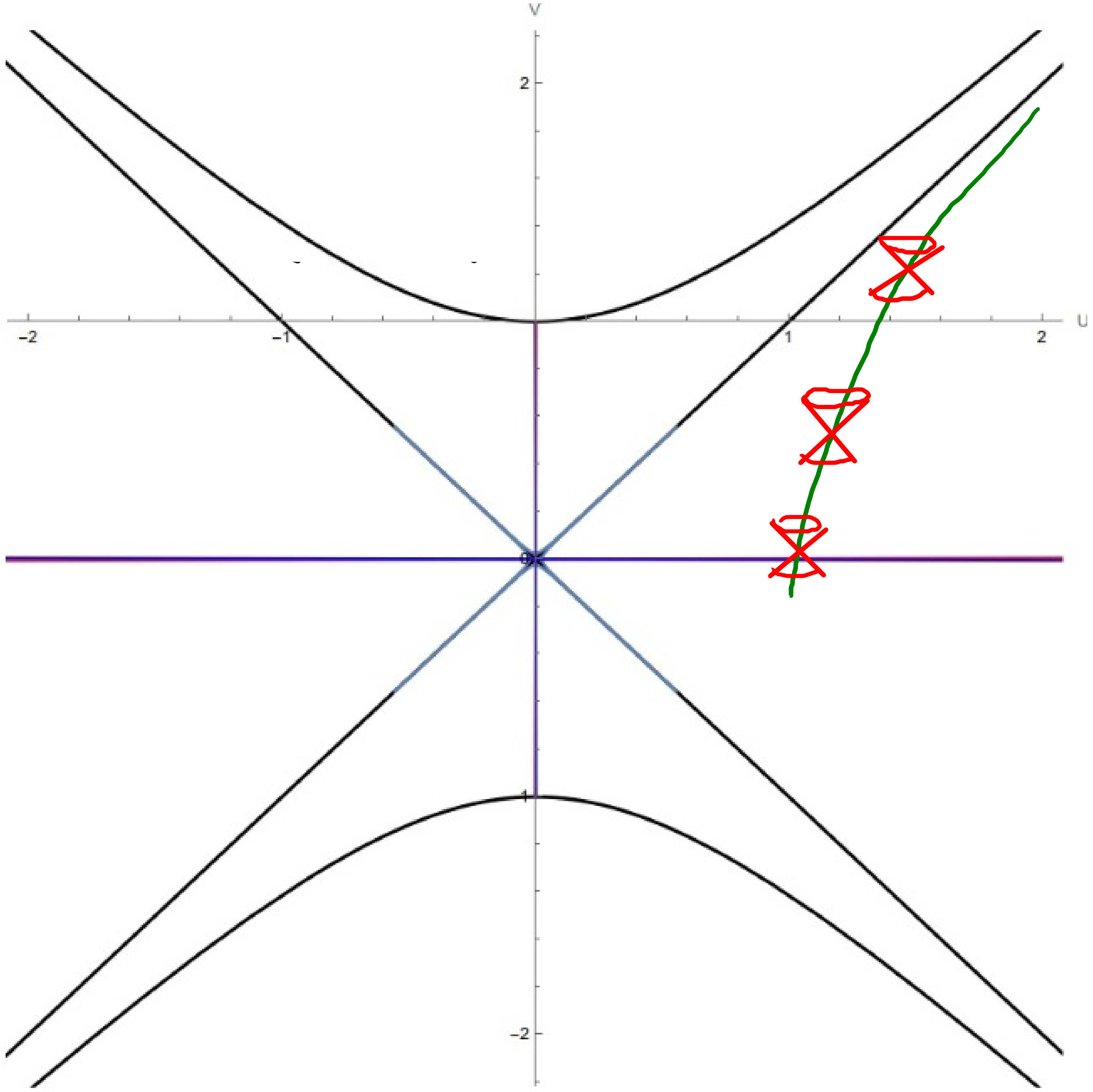
$$V = U + U_0 \Rightarrow \quad V - U = U_0 = \text{const}$$

