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Final Exam - General Theory of Relativity

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Problem 1

Problem 1

Consider the Schwarzschild metric in the $r > 2M$ region, and the coordinate transformation $(t, r, \theta, \phi) \to (t, \xi, \theta, \phi)$, so that

$$r - 2M = \frac{\xi^2}{8M}\,, \quad \xi > 0\,. \tag{1}$$

The Metric 10

Show that in the (t, ξ, θ, ϕ) coordinate system

$$ds^2 = -\frac{\kappa^2\xi^2}{\kappa^2\xi^2+1}dt^2 + (\kappa^2\xi^2+1)d\xi^2 + \frac{1}{4\kappa^2}(\kappa^2\xi^2+1)^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{2}$$

where

$$\kappa = \frac{1}{4M}\,. \tag{3}$$

Tangent Space 5

Consider the coordinate basis

$$\{\partial_\mu\} = \{\partial_0, \partial_1, \partial_2, \partial_3\} = \{\partial_t, \partial_\xi, \partial_\theta, \partial_\phi\}\,. \tag{4}$$

Determine the type (spacelike, timelike or null) of the coordinate basis vectors.
Compute the orthocanonical basis

$$\{\hat{e}_\mu\} = \{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{e}_t, \hat{e}_\xi, \hat{e}_\theta, \hat{e}_\phi\}\,. \tag{5}$$

Compute a null vector, and write it as a linear combination of the coordinate basis elements $\{\partial_\mu\}$.

Killing Vector Fields (KVF) 5

Show that the vector fields ∂_t and ∂_ϕ are KVF of the metric ($\boxed{2}$).
A massive particle is falling freely following a trajectory $(t(\tau), \xi(\tau), \theta(\tau), \phi(\tau))$. Write down the equations that give the respective conserved quantities during the particle's motion.

Curvature 15

The components of the Riemann tensor in the $\{\partial_\mu\}$ basis are:

$$R_{1010} = -\frac{4\kappa^4\xi^2}{(\kappa^2\xi^2+1)^3} \tag{6}$$

$$R_{2020} = \frac{\kappa^2\xi^2}{2(\kappa^2\xi^2+1)^2} \tag{7}$$

$$R_{2121} = -\frac{1}{2} \tag{8}$$

$$R_{3030} = \frac{\kappa^2\xi^2\sin^2\theta}{2(\kappa^2\xi^2+1)^2} \tag{9}$$

$$R_{3131} = -\frac{1}{2}\sin^2\theta \tag{10}$$

$$R_{3232} = \frac{(\kappa^2\xi^2+1)\sin^2\theta}{4\kappa^2}\,. \tag{11}$$

Compute the components of $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}$ in the $\{\hat{e}_\mu\}$ basis.

Free fall, fixed ξ 25

An observer is falling freely, following a trajectory with fixed $\xi = \xi_0$.
Compute the angular velocity $\Omega = \frac{d\phi}{dt}$.
How much time elapses according to the observer (her proper time) during one revolution?
The geodesic equations are:

$$\ddot{t} + \frac{2}{\kappa^2\xi^3+\xi}\dot{t}\dot{\xi} = 0 \tag{12}$$

$$\ddot{\xi} - \frac{1}{2}\xi\left[\dot{\theta}^2 - \frac{2\kappa^2}{(\kappa^2\xi^2+1)^3}\left(\dot{t}^2 + (\kappa^2\xi^2+1)^2\dot{\xi}^2\right) + \sin^2\theta\,\dot{\phi}^2\right] = 0 \tag{13}$$

$$\ddot{\theta} + \frac{4\kappa^2\xi}{\kappa^2\xi^2+1}\dot{\theta}\dot{\xi} - \cos\theta\sin\theta\,\dot{\phi}^2 = 0 \tag{14}$$

$$\ddot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} + \frac{4\kappa^2\xi}{\kappa^2\xi^2+1}\dot{\xi}\dot{\phi} = 0\,. \tag{15}$$

where

$$\dot{t} = \frac{dt}{d\tau}\,, \quad \dot{\xi} = \frac{d\xi}{d\tau}\,, \quad \ldots \tag{16}$$

