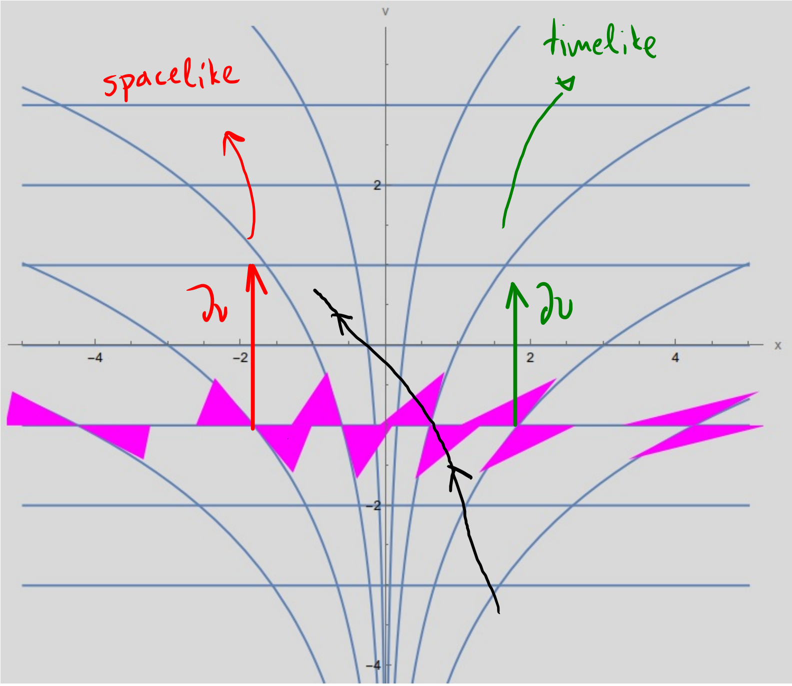


Light cones are in the directions for which $ds^2 < 0 \Rightarrow$

$dv > 0$	$dv < 0$
and	or
$-x\,dv + 2dx < 0$	$-x\,dv + 2dx > 0$
(1)	(2)

For $x<0$, $(1) \Rightarrow 2\,dx < x\,dv < 0$, so light cones are tilted to the left, and we can't cross from $x<0$ to $x>0$.

We notice that ∂_v has norm $\partial_v \cdot \partial_v = g_{vv} = -x \begin{cases} <0 \text{ for } x>0 \text{ timelike} \\ >0 \text{ for } x<0 \text{ spacelike} \end{cases}$

So for $x>0$, ∂_v is <u>in</u> the light cone

$x<0$ " " outside "

18. Consider the three-dimensional space with the line element

$$dS^2 = \frac{dr^2}{(1-2M/r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

(a) Calculate the radial distance between the sphere $r = 2M$ and the sphere $r = 3M$.

(b) Calculate the spatial volume between the two spheres in part (a).

(a) As we move along the radial distance $d\theta = d\phi = 0$

$$S = \int_{2M}^{3M} \frac{dr}{\left(1-\frac{2M}{r}\right)^{1/2}} = r\left(1-\frac{2M}{r}\right)^{1/2} + 2M\tan^{-1}\left[\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\right]\Bigg|_{2M}^{3M}$$

$$= M\left[\sqrt{3} + 2\tan^{-1}\sqrt{3}\right] \approx 3.049\, M$$

(b) The determinant of the metric is

$$g = \frac{1}{1-\frac{2M}{r}} \cdot r^2 \cdot r^2\sin^2\theta \Rightarrow \sqrt{g} = \left(1-\frac{2M}{r}\right)^{-1/2} r^2 \sin\theta$$

$$V = \int \sqrt{g}\, dr\, d\theta\, d\phi = \int_{2M}^{3M} dr\left(1-\frac{2M}{r}\right)^{-1/2} r^2 \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi\int_{2M}^{3M} dr\, r^2\left(1-\frac{2M}{r}\right)^{-1/2}$$

$$V = 4\pi \left. \frac{1}{6}\left(1-\frac{2M}{r}\right)^{-1/2}\left(2r^3 + Mr^2 + 5M^2 r - 30M^3\right) + 5M^3 \tan^{-1}\left[\left(1-\frac{2M}{r}\right)^{-1/2}\right] \right|_{2M}^{3M}$$

$$= 2\pi M^3\left[16\sqrt{3} + \ln\left(362 + 209\sqrt{3}\right)\right] \approx 215.50\, M^3$$

19. The surface of a sphere of radius R in four flat Euclidean dimensions is given by

$$X^2 + Y^2 + Z^2 + W^2 = R^2.$$

(a) Show that points on the sphere may be located by coordinates (χ, θ, ϕ), where

$$X = R \sin\chi \sin\theta \cos\phi, \qquad Z = R \sin\chi \cos\theta,$$

$$Y = R \sin\chi \sin\theta \sin\phi, \qquad W = R \cos\chi.$$

(b) Find the metric describing the geometry on the surface of the sphere in these coordinates.