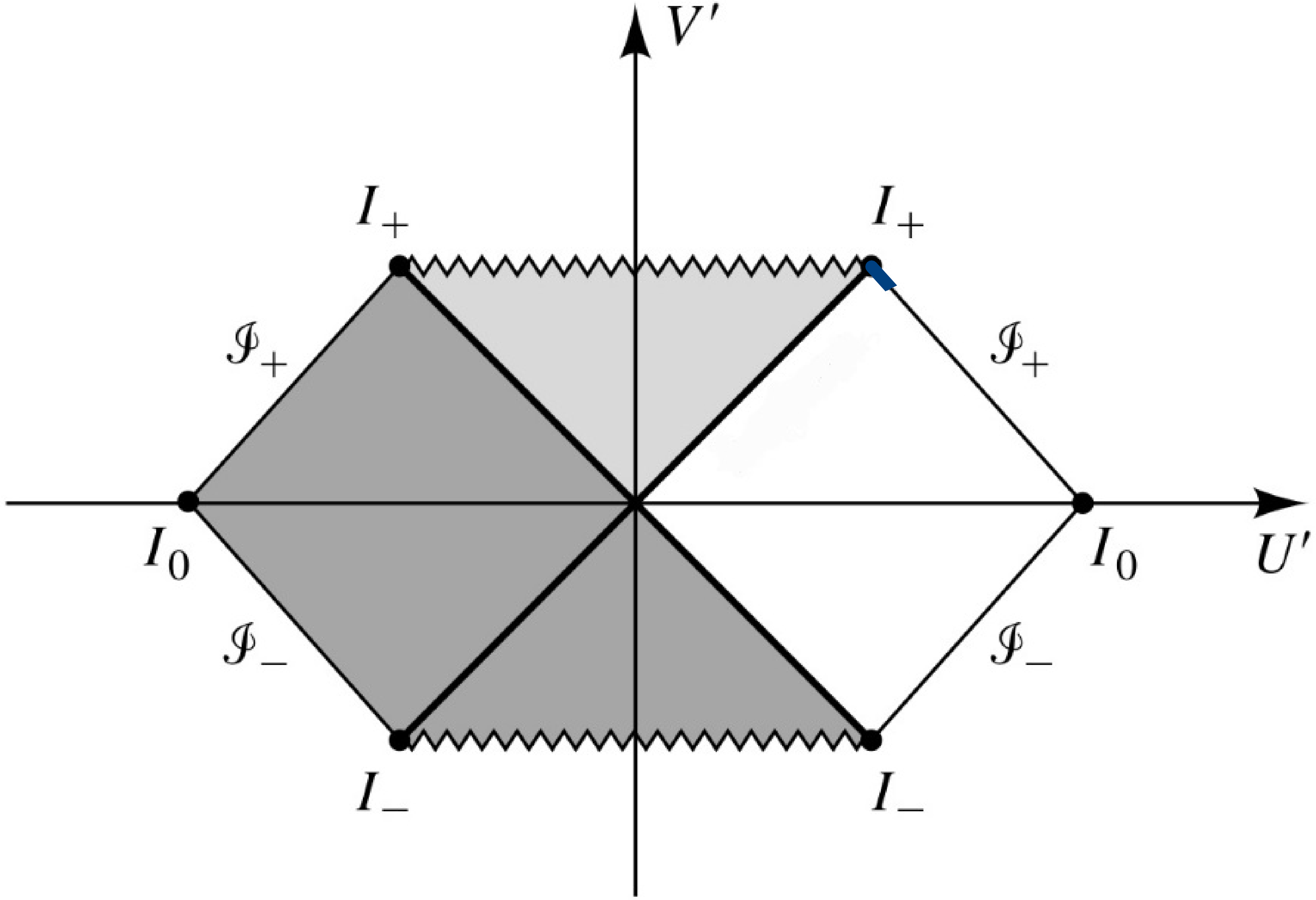
$$\Rightarrow V' - U' = \tan^{-1}(V-U) = \tan^{-1} U_0 \equiv U_0', \quad 0 < U_0' < \frac{\pi}{2}$$