The Metric

$$k = \frac{1}{4M} \Rightarrow 2M = \frac{1}{2k}$$

$$r - 2M = \frac{\xi^2}{8M} \Rightarrow r = 2M + \frac{\xi^2}{8M} = 2M\left(1 + \frac{\xi^2}{16M^2}\right) \Rightarrow r = \frac{1}{2k}\left(k^2\xi^2+1\right)$$

$$1 - \frac{2M}{r} = 1 - \frac{1}{2k} \cdot \frac{2k}{k^2\xi^2+1} = \frac{k^2\xi^2}{k^2\xi^2+1}$$

$$dr = \frac{1}{2k} k^2 \, 2\xi \, d\xi = k\xi \, d\xi \qquad dr^2 = k^2\xi^2 d\xi^2$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$= -\frac{k^2\xi^2}{k^2\xi^2+1} dt^2 + \frac{k^2\xi^2+1}{k^2\xi^2} k^2\xi^2 d\xi^2 + \left[\frac{1}{2k}\left(k^2\xi^2+1\right)\right]^2 d\Omega^2$$

$$= -\frac{k^2\xi^2}{k^2\xi^2+1} dt^2 + \left(k^2\xi^2+1\right) d\xi^2 + \frac{1}{4k^2}\left(k^2\xi^2+1\right)^2 d\Omega^2$$

Tangent Space $\quad r > 2M \Rightarrow \xi > 0$

∂_t is timelike : $\quad \partial_t \cdot \partial_t = g_{tt} = -\dfrac{k^2\xi^2}{k^2\xi^2+1} < 0$

∂_ξ is spacelike : $\quad \partial_\xi \cdot \partial_\xi = g_{\xi\xi} = (k^2\xi^2+1) > 0$

∂_θ " " $\quad \partial_\theta \cdot \partial_\theta = g_{\theta\theta} = \dfrac{1}{4k^2}(k^2\xi^2+1)^2 > 0$

∂_ϕ " " $\quad \partial_\phi \partial_\phi = g_{\phi\phi} = g_{\theta\theta} \sin^2\theta > 0$

If $V = \alpha\partial_t + \beta\partial_\xi$ is null, then

$$0 = V \cdot V = \alpha^2 \partial_t \cdot \partial_t + \beta^2 \partial_\xi \cdot \partial_\xi = -\alpha^2 \frac{k^2\xi^2}{k^2\xi^2+1} + \beta^2 (k^2\xi^2+1) \Rightarrow \frac{\alpha^2}{\beta^2} = \frac{(k^2\xi^2+1)^2}{k^2\xi^2}$$

So, we can choose: $V = (k^2\xi^2+1)\partial_t + k\xi\,\partial_\xi$. Indeed:

$$V \cdot V = -(k^2\xi^2+1)^2 \frac{k^2\xi^2}{(k^2\xi^2+1)} + k^2\xi^2(k^2\xi^2+1) = 0$$

The metric is in a diagonal form, so the basis

$$\hat{e}_t = |g_{tt}|^{-1/2} \partial_t \qquad (\hat{e}_t)^\mu = \left[\frac{(k^2\zeta^2+1)^{1/2}}{k\zeta}, 0, 0, 0 \right]$$

$$\hat{e}_\zeta = |g_{\zeta\zeta}|^{-1/2} \partial_\zeta \qquad (\hat{e}_\zeta)^\mu = \left[0, (k^2\zeta^2+1)^{-1/2}, 0, 0 \right]$$

$$\hat{e}_\theta = |g_{\theta\theta}|^{-1/2} \partial_\theta \qquad (\hat{e}_\theta)^\mu = \left[0, 0, 2k(k^2\zeta^2+1)^{-1}, 0 \right]$$

$$\hat{e}_\phi = |g_{\phi\phi}|^{-1/2} \partial_\phi \qquad (\hat{e}_\phi)^\mu = \left[0, 0, 0, 2k(k^2\zeta^2+1)^{-1} \sin^{-1}\theta \right]$$

is orthonormal.

Killing Vector Fields

The metric components $g_{\mu\nu}$ are independent of t and ϕ.
Therefore $\partial_t, \partial_\phi$ are KVF. Then the following quantities are conserved along a geodesic $(t(z), \xi(z), \theta(z), \phi(z))$, where $u^\mu = (\dot{t}, \dot{\xi}, \dot{\theta}, \dot{\phi})$, $\dot{} = \frac{d}{dz}$

$$e = -\partial_t \cdot u = -g_{tt} (\partial_t)^t u^t = -\left[-\frac{k^2\xi^2}{k^2\xi^2+1} \right] \cdot 1 \cdot \dot{t} = \frac{k^2\xi^2}{k^2\xi^2+1} \cdot \dot{t}$$

$$l = \partial_\phi \cdot u = g_{\phi\phi} (\partial_\phi)^\phi u^\phi = \frac{1}{4k^2} (k^2\xi^2+1)^2 \sin^2\theta \, \dot{\phi}$$

Curvature

See Lecture10_BlackHoles_Slides.pdf p. 52-...

