Schwarzschild Black Holes

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- spherically symmetric

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- spherically symmetric

$$R = \partial_\varphi$$ $\to$ easy to see: $\partial_\varphi g_{\mu\nu} = 0$

$$S = \cos\phi\, \partial_\theta - \cot\theta \sin\phi\, \partial_\varphi$$

$$T = -\sin\phi\, \partial_\theta - \cot\theta \cos\phi\, \partial_\varphi$$

are Killing vector fields (KVF)

$\to$ generate isometries

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  $[R, S] = T \qquad [S, T] = R \qquad [T, R] = S$

  SO(3) algebra

- ∂_t is a KVF $\Leftarrow \partial_t g_{\mu\nu} = 0$

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$\boxed{\text{stationary!}}$

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The Schwarzschild metric is the unique spherically symmetric solution of the vacuum Einstein field equations

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- the metric inside a spherical cavity is Minkowski (flat!)

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time direction!

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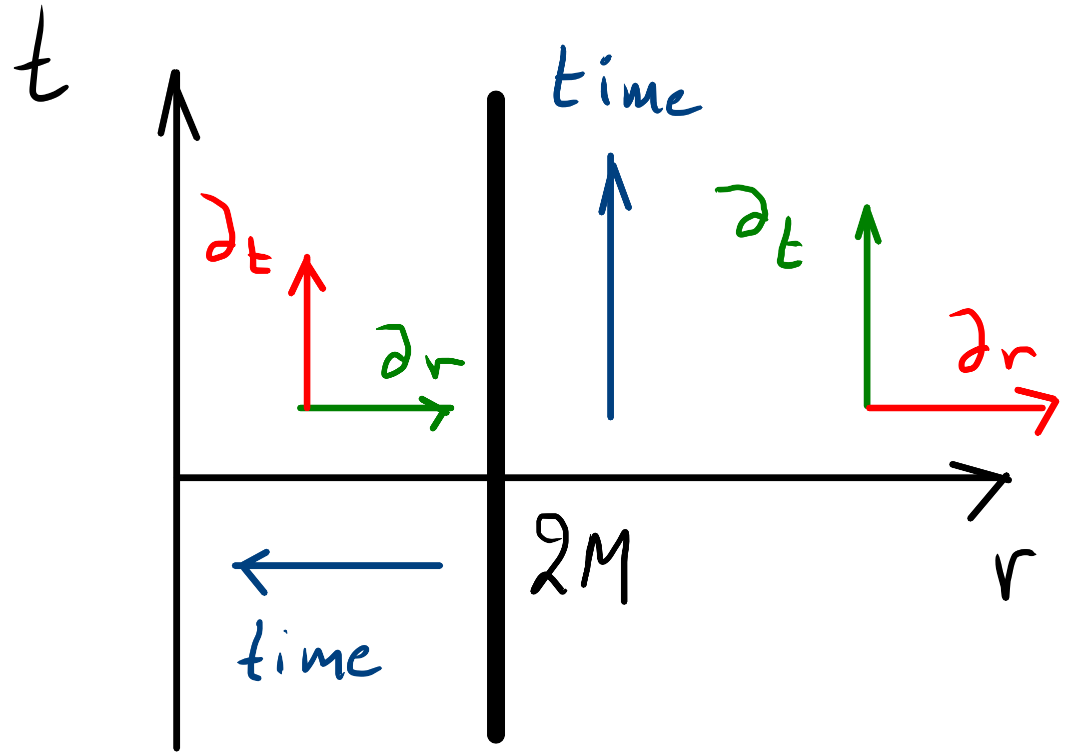

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– ∂_r	spacelike	timelike

  - causal geodesics starting at $r_0 > 2M$ need infinite t to approach $r = 2M$ – seem to never cross it

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$$\Rightarrow t = \pm 2M\left[\frac{r}{2M} + \ln\left(\frac{r}{2M}-1\right)\right] + t_0$$