(a) $X^2 + Y^2 = R \sin^2\chi \sin^2\theta$

$X^2 + Y^2 + Z^2 = R \sin^2\chi \,(\sin^2\theta + \cos^2\theta) = R^2 \sin^2\chi$

$X^2 + Y^2 + Z^2 + W^2 = R^2 (\sin^2\chi + \cos^2\chi) = R^2$

(b) $dX = \frac{\partial X}{\partial \chi} d\chi + \frac{\partial X}{\partial \theta} d\theta + \frac{\partial X}{\partial \phi} d\phi = R\cos\chi \sin\theta \cos\phi \, d\chi + R \sin\chi \cos\theta \cos\phi \, d\theta - R \sin\chi \sin\theta \sin\phi \, d\phi$

$dY = R \cos\chi \sin\theta \sin\phi \, d\chi + R \sin\chi \cos\theta \sin\phi \, d\theta + R \sin\chi \sin\theta \cos\phi \, d\phi$

$dZ = R \cos\chi \cos\theta \, d\chi - R \sin\chi \sin\theta \, d\theta$

$dW = -R \sin\chi \, d\chi$

$$ds^2 = dX^2 + dY^2 + dZ^2 + dW^2$$

$$dW^2 = R^2 \sin^2\chi \, d\chi^2$$

$$dZ^2 = R^2 \left[\cos^2\chi \cos^2\theta \, d\chi^2 + \sin^2\chi \sin^2\theta \, d\theta^2 - 2 \cos\chi \sin\cancel{\chi} \cos\theta \sin\theta \, d\chi \, d\theta \right]$$

$$\begin{aligned} dY^2 = R^2 \{ & \cos^2\chi \sin^2\theta \sin^2\phi \, d\chi^2 + \sin^2\chi \cos^2\theta \sin^2\phi \, d\theta^2 + \sin^2\chi \sin^2\theta \cos^2\phi \, d\phi^2 \\ & + 2 \cos\chi \sin\chi \cancel{\sin\theta} \cos\theta \sin^2\phi \, d\chi \, d\theta \\ & + 2 \cos\chi \sin\chi \cancel{\sin^2\theta} \sin\phi \cos\phi \, d\chi \, d\phi \\ & + 2 \sin^2\chi \sin\theta \cancel{\cos\theta} \sin\phi \cos\phi \, d\theta \, d\phi \,] \end{aligned}$$

$$\begin{aligned} dX^2 = R^2 [& \cos^2\chi \sin^2\theta \cos^2\phi \, d\chi^2 + \sin^2\chi \cos^2\theta \cos^2\phi \, d\theta^2 + \sin^2\chi \sin^2\theta \sin^2\phi \, d\phi^2 \\ & + 2 \sin\chi \cos\chi \cancel{\sin\theta} \cos\theta \cos^2\phi \, d\chi \, d\theta \\ & - 2 \sin\chi \cos\chi \cancel{\sin^2\theta} \sin\phi \cos\phi \, d\chi \, d\phi \\ & - 2 \sin^2\chi \sin\theta \cancel{\cos\theta} \sin\phi \cos\phi \, d\theta \, d\phi \,] \end{aligned}$$

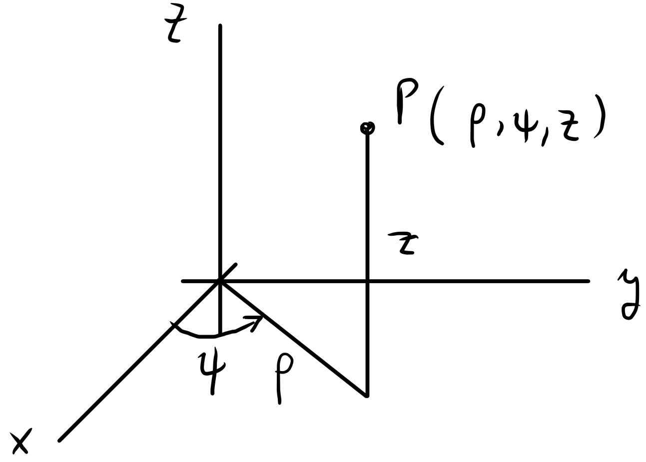
$$ds^2 = R^2 \left[d\chi^2 + \sin^2\chi \, d\theta^2 + \sin^2\chi \sin^2\theta \, d\phi^2 \right]$$

20. *Make the cover* Consider the two-dimensional geometry with the line element

$$d\Sigma^2 = \frac{dr^2}{(1-2M/r)} + r^2 d\phi^2.$$

Find a two-dimensional surface in three-dimensional flat space that has the *same* intrinsic geometry as this slice. Sketch a picture of your surface. (*Comment*: This is a slice of the Schwarzschild black-hole geometry to be discussed in Chapter 12. It is also the surface on the cover of this book.)