The metric is in diagonal form, therefore

$$R_{\hat{1}\hat{0}\hat{1}\hat{0}} = |g_{11}|^{-1/2} |g_{00}|^{-1/2} |g_{11}|^{-1/2} |g_{00}|^{-1/2} R_{1010} = |g_{11} g_{00}|^{-1} R_{1010}$$

$$= \left| - \frac{k^2\xi^2+1}{k^2\xi^2} \cdot \frac{1}{k^2\xi^2+1} \right| \left(- \frac{4k^4\xi^2}{(k^2\xi^2+1)^3} \right) = - \frac{4k^2}{(k^2\xi^2+1)^3}$$

$$R_{\hat{2}\hat{0}\hat{2}\hat{0}} = |g_{22} g_{00}|^{-1} R_{2020} = \frac{k^2\xi^2+1}{k^2\xi^2} \frac{4k^2}{(k^2\xi^2+1)^2} \frac{k^2\xi^2}{2(k^2\xi^2+1)^2} = \frac{2k^2}{(k^2\xi^2+1)^3}$$

$$R_{\hat{2}\hat{1}\hat{2}\hat{1}} = |g_{22} g_{11}|^{-1} R_{2121} = \frac{1}{(k^2\xi+1)} \frac{4k^2}{(k^2\xi^2+1)^2} \left(-\frac{1}{2} \right) = - \frac{2k^2}{(k^2\xi^2+1)^3}$$

$$R_{\hat{3}\hat{0}\hat{3}\hat{0}} = |g_{33} g_{00}|^{-1} R_{3030} = \frac{k^2\xi^2+1}{k^2\xi^2} \cdot \frac{4k^2}{(k^2\xi^2+1)^2 \sin^2\theta} \frac{k^2\xi^2 \sin^2\theta}{2(k^2\xi^2+1)^2} = \frac{2k^2}{(k^2\xi^2+1)^3}$$

$$R_{\hat{3}\hat{1}\hat{3}\hat{1}} = |g_{33} g_{11}|^{-1} R_{3131} = \frac{1}{k^2\xi^2+1} \frac{4k^2}{(k^2\xi^2+1)^2 \sin^2\theta} \left(-\frac{1}{2} \sin^2\theta \right) = - \frac{2k^2}{(k^2\xi^2+1)^3}$$

$$R_{\hat{3}\hat{2}\hat{3}\hat{2}} = |g_{33}\, g_{22}|^{-1} = \frac{4k^2}{(k^2\xi^2+1)^2}\,\frac{4k^2}{(k^2\xi^2+1)^2\sin^2\theta}\quad\frac{(k^2\xi^2+1)\sin^2\theta}{4k^2} = \frac{4k^2}{(k^2\xi^2+1)^3}$$

Free Fall, fixed ξ

- the trajectory lies on a plane (ang. momentum conservation)

  choose θ, φ so that

  $$\theta = \frac{\pi}{2} \quad \sin\theta = 1 \quad \dot{\theta} = 0 \quad \ddot{\theta} = 0$$

- $\xi = \xi_0 \Rightarrow \dot{\xi} = 0 \quad \ddot{\xi} = 0$

Then the geodesic equations become:

(12): $\ddot{t} = 0 \Rightarrow \dot{t} = \gamma \quad , t = \gamma\tau + t_0$

(14) is consistent

(15): $\ddot{\phi} = 0 \Rightarrow \dot{\phi} = \omega \quad \phi = \omega\tau + \phi_0$

We choose $\omega, \gamma > 0$
(sign determines direction of motion)

$$(13) \Rightarrow 0 - \frac{1}{2}\zeta_0\left[-\frac{2k^2}{(k^2\zeta_0^2+1)^3}\dot{t}^2 + \dot{\phi}^2\right] = 0 \Rightarrow$$

$$\dot{\phi} = +\left[2k^2(k^2\zeta_0^2+1)^{-3}\right]^{1/2}\dot{t}$$ (we have chosen +)

$$\Omega = \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{\dot{\phi}}{\dot{t}} = \sqrt{2}\,k\,(k^2\zeta_0^2+1)^{-3/2} = \text{constant}$$