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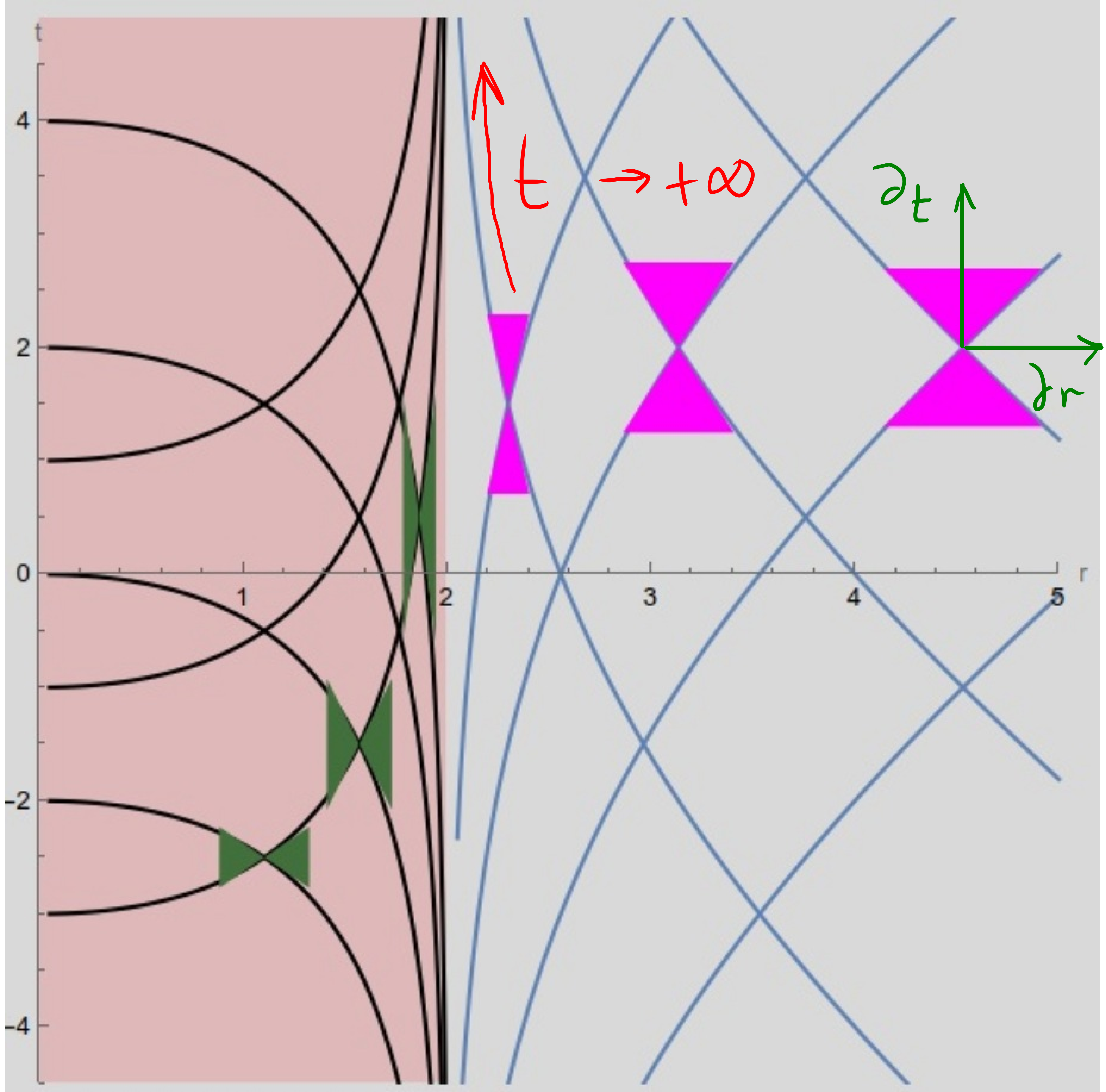

