Lecture 5 - Geodesics - Problem Solutions (Hartle chap 8 and Carroll, chap 3)

2. In usual spherical coordinates the metric on a two-dimensional sphere is [cf. (2.15)]

$$dS^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right),$$

where a is a constant.

(a) Calculate the Christoffel symbols "by hand".

(b) Show that a great circle is a solution of the geodesic equation. (*Hint*: Make use of the freedom to orient the coordinates so the equation of a great circle is simple.)

$$g_{\mu\nu} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{\mathrm{Csc}[\theta]^2}{a^2} \end{pmatrix}$$

$$g = a^4 \mathrm{Sin}[\theta]^2$$

Christoffel Symbols:

$\Gamma^1{}_{2,2}$ $\quad -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$

$\Gamma^2{}_{2,1}$ $\quad \mathrm{Cot}[\theta]$

Geodesic Equations:

$$\theta_{\tau\tau} + -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]\,\phi_\tau^2 = 0$$

$$\phi_{\tau\tau} + 2\,\mathrm{Cot}[\theta]\,\theta_\tau\,\phi_\tau = 0$$

We choose coordinates, so that the great circle is the equator $\theta = \frac{\pi}{2}$ $\quad 0 < \varphi < 2\pi$. Then $\theta_\tau = \theta_{\tau\tau} = 0$ $\quad \cos\theta = 0$ are left with $\quad \phi_{\tau\tau} = 0 \Rightarrow \phi = C_1 \tau + C_2$

3. A three-dimensional spacetime has the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2.$$

(a) Find the explicit Lagrangian for the variational principle for geodesics in this spacetime in these coordinates.

(b) Using the results of (a) write out the components of the geodesic equation by computing them from the Lagrangian.

(c) Read off the nonzero Christoffel symbols for this metric from your results in (b).

(a) $L(\dot t, \dot r, \dot\phi, r) = \frac{1}{2}\left\{\left(1-\frac{2M}{r}\right)(\dot t)^2 - \left(1-\frac{2M}{r}\right)^{-1}(\dot r)^2 - r^2(\dot\phi)^2\right\}$

(the extrema of $\int L\, d\tau$ are the same as those of $\int \sqrt{|g_{\mu\nu}\dot x^\mu \dot x^\nu|}\, d\tau$)

$S = \int L\, d\tau$, $\delta S = 0 \Rightarrow \frac{d}{d\tau}\frac{\partial L}{\partial \dot x^\mu} - \frac{\partial L}{\partial x^\mu} = 0$

$\frac{\partial L}{\partial \dot t} = \frac{1}{2}\frac{\partial}{\partial \dot t}\left[\left(1-\frac{2M}{r}\right)(\dot t)^2\right] = \frac{1}{2}\left(1-\frac{2M}{r}\right)2\dot t \Rightarrow \frac{d}{d\tau}\frac{\partial L}{\partial \dot t} = \frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)\dot t\right]$

$\frac{\partial L}{\partial t} = 0$

$\rightsquigarrow \frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)\dot t\right] = 0$

$$\frac{\partial L}{\partial \dot{r}} = -\frac{1}{2}\left(1-\frac{2M}{r}\right)^{-1} 2\dot{r} = -\left(1-\frac{2M}{r}\right)^{-1}\dot{r} \Rightarrow \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = -\frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)^{-1}\dot{r}\right]$$

$$\frac{\partial L}{\partial r} = \frac{M}{r^2}(\dot{t})^2 + \left(1-\frac{2M}{r}\right)^{-2}\frac{M}{r^2}(\dot{r})^2 - r(\dot{\phi})^2$$

$$\Rightarrow \frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)^{-1}\dot{r}\right] + \frac{M}{r^2}(\dot{t})^2 + \left(1-\frac{2M}{r}\right)^{-2}\frac{M}{r^2}(\dot{r})^2 - r(\dot{\phi})^2 = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = -r^2\dot{\phi} \Rightarrow \frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}} = -\frac{d}{d\tau}(r^2\dot{\phi})$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow -\frac{d}{d\tau}(r^2\dot{\phi}) = 0$$

$$\frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)\dot t\right]=0 \Rightarrow \frac{M}{r^2}\dot r\dot t + \left(1-\frac{2M}{r}\right)\ddot t = 0$$