But $u_\mu u^\mu = -1 \Rightarrow g_{tt}\dot{t}^2 + g_{44}\dot{\phi}^2 = -1 \Rightarrow -\frac{k^2\zeta_0^2}{k^2\zeta_0^2+1}\gamma^2 + \frac{1}{4k^2}(1+k^2\zeta_0^2)^2\omega^2 = -1$

$\Omega = \frac{\dot{\phi}}{\dot{t}} = \frac{\omega}{\gamma} \Rightarrow \gamma = \frac{\omega}{\Omega}$, so

$$-\frac{k^2\zeta_0^2}{1+k^2\zeta_0^2}\frac{\omega^2}{\Omega^2} + \frac{1}{4k^2}(1+k^2\zeta_0)^2\omega^2 = -1 \Rightarrow \omega^2 = 4k^2(k^2\zeta_0^2+1)^{-2}(2k^2\zeta_0^2-1)^{-1}$$

For a full revolution $2\pi = \omega T \Rightarrow T = 2\pi/\omega$

Problem 2

Consider the electromagnetic (EM) field, whose Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \tag{17}$$

where

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{18}$$

The following questions should be answered for the case of a Minkowski (flat) metric. You may use equations which are valid for a more general, curved spacetime.

Energy-Momentum Tensor

Show that the energy-momentum tensor of the EM field can be written in the form:

$$T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} \tag{19}$$

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Given that

$$E_i = -F_{0i} \tag{20}$$

$$B_k = \frac{1}{2} \epsilon_{kij} F^{ij} , \tag{21}$$

compute the $T_{\mu\nu}$ components in terms of the E_i, B_i. 7

Compute the Lagrangian density ($\boxed{17}$) in terms of the E_i, B_i. 5

You may use the relations:

$$\epsilon_{kij} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \tag{22}$$

$$\delta g = -g \, g_{\mu\nu} \delta g^{\mu\nu} . \tag{23}$$

Energy-Momentum Conservation

Show that when the equations of motion for ($\boxed{17}$) (Maxwell's equations) are satified, then

$$\partial_\mu T^{\mu\nu} = 0 . \tag{24}$$

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Energy-Momentum Tensor

$$S = \int \sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = \int \sqrt{-g}\left(-\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\right) = \int \sqrt{-g}\left(-\frac{1}{4}F^2\right)$$

$$\delta S = -\frac{1}{4}\int \delta\sqrt{-g}\,F^2 - \frac{1}{4}\int \sqrt{-g}\,\delta g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{4}\int \sqrt{-g}\,g^{\mu\rho}\delta g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$\rho \to \nu \quad \nu \to \rho \qquad\qquad \sigma \to \mu \quad \mu \to \sigma$

$$= -\frac{1}{4}\int \sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\right) - \frac{1}{4}\int \sqrt{-g}\,\delta g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}\int \sqrt{-g}\,g^{\sigma\rho}\delta g^{\nu\mu}\underbrace{F_{\sigma\nu}}_{=-F_{\nu\sigma}}\underbrace{F_{\rho\mu}}_{=-F_{\mu\rho}}$$

$$= -\frac{1}{2}\int \sqrt{-g}\left[g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^2\right]\delta g^{\mu\nu}$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F^2$$

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 \\ -\mathcal{E}_1 & 0 & \mathcal{B}_3 & -\mathcal{B}_2 \\ -\mathcal{E}_2 & -\mathcal{B}_3 & 0 & \mathcal{B}_1 \\ -\mathcal{E}_3 & \mathcal{B}_2 & -\mathcal{B}_1 & 0 \end{pmatrix} \qquad (F_{\mu\nu}) = \begin{pmatrix} 0 & -\mathcal{E}_1 & -\mathcal{E}_2 & -\mathcal{E}_3 \\ \mathcal{E}_1 & 0 & \mathcal{B}_3 & -\mathcal{B}_2 \\ \mathcal{E}_2 & -\mathcal{B}_3 & 0 & \mathcal{B}_1 \\ \mathcal{E}_3 & \mathcal{B}_2 & -\mathcal{B}_1 & 0 \end{pmatrix}$$

$$F^{0i} = -F^{i0} = \mathcal{E}^i \qquad F_{0i} = -F_{i0} = -\mathcal{E}_i$$

$$F_{ij} = \epsilon_{ijk}\mathcal{B}_k = F^{ij} \qquad \mathcal{B}_k = \frac{1}{2}\epsilon_{kij}F_{ij}$$