$$\text{But } V' + U' = \tan^{-1}(V+U) < \frac{\pi}{2} \Rightarrow V' < -U' + \frac{\pi}{2}$$

so $V' = -U' + \frac{\pi}{2}$ defines $\mathscr{I}_+$

Penrose Diagram of Schwarzschild Geometry





Timelike curves: must be inside lightcones

$\Rightarrow$ – fall into singularity

or

– asymptote to $r = 2M$ to escape to ∞

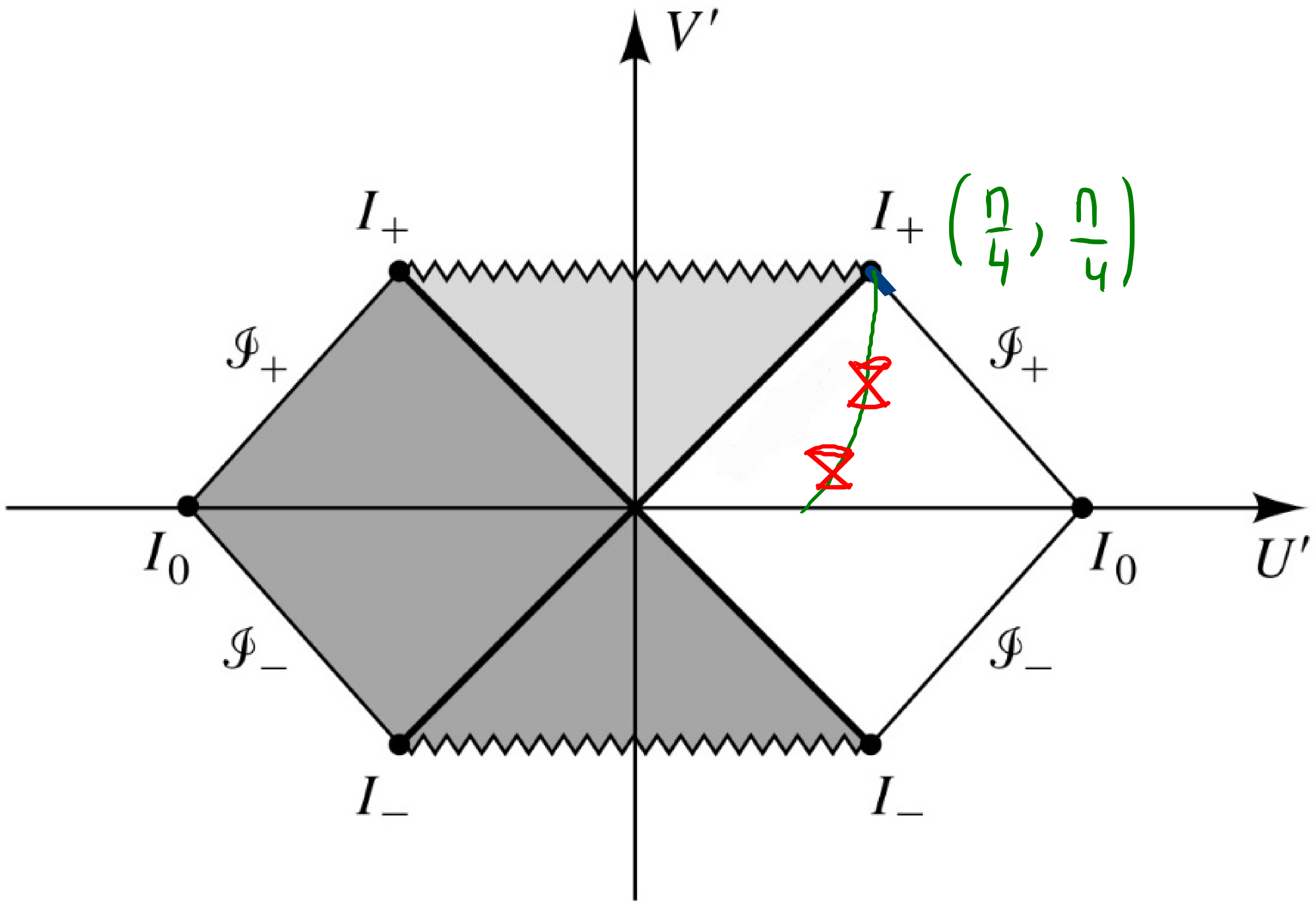
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V' - U')$$