$$\Rightarrow \ddot t + 2\left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}\dot r\dot t = 0 \Rightarrow \Gamma^t{}_{rt} = \left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}$$

$$\Gamma^t{}_{rt}\dot r\dot t + \Gamma^t{}_{tr}\dot t\dot r = 2\Gamma^t{}_{rt}\dot r\dot t$$

no "2"

$$\frac{d}{d\tau}\left(r^2\dot\phi\right)=0 \Rightarrow 2r\dot r\dot\phi + r^2\ddot\phi = 0 \Rightarrow$$

$$\ddot\phi + \frac{2}{r}\dot r\dot\phi = 0 \Rightarrow \Gamma^\phi{}_{r\phi} = \frac{1}{r}$$

$$\frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)^{-1}\dot{r}\right] + \frac{M}{r^2}(\dot{t})^2 + \cancel{\left(1-\frac{2M}{r}\right)^{-2}} \frac{M}{r^2}(\dot{r})^2 - r(\dot{\phi})^2 = 0$$

$$\parallel$$

$$-\left(1-\frac{2M}{r}\right)^{-2}\left(-\left(-\frac{2M}{r^2}\right)\right)\dot{r}\,\dot{r} + \left(1-\frac{2M}{r}\right)^{-1}\ddot{r} = -\left(1-\frac{2M}{r}\right)^{-2}\frac{\cancel{2}M}{r^2}\dot{r}^2 + \left(1-\frac{2M}{r}\right)^{-1}\ddot{r}$$

$$\Rightarrow \left(1-\frac{2M}{r}\right)^{-1}\ddot{r} + \frac{M}{r^2}(\dot{t})^2 - \left(1-\frac{2M}{r}\right)^{-2}\frac{M}{r^2}(\dot{r})^2 - r(\dot{\phi})^2 = 0$$

$$\Rightarrow \ddot{r} + \left(1-\frac{2M}{r}\right)\frac{M}{r^2}(\dot{t})^2 - \left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}(\dot{r})^2 - r\left(1-\frac{2M}{r}\right)(\dot{\phi})^2 = 0$$

$$\Rightarrow \Gamma^r{}_{tt} = \left(1-\frac{2M}{r}\right)\frac{M}{r^2} \qquad \Gamma^r{}_{rr} = -\left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2} \qquad \Gamma^r{}_{\phi\phi} = -r\left(1-\frac{2M}{r}\right)$$

4. [A] *Rotating Frames* The line element of *flat spacetime* in a frame (t, x, y, z) that is rotating with an angular velocity Ω about the z-axis of an inertial frame is

$$ds^2 = -[1 - \Omega^2(x^2 + y^2)]\,dt^2 + 2\Omega(y\,dx - x\,dy)\,dt + dx^2 + dy^2 + dz^2.$$

(a) Verify this by transforming to polar coordinates and checking that the line element is (7.4) with the substitution $\phi \to \phi - \Omega t$.

(b) Find the geodesic equations for x, y, and z in the rotating frame.

(a) $x = r\sin\theta\cos\phi \Rightarrow dx = \sin\theta\cos\phi\,dr + r\cos\theta\cos\phi\,d\theta - r\sin\theta\sin\phi\,d\phi$

$y = r\sin\theta\sin\phi \Rightarrow dy = \sin\theta\sin\phi\,dr + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi$

$z = r\cos\theta \Rightarrow dz = \cos\theta\,dr - r\sin\theta\,d\theta$

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

$$y\,dx - x\,dy = r\sin^2\theta\,\cancel{\sin\phi}\cos\phi\,dr + r^2\sin\theta\,\cancel{\cos\theta}\sin\phi\cos\phi\,d\theta - r^2\sin^2\theta\,\underline{\sin^2\phi}\,d\phi$$
$$- r\sin^2\theta\,\cancel{\sin\phi}\cos\phi\,dr - r^2\sin\theta\,\cancel{\cos\theta}\sin\phi\cos\phi\,d\theta - r^2\sin^2\theta\,\underline{\cos^2\phi}\,d\phi$$
$$= -r^2\sin^2\theta\,d\phi$$

$x^2 + y^2 = r^2\sin^2\theta$

$$\Rightarrow ds^2 = -\left[1 - \Omega^2 r^2 \sin^2\theta\right] dt^2 - 2\Omega r^2 \sin^2\theta \, d\phi \, dt + dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$$= -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \left(d\phi - \Omega dt\right)^2$$

$$= -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \left[d(\phi - \Omega t)\right]^2$$

