Hartle ch 22

Einstein Equations

Carroll ch 4

Ferrari ch 6+7

- the dynamics

- the
  „matter tells spacetime how to curve"
  „curvature tells matter how to move"
  equations

- difficult: highly non linear (unlike EM)
- can formulate initial value problem (with limitations ...)

Newtonian Gravity

$$\frac{dv^i}{dt} = -\nabla^i \Phi \qquad \left(\Phi = G\frac{M}{R} \right)$$

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$$\frac{dv^i}{dt} = -\nabla^i \Phi$$

$\hookrightarrow$ a force

$$\nabla^2 \Phi = 4\pi G \rho$$

$\hookrightarrow$ Gauss' law

Valid when:
- $v^i \ll 1$
- static field
- weak field

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build using Minimal Coupling Principle: (MCP)

- small enough region described by SR physics
- we can't detect gravity with a local experiment
  ($\Rightarrow$ no direct coupling to curvature)
- laws of physics can be expressed in coordinate invariant form
  ($\Rightarrow$ tensorial equations)

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locally $\partial_\mu T^{\mu\nu} = 0$

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MCP is strong! why not $\frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho = \alpha \nabla_\nu R \, u^\nu u^\mu$?

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$\ell_p \sim 1.6 \times 10^{-35}\,\mathrm{m} \sim (1.2 \times 10^{19}\,\mathrm{GeV})^{-1}$

so is it because $\alpha \sim \ell_p^2$?

Slow motion, static weak field limit:

slow: $\frac{dx^i}{c\, d\tau} \ll \frac{c\, dt}{c\, d\tau} \approx 1$

Slow motion, static+weak field limit:

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$\Phi = -\dfrac{GM}{r}$

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Determine κ in weak+static field, slow moving matter:

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- 6 independent equations (Bianchi equation)
- 6 independent degrees of freedom ($g_{\mu\nu}$)
- can formulate initial value problem

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$T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$ Einstein equation in the vacuum (and $\Lambda = 0$)

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Cosmological Constant

If there is vacuum energy, must be Lorentz invariant

$$T^{(vac)}_{\hat{\mu}\hat{\nu}} = -\rho_{vac}\,\eta_{\hat{\mu}\hat{\nu}} \rightarrow T^{(vac)}_{\mu\nu} = -\rho_{vac}\,g_{\mu\nu} \qquad \rho_{vac} = \text{constant}$$

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Cosmological Constant

- $|\rho_{vac}| \lesssim (10^{-12}\,\text{GeV})^4 \ll (10^{18}\,\text{GeV})^4$

$$(10^{18}\,\text{GeV}) = E_p$$

- Λ has been introduced in cosmology by hand as a geometric term to produce static universe (Einstein's "biggest blunder")

In principle a free parameter to be determined by observation

$$G_{\mu\nu} + (8\pi G \rho_{vac})\, g_{\mu\nu} = 8\pi G\, T^{M}_{\mu\nu} \Rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T^{M}_{\mu\nu}$$

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Derive Einstein equations from action principle

- explicit manifestation of symmetries
- can be used to build extensions to GR, or study connection of GR to other theories (e.g. strings)
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Simple:
$$S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\,(R - 2\Lambda) + S^M$$

$$\delta S = 0 \Rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

Functional Differentiation

$$F : \mathbb{R}^n \to \mathbb{R} \qquad q_i \to F(q_i) \in \mathbb{R}$$

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$S : F(\mathbb{R}^4) \to \mathbb{R} \qquad \phi(x) \to S[\phi(x)] \in \mathbb{R}$

$$dF = \sum_i \frac{\partial F}{\partial q_i} dq_i = F(q_i + dq_i) - F(q_i) + \mathcal{O}(dq_i^2)$$

$$\delta S = \int dx \frac{\delta S}{\delta \phi(x)} \delta\phi(x) = S[\phi(x) + \delta\phi(x)] - S[\phi(x)] + \mathcal{O}(\delta\phi^2)$$

Functional Differentiation

$F: \mathbb{R}^n \to \mathbb{R}$ $\qquad q_i \to F(q_i) \in \mathbb{R}$

$i \leftrightarrow x$

$S: F(\mathbb{R}^n) \to \mathbb{R}$ $\qquad \phi(x) \to S[\phi(x)] \in \mathbb{R}$

$$dF = \sum_i \frac{\partial F}{\partial q_i} dq_i = F(q_i + dq_i) - F(q_i) + \mathcal{O}(dq_i^2)$$

↪ partial derivative w.r.t q_i at i

$$\delta S = \int dx \frac{\delta S}{\delta \phi(x)} \delta\phi(x) = S[\phi(x) + \delta\phi(x)] - S[\phi(x)] + \mathcal{O}(\delta\phi^2)$$

↳ functional derivative w.r.t. $\phi(x)$ at x

Action Principle

$$S[q_i(t)] = \int dt \, \underbrace{L(q_i, \dot{q}_i)}_{\text{function of } q_i, \dot{q}_i}$$

independent variables

$\Rightarrow$ it makes sense to calculate

$$\frac{\partial L}{\partial q_i}, \quad \frac{\partial L}{\partial \dot{q}_i}$$

Action Principle

$$S[q_i(t)] = \int dt\, L(q_i, \dot{q}_i)$$

functional of $q_i(t)$

$$q_i(t) \rightarrow S[q_i(t)] \in \mathbb{R}$$

Action Principle

$$S[q_i(t)] = \int dt\, L(q_i, \dot{q}_i)$$

functional of $q_i(t)$

<u>number</u> $S[q_i(t)]$ calculated by substituting $q_i(t)$. <u>Then</u> $\dot{q}_i = \frac{d q_i(t)}{dt}$

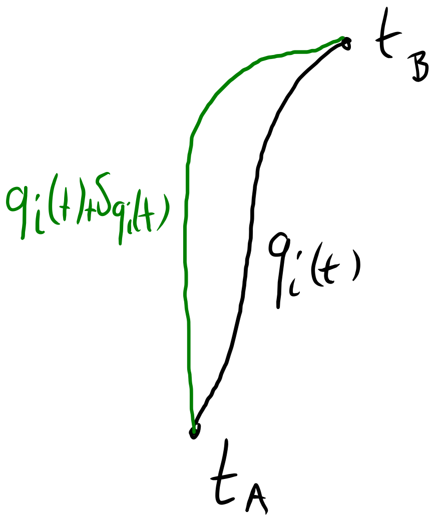
$$q_i(t) \to S[q_i(t)] \in \mathbb{R}$$

Action Principle

$$S[q_i(t)] = \int dt \, L(q_i, \dot{q}_i)$$

Consider variations $q_i(t) \to q_i(t) + \delta q_i(t)$

s.t. $\delta q_i(t_A) = \delta q_i(t_B) = 0$

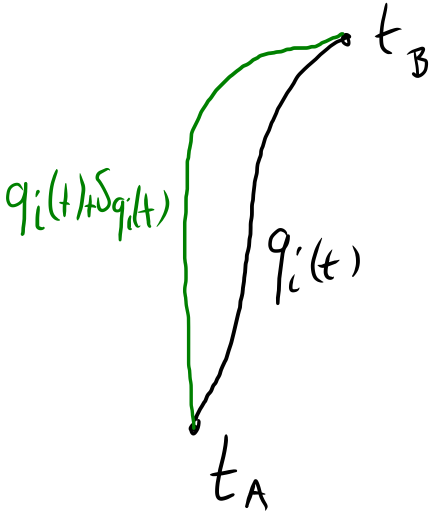


Action Principle

$$S[q_i(t)] = \int dt\, L(q_i, \dot{q}_i)$$

Consider variations $q_i(t) \to q_i(t) + \delta q_i(t)$

s.t. $\delta q_i(t_A) = \delta q_i(t_B) = 0$, then



$$\delta S = 0 \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad \text{eom}$$