Embed in 3d flat space with cylindrical coordinates (ρ, ψ, z)



$$ds^2 = dx^2 + dy^2 + dz^2$$
$$= d\rho^2 + \rho^2 d\psi^2 + dz^2$$

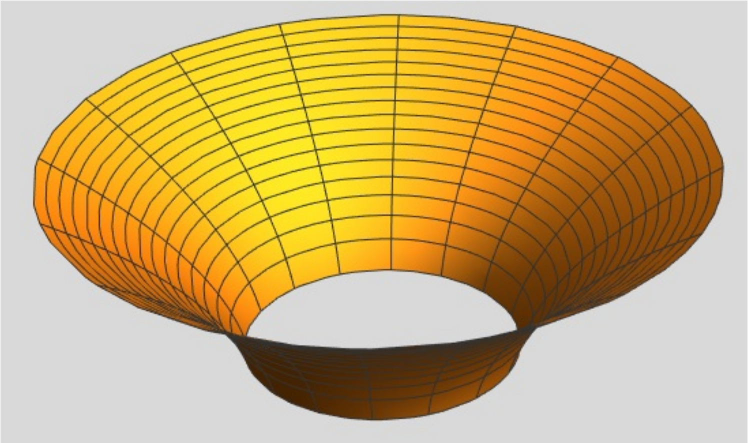
We will make an axisymmetric embedding $z = z(\rho)$, so

$$d\Sigma^2 = d\rho^2 + \rho^2 d\psi^2 + \left(\frac{\partial z}{\partial \rho}\right)^2 d\rho^2 = \left[1 + \left(\frac{\partial z}{\partial \rho}\right)^2\right] d\rho^2 + \rho^2 d\psi^2$$

So we should have $\psi = \phi$, $\rho = r$ and $1 + \left(\frac{\partial z}{\partial \rho}\right)^2 = \left(1 - \frac{2M}{\rho}\right)^{-1} \Rightarrow \left(\frac{\partial z}{\partial \rho}\right)^2 = -1 + \frac{1}{1-\frac{2M}{\rho}} \Rightarrow$

$$\left(\frac{\partial z}{\partial \rho}\right)^2 = \frac{2M/\rho}{1-\frac{2M}{\rho}} = \frac{2M}{(\rho-2M)} \Rightarrow \frac{\partial z}{\partial \rho} = \sqrt{2M}\,(\rho-2M)^{-1/2}$$

$$\Rightarrow z = \int d\rho \sqrt{2M}\,(\rho-2M)^{-1/2} = 2\sqrt{2M}\,(\rho-2M)^{1/2}$$



Carroll 3.4 $\quad x = uv\cos\phi \quad y = uv\sin\phi \quad z = \frac{1}{2}(u^2 - v^2)$

$$ds^2 = dx^2 + dy^2 + dz^2$$

- Compute $g_{\mu\nu}$ in the (u, v, ϕ) coordinate system
- if $V^\mu = v\partial_u - u\partial_v$ compute the components of V_μ and $V_\mu V^\mu$
- if $U^\mu = \sin\phi\,\partial_u - \cos\phi\,\partial_v$ compute $V_\mu U^\mu$

$$dx = v\cos\phi\, du + u\cos\phi\, dv - uv\sin\phi\, d\phi$$

$$dy = v\sin\phi\, du + u\sin\phi\, dv + uv\cos\phi\, d\phi$$

$$dz = u\,du - v\,dv$$

$$dx^2 = v^2\cos^2\phi\, du^2 + u^2\cos^2\phi\, dv^2 + u^2v^2\sin^2\phi\, d\phi^2 + 2vu\cos^2\phi\, du\,dv - 2uv^2\cancel{\cos\phi\sin\phi}\, du\,d\phi - 2u^2v\cancel{\cos\phi}\sin\phi\, dv\,d\phi$$

$$dy^2 = v^2\sin^2\phi\, du^2 + u^2\sin^2\phi\, dv^2 + u^2v^2\cos^2\phi\, d\phi^2 + 2vu\sin^2\phi\, du\,dv + 2uv^2\sin\phi\cancel{\cos\phi}\, du\,d\phi + 2u^2v\cancel{\sin\phi}\cos\phi\, dv\,d\phi$$