(b)

$$g_{\mu\nu} = \begin{pmatrix} -1 + (x^2+y^2)\,\Omega^2 & y\,\Omega & -x\,\Omega & 0 \\ y\,\Omega & 1 & 0 & 0 \\ -x\,\Omega & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & y\,\Omega & -x\,\Omega & 0 \\ y\,\Omega & 1-y^2\,\Omega^2 & x\,y\,\Omega^2 & 0 \\ -x\,\Omega & x\,y\,\Omega^2 & 1-x^2\,\Omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$g = -1$

Christoffel Symbols:

$\Gamma^2{}_{1,1}$ $\quad -x\,\Omega^2$

$\Gamma^2{}_{3,1}$ $\quad \Omega$

$\Gamma^3{}_{1,1}$ $\quad -y\,\Omega^2$

$\Gamma^3{}_{2,1}$ $\quad -\Omega$

Geodesic Equations:

$t_{\tau\tau} + 0 = 0$

$x_{\tau\tau} + -\Omega\, t_\tau\,(x\,\Omega\, t_\tau - 2\, y_\tau) = 0$

$y_{\tau\tau} + -\Omega\, t_\tau\,(y\,\Omega\, t_\tau + 2\, x_\tau) = 0$

$z_{\tau\tau} + 0 = 0$

Note In the non relativistic limit $t_\tau = \frac{dt}{d\tau} \approx 1$ $\quad x_{\tau\tau} \approx \frac{d^2x}{dt^2}$ etc

Coriolis force $2\vec{\Omega}\times\frac{d\vec{x}}{dt}$, $\vec{\Omega} = \Omega\,\hat{z}$

$$\frac{d^2x}{dt^2} = -2\Omega\frac{dy}{dt} + \Omega^2 x$$

centrifugal force $\vec{\Omega}\times(\vec{\Omega}\times\vec{x})$

$$\frac{d^2y}{dt^2} = +2\Omega\frac{dx}{dt} + \Omega^2 y$$

9. Consider the two-dimensional spacetime with the line element

$$ds^2 = -X^2 dT^2 + dX^2.$$

Find the shapes $X(T)$ of all the timelike geodesics in this spacetime.

The components of the metric are

$$g_{\mu\nu} = \begin{pmatrix} -X^2 & 0 \\ 0 & 1 \end{pmatrix}$$

and they are independent of T. Therefore $\xi = \partial_T$ is a Killing vector field, with $\xi^\mu = [1, 0]$. Along geodesics with tangent vector u, $\xi_\mu u^\mu$ is conserved. We define

$$e = -\xi_\mu u^\mu = -g_{\mu\nu}\xi^\mu u^\nu = -g_{00}\xi^0 u^0 = +X^2 \cdot 1 \cdot \frac{dT}{d\tau}$$

For timelike geodesics

$$U_\mu U^\mu = -1 \Rightarrow g_{00}(U^0)^2 + g_{11}(U^1)^2 = -1 \Rightarrow -X^2\left(\frac{dT}{d\tau}\right)^2 + \left(\frac{dX}{d\tau}\right)^2 = -1$$

$$e = X^2 \frac{dT}{d\tau} \Rightarrow \frac{dT}{d\tau} = X^{-2} e \quad , \text{ so}$$

$$-X^2 X^{-4} e^2 + \left(\frac{dX}{d\tau}\right)^2 = -1 \Rightarrow \left(\frac{dX}{d\tau}\right)^2 = \frac{e^2}{X^2} - 1 \Rightarrow \frac{dX}{d\tau} = \pm\left(\frac{e^2}{X^2} - 1\right)^{\frac{1}{2}}$$

Then $$\frac{dT}{dX} = \frac{dT/d\tau}{dX/d\tau} = \frac{e/X^2}{\pm\left(\frac{e^2}{X^2} - 1\right)^{\frac{1}{2}}} = \pm \frac{e}{X^2}\left(\frac{e^2}{X^2} - 1\right)^{-1/2}$$

$$\Rightarrow T(X) = \pm \cosh^{-1}\left(\frac{e}{X}\right) + T_0$$

12. *The Hyperbolic Plane* The hyperbolic plane defined by the metric

$$dS^2 = y^{-2}(dx^2 + dy^2), \quad y \geq 0$$

is a classic example of a two-dimensional surface.