Action Principle

Field Theory:

$$S[\phi(x)] = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

Action Principle

Field Theory:

$$S[\phi(x)] = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$S[\phi(x) + \delta\phi(x)] = \int d^4x \, \mathcal{L}(\phi + \delta\phi, \partial_\mu\phi + \delta\partial_\mu\phi)$$

Action Principle

Field Theory:

$$S[\phi(x)] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$S[\phi(x)+\delta\phi(x)] = \int d^4x\, \mathcal{L}(\phi+\delta\phi, \partial_\mu\phi + \delta\partial_\mu\phi)$$

$$= \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi) + \int d^4x \left[\frac{\partial \mathcal{L}}{\partial\phi(x)} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi(x))}\, \delta(\partial_\mu\phi(x)) \right] + \mathcal{O}(\delta\phi^2)$$

Action Principle

Field Theory:

$$S[\phi(x)] = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$S[\phi(x) + \delta\phi(x)] = \int d^4x \, \mathcal{L}(\phi + \delta\phi, \partial_\mu\phi + \delta\partial_\mu\phi)$$

$$= \int d^4x \, \mathcal{L}(\phi, \partial_\mu\phi) + \int d^4x \left[\frac{\partial \mathcal{L}}{\partial\phi(x)} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi(x))} \delta(\partial_\mu\phi(x)) \right] + \mathcal{O}(\delta\phi^2)$$

$$= S[\phi(x)] + \underbrace{\int d^4x \, \frac{\delta S}{\delta\phi(x)} \delta\phi(x)}_{= \delta S}$$

Action Principle

$$\delta S = \int d^4x \, \frac{\delta S}{\delta \phi(x)} \, \delta \phi(x)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \, \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \, \underbrace{\delta(\partial_\mu \phi(x))} \right]$$

get rid: trade for $\delta\phi(x)$
to calculate $\frac{\delta S}{\delta \phi}$

$$\underbrace{\int d^4x \, \frac{\delta S}{\delta \phi(x)} \, \delta \phi(x)} = \delta S$$

Action Principle

$$\delta S = \int d^4x \, \frac{\delta S}{\delta \phi(x)} \, \delta\phi(x)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \delta(\partial_\mu \phi(x)) \right]$$

$$\phi(x) \to \phi(x) + \delta\phi(x) \Rightarrow \partial_\mu \phi(x) \to \partial_\mu \phi(x) + \partial_\mu(\delta\phi(x))$$

Action Principle

$$\delta S = \int d^4x \, \frac{\delta S}{\delta \phi(x)} \, \delta\phi(x)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \delta(\partial_\mu \phi(x)) \right]$$

$$\phi(x) \to \phi(x) + \delta\phi(x) \Rightarrow \partial_\mu \phi(x) \to \partial_\mu \phi(x) + \underbrace{\partial_\mu(\delta\phi(x))}_{\delta(\partial_\mu \phi(x))}$$

$$\Rightarrow \delta(\partial_\mu \phi(x)) = \partial_\mu(\delta\phi(x))$$

Action Principle

$$\delta S = \int d^4x \, \frac{\delta S}{\delta \phi(x)} \, \delta \phi(x)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \delta(\partial_\mu \phi(x)) \right]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \partial_\mu(\delta \phi(x)) \right]$$

$$\Rightarrow \delta(\partial_\mu \phi(x)) = \partial_\mu(\delta \phi(x))$$

Action Principle

$$\delta S = \int_V d^4x \, \frac{\delta S}{\delta \phi(x)} \, \delta \phi(x)$$

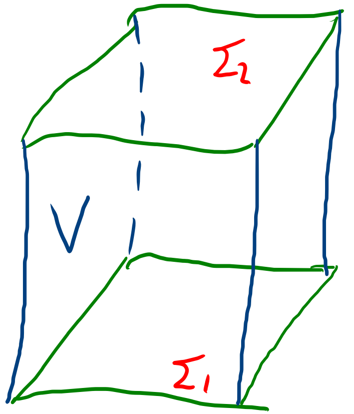
$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \, \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \, \delta(\partial_\mu \phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \, \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \, \partial_\mu(\delta \phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \, \delta\phi(x) - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \right] \delta \phi(x) \right]$$

$$+ \int_V d^4x \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \, \delta \phi(x) \right]$$

Action Principle



$$\delta S = \int_V d^4x \, \frac{\delta S}{\delta\phi(x)} \, \delta\phi(x)$$

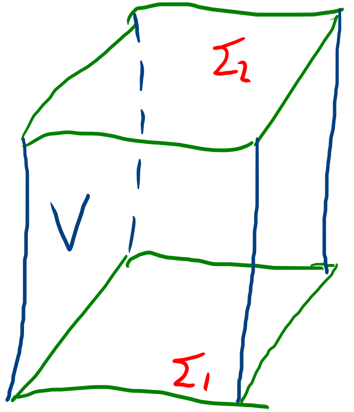
$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial\phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi(x))} \delta(\partial_\mu\phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial\phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi(x))} \partial_\mu(\delta\phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial\phi(x)} \delta\phi(x) - \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi(x))}\right] \delta\phi(x) \right]$$

$$+ \int_{\partial V} d^3y \, n^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi(x))} \delta\phi(x) \right]$$

Action Principle



$$\delta S = \int_V d^4x \, \frac{\delta S}{\delta \phi(x)} \, \delta\phi(x)$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \delta(\partial_\mu \phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \partial_\mu(\delta\phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta\phi(x) - \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\right] \delta\phi(x) \right]$$

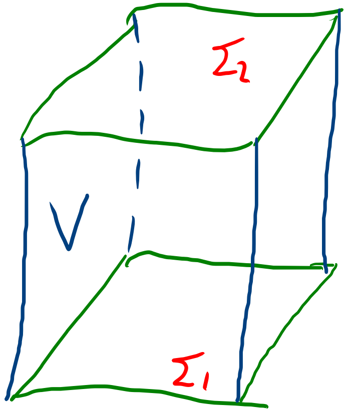
$$+ \int_{\partial V} d^3y \, n^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \delta\phi(x) \right]$$

$$\delta\phi\big|_{\Sigma_1} = \delta\phi\big|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi\Big|_{\partial V} = 0$$

Choose $\delta\phi$ to make this zero

Action Principle



$$\delta\phi\big|_{\Sigma_1} = \delta\phi\big|_{\Sigma_2} = 0$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\Big|_{\partial V} = 0$$

$$\delta S = \int_V d^4x\ \frac{\delta S}{\delta\phi(x)}\ \delta\phi(x)$$

$$= \int_V d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi(x)}\,\delta\phi(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\,\delta(\partial_\mu\phi(x)) \right]$$

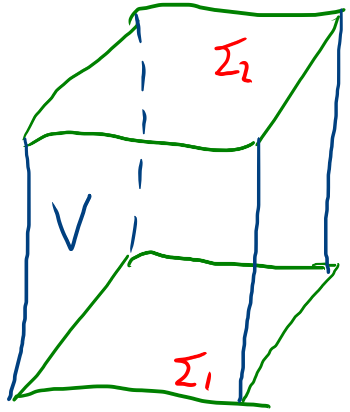
$$= \int_V d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi(x)}\,\delta\phi(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\,\partial_\mu(\delta\phi(x)) \right]$$

$$= \int_V d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi(x)}\,\delta\phi(x) - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\right]\delta\phi(x) \right]$$

$$= \int_V d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi(x)} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\right] \right]\delta\phi(x)$$

Action Principle

$$\delta S = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\right] = 0$$



$$\delta\phi\big|_{\Sigma_1} = \delta\phi\big|_{\Sigma_2} = 0$$

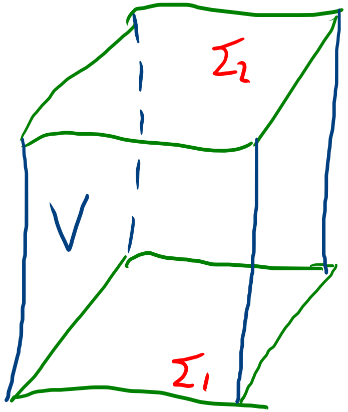
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta\phi\Big|_{\partial V} = 0$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\right]\right]\delta\phi(x)$$

Action Principle

$$\delta S = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \right] = 0$$

classical eom!