$$V' + U' = \tan(V' + U')$$



Timelike curves that escape to ∞

asymptotically $V = U$, so

$$U' = \frac{1}{2}\tan^{-1}(2U) - \frac{1}{2}\tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$V' = \frac{1}{2}\tan^{-1}(2U) + \frac{1}{2}\tan^{-1}(0) = \frac{\pi}{4} + 0 = \frac{\pi}{4}$$

$\rightsquigarrow$ all end up at $I_+ \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

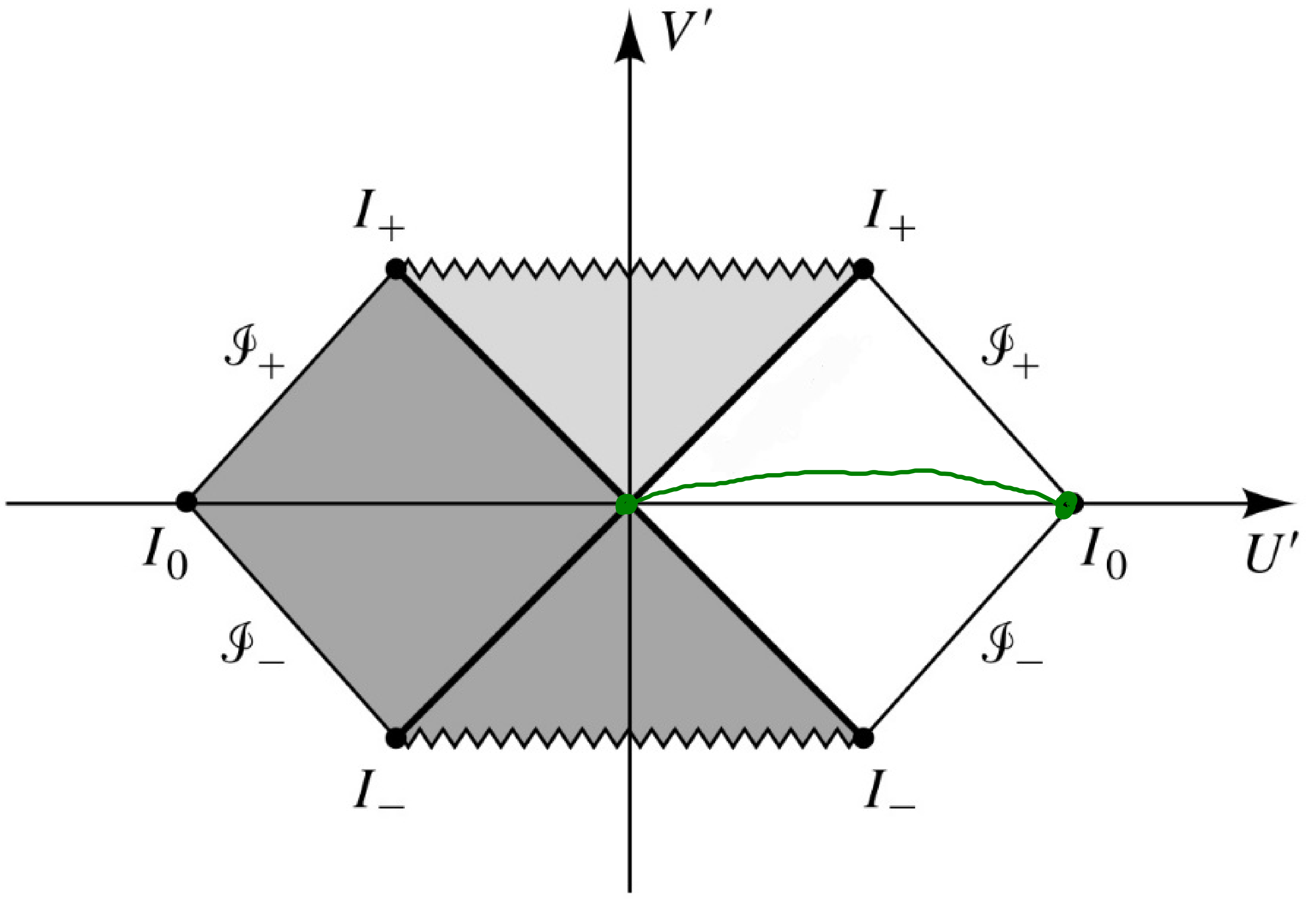
Penrose Diagram of Schwarzschild Geometry