(a) Show that points on the x-axis are an infinite distance from any point (x, y) in the upper half-plane.

(b) Write out the geodesic equations.

(c) Show that the geodesics are semicircles centered on the x-axis or vertical lines, as illustrated.

(d) Solve the geodesic equations to find x and y as functions of the length S along these curves.

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$$

$$g = \frac{1}{y^4}$$

Christoffel Symbols:

$\Gamma^1{}_{2,1} \quad -\frac{1}{y}$

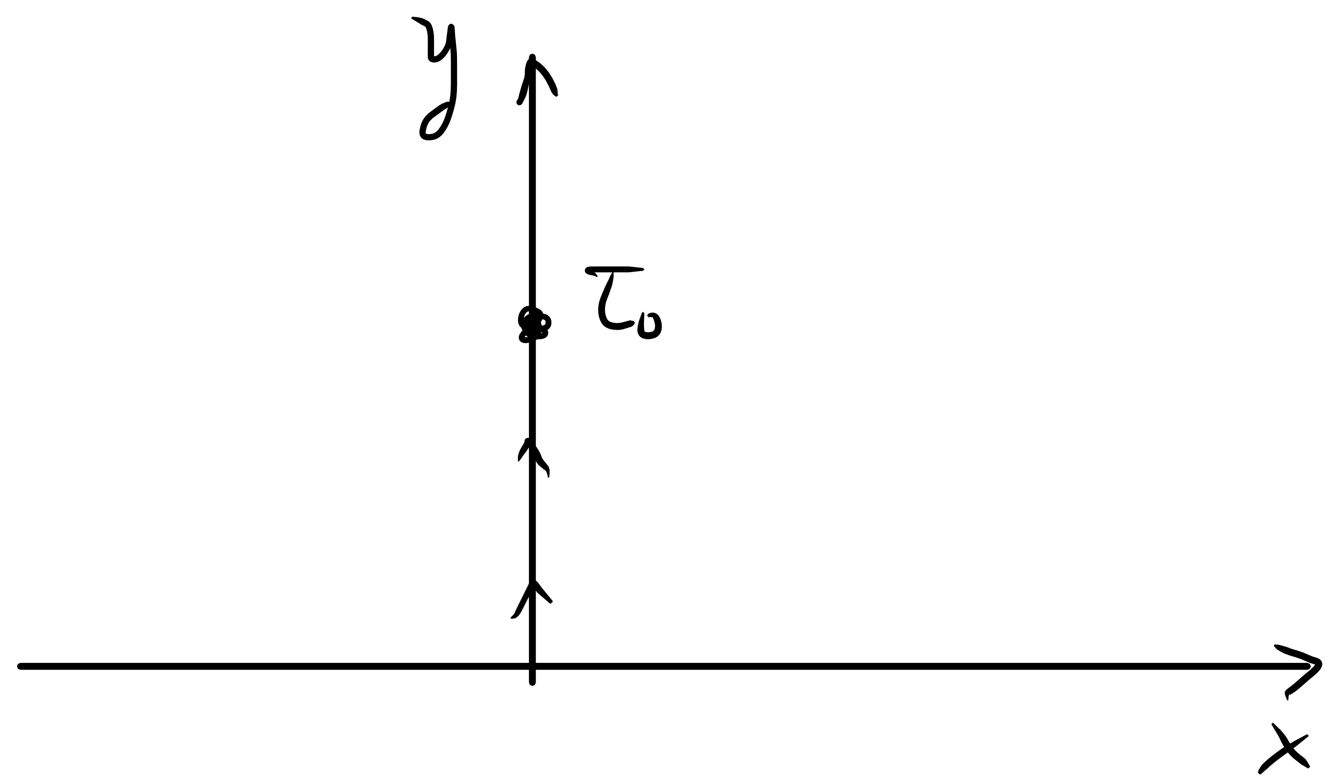
$\Gamma^2{}_{1,1} \quad \frac{1}{y}$

$\Gamma^2{}_{2,2} \quad -\frac{1}{y}$

Geodesic Equations:

$$x_{\tau\tau} \quad + \quad -\frac{2\, x_\tau\, y_\tau}{y} \quad = 0$$

$$y_{\tau\tau} \quad + \quad -\frac{-x_\tau^2 + y_\tau^2}{y} \quad = 0$$



(a) We can choose the $x=0$ axis so that $(x(\tau), y(\tau)) = (0, \tau)$, $\tau > 0$

$$S = \int (|g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu|)^{1/2} d\tau = \int_0^{\tau_0} [g_{yy}\, \dot{y}(\tau)^2]^{1/2} d\tau = \int_0^{\tau_0} \left(\frac{1}{y^2} \cdot 1\right)^{1/2} d\tau = \int_0^{\tau_0} \frac{d\tau}{\tau}$$

$$= \ln \tau \Big|_0^{\tau_0} = +\infty$$

(b) $\ddot{x} = \frac{1}{y}\dot{x}\dot{y} \Rightarrow y^2\frac{d}{d\tau}\left(\frac{1}{y^2}\frac{dx}{d\tau}\right) = 0 \Rightarrow \frac{dx}{d\tau} = \frac{y^2}{R}$ $\quad R = \text{const}$ (1)

$$\ddot{y} = \frac{1}{y}\left(-\dot{x}^2 + \dot{y}^2\right) \qquad (2)$$

(c)

But $u_\mu u^\mu = 1 \Rightarrow g_{xx}\dot{x}^2 + g_{yy}\dot{y}^2 = 1 \Rightarrow \dot{x}^2 + \dot{y}^2 = y^2$ (3)

$$(3) \overset{(1)}{\Longrightarrow} \frac{y^4}{R^2} + \dot{y}^2 = y^2 \Rightarrow \dot{y}^2 = y^2 - \frac{y^4}{R^2} \Rightarrow \dot{y} = \pm\left(y^2 - \frac{y^4}{R^2}\right)^{1/2}$$

So $\frac{dx}{dy} = \frac{dx/d\tau}{dy/d\tau} = \frac{y^2/R}{\pm y\left(1-\frac{y^2}{R^2}\right)^{1/2}} = \pm\frac{y}{R}\left(1-\frac{y^2}{R^2}\right)^{-1/2} \Rightarrow$

$x = \pm\sqrt{R^2 - y^2} + x_0 \Rightarrow (x-x_0)^2 + y^2 = R^2$ $\quad$ circles w/center $(x_0, 0)$, radius R

(d) $\dot{y} = \pm\, y\left(1 - \frac{y^2}{R^2}\right)^{1/2} \Rightarrow y^{-1}\left(1 - \frac{y^2}{R^2}\right)^{-1/2} dy = \pm\, dz \Rightarrow$

$$\frac{1}{2}\left[\ln\left[R^2\left(-1 + \sqrt{1 - \left(\frac{y}{R}\right)^2}\right)\right] - \ln\left[R^2\left(1 + \sqrt{1 - \left(\frac{y}{R}\right)^2}\right)\right]\right] = \pm z \Rightarrow$$

$$y(z) = \frac{R}{\cosh(z)}$$

$$\dot{x} = \frac{y^2}{R} \Rightarrow \dot{x} = \frac{1}{R}\frac{R^2}{\cosh^2(z)} \Rightarrow \frac{1}{R}\, dx = \frac{dz}{\cosh^2(z)} \Rightarrow x(z) = R\tanh(z)$$

8. The metric for the three-sphere in coordinates $x^\mu = (\psi, \theta, \phi)$ can be written

$$ds^2 = d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (3.220)$$

(a) Calculate the Christoffel connection coefficients.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Sin}[\psi]^2 & 0 \\ 0 & 0 & \mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Csc}[\psi]^2 & 0 \\ 0 & 0 & \mathrm{Csc}[\theta]^2\,\mathrm{Csc}[\psi]^2 \end{pmatrix}$$

$$g = \mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^4$$

Christoffel Symbols:

$\Gamma^1{}_{2,2}$ $-\mathrm{Cos}[\psi]\,\mathrm{Sin}[\psi]$

$\Gamma^1{}_{3,3}$ $-\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]$

$\Gamma^2{}_{2,1}$ $\mathrm{Cot}[\psi]$

$\Gamma^2{}_{3,3}$ $-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$

$\Gamma^3{}_{3,1}$ $\mathrm{Cot}[\psi]$

$\Gamma^3{}_{3,2}$ $\mathrm{Cot}[\theta]$

6. A good approximation to the metric outside the surface of the Earth is provided by

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{3.218}$$

where

$$\Phi = -\frac{GM}{r} \tag{3.219}$$

(a) Imagine a clock on the surface of the Earth at distance R_1 from the Earth's center, and another clock on a tall building at distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster?