$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \Big|_{\partial V} = 0$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \right] \right] \delta\phi(x)$$

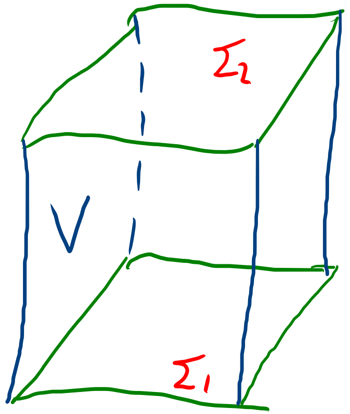
Action Principle

In curved space:

$$\mathcal{L}(\phi, \partial_\mu \phi) \rightarrow \mathcal{L}(\phi, \nabla_\mu \phi)$$

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) \rightarrow S = \int d^4x \sqrt{-g} \, \mathcal{L}(\phi, \nabla_\mu \phi)$$

minimal coupling principle!



$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

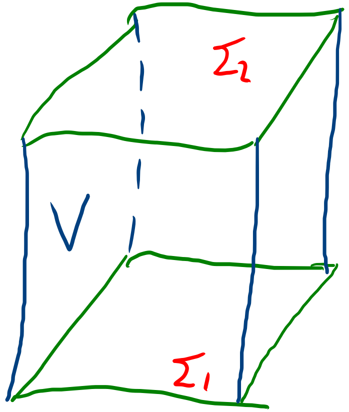
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \Big|_{\partial V} = 0$$

Action Principle

In curved space:

$$\mathcal{L}(\phi, \partial_\mu \phi) \to \mathcal{L}(\phi, \nabla_\mu \phi)$$

$$S = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi) \to S = \int d^4x \sqrt{-g}\, \mathcal{L}(\phi, \nabla_\mu \phi)$$



$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \Big|_{\partial V} = 0$$

Partial integration: use Stokes' theorem!

$$\int_V d^4x \sqrt{-g}\, \nabla_\mu V^\mu = \int_{\partial V} d^3y \sqrt{\gamma}\, n_\mu V^\mu$$

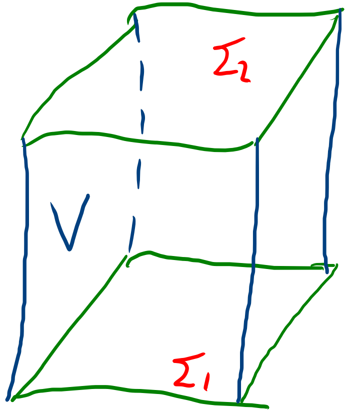
induced metric on ∂V by $g_{\mu\nu}$

normal to the 3-surface ∂V

Action Principle

Then:

$$\int_V d^4x \sqrt{-g}\, \nabla_\mu (A^\mu B) = \underbrace{\int_V d^4x \sqrt{-g}\, \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g}\, A^\mu \nabla_\mu B}_{\text{... Leibniz rule ...}}$$



$$\delta\phi\big|_{\Sigma_1} = \delta\phi\big|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\, \delta\phi\Big|_{\partial V} = 0$$

Partial integration: use Stokes' theorem!

$$\int_V d^4x \sqrt{-g}\, \nabla_\mu V^\mu = \int_{\partial V} d^3y \sqrt{\gamma}\; n_\mu V^\mu$$

$\sqrt{\gamma}$: induced metric on ∂V by $g_{\mu\nu}$

n_μ: normal to the 3-surface ∂V

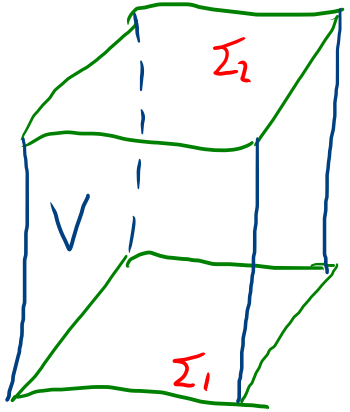
Action Principle

Then:

$$\int_V d^4x \sqrt{-g}\, \nabla_\mu (A^\mu B) = \int_V d^4x \sqrt{-g}\, \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g}\, A^\mu \nabla_\mu B$$

Stokes

$$\int_{\partial V} d^3y \sqrt{\gamma}\, n_\mu A^\mu B = \int_V d^4x \sqrt{-g}\, \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g}\, A^\mu \nabla_\mu B$$



$$\delta\phi\big|_{\Sigma_1} = \delta\phi\big|_{\Sigma_2} = 0$$

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normal to the 3-surface ∂V

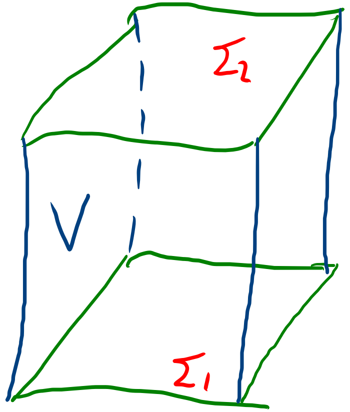
Action Principle

Then:

$$\int_V d^4x \sqrt{-g}\, \nabla_\mu (A^\mu B) = \int_V d^4x \sqrt{-g}\, \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g}\, A^\mu \nabla_\mu B$$

$$\int_{\partial V} d^3y \sqrt{\gamma}\, n_\mu A^\mu B = \int_V d^4x \sqrt{-g}\, \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g}\, A^\mu \nabla_\mu B$$

$$\Rightarrow \int_V d^4x \sqrt{-g}\, A^\mu \nabla_\mu B = - \int_V d^4x \sqrt{-g}\, \nabla_\mu A^\mu B + \int_{\partial V} d^3y \sqrt{\gamma}\, n_\mu A^\mu B$$

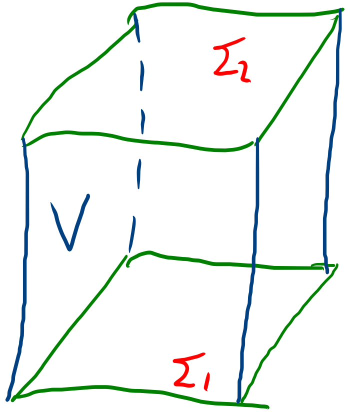


$$\delta\phi\big|_{\Sigma_1} = \delta\phi\big|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\, \delta\phi\Big|_{\partial V} = 0$$

Action Principle

Then:



$$\int_V d^4x \sqrt{-g}\, \nabla_\mu (A^\mu B) = \int_V d^4x \sqrt{-g}\, \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g}\, A^\mu \nabla_\mu B$$

$$\delta\phi\big|_{\Sigma_1} = \delta\phi\big|_{\Sigma_2} = 0$$

$$\int_{\partial V} d^3y \sqrt{\gamma}\, n_\mu A^\mu B = \int_V d^4x \sqrt{-g}\, \nabla_\mu A^\mu B + \int_V d^4x \sqrt{-g}\, A^\mu \nabla_\mu B$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \Big|_{\partial V} = 0$$

$$\Rightarrow \int_V d^4x \sqrt{-g}\, A^\mu \nabla_\mu B = - \int_V d^4x \sqrt{-g}\, \nabla_\mu A^\mu B + \underbrace{\int_{\partial V} d^3y \sqrt{\gamma}\, n_\mu A^\mu B}$$

"passed over"
ignoring $\sqrt{-g}$
(magic of ∇_μ!) $\rightsquigarrow$ Stokes ...