$$A_\mu = (-\phi, \vec{A}) \qquad A^\mu = (\phi, \vec{A})$$

$$\mathcal{E}_i = -F_{0i} = -(\partial_0 A_i - \partial_i A_0) = -\frac{\partial A_i}{\partial t} + \frac{\partial A_0}{\partial x^i} = -\frac{\partial A_i}{\partial t} - \frac{\partial \phi}{\partial x^i} = -(\vec{\nabla}\phi)_i - \frac{\partial A_i}{\partial t}$$

$$\mathcal{B}_k = \frac{1}{2}\epsilon_{kij}F_{ij} = \frac{1}{2}\epsilon_{kij}(\partial_i A_j - \partial_j A_i) = \frac{1}{2}\epsilon_{kij}\, 2\partial_i A_j = (\vec{\nabla}\times\vec{A})_k$$

$$F^2 = F_{\mu\nu}F^{\mu\nu} = F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} = 2F_{0i}F^{0i} + F_{ij}F^{ij}$$

$$= 2(-\mathcal{E}_i)(\mathcal{E}_i) + \epsilon_{ijk}B_k\,\epsilon_{ijl}B_l$$

$$= -2\mathcal{E}^2 + (\delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj})B_kB_l$$

$$= -2\mathcal{E}^2 + 3\delta_{kl}B_kB_l - \delta_{jl}\delta_{kj}B_kB_l$$

$$= -2\mathcal{E}^2 + 3B_kB_k - B_kB_k$$

$$= -2(\mathcal{E}^2 - B^2)$$

$$T_{00} = \frac{1}{2}(\mathcal{E}^2 + B^2) = T^{00}$$

$$T_{0i} = F_{0\rho}F_i{}^{\rho} - \frac{1}{4}\cancel{\eta_{0i}}^{\,0}F^2 = F_{0j}F_{ij} = (-\mathcal{E}_j)\epsilon_{ijk}B_k = -\epsilon_{ijk}\mathcal{E}_jB_k = -(\vec{\mathcal{E}}\times\vec{B})_i$$

$$T^{0i} = -T_{0i} = (\vec{\mathcal{E}}\times\vec{B})_i$$

$$T_{ij} = F_{i\rho}F_j{}^{\rho} - \frac{1}{4}\eta_{ij}F^2 = F_{i0}F_j{}^0 + F_{ik}F_j{}^k - \frac{1}{4}\delta_{ij}F^2 = -\mathcal{E}_i\mathcal{E}_j - B_iB_j + \delta_{ij}B^2 + \frac{1}{4}\delta_{ij}2(\mathcal{E}^2 - B^2)$$

$$= \left(-\mathcal{E}_i\mathcal{E}_j + \frac{1}{2}\delta_{ij}\mathcal{E}^2\right) + \left(-B_iB_j + \frac{1}{2}\delta_{ij}B^2\right)$$

$$F_{i0}F_j{}^0 = -F_{i0}F_{j0} = -\mathcal{E}_i\mathcal{E}_j$$

$$F_{ik}F_j{}^k = F_{ik}F_{jk} = \epsilon_{ikl}B_l\,\epsilon_{jkm}B_m = \epsilon_{kli}\,\epsilon_{kmj}B_lB_m$$

$$= (\delta_{lm}\delta_{ij} - \delta_{lj}\delta_{im})B_lB_m = \delta_{ij}B^2 - B_iB_j$$

Energy - Momentum Conservation

$$\partial_\mu T^\mu{}_\nu = \partial_\mu \left(F^{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta^\mu{}_\nu F^2 \right) =$$

$$= \cancel{\partial_\mu F^{\mu\rho}} F_{\nu\rho} + F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} \delta^\mu{}_\nu F^{\rho\sigma} \partial_\mu F_{\rho\sigma}$$

$$= F^{\mu\rho} \partial_\mu F_{\nu\rho} - \frac{1}{2} F^{\rho\sigma} \partial_\nu F_{\rho\sigma}$$

on shell (Maxwell's eq)

$$\partial_\mu F^{\mu\rho} = 0$$

Bianchi identities (Maxwell's eqs)