$$dz^2 = u^2 du^2 + v^2 dv^2 - \cancel{2uv\, du\, dv}$$

$$dx^2 + dy^2 = v^2 du^2 + u^2 dv^2 + u^2 v^2 d\phi^2 + \cancel{2uv\, du\, dv}$$

$$dx^2 + dy^2 + dz^2 = (u^2+v^2)(du^2 + dv^2) + u^2 v^2 d\phi^2$$

$$(g_{\mu\nu}) = \begin{pmatrix} u^2+v^2 & & \\ & u^2+v^2 & \\ & & u^2v^2 \end{pmatrix} \qquad (g^{\mu\nu}) = \begin{pmatrix} \frac{1}{u^2+v^2} & & \\ & \frac{1}{u^2+v^2} & \\ & & \frac{1}{u^2v^2} \end{pmatrix}$$

$$V^\mu = [v, -u, 0]$$

$$V_u = g_{uu} V^u = (u^2+v^2)\, v$$

$$V_v = g_{vv} V^v = -(u^2+v^2)\, u$$

$$V_\phi = g_{\phi\phi} V^\phi = 0$$

$$V^\mu V_\mu = g_{\mu\nu} V^\mu V^\nu = (u^2+v^2)\, v^2 + (u^2+v^2)\, u^2 = (u^2+v^2)^2$$

$$U^\mu = [\sin\phi, -\cos\phi, 0]$$

$$V_\mu U^\mu = g_{\mu\nu} V^\mu U^\nu = g_{uu} V^u U^u + g_{vv} V^v U^v$$

$$= (u^2+v^2)\, v \sin\phi + (u^2+v^2)(-u)(-\cos\phi)$$

$$= (u^2+v^2)\,[v\sin\phi + u\cos\phi]$$

Carroll §2.7, Misner Space

(i) $ds^2 = 0 \Rightarrow -dt^2 - \frac{2}{t}\,dt\,dx + dx^2 = 0 \Rightarrow \left(\frac{dt}{dx}\right)^2 + \frac{2}{t}\left(\frac{dt}{dx}\right) - 1 = 0 \Rightarrow \frac{dt}{dx} = -\frac{1}{t} \pm \sqrt{1+\frac{1}{t^2}}$

Define $\rho_1(t) = -\frac{1}{t} + \left(1+\frac{1}{t^2}\right)^{1/2}$ $\qquad \frac{dt}{dx} = \rho_1(t)$

or

$\rho_2(t) = -\frac{1}{t} - \left(1+\frac{1}{t^2}\right)^2$ $\qquad \frac{dt}{dx} = \rho_2(t)$

Two null lines starting at each event

(ii) Consider a null vector $V^\mu = (V^0, V^1) \equiv (V^t, V^x)$

$$V^\mu V_\mu = g_{\mu\nu} V^\mu V^\nu = 0 \Rightarrow -\cos\lambda\,(V^0)^2 - 2\sin\lambda\, V^0 V^1 + \cos\lambda\,(V^1)^2 = 0$$

$$\Rightarrow \left(\frac{V^0}{V^1}\right)^2 + \frac{2}{t}\left(\frac{V^0}{V^1}\right) - 1 = 0 \Rightarrow \frac{V^0}{V^1} = \rho_1(t)$$

or

$$\frac{V^0}{V^1} = \rho_2(t)$$

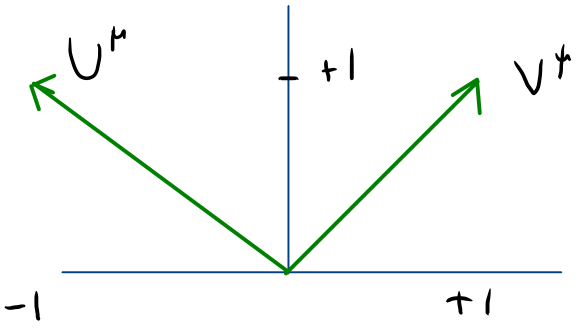
Notice that $\lim_{t\to-\infty} p_1(t) = -0 + (1+0)^{1/2} = 1$

$$\lim_{t\to-\infty} p_2(t) = -0 - (1+0)^{1/2} = -1$$

So we take $V^\mu = (p_1, 1) \xrightarrow{t\to-\infty} (1,1)$

$$U^\mu = (-p_2, -1) \xrightarrow[t\to-\infty]{} (1,-1)$$

We want



for $t<0$

so $$V^\mu = \left(-\frac{1}{t} + \left(1+\frac{1}{t^2}\right)^{1/2}, 1\right)$$

$t<0$

$$U^\mu = \left(\frac{1}{t} + \left(1+\frac{1}{t^2}\right)^{1/2}, -1\right)$$

Now we will try to define (V, U) to be continuous accross the $t=0$ line

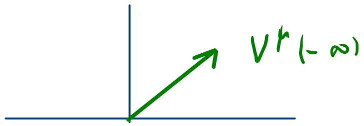
For $t<0$ $$V^\mu = \left(-\frac{1}{t} - \frac{1}{t}(1+t^2)^{1/2}, 1\right)$$