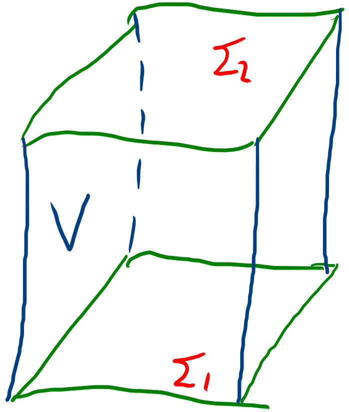
Boundary term:
set to 0 or cancel otherwise
(careful ...)

Action Principle

Example: scalar field

$$S = \int \sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\phi\,\nabla^\mu\phi - V(\phi)\right)$$

minimal coupling, no e.g. Re^ϕ term



$$\delta\phi\big|_{\Sigma_1} = \delta\phi\big|_{\Sigma_2} = 0$$

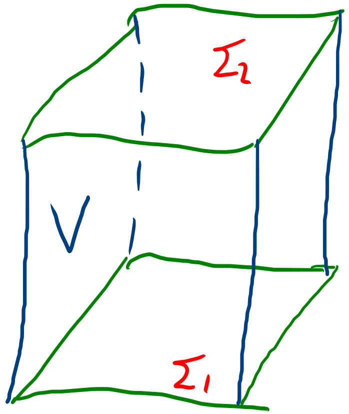
$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\Big|_{\partial V} = 0$$

Action Principle

Example: scalar field

$$S = \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right)$$

$$S[\phi + \delta\phi] = \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu (\phi + \delta\phi) \nabla^\mu (\phi + \delta\phi) - V(\phi + \delta\phi) \right)$$



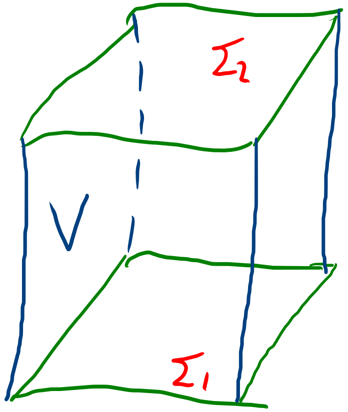
$$\delta\phi \big|_{\Sigma_1} = \delta\phi \big|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \Big|_{\partial V} = 0$$

Action Principle

Example: scalar field

$$S = \int \sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\phi\,\nabla^\mu\phi - V(\phi)\right)$$



$$\delta\phi\big|_{\Sigma_1} = \delta\phi\big|_{\Sigma_2} = 0$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\Big|_{\partial V} = 0$$

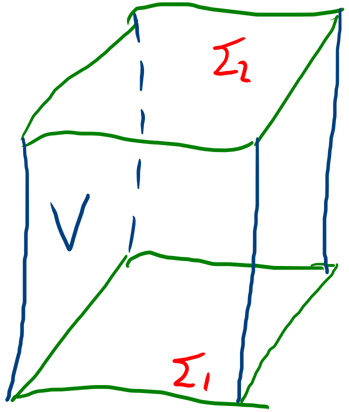
$$S[\phi+\delta\phi] = \int\sqrt{-g}\left(-\frac{1}{2}\nabla_\mu(\phi+\delta\phi)\,\nabla^\mu(\phi+\delta\phi) - V(\phi+\delta\phi)\right)$$

$$= \int\sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\phi\,\nabla^\mu\phi - V(\phi)\right) + \int\sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\delta\phi\,\nabla^\mu\phi - \frac{1}{2}\nabla_\mu\phi\,\nabla^\mu\delta\phi - \frac{dV}{d\phi}\,\delta\phi\right) + \cdots$$

Action Principle

Example: scalar field

$$S = \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right)$$



$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$

$\frac{\partial \mathcal{L}}{\partial(\nabla_\mu \phi)} \delta\phi \Big|_{\partial V} = 0$

$$S[\phi + \delta\phi] = \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu (\phi + \delta\phi) \nabla^\mu (\phi + \delta\phi) - V(\phi + \delta\phi) \right)$$

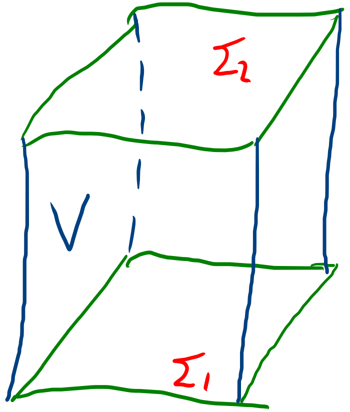
$$= \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right) + \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \delta\phi \nabla^\mu \phi - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \delta\phi - \frac{dV}{d\phi} \delta\phi \right) + \cdots$$

$$= S[\phi] + \int \sqrt{-g} \left(- g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \delta\phi - \frac{dV}{d\phi} \delta\phi \right) + \mathcal{O}(\delta\phi^2)$$

Action Principle

Example: scalar field

$$S = \int \sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right)$$



$$\delta\phi\big|_{\Sigma_1} = \delta\phi\big|_{\Sigma_2} = 0$$

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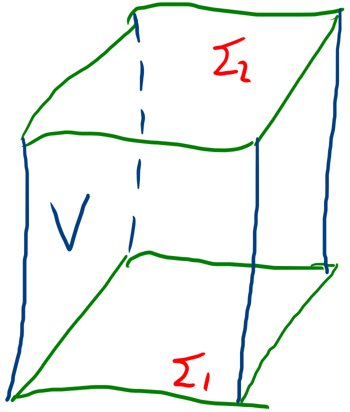
$$\Rightarrow \delta S = \int_V \sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{dV}{d\phi}\right)\delta\phi + \int_{\partial V}\sqrt{\gamma}\, n^\mu \nabla_\mu\phi\ \delta\phi$$



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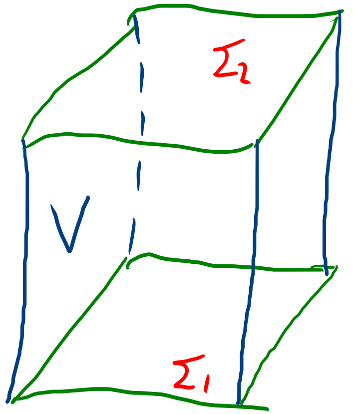

Take:

$$\int_{\partial V}\sqrt{\gamma}\, n^\mu \nabla_\mu\phi\ \delta\phi = 0$$

$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$

ϕ has compact support ($\phi = 0$ at ∞)

or

$\phi, \delta\phi \to 0$ at ∞ fast enough

$$\delta\phi|_{\Sigma_1} = \delta\phi|_{\Sigma_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big|_{\partial V} = 0$$

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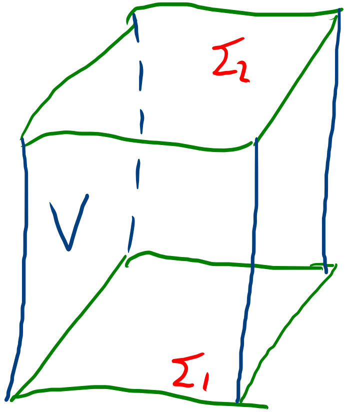

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$$\delta S = 0 \quad \forall\, \delta\phi \quad \Rightarrow \quad \nabla^2 \phi - \frac{dV}{d\phi} = 0$$

or $$\Box \phi - \frac{dV}{d\phi} = 0$$

Gravity:

Ingredients:
1. $g_{\mu\nu}$ (degrees of freedom - not all independent)
2. scalar Lagrangian (diffeomorphism invariance)
3. at most $\partial^2 g$
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ε. enjoy! $\delta S = 0$

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Einstein eqs in vacuum
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careful: includes variation of $\sqrt{-g}$!