$$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$$

$$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$$

$$V-U = \tan(V'-U')$$

$$V'+U' = \tan(V'+U')$$



Spacelike infinity

Consider $V = \alpha U$, $\alpha < 1$ (const. t)

$$U' = \frac{1}{2}\tan^{-1}[(1+\alpha)U] - \frac{1}{2}\tan^{-1}[(1-\alpha)U]$$

$(1-\alpha) < 0$

$$V' = \frac{1}{2}\tan^{-1}[(1+\alpha)U] + \frac{1}{2}\tan^{-1}[(1-\alpha)U]$$

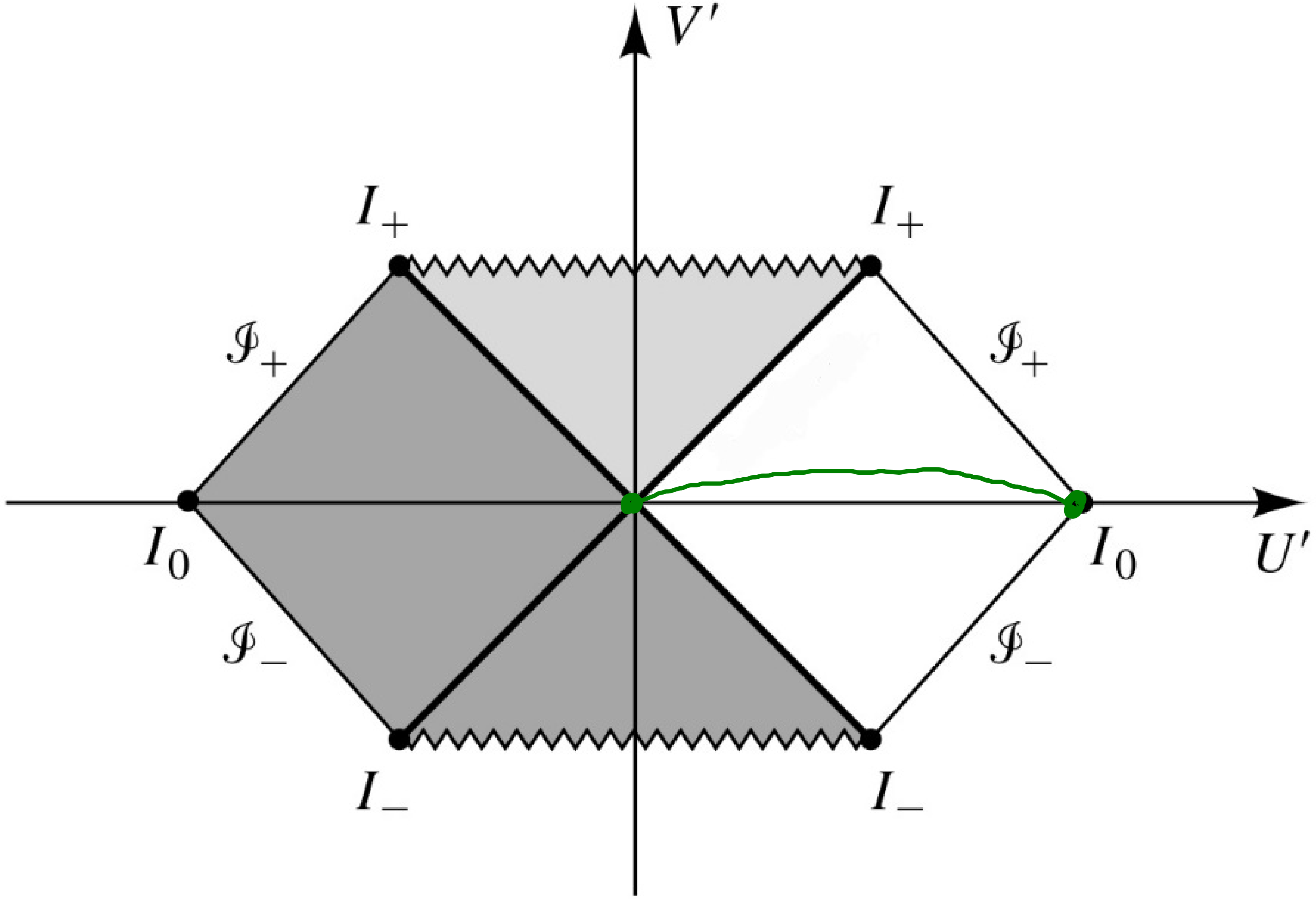
Penrose Diagram of Schwarzschild Geometry