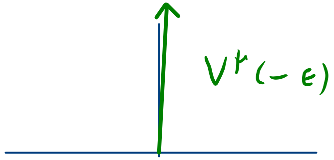
$$U^\mu = \left(\frac{1}{t} - \frac{1}{t}(1+t^2)^{1/2}, -1\right)$$

For $t=-\epsilon$, $\epsilon>0$ infinitesimal

$$V^\mu = \left(\frac{1}{\epsilon} + \frac{1}{\epsilon}(1+\epsilon^2)^{1/2}, 1\right) \approx \left(\frac{1}{\epsilon} + \frac{1}{\epsilon}\left(1+\frac{\epsilon^2}{2}\right), 1\right) = \left(\frac{2}{\epsilon}+\frac{\epsilon}{2}, 1\right)$$

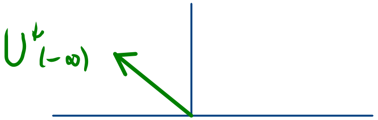
As $\epsilon\to 0^+$ $V^0\to+\infty$ and V^μ points in the $+t$-direction

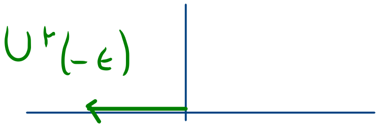




$$U^\mu = \left(-\frac{1}{\epsilon} + \frac{1}{\epsilon}(1+\epsilon^2)^{1/2}, -1\right) \approx \left(-\frac{1}{\epsilon} + \frac{1}{\epsilon}\left(1+\frac{\epsilon^2}{2}\right), -1\right) \approx \left(\frac{\epsilon}{2}, -1\right)$$

As $\epsilon\to 0^+ \Rightarrow U^0\to 0^+$





In order to have continuous vector fields at $t=0$, we define

$$V^\mu = (-p_2, -1) = \left(\frac{1}{t} + \left(1+\frac{1}{t^2}\right)^{1/2}, -1\right)$$

$$U^\mu = (-p_1, -1) = \left(\frac{1}{t} - \left(1+\frac{1}{t^2}\right)^{1/2}, -1\right) \qquad t>0$$

Taking the limit $t \to 0^+$, set $t = \epsilon > 0$

$$V^\mu = \left(\frac{1}{\epsilon} + \frac{1}{\epsilon}(1+\epsilon^2)^{1/2}, -1 \right) \approx \left(\frac{2}{\epsilon} + \frac{\epsilon}{2}, -1 \right)$$

$$U^\mu = \left(\frac{1}{\epsilon} - \frac{1}{\epsilon}(1+\epsilon^2)^{1/2}, -1 \right) \approx \left(-\frac{\epsilon}{2}, -1 \right)$$

So
$$\underset{t<0}{V^\mu \approx \left(\frac{2}{\epsilon} + \frac{\epsilon}{2}, +1 \right)} \to \underset{t>0}{\left(\frac{2}{\epsilon} + \frac{\epsilon}{2}, -1 \right)}$$

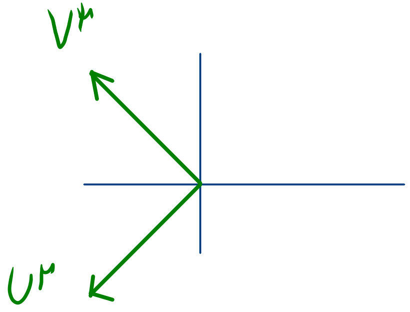
$$U^\mu \approx \left(\frac{\epsilon}{2}, -1 \right) \to \left(-\frac{\epsilon}{2}, -1 \right)$$

and the light cone rotates continuously

When $t \to +\infty$

$$V^\mu \approx \left(\frac{1}{t} + 1 + \frac{1}{2t^2} + \cdots, -1 \right) \to (1, -1)$$

$$U^\mu \approx \left(\frac{1}{t} - 1 - \frac{1}{2t^2} + \cdots, -1 \right) \to (-1, -1)$$