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$$\delta\left(S^{EH} + S^{\Lambda} + S^{M}\right) = 0 \Rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

Compute metric variations:

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not inverses to each other!

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almost like lowering indices

Variation of $\sqrt{-g}$

Temporary notation

g : the matrix $g_{\mu\nu}$

$\det g$: its determinant

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$\hookrightarrow$
- diagonalize $g \to \Lambda g_D \Lambda^{-1}$
- $\operatorname{tr} g = \operatorname{tr} \Lambda g_D \Lambda^{-1} = \operatorname{tr} \Lambda^{-1} \Lambda g_D = \operatorname{tr} g_D$

$$= \sum g_\mu$$

$$g_D = \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & g_3 & \\ & & & g_4 \end{pmatrix}$$

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$\hookrightarrow \frac{1}{g_\mu}$ evs of g^{-1} !

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$$= \frac{\Lambda}{16\pi G}\int \sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}$$

as promised!

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$$= \int \sqrt{-g} \left(-\tfrac{1}{2} g_{\rho\sigma}\, \delta g^{\rho\sigma}\right) R + \int \sqrt{-g}\; R_{\mu\nu}\, \delta g^{\mu\nu} + \int \sqrt{-g}\; g^{\mu\nu}\, \delta R_{\mu\nu}$$

Variation of S^{EH}

$$16\pi G\, S^{EH} = \int \sqrt{-g}\, R = \int \sqrt{-g}\; g^{\mu\nu} R_{\mu\nu} \quad \Rightarrow$$

$$16\pi G\, \delta S^{EH} = \int \delta\sqrt{-g}\; g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g}\; \delta g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu}$$

$$= \int \sqrt{-g} \left(-\frac{1}{2} g_{\rho\sigma}\, \delta g^{\rho\sigma}\right) R + \int \sqrt{-g}\; R_{\mu\nu}\, \delta g^{\mu\nu} + \int \sqrt{-g}\; g^{\mu\nu}\, \delta R_{\mu\nu}$$

$$= \int \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) \delta g^{\mu\nu} + \int \sqrt{-g}\; g^{\mu\nu}\, \delta R_{\mu\nu}$$

Variation of S^{EH}

$$16\pi G\, S^{EH} = \int \sqrt{-g}\, R = \int \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu} \quad \Rightarrow$$

$$16\pi G\, \delta S^{EH} = \int \delta\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g}\, \delta g^{\mu\nu} R_{\mu\nu} + \int \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu}$$

$$= \int \sqrt{-g} \left(-\tfrac{1}{2} g_{\rho\sigma}\, \delta g^{\rho\sigma}\right) R + \int \sqrt{-g}\, R_{\mu\nu}\, \delta g^{\mu\nu} + \int \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu}$$

$$= \int \sqrt{-g} \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} + \int \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu}$$

$$= \int \sqrt{-g}\, G_{\mu\nu}\, \delta g^{\mu\nu} + \int \sqrt{-g}\, \underbrace{g^{\mu\nu} \delta R_{\mu\nu}}$$

↳ almost there... ↳ get rid of this!

Compute $\delta R_{\mu\nu}$:

$$R^{\rho}{}_{\mu\sigma\nu} = \partial_{\sigma}\Gamma_{\nu}{}^{\rho}{}_{\mu} - \partial_{\nu}\Gamma_{\sigma}{}^{\rho}{}_{\mu} + \Gamma_{\sigma}{}^{\rho}{}_{\lambda}\Gamma_{\nu}{}^{\lambda}{}_{\mu} - \Gamma_{\nu}{}^{\rho}{}_{\lambda}\Gamma_{\sigma}{}^{\lambda}{}_{\mu}$$

Compute $\delta R_{\mu\nu}$:

$$R^{\rho}{}_{\mu\sigma\nu} = \partial_{\sigma}\Gamma^{\rho}_{\nu\mu} - \partial_{\nu}\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}$$

$$R^{\rho}{}_{\mu\rho\nu} = \partial_{\rho}\Gamma^{\rho}_{\nu\mu} - \partial_{\nu}\Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu}$$

Compute $\delta R_{\mu\nu}$:

$$R^{\rho}{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu}$$

For $\delta R_{\mu\nu}$ we need $\delta \Gamma^{\mu}_{\nu\rho}$

Compute $\delta \Gamma^{\mu}{}_{\nu\rho}$

Consider $\Gamma_{\mu\nu\rho} = g_{\mu\lambda} \Gamma^{\lambda}{}_{\nu\rho} = \frac{1}{2} \left(\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\nu\mu} - \partial_{\mu} g_{\nu\rho} \right)$

Compute $\delta \Gamma^{\mu}{}_{\nu\rho}$

Consider $\Gamma_{\mu\nu\rho} = g_{\mu\lambda} \Gamma^{\lambda}{}_{\nu\rho} = \frac{1}{2}\left(\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\nu\mu} - \partial_{\mu} g_{\nu\rho} \right)$

$\Rightarrow \quad \delta \Gamma_{\mu\nu\rho} = \frac{1}{2}\left(\partial_{\nu} \delta g_{\rho\mu} + \partial_{\rho} \delta g_{\nu\mu} - \partial_{\mu} \delta g_{\nu\rho} \right)$

Compute $\delta \Gamma^{\mu}{}_{\nu\rho}$

Consider $\Gamma_{\mu\nu\rho} = g_{\mu\lambda} \Gamma^{\lambda}{}_{\nu\rho} = \frac{1}{2}\left(\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\nu\mu} - \partial_{\mu} g_{\nu\rho} \right)$

$$\Rightarrow \quad \delta \Gamma_{\mu\nu\rho} = \frac{1}{2}\left(\partial_{\nu} \delta g_{\rho\mu} + \partial_{\rho} \delta g_{\nu\mu} - \partial_{\mu} \delta g_{\nu\rho} \right)$$

$$\Gamma^{\lambda}{}_{\nu\rho} = g^{\lambda\mu} \Gamma_{\mu\nu\rho} \Rightarrow \delta \Gamma^{\lambda}{}_{\nu\rho} = \delta g^{\lambda\mu} \Gamma_{\mu\nu\rho} + g^{\lambda\mu} \delta \Gamma_{\mu\nu\rho}$$

Compute $\delta \Gamma^{\mu}{}_{\nu\rho}$

Consider $\Gamma_{\mu\nu\rho} = g_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\rho} = \frac{1}{2}\left(\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\nu\mu} - \partial_{\mu} g_{\nu\rho}\right)$

$$\Rightarrow \quad \delta\Gamma_{\mu\nu\rho} = \frac{1}{2}\left(\partial_{\nu}\delta g_{\rho\mu} + \partial_{\rho}\delta g_{\nu\mu} - \partial_{\mu}\delta g_{\nu\rho}\right)$$

$$\Gamma^{\lambda}{}_{\nu\rho} = g^{\lambda\mu}\Gamma_{\mu\nu\rho} \Rightarrow \delta\Gamma^{\lambda}{}_{\nu\rho} = \delta g^{\lambda\mu}\,\Gamma_{\mu\nu\rho} + g^{\lambda\mu}\,\delta\Gamma_{\mu\nu\rho}$$

$$= -g^{\lambda\alpha}g^{\mu\beta}\delta g_{\alpha\beta}\,\Gamma_{\mu\nu\rho} + g^{\lambda\mu}\,\delta\Gamma_{\mu\nu\rho}$$