(c) How much proper time elapses while a satellite at radius R_1 (skimming along the surface of the earth, neglecting air resistance) completes one orbit? You can work

(b) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?

$$g_{\mu\nu} = \begin{pmatrix} -1 + \frac{2M}{r} & 0 & 0 & 0 \\ 0 & 1 + \frac{2M}{r} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} \frac{r}{2M-r} & 0 & 0 & 0 \\ 0 & \frac{r}{2M+r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

$$g = -r^2 \left(-4M^2 + r^2\right) \mathrm{Sin}[\theta]^2$$

Christoffel Symbols:

$\Gamma^1{}_{2,1}$	$\frac{M}{r(-2M+r)}$
$\Gamma^2{}_{1,1}$	$\frac{M}{2Mr+r^2}$
$\Gamma^2{}_{2,2}$	$-\frac{M}{2Mr+r^2}$
$\Gamma^2{}_{3,3}$	$-\frac{r^2}{2M+r}$
$\Gamma^2{}_{4,4}$	$-\frac{r^2 \mathrm{Sin}[\theta]^2}{2M+r}$
$\Gamma^3{}_{3,2}$	$\frac{1}{r}$
$\Gamma^3{}_{4,4}$	$-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$
$\Gamma^4{}_{4,2}$	$\frac{1}{r}$
$\Gamma^4{}_{4,3}$	$\mathrm{Cot}[\theta]$

Geodesic Equations:

$$t_{\tau\tau} + -\frac{2M r_\tau t_\tau}{2Mr - r^2} = 0$$

$$r_{\tau\tau} + -\frac{M r_\tau^2 - M t_\tau^2 + r^3 \theta_\tau^2 + r^3 \mathrm{Sin}[\theta]^2 \phi_\tau^2}{2Mr + r^2} = 0$$

$$\theta_{\tau\tau} + \frac{2 r_\tau \theta_\tau}{r} - \mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]\,\phi_\tau^2 = 0$$

$$\phi_{\tau\tau} + \frac{2(r_\tau + r\,\mathrm{Cot}[\theta]\,\theta_\tau)\,\phi_\tau}{r} = 0$$

(a) $dr = d\theta = d\phi = 0$

$$d\tau^2 = -g_{00}\, dt^2 \Rightarrow \tau = \left(1 - \frac{2M}{r}\right)^{1/2} t$$

For same t:

$$\frac{\tau_2}{\tau_1} = \frac{(1 - 2M/R_2)^{1/2}}{(1-2M/R_1)^{1/2}} \approx \left(1 - \frac{M}{R_2} + \frac{M}{R_1}\right) = 1 + M\left(\frac{1}{R_1} - \frac{1}{R_2}\right) > 1$$

$$R_1 < R_2 \Rightarrow \frac{1}{R_1} - \frac{1}{R_2} > 0$$

(b) $dr = 0$, $d\theta = 0$, $\theta = \frac{\pi}{2}$, $\sin\theta = 1$

Conserved quantities, since $\xi^\mu = [1,0,0,0]$ $\eta^\mu = [0,0,0,1]$ are KVF

$$e = -\xi_\mu \cdot u^\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1} e$$

$$l = +\eta_\mu u^\mu = r^2 \sin^2\theta \frac{d\phi}{d\tau} = r^2 \frac{d\phi}{d\tau} \Rightarrow \frac{d\phi}{d\tau} = \frac{l}{r^2}$$

From $\ddot{r}$ geodesic equation: set $\dot{r}=0,\ \dot{\theta}=0,\ \sin\theta=1,\ \ddot{r}=0$

$$0-\frac{1}{r^2+2Mr}\left[-M\dot{t}^2+r^3\dot{\phi}^2\right]=0 \Rightarrow \dot{\phi}^2=\frac{M}{r^3}\dot{t}^2$$

$$\Omega=\frac{d\phi}{dt}=\frac{\dot{\phi}}{\dot{t}}=\pm\left(\frac{M}{r^3}\right)^{1/2} \qquad \text{(angular velocity)}$$