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0 \Rightarrow$$

$$F^{\mu\rho} \left(\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} \right) = 0 \Rightarrow F^{\mu\rho} \partial_\mu F_{\nu\rho} + F^{\mu\rho} \underset{\mu \leftrightarrow \rho}{\partial_\rho F_{\mu\nu}} + F^{\mu\rho} \partial_\nu \underset{F_{\rho\mu} = -F_{\mu\rho}}{F_{\rho\mu}} = 0$$

$$F^{\mu\rho} \partial_\mu F_{\nu\rho} + \underset{F^{\rho\mu} = -F^{\mu\rho}}{F^{\rho\mu}} \partial_\mu \underset{F_{\rho\nu} = -F_{\nu\rho}}{F_{\rho\nu}} - F^{\mu\rho} \partial_\nu F_{\mu\rho} = 0$$

$$\Rightarrow 2 F^{\mu\rho} \partial_\mu F_{\nu\rho} - F^{\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$$

Problem 3

Consider the covariant derivative ∇_μ of the Levi-Civita connection compatible with the metric $g_{\mu\nu}$.

You may consider the following equations to be given:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda \tag{25}$$

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda \tag{26}$$

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\lambda} - \partial_\nu \Gamma^\rho{}_{\mu\lambda} + \Gamma^\rho{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\lambda} - \Gamma^\rho{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\lambda}\,. \tag{27}$$

Connections

The vectors W^μ, U^μ are parallel transported along a curve, whose tangent vector is V^μ. Show that the inner products $W^\mu W_\mu$, $U^\mu U_\mu$, $W^\mu U_\mu$, remain constant along the curve. 5

Show that, if ω_μ is a one-form field, then

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu}\omega_\lambda\,. \quad 5 \tag{28}$$

Curvature

Show that

$$[\nabla_\mu, \nabla_\nu]\omega_\rho = -R^\lambda{}_{\rho\mu\nu}\omega_\lambda \quad 6 \tag{29}$$

$$[\nabla_\mu, \nabla_\nu] F^\sigma{}_\rho = R^\sigma{}_{\lambda\mu\nu} F^\lambda{}_\rho - R^\lambda{}_{\rho\mu\nu} F^\sigma{}_\lambda\,. \quad 9 \tag{30}$$

Symmetries of the Riemann Tensor

Show that

$$R^\mu{}_{[\nu\rho\sigma]} = 0\,. \quad 15 \tag{31}$$

Connections

If W^μ, U^μ parallel-transported along V^μ, then

$$D_V W^\mu = D_V U^\mu = 0 \iff$$

$$V^\rho \nabla_\rho W^\mu = V^\rho \nabla_\rho U^\mu = 0$$

metric compatibility

$$\text{then } \frac{d}{d\tau}(W^\mu W_\mu) = V^\rho \nabla_\rho (g_{\mu\nu} W^\mu W^\nu) = V^\rho \cancel{\nabla_\rho g_{\mu\nu}} W^\mu W^\nu + V^\rho g_{\mu\nu} \nabla_\rho W^\mu\, W^\nu + V^\rho g_{\mu\nu} W^\mu \nabla_\rho W^\nu$$

Leibnitz rule

$$= g_{\mu\nu}(\cancel{V^\rho \nabla_\rho W^\mu}) W^\nu + g_{\mu\nu} W^\mu (\cancel{V^\rho \nabla_\rho W^\nu}) = 0 \Rightarrow W^\mu W_\mu = \text{constant}$$

parallel transport

Similarly for $W^\mu U_\mu$, $U^\mu U_\mu$

Consider the contraction $V^\mu \omega_\mu$, which is a function. Then

$$\nabla_\mu (\omega_\rho V^\rho) = \partial_\mu (\omega_\rho V^\rho) \Rightarrow$$

$$(\nabla_\mu \omega_\rho) V^\rho + \omega_\rho (\nabla_\mu V^\rho) = (\partial_\mu \omega_\rho) V^\rho + \omega_\rho \partial_\mu V^\rho \Rightarrow$$

$$(\nabla_\mu \omega_\rho) V^\rho + \omega_\rho (\cancel{\partial_\mu V^\rho} + \Gamma^\rho_{\mu\nu} V^\nu) = (\partial_\mu \omega_\rho) V^\rho + \omega_\rho \cancel{\partial_\mu V^\rho} \Rightarrow$$

$$[\nabla_\mu \omega_\rho - (\partial_\mu \omega_\rho - \Gamma^\nu_{\mu\rho} \omega_\nu)] V^\rho = 0 \qquad \forall V^\rho \Rightarrow$$

$$\nabla_\mu \omega_\rho = \partial_\mu \omega_\rho - \Gamma^\nu{}_{\mu\rho} \omega_\nu$$