Compute $\delta \Gamma^{\mu}{}_{\nu\rho}$

Consider $\Gamma_{\mu\nu\rho} = g_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\rho} = \frac{1}{2}\left(\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\nu\mu} - \partial_{\mu} g_{\nu\rho}\right)$

$$\Rightarrow \quad \delta\Gamma_{\mu\nu\rho} = \frac{1}{2}\left(\partial_{\nu}\delta g_{\rho\mu} + \partial_{\rho}\delta g_{\nu\mu} - \partial_{\mu}\delta g_{\nu\rho}\right)$$

$$\Gamma^{\lambda}{}_{\nu\rho} = g^{\lambda\mu}\Gamma_{\mu\nu\rho} \Rightarrow \delta\Gamma^{\lambda}{}_{\nu\rho} = \delta g^{\lambda\mu}\,\Gamma_{\mu\nu\rho} + g^{\lambda\mu}\,\delta\Gamma_{\mu\nu\rho}$$

$$= -g^{\lambda\alpha} g^{\mu\beta}\,\delta g_{\alpha\beta}\,\Gamma_{\mu\nu\rho} + g^{\lambda\mu}\,\delta\Gamma_{\mu\nu\rho}$$

$$= -g^{\lambda\alpha}\,\delta g_{\alpha\beta}\,\Gamma^{\beta}{}_{\nu\rho} + g^{\lambda\mu}\,\frac{1}{2}\left(\partial_{\nu}\delta g_{\rho\mu} + \partial_{\rho}\delta g_{\nu\mu} - \partial_{\mu}\delta g_{\nu\rho}\right)$$

Compute $\delta \Gamma^{\mu}{}_{\nu\rho}$

Consider $\Gamma_{\mu\nu\rho} = g_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\nu\rho}\right)$

$$\Rightarrow \quad \delta\Gamma_{\mu\nu\rho} = \frac{1}{2}\left(\partial_\nu \delta g_{\rho\mu} + \partial_\rho \delta g_{\nu\mu} - \partial_\mu \delta g_{\nu\rho}\right)$$

$$\Gamma^{\lambda}{}_{\nu\rho} = g^{\lambda\mu}\Gamma_{\mu\nu\rho} \Rightarrow \delta\Gamma^{\lambda}{}_{\nu\rho} = \delta g^{\lambda\mu}\,\Gamma_{\mu\nu\rho} + g^{\lambda\mu}\,\delta\Gamma_{\mu\nu\rho}$$

$$= -g^{\lambda\alpha} g^{\mu\beta}\,\delta g_{\alpha\beta}\,\Gamma_{\mu\nu\rho} + g^{\lambda\mu}\,\delta\Gamma_{\mu\nu\rho}$$

$$= -g^{\lambda\alpha}\,\delta g_{\alpha\beta}\,\Gamma^{\beta}{}_{\nu\rho} + g^{\lambda\mu}\,\frac{1}{2}\left(\partial_\nu \delta g_{\rho\mu} + \partial_\rho \delta g_{\nu\mu} - \partial_\mu \delta g_{\nu\rho}\right)$$

$$= \frac{1}{2} g^{\lambda\mu}\left[\partial_\nu \delta g_{\rho\mu} + \partial_\rho \delta g_{\nu\mu} - \partial_\mu \delta g_{\nu\rho} - 2\,\Gamma^{\sigma}{}_{\nu\rho}\,\delta g_{\mu\sigma}\right]$$

$$= \frac{1}{2} g^{\lambda\mu} \Big[\left(\partial_\nu \delta g_{\rho\mu} - \Gamma^\sigma_{\nu\rho} \delta g_{\sigma\mu} - \Gamma^\sigma_{\nu\mu} \delta g_{\rho\sigma} \right)$$

$$+ \left(\partial_\rho \delta g_{\nu\mu} - \Gamma^\sigma_{\rho\nu} \delta g_{\sigma\mu} - \Gamma^\sigma_{\rho\mu} \delta g_{\nu\sigma} \right)$$

$$- \left(\partial_\mu \delta g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} \delta g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} \delta g_{\nu\sigma} \right) \Big]$$

$$\delta \Gamma^\lambda_{\nu\rho} = \frac{1}{2} g^{\lambda\mu} \left[\partial_\nu \delta g_{\rho\mu} + \partial_\rho \delta g_{\nu\mu} - \partial_\mu \delta g_{\nu\rho} - 2 \Gamma^\sigma_{\nu\rho} \delta g_{\mu\sigma} \right]$$

$$= \frac{1}{2} g^{\lambda\mu} \Big[\big(\partial_\nu \delta g_{\rho\mu} - \Gamma^\sigma_{\nu\rho} \delta g_{\sigma\mu} - \Gamma^\sigma_{\nu\mu} \delta g_{\rho\sigma} \big)$$

$$+ \big(\partial_\rho \delta g_{\nu\mu} - \Gamma^\sigma_{\rho\nu} \delta g_{\sigma\mu} - \Gamma^\sigma_{\rho\mu} \delta g_{\nu\sigma} \big)$$

$$- \big(\partial_\mu \delta g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} \delta g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} \delta g_{\nu\sigma} \big) \Big]$$

$$= \frac{1}{2} g^{\lambda\mu} \Big[\nabla_\nu \delta g_{\rho\mu} + \nabla_\rho \delta g_{\nu\mu} - \nabla_\mu \delta g_{\nu\rho} \Big]$$

$$\delta\Gamma^\lambda_{\nu\rho} = \frac{1}{2} g^{\lambda\mu} \Big[\partial_\nu \delta g_{\rho\mu} + \partial_\rho \delta g_{\nu\mu} - \partial_\mu \delta g_{\nu\rho} - 2 \Gamma^\sigma_{\nu\rho} \delta g_{\mu\sigma} \Big]$$

$$
\begin{aligned}
= \frac{1}{2} g^{\lambda\mu} \Big[& \left(\partial_\nu \delta g_{\rho\mu} - \Gamma^\sigma{}_{\nu\rho} \delta g_{\sigma\mu} - \Gamma^\sigma{}_{\nu\mu} \delta g_{\rho\sigma} \right) \\
& + \left(\partial_\rho \delta g_{\nu\mu} - \Gamma^\sigma{}_{\rho\nu} \delta g_{\sigma\mu} - \Gamma^\sigma{}_{\rho\mu} \delta g_{\nu\sigma} \right) \\
& - \left(\partial_\mu \delta g_{\nu\rho} - \Gamma^\sigma{}_{\mu\nu} \delta g_{\sigma\rho} - \Gamma^\sigma{}_{\mu\rho} \delta g_{\nu\sigma} \right) \Big]
\end{aligned}
$$

$$
= \frac{1}{2} g^{\lambda\mu} \left[\nabla_\nu \delta g_{\rho\mu} + \nabla_\rho \delta g_{\nu\mu} - \nabla_\mu \delta g_{\nu\rho} \right]
$$

$\Rightarrow \delta \Gamma^\lambda{}_{\nu\rho}$ is a tensor!

$$= \frac{1}{2} g^{\lambda\mu} \Big[\big(\partial_\nu \delta g_{\rho\mu} - \Gamma^\sigma_{\nu\rho} \delta g_{\sigma\mu} - \Gamma^\sigma_{\nu\mu} \delta g_{\rho\sigma} \big)$$

$$+ \big(\partial_\rho \delta g_{\nu\mu} - \Gamma^\sigma_{\rho\nu} \delta g_{\sigma\mu} - \Gamma^\sigma_{\rho\mu} \delta g_{\nu\sigma} \big)$$

$$- \big(\partial_\mu \delta g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} \delta g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} \delta g_{\nu\sigma} \big) \Big]$$

$$= \frac{1}{2} g^{\lambda\mu} \big[\nabla_\nu \delta g_{\rho\mu} + \nabla_\rho \delta g_{\nu\mu} - \nabla_\mu \delta g_{\nu\rho} \big]$$

$\Rightarrow$ $\delta \Gamma^\lambda{}_{\nu\rho}$ is a tensor!