From $\ddot{\phi}$ geodesic eq:

$$\ddot{\phi}=0 \Rightarrow \phi=\omega\tau+\phi_0\ , \quad \dot{\phi}=\omega=\frac{l}{r^2}$$

$$\ddot{t}=0 \Rightarrow t=\gamma\tau+t_0\ , \quad \dot{t}=\gamma=\left(1-\frac{2M}{r}\right)^{-1}e$$

$$\Omega=\frac{\dot{\phi}}{\dot{t}}=\frac{\omega}{\gamma}=\frac{l/r^2}{\left(1-\frac{2M}{r}\right)^{-1}e}=\frac{1}{r^2}\left(1-\frac{2M}{r}\right)\frac{l}{e}=\left(\frac{M}{r^3}\right)^{1/2} \Rightarrow \frac{l}{e}=r^2\left(1-\frac{2M}{r}\right)^{-1}\overbrace{\left(\frac{M}{r^3}\right)^{1/2}}^{\Omega}$$

$$u_\mu u^\mu = -1 \Rightarrow \left(-1+\frac{2M}{r}\right)\dot{t}^2 + \left(1+\frac{2M}{r}\right)\cancel{\dot{r}^2} + r^2\dot{\phi}^2 = -1$$

$$\Rightarrow \left(-1+\frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)^{-2} e^2 + r^2\frac{l^2}{r^4} = -1$$

$$\Rightarrow -\left(1-\frac{2M}{r}\right)^{-1} e^2 + \frac{l^2}{r^2} = -1 \Rightarrow \frac{l^2}{r^2} = \left(1-\frac{2M}{r}\right)^{-1} e^2 - 1 \Rightarrow l = r\left[\left(1-\frac{2M}{r}\right)^{-1} e^2 - 1\right]^{1/2}$$

But $\frac{l}{e} = r^2\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{M}{r^3}\right)^{1/2} = (Mr)^{1/2}\left(1-\frac{2M}{r}\right)^{-1}$

$$l^2 = Mr\left(1-\frac{2M}{r}\right)^{-2} e^2$$, and

$$-\left(1-\frac{2M}{r}\right)^{-1} e^2 + \frac{1}{r^2} Mr\left(1-\frac{2M}{r}\right)^{-2} e^2 = -1 \Rightarrow$$

$$e^2\left(1-\frac{2M}{r}\right)^{-1}\left[-1+\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1}\right] = -1 \Rightarrow e^2 = -\frac{1-\frac{2M}{r}}{\frac{M/r}{1-\frac{2M}{r}} - 1} = \frac{\left(1-\frac{2M}{r}\right)^2}{1-\frac{3M}{r}}$$

$$\Rightarrow \ell^2 = Mr\left(1-\frac{2M}{r}\right)^{-2} \cdot \frac{\left(1-\frac{2M}{r}\right)^2}{1-\frac{3M}{r}} = \frac{Mr}{1-\frac{3M}{r}}$$

$$\omega = \frac{\ell}{r^2} = \frac{(Mr)^{1/2}}{r^2\left(1-\frac{3M}{r}\right)^{1/2}} = \left(\frac{M}{r^3}\right)^{1/2}\left(1-\frac{3M}{r}\right)^{-1/2} = \Omega\left(1-\frac{3M}{r}\right)^{-1/2}$$

$$\gamma = \left(1-\frac{2M}{r}\right)^{-1} e = \left(1-\frac{2M}{r}\right)^{-1} \cdot \frac{1-\frac{2M}{r}}{\left(1-\frac{3M}{r}\right)^{1/2}} = \left(1-\frac{3M}{r}\right)^{-1/2}$$

So the geodesic is

$$\phi = \Omega\left(1-\frac{3M}{r}\right)^{-1/2}\tau + \phi_0$$

$$t = \left(1-\frac{3M}{r}\right)^{-1/2}\tau + t_0$$

$$\Omega = \left(\frac{M}{r^3}\right)^{1/2}$$

(c) A full orbit corresponds to $\phi = 2\pi$, so

$$2\pi = \omega T \Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{\Omega}\left(1 - \frac{3M}{r}\right) \qquad \text{(proper period)}$$

$$= T_0 \left(1 - \frac{3M}{r}\right)$$

↳ period in coordinate time