$$U' = \tfrac{1}{2}\tan^{-1}(V+U) - \tfrac{1}{2}\tan^{-1}(V-U)$$

$$V' = \tfrac{1}{2}\tan^{-1}(V+U) + \tfrac{1}{2}\tan^{-1}(V-U)$$

$$V - U = \tan(V'-U')$$

$$V'+U' = \tan(V'+U')$$



Spacelike infinity

Consider $V = \alpha U$, $\alpha < 1$ (const. t)

$$U' = \tfrac{1}{2}\tan^{-1}\left[(1+\alpha)U\right] - \tfrac{1}{2}\tan^{-1}\left[(1-\alpha)U\right]$$

$$= \tfrac{1}{2}\left(\tfrac{\pi}{2}\right) - \tfrac{1}{2}\left(-\tfrac{\pi}{2}\right) = \tfrac{\pi}{2} \qquad (U \to +\infty)$$

$$V' = \tfrac{1}{2}\tan^{-1}\left[(1+\alpha)U\right] + \tfrac{1}{2}\tan^{-1}\left[(1-\alpha)U\right]$$

$$= \tfrac{1}{2}\left(\tfrac{\pi}{2}\right) + \tfrac{1}{2}\left(-\tfrac{\pi}{2}\right) = 0 \qquad (U \to +\infty)$$