Curvature

See Lecture06_Curvature_Slides.pdf p. 35-

The contraction $\omega_\mu V^\mu$ is a function, therefore

$$[\nabla_\mu, \nabla_\nu](\omega_\lambda V^\lambda) = 0$$

$$\nabla_\mu \nabla_\nu (\omega_\lambda V^\lambda) = \nabla_\mu \left[(\nabla_\nu \omega_\lambda) V^\lambda + \omega_\lambda (\nabla_\nu V^\lambda) \right]$$

$$= (\nabla_\mu \nabla_\nu \omega_\lambda) V^\lambda + \cancel{(\nabla_\nu \omega_\lambda)(\nabla_\mu V^\lambda)} + \cancel{(\nabla_\mu \omega_\lambda)(\nabla_\nu V^\lambda)} + \omega_\lambda \nabla_\mu \nabla_\nu V^\lambda$$

$\mu \leftrightarrow \nu$

$$\nabla_\nu \nabla_\mu (\omega_\lambda V^\lambda) = (\nabla_\nu \nabla_\mu \omega_\lambda) V^\lambda + \cancel{(\nabla_\mu \omega_\lambda)(\nabla_\nu V^\lambda)} + \cancel{(\nabla_\nu \omega_\lambda)(\nabla_\mu V^\lambda)} + \omega_\lambda \nabla_\nu \nabla_\mu V^\lambda$$

$$\Rightarrow [\nabla_\mu, \nabla_\nu](\omega_\lambda V^\lambda) = (\underbrace{[\nabla_\mu, \nabla_\nu]\omega_\lambda}_{\text{we want this}}) V^\lambda + \omega_\lambda (\underbrace{[\nabla_\mu, \nabla_\nu] V^\lambda}_{\text{we know that...}})$$

$$\Rightarrow \quad 0 = ([\nabla_\mu, \nabla_\nu]\omega_\lambda) V^\lambda + \omega_\sigma R^\sigma{}_{\lambda\mu\nu} V^\lambda$$

$$= \{ \underbrace{[\nabla_\mu, \nabla_\nu]\omega_\lambda + R^\sigma{}_{\lambda\mu\nu}\,\omega_\sigma}_{= 0} \} V^\lambda$$

Now consider $F^{\sigma}{}_{\rho}\,\omega_\sigma V^\rho$, a function. Then, the torsion free condition is:

$$[\nabla_\mu, \nabla_\nu](F^{\sigma}{}_{\rho}\,\omega_\sigma V^\rho) = 0$$

$$\nabla_\mu\nabla_\nu(F^{\sigma}{}_{\rho}\,\omega_\sigma V^\rho) = \nabla_\mu\left[(\nabla_\nu F^{\sigma}{}_{\rho})\,\omega_\sigma V^\rho + F^{\sigma}{}_{\rho}(\nabla_\nu \omega_\sigma)V^\rho + F^{\sigma}{}_{\rho}\,\omega_\sigma(\nabla_\nu V^\rho)\right]$$

$$= (\nabla_\mu\nabla_\nu F^{\sigma}{}_{\rho})\,\omega_\sigma V^\rho + \cancel{(\nabla_\nu F^{\sigma}{}_{\rho})(\nabla_\mu \omega_\sigma)}V^\rho + \cancel{(\nabla_\nu F^{\sigma}{}_{\rho})\,\omega_\sigma}(\nabla_\mu V^\rho)$$

$$+ (\nabla_\mu F^{\sigma}{}_{\rho})\cancel{(\nabla_\nu \omega_\sigma)}V^\rho + F^{\sigma}{}_{\rho}(\nabla_\mu\nabla_\nu \omega_\sigma)V^\rho + F^{\sigma}{}_{\rho}\cancel{(\nabla_\nu \omega_\sigma)}(\nabla_\mu V^\rho)$$

$$+ (\nabla_\mu F^{\sigma}{}_{\rho})\cancel{\omega_\sigma}\,\nabla_\nu V^\rho + F^{\sigma}{}_{\rho}\cancel{(\nabla_\mu \omega_\sigma)}(\nabla_\nu V^\rho) + F^{\sigma}{}_{\rho}\,\omega_\sigma(\nabla_\mu\nabla_\nu V^\rho)$$