$t \to +\infty$

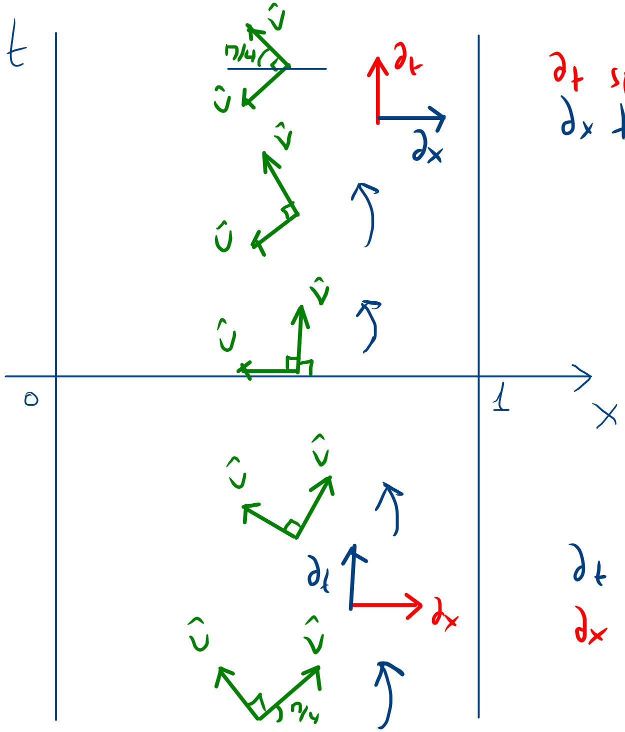
To avoid infinities you may define

$$\hat{V}^\mu = \begin{cases} \left(\frac{\rho_1}{\sqrt{1+\rho_1^2}} , \frac{1}{\sqrt{1+\rho_1^2}} \right) & t<0 \\ (1, 0) & t=0 \\ \left(-\frac{\rho_2}{\sqrt{1+\rho_2^2}} , \frac{-1}{\sqrt{1+\rho_2^2}} \right) & t>0 \end{cases} \qquad \hat{U}^\mu = \begin{cases} \left(-\frac{\rho_2}{\sqrt{1+\rho_2^2}} , \frac{-1}{\sqrt{1+\rho_2^2}} \right) & t<0 \\ (0, -1) & t=0 \\ \left(-\frac{\rho_1}{\sqrt{1+\rho_1^2}} , \frac{-1}{\sqrt{1+\rho_1^2}} \right) & t>0 \end{cases}$$

Notice that when we draw $\hat{V}^\mu, \hat{U}^\mu$ in the $t-x$ diagram, the Euclidean angle between them is always 90°:

$$\cos\theta = \frac{\hat{V}\cdot\hat{U}}{|\hat{V}||\hat{U}|} = \frac{V^0U^0 + V^1U^1}{1\cdot 1} = \frac{-1-\rho_1\rho_2}{\sqrt{(1+\rho_1^2)(1+\rho_2^2)}} = \frac{(-1)-(-1)}{\sqrt{\cdots}} = 0 \qquad t<0$$

$$\frac{1+\rho_1\rho_2}{\sqrt{(1+\rho_1^2)(1+\rho_2^2)}} = \frac{1+(-1)}{\sqrt{\cdots}} = 0 \qquad \text{since } \rho_1\rho_2 = -1$$