$$\Rightarrow \nabla_\mu \delta \Gamma^\lambda{}_{\nu\rho} = \partial_\mu \delta \Gamma^\lambda{}_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} \delta \Gamma^\sigma{}_{\nu\rho} - \Gamma^\sigma_{\mu\nu} \delta \Gamma^\lambda{}_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} \delta \Gamma^\lambda{}_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\nu\mu} - \partial_\nu \Gamma^\rho{}_{\rho\mu} + \Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu}$$

$$\Rightarrow \nabla_\mu \delta \Gamma^\lambda{}_{\nu\rho} = \partial_\mu \delta \Gamma^\lambda{}_{\nu\rho} + \Gamma^\lambda{}_{\mu\sigma} \delta \Gamma^\sigma{}_{\nu\rho} - \Gamma^\sigma{}_{\mu\nu} \delta \Gamma^\lambda{}_{\sigma\rho} - \Gamma^\sigma{}_{\mu\rho} \delta \Gamma^\lambda{}_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\nu\mu} - \partial_\nu \Gamma^\rho{}_{\rho\mu} + \Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu}$$

$$\begin{aligned} \delta R_{\mu\nu} = {} & \partial_\rho \delta\Gamma^\rho{}_{\nu\mu} - \partial_\nu \delta\Gamma^\rho{}_{\rho\mu} \\ & + \delta\Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} + \Gamma^\rho{}_{\rho\lambda} \delta\Gamma^\lambda{}_{\nu\mu} \\ & - \delta\Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu} - \Gamma^\rho{}_{\nu\lambda} \delta\Gamma^\lambda{}_{\rho\mu} \end{aligned}$$

$$\Rightarrow \nabla_\mu \delta\Gamma^\lambda{}_{\nu\rho} = \partial_\mu \delta\Gamma^\lambda{}_{\nu\rho} + \Gamma^\lambda{}_{\mu\sigma} \delta\Gamma^\sigma{}_{\nu\rho} - \Gamma^\sigma{}_{\mu\nu} \delta\Gamma^\lambda{}_{\sigma\rho} - \Gamma^\sigma{}_{\mu\rho} \delta\Gamma^\lambda{}_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\nu\mu} - \partial_\nu \Gamma^\rho{}_{\rho\mu} + \Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu}$$

$$\begin{aligned}
\delta R_{\mu\nu} = {} & \partial_\rho \delta\Gamma^\rho{}_{\nu\mu} - \partial_\nu \delta\Gamma^\rho{}_{\rho\mu} \\
& + \delta\Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} + \Gamma^\rho{}_{\rho\lambda} \delta\Gamma^\lambda{}_{\nu\mu} \\
& - \delta\Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu} - \Gamma^\rho{}_{\nu\lambda} \delta\Gamma^\lambda{}_{\rho\mu} \\
& \underbrace{- \Gamma^\lambda{}_{\rho\nu} \delta\Gamma^\rho{}_{\lambda\mu} + \Gamma^\lambda{}_{\nu\rho}}_{\text{add + subtract the same term}} \delta\Gamma^\rho{}_{\lambda\mu}
\end{aligned}$$

$$\Rightarrow \nabla_\mu \delta\Gamma^\lambda{}_{\nu\rho} = \partial_\mu \delta\Gamma^\lambda{}_{\nu\rho} + \Gamma^\lambda{}_{\mu\sigma} \delta\Gamma^\sigma{}_{\nu\rho} - \Gamma^\sigma{}_{\mu\nu} \delta\Gamma^\lambda{}_{\sigma\rho} - \Gamma^\sigma{}_{\mu\rho} \delta\Gamma^\lambda{}_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}$$

$$\begin{aligned}
\delta R_{\mu\nu} = {} & \partial_\rho \delta\Gamma^\rho_{\nu\mu} - \partial_\nu \delta\Gamma^\rho_{\rho\mu} \\
& + \delta\Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} + \Gamma^\rho_{\rho\lambda}\delta\Gamma^\lambda_{\nu\mu} \\
& - \delta\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\rho\mu} \\
& - \Gamma^\lambda_{\rho\nu}\delta\Gamma^\rho_{\lambda\mu} + \Gamma^\lambda_{\nu\rho}\delta\Gamma^\rho_{\lambda\mu}
\end{aligned}$$

$$\Rightarrow \nabla_\mu \delta\Gamma^\lambda_{\nu\rho} = \partial_\mu \delta\Gamma^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma}\delta\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\mu\nu}\delta\Gamma^\lambda_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\delta\Gamma^\lambda_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\nu\mu} - \partial_\nu \Gamma^\rho{}_{\rho\mu} + \Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu}$$

$$\begin{aligned}
\delta R_{\mu\nu} = {} & \partial_\rho \delta \Gamma^\rho{}_{\nu\mu} - \partial_\nu \delta \Gamma^\rho{}_{\rho\mu} \\
& + \delta\Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} + \Gamma^\rho{}_{\rho\lambda} \delta \Gamma^\lambda{}_{\nu\mu} \\
& - \delta\Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu} - \Gamma^\rho{}_{\nu\lambda} \delta\Gamma^\lambda{}_{\rho\mu} \\
& - \Gamma^\lambda{}_{\rho\nu} \delta \Gamma^\rho{}_{\lambda\mu} + \Gamma^\lambda{}_{\nu\rho} \delta \Gamma^\rho{}_{\lambda\mu} \\
= {} & \nabla_\rho \delta \Gamma^\rho{}_{\nu\mu}
\end{aligned}$$

$$\Rightarrow \nabla_\mu \delta \Gamma^\lambda{}_{\nu\rho} = \partial_\mu \delta \Gamma^\lambda{}_{\nu\rho} + \Gamma^\lambda{}_{\mu\sigma} \delta \Gamma^\sigma{}_{\nu\rho} - \Gamma^\sigma{}_{\mu\nu} \delta \Gamma^\lambda{}_{\sigma\rho} - \Gamma^\sigma{}_{\mu\rho} \delta \Gamma^\lambda{}_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\nu\mu} - \partial_\nu \Gamma^\rho{}_{\rho\mu} + \Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu}$$

$$\begin{aligned}
\delta R_{\mu\nu} = {} & \partial_\rho \delta \Gamma^\rho{}_{\nu\mu} - \partial_\nu \delta \Gamma^\rho{}_{\rho\mu} \\
& + \delta \Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} + \Gamma^\rho{}_{\rho\lambda} \delta \Gamma^\lambda{}_{\nu\mu} \\
& - \delta \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu} - \Gamma^\rho{}_{\nu\lambda} \delta \Gamma^\lambda{}_{\rho\mu} \\
& - \Gamma^\lambda{}_{\rho\nu} \delta \Gamma^\rho{}_{\lambda\mu} + \Gamma^\lambda{}_{\nu\rho} \delta \Gamma^\rho{}_{\lambda\mu} \\
= {} & \nabla_\rho \delta \Gamma^\rho{}_{\nu\mu} - \nabla_\nu \delta \Gamma^\rho{}_{\rho\mu}
\end{aligned}$$

$$\Rightarrow \nabla_\mu \delta \Gamma^\lambda{}_{\nu\rho} = \partial_\mu \delta \Gamma^\lambda{}_{\nu\rho} + \Gamma^\lambda{}_{\mu\sigma} \delta \Gamma^\sigma{}_{\nu\rho} - \Gamma^\sigma{}_{\mu\nu} \delta \Gamma^\lambda{}_{\sigma\rho} - \Gamma^\sigma{}_{\mu\rho} \delta \Gamma^\lambda{}_{\nu\sigma}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}$$