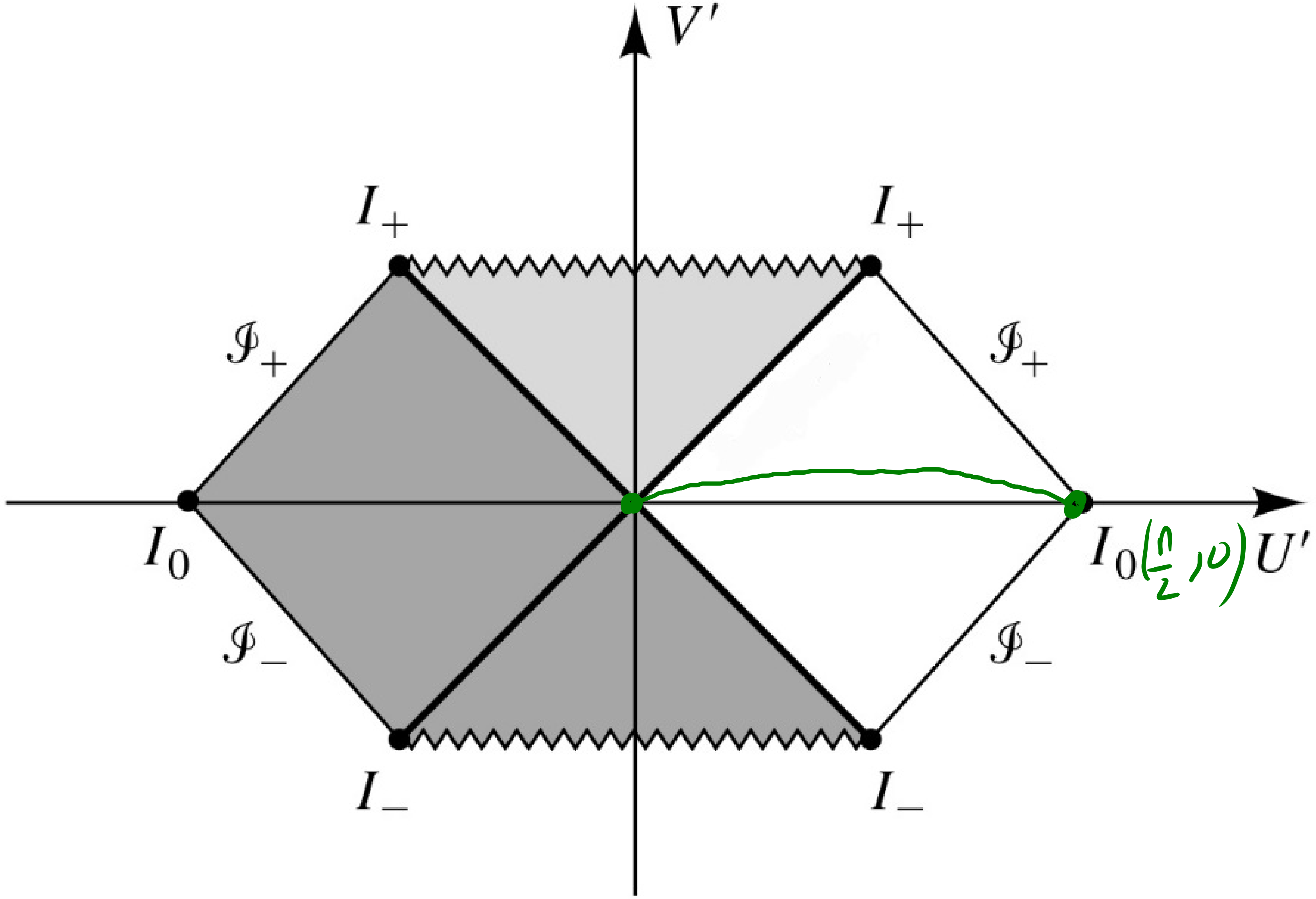
Penrose Diagram of Schwarzschild Geometry

$U' = \frac{1}{2}\tan^{-1}(V+U) - \frac{1}{2}\tan^{-1}(V-U)$

$V' = \frac{1}{2}\tan^{-1}(V+U) + \frac{1}{2}\tan^{-1}(V-U)$

$V - U = \tan(V'-U')$

$V'+U' = \tan(V'+U')$



Spacelike infinity

Consider $V = \alpha U$, $\alpha < 1$ (const. t)

$U' = \frac{1}{2}\tan^{-1}[(1+\alpha)U] - \frac{1}{2}\tan^{-1}[(1-\alpha)U]$

$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2} \qquad (U \to +\infty)$

$V' = \frac{1}{2}\tan^{-1}[(1+\alpha)U] + \frac{1}{2}\tan^{-1}[(1-\alpha)U]$

$= \frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{2}\left(-\frac{\pi}{2}\right) = 0 \qquad (U \to +\infty)$

$\Rightarrow$ end up at $I_0(\frac{\pi}{2}, 0)$