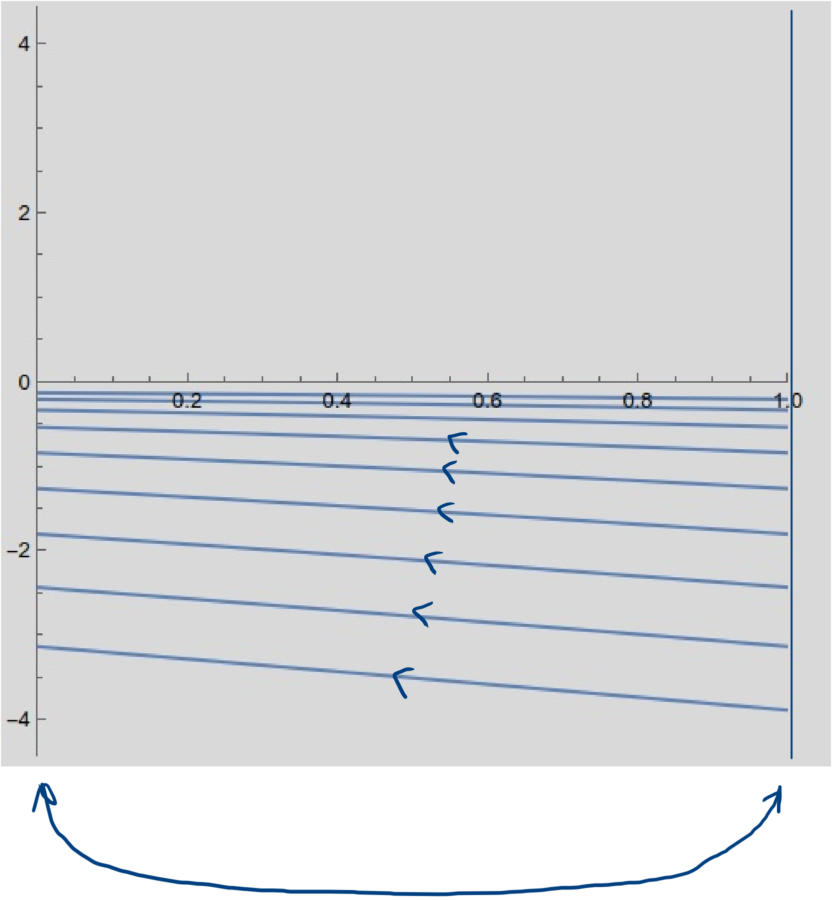
∂_t spacelike
∂_x timelike

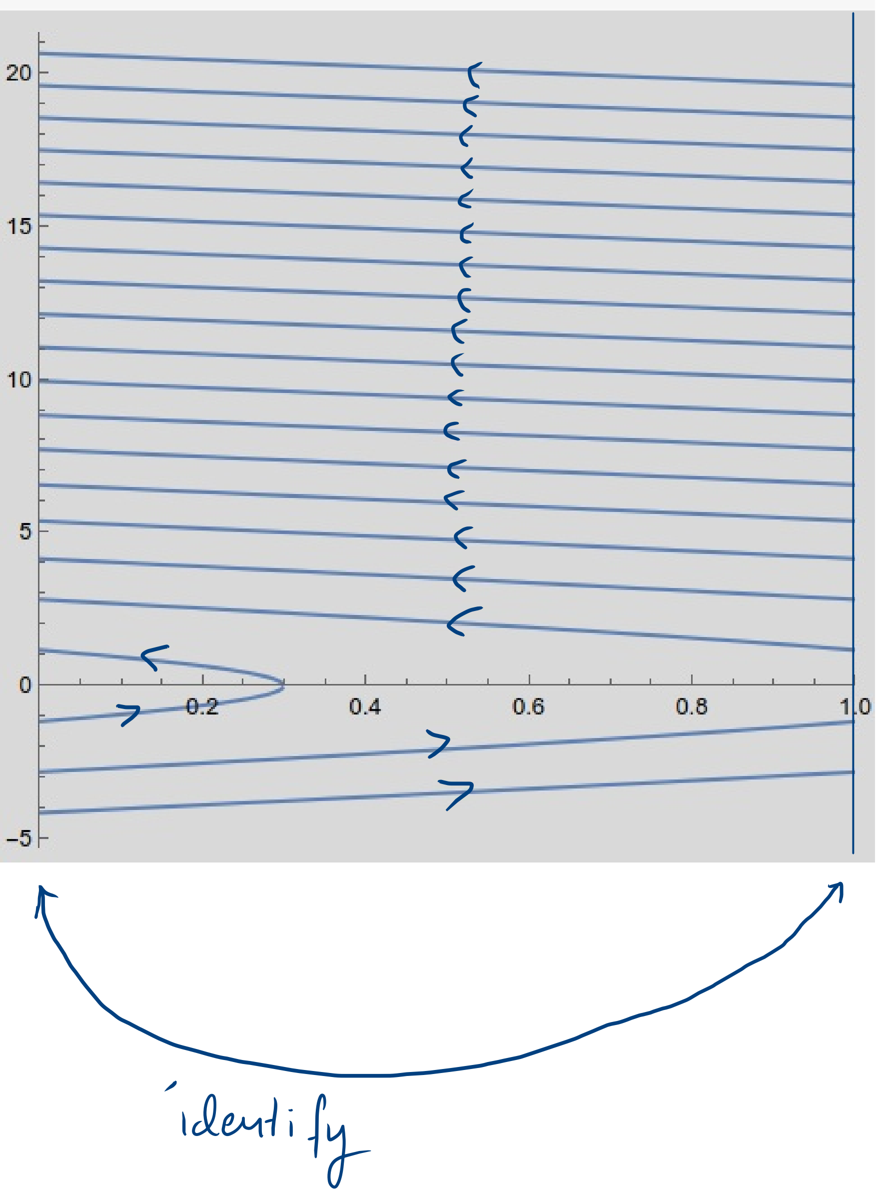
flow of time in $-\partial_x$ direction

∂_t timelike
∂_x spacelike

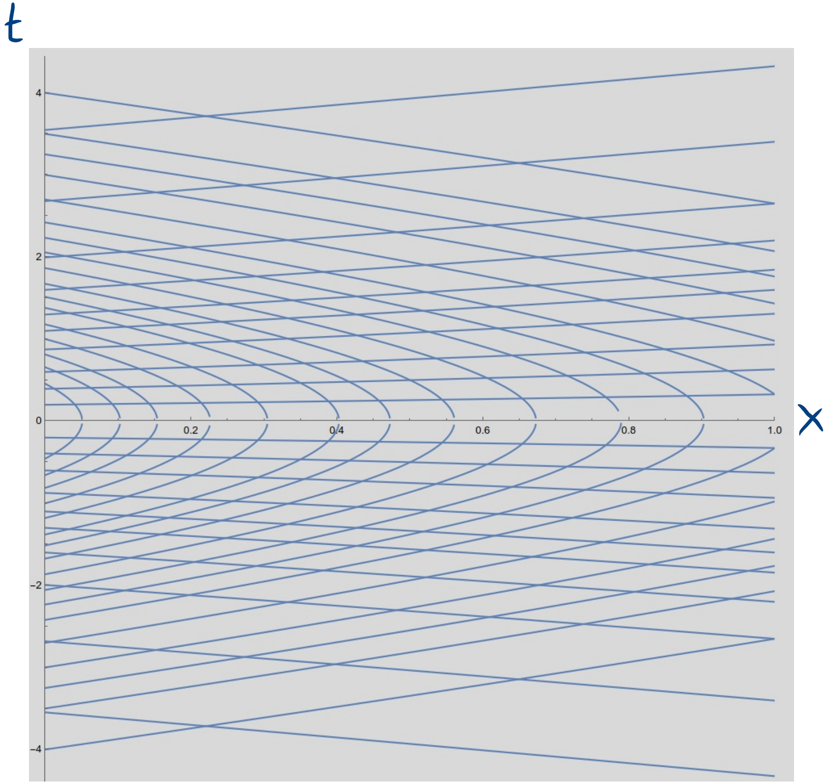
flow of time in $+\partial_t$ direction

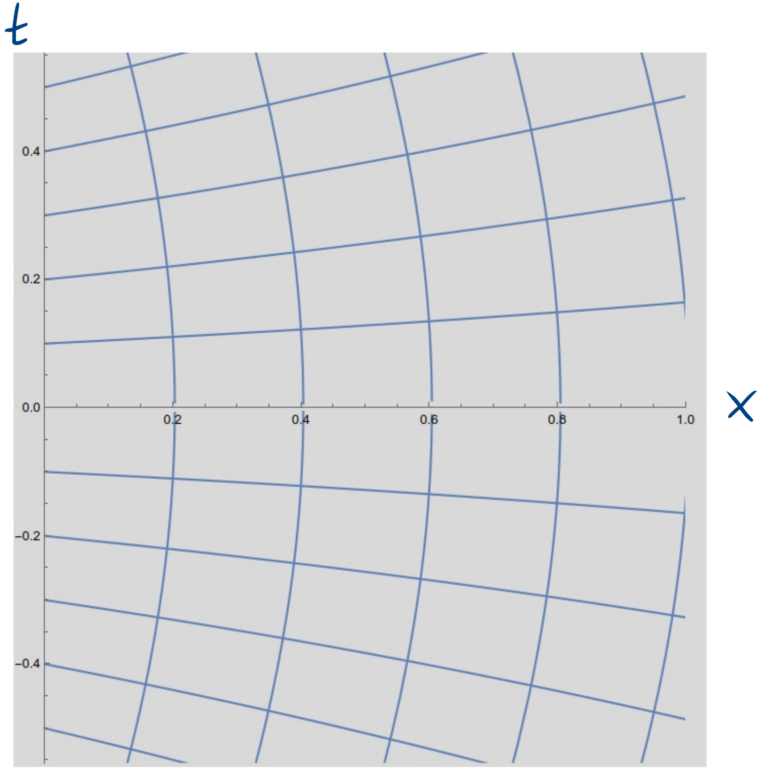
Light ray that starts in the $\hat{U}^\mu$ direction for $t<0$. The slope approches zero and the curve has as limit the $t=0$ circle





Light ray starting at $t<0$ in the direction of $\hat{V}^\mu$. Starts at slope $+45^\circ$. When approaching the $t=0$ circle, at some point $\hat{V}^\mu$ becomes vertical and crosses to the $t>0$ region. Then it moves in the $-x$ direction, as V^μ does! For large enough t, it moves in the 135° direction.





Samples of light rays in spacetime. The left figure is not to scale, and the 90° light cone is not obvious. The right figure is a detail, and you can see that the lines cross at 90°.
↳ do scale