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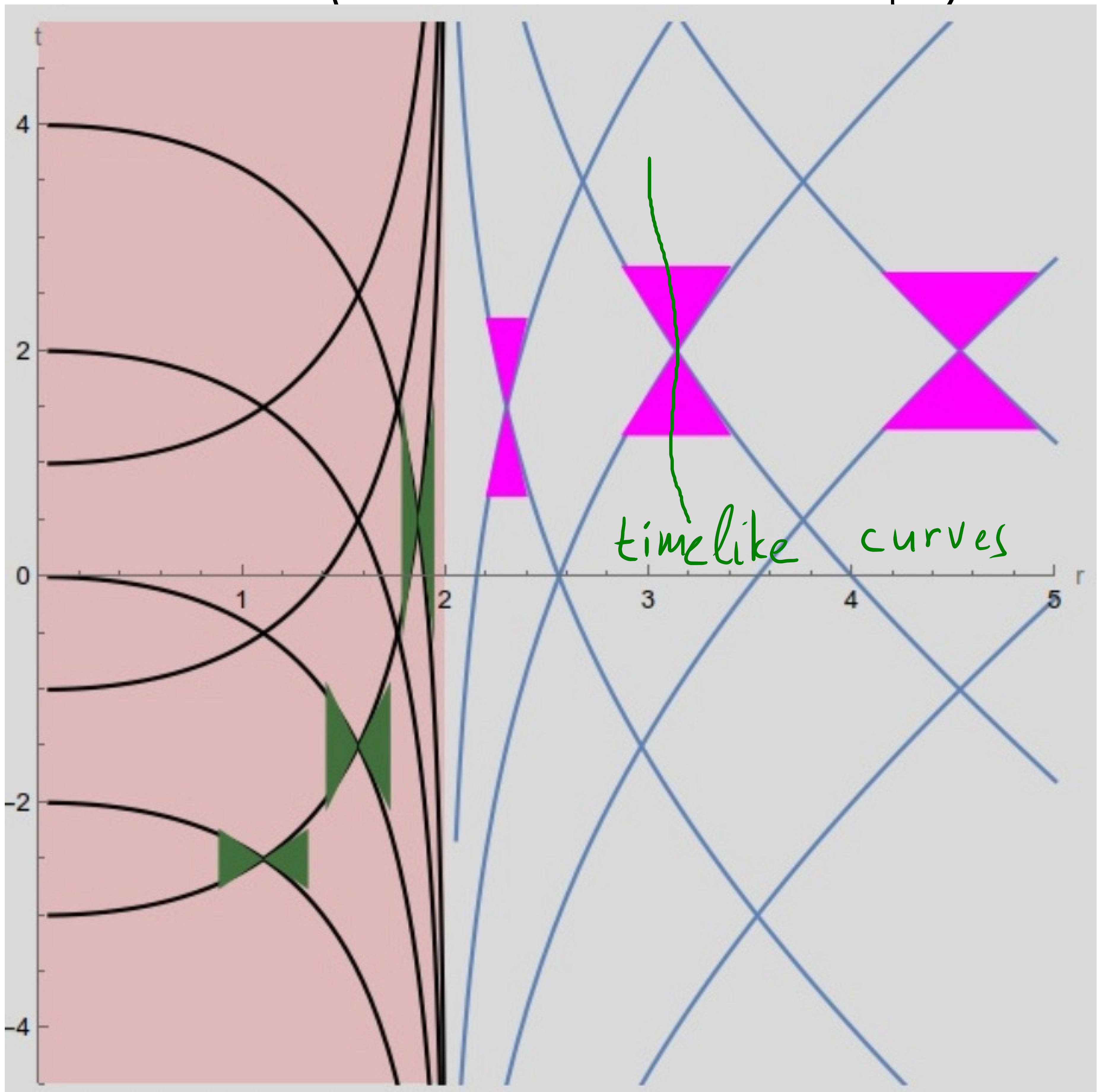

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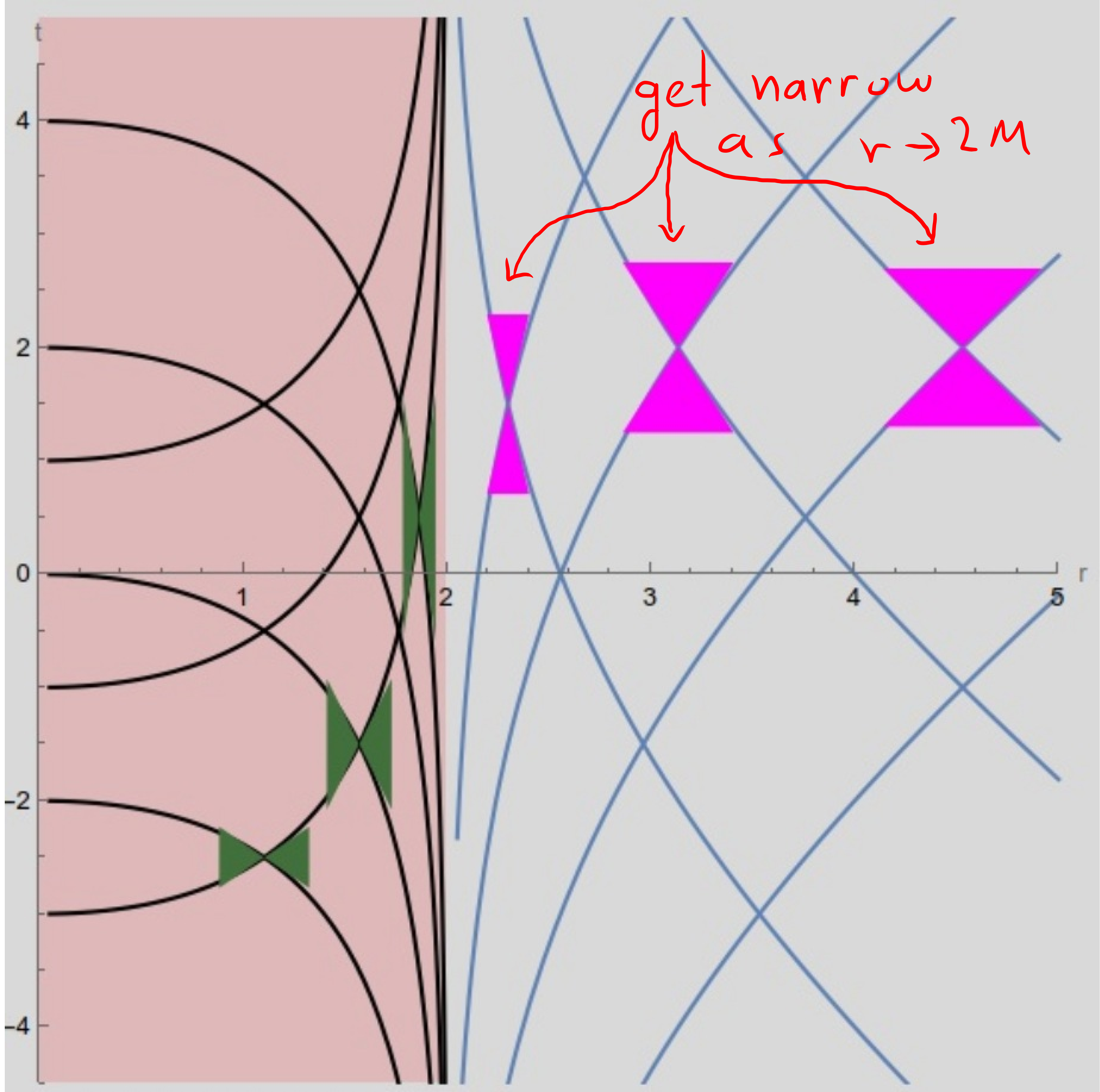

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