$$\begin{aligned}
\delta R_{\mu\nu} = {} & \partial_\rho \delta\Gamma^\rho_{\nu\mu} - \partial_\nu \delta\Gamma^\rho_{\rho\mu} \\
& + \delta\Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} + \Gamma^\rho_{\rho\lambda} \delta\Gamma^\lambda_{\nu\mu} \\
& - \delta\Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu} - \Gamma^\rho_{\nu\lambda} \delta\Gamma^\lambda_{\rho\mu} \\
& - \Gamma^\lambda_{\rho\nu} \delta\Gamma^\rho_{\lambda\mu} + \Gamma^\lambda_{\nu\rho} \delta\Gamma^\rho_{\lambda\mu} \\
= {} & \nabla_\rho \delta\Gamma^\rho_{\nu\mu} - \nabla_\nu \delta\Gamma^\rho_{\rho\mu}
\end{aligned}$$

$$\Rightarrow g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} \nabla_\rho \delta\Gamma^\rho_{\nu\mu} - g^{\mu\nu} \nabla_\nu \delta\Gamma^\rho_{\rho\mu}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}$$

$$\begin{aligned}
\delta R_{\mu\nu} = {} & \partial_\rho \delta\Gamma^\rho_{\nu\mu} - \partial_\nu \delta\Gamma^\rho_{\rho\mu} \\
& + \delta\Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} + \Gamma^\rho_{\rho\lambda}\delta\Gamma^\lambda_{\nu\mu} \\
& - \delta\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\rho\mu} \\
& - \Gamma^\lambda_{\rho\nu}\delta\Gamma^\rho_{\lambda\mu} + \Gamma^\lambda_{\nu\rho}\delta\Gamma^\rho_{\lambda\mu} \\
= {} & \nabla_\rho \delta\Gamma^\rho_{\nu\mu} - \nabla_\nu \delta\Gamma^\rho_{\rho\mu}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\nabla_\rho \delta\Gamma^\rho_{\nu\mu} - g^{\mu\nu}\nabla_\nu \delta\Gamma^\rho_{\rho\mu} \\
&= \nabla_\rho\left[g^{\mu\nu}\delta\Gamma^\rho_{\nu\mu}\right] - \nabla_\nu\left[g^{\mu\nu}\delta\Gamma^\rho_{\rho\mu}\right]
\end{aligned}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\nu\mu} - \partial_\nu \Gamma^\rho{}_{\rho\mu} + \Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu}$$

$$\begin{aligned}
\delta R_{\mu\nu} = \; & \partial_\rho \delta\Gamma^\rho{}_{\nu\mu} - \partial_\nu \delta\Gamma^\rho{}_{\rho\mu} \\
& + \delta\Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} + \Gamma^\rho{}_{\rho\lambda} \delta\Gamma^\lambda{}_{\nu\mu} \\
& - \delta\Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu} - \Gamma^\rho{}_{\nu\lambda} \delta\Gamma^\lambda{}_{\rho\mu} \\
& - \Gamma^\lambda{}_{\rho\nu} \delta\Gamma^\rho{}_{\lambda\mu} + \Gamma^\lambda{}_{\nu\rho} \delta\Gamma^\rho{}_{\lambda\mu} \\
= \; & \nabla_\rho \delta\Gamma^\rho{}_{\nu\mu} - \nabla_\nu \delta\Gamma^\rho{}_{\rho\mu}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow g^{\mu\nu} \delta R_{\mu\nu} &= g^{\mu\nu} \nabla_\rho \delta\Gamma^\rho{}_{\nu\mu} - g^{\mu\nu} \nabla_\nu \delta\Gamma^\rho{}_{\rho\mu} \\
&= \nabla_\rho \left[g^{\mu\nu} \delta\Gamma^\rho{}_{\nu\mu} \right] - \nabla_\nu \left[g^{\mu\nu} \delta\Gamma^\rho{}_{\rho\mu} \right] \\
&= \nabla_\rho \left[g^{\mu\nu} \delta\Gamma^\rho{}_{\nu\mu} - g^{\mu\rho} \delta\Gamma^\nu{}_{\nu\mu} \right]
\end{aligned}$$

almost there...

$$\int_V \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} = \int_V \sqrt{-g}\, \nabla_\rho \left[g^{\mu\nu} \delta \Gamma^\rho{}_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu{}_{\nu\mu} \right]$$

$$= \int_{\partial V} \sqrt{\gamma}\, n_\rho \left[g^{\mu\nu} \delta \Gamma^\rho{}_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu{}_{\nu\mu} \right]$$

$$\Rightarrow g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} \nabla_\rho \delta \Gamma^\rho{}_{\nu\mu} - g^{\mu\nu} \nabla_\nu \delta \Gamma^\rho{}_{\rho\mu}$$

$$= \nabla_\rho \left[g^{\mu\nu} \delta \Gamma^\rho{}_{\nu\mu} \right] - \nabla_\nu \left[g^{\mu\nu} \delta \Gamma^\rho{}_{\rho\mu} \right]$$

$$= \nabla_\rho \left[g^{\mu\nu} \delta \Gamma^\rho{}_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu{}_{\nu\mu} \right]$$

almost there...

$$\int_V \sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu} = \int_V \sqrt{-g}\, \nabla_\rho \left[g^{\mu\nu}\, \delta\Gamma^\rho{}_{\nu\mu} - g^{\mu\rho}\, \delta\Gamma^\nu{}_{\nu\mu} \right]$$

$$= \int_{\partial V} \sqrt{\gamma}\, n_\rho \left[g^{\mu\nu}\, \delta\Gamma^\rho{}_{\nu\mu} - g^{\mu\rho}\, \delta\Gamma^\nu{}_{\nu\mu} \right]$$

oops! ... not zero!

(can't have $\delta g|_{\partial V} = 0$ and $\delta\Gamma|_{\partial V} = 0$ at the same time)

$$\Rightarrow g^{\mu\nu}\, \delta R_{\mu\nu} = g^{\mu\nu}\, \nabla_\rho \delta\Gamma^\rho{}_{\nu\mu} - g^{\mu\nu}\, \nabla_\nu \delta\Gamma^\rho{}_{\rho\mu}$$

$$= \nabla_\rho \left[g^{\mu\nu}\, \delta\Gamma^\rho{}_{\nu\mu} \right] - \nabla_\nu \left[g^{\mu\nu}\, \delta\Gamma^\rho{}_{\rho\mu} \right]$$

$$= \nabla_\rho \left[g^{\mu\nu}\, \delta\Gamma^\rho{}_{\nu\mu} - g^{\mu\rho}\, \delta\Gamma^\nu{}_{\nu\mu} \right]$$

almost there...

$$\int_V \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} = \int_V \sqrt{-g}\, \nabla_\rho \left[g^{\mu\nu} \delta \Gamma^\rho{}_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu{}_{\nu\mu} \right]$$

$$= \int_{\partial V} \sqrt{\gamma}\, n_\rho \left[g^{\mu\nu} \delta \Gamma^\rho{}_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu{}_{\nu\mu} \right]$$

oops! ... not zero!

(can't have $\delta g|_{\partial V} = 0$ and $\delta \Gamma|_{\partial V} = 0$ at the same time)

Can get rid of this term by adding a boundary term:

$S^k \propto \frac{1}{2} \int_{\partial V} \sqrt{\gamma}\, K$, $K = \gamma_{ij} K^{ij}$ K^{ij} the extrinsic curvature!

$$\int_V \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} = \int_V \sqrt{-g}\, \nabla_\rho \left[g^{\mu\nu} \delta \Gamma^\rho{}_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu{}_{\nu\mu} \right]$$

$$= \int_{\partial V} \sqrt{\gamma}\, n_\rho \left[g^{\mu\nu} \delta \Gamma^\rho{}_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu{}_{\nu\mu} \right]$$

Then $S = S^{EH} + S^{\Lambda} + S^{M} + S^{k}$,

and δS^k cancels the $\int_{\partial V} \sqrt{\gamma}\, n_\rho [g^{\mu\nu}\delta\Gamma^\rho{}_{\nu\mu} - g^{\mu\rho}\delta\Gamma^\nu{}_{\nu\mu}]$ term

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