$$\nabla_\nu\nabla_\mu(F^{\sigma}{}_{\rho}\,\omega_\sigma V^\rho) = \nabla_\nu\nabla_\mu F^{\sigma}{}_{\rho}\,\omega_\sigma V^\rho + (\nabla_\mu F^{\sigma}{}_{\rho})\cancel{(\nabla_\nu \omega_\sigma)}V^\rho + (\nabla_\mu F^{\sigma}{}_{\rho})\cancel{\omega_\sigma}\,\nabla_\nu V^\rho$$

$$+ (\nabla_\nu F^{\sigma}{}_{\rho})\cancel{(\nabla_\mu \omega_\sigma)}V^\rho + F^{\sigma}{}_{\rho}(\nabla_\nu\nabla_\mu \omega_\sigma)V^\rho + F^{\sigma}{}_{\rho}\cancel{(\nabla_\mu \omega_\sigma)}(\nabla_\nu V^\rho)$$

$$+ (\nabla_\nu F^{\sigma}{}_{\rho})\cancel{\omega_\sigma}\,\nabla_\mu V^\rho + F^{\sigma}{}_{\rho}\cancel{(\nabla_\nu \omega_\sigma)}(\nabla_\mu V^\rho) + F^{\sigma}{}_{\rho}\,\omega_\sigma(\nabla_\nu\nabla_\mu V^\rho)$$

$$\Rightarrow 0 = [\nabla_\mu, \nabla_\nu](F^\sigma{}_\rho \omega_\sigma V^\rho) =$$

$$= \left([\nabla_\mu, \nabla_\nu] F^\sigma{}_\rho\right) \omega_\sigma V^\rho + F^\sigma{}_\rho \left([\nabla_\mu, \nabla_\nu] \omega_\sigma\right) V^\rho + F^\sigma{}_\rho \omega_\sigma \left([\nabla_\mu, \nabla_\nu] V^\rho\right)$$

$$= \left([\nabla_\mu, \nabla_\nu] F^\sigma{}_\rho\right) \omega_\sigma V^\rho - F^\sigma{}_\rho R^\lambda{}_{\sigma\mu\nu} \omega_\lambda V^\rho + F^\sigma{}_\rho \omega_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$= \left([\nabla_\mu, \nabla_\nu] F^\sigma{}_\rho\right) \omega_\sigma V^\rho - F^\lambda{}_\rho R^\sigma{}_{\lambda\mu\nu} \omega_\sigma V^\rho + F^\sigma{}_\lambda \omega_\sigma R^\lambda{}_{\rho\mu\nu} V^\rho$$

$$= \left[[\nabla_\mu, \nabla_\nu] F^\sigma{}_\rho - R^\sigma{}_{\lambda\mu\nu} F^\lambda{}_\rho + R^\lambda{}_{\rho\mu\nu} F^\sigma{}_\lambda\right] \omega_\sigma V^\rho \qquad \forall \omega, V$$

$$\Rightarrow [\nabla_\mu, \nabla_\nu] F^\sigma{}_\rho = R^\sigma{}_{\lambda\mu\nu} F^\lambda{}_\rho - R^\lambda{}_{\rho\mu\nu} F^\sigma{}_\lambda$$

Symmetries of the Riemann Tensor

See Lecture06_Curvature_Slides.pdf p94

$$R^\rho{}_{[\sigma\mu\nu]} = 0 \Leftrightarrow \left(R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R^\rho{}_{\mu\nu\sigma} - R^\rho{}_{\sigma\nu\mu} - R^\rho{}_{\mu\sigma\nu} - R^\rho{}_{\nu\mu\sigma} \right) = 0$$

$$\Leftrightarrow R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R_{\mu\nu\sigma} = 0$$

Choose inertial frame: $\Gamma^\mu{}_{\nu\rho} = 0$ at P (torsion free)

$$\left.\begin{aligned}
R^\rho{}_{\sigma\mu\nu} &= \partial_\mu \cancel{\Gamma^\rho{}_{\nu\sigma}} - \partial_\nu \cancel{\Gamma^\rho{}_{\mu\sigma}} + \\
R^\rho{}_{\nu\sigma\mu} &= \partial_\sigma \cancel{\Gamma^\rho{}_{\mu\nu}} - \partial_\mu \cancel{\Gamma^\rho{}_{\sigma\nu}} + \\
R^\rho{}_{\mu\nu\sigma} &= \partial_\nu \cancel{\Gamma^\rho{}_{\sigma\mu}} - \partial_\sigma \cancel{\Gamma^\rho{}_{\nu\mu}} +
\end{aligned}\right\} = 0$$

If a tensor is 0 at one frame, it is 0 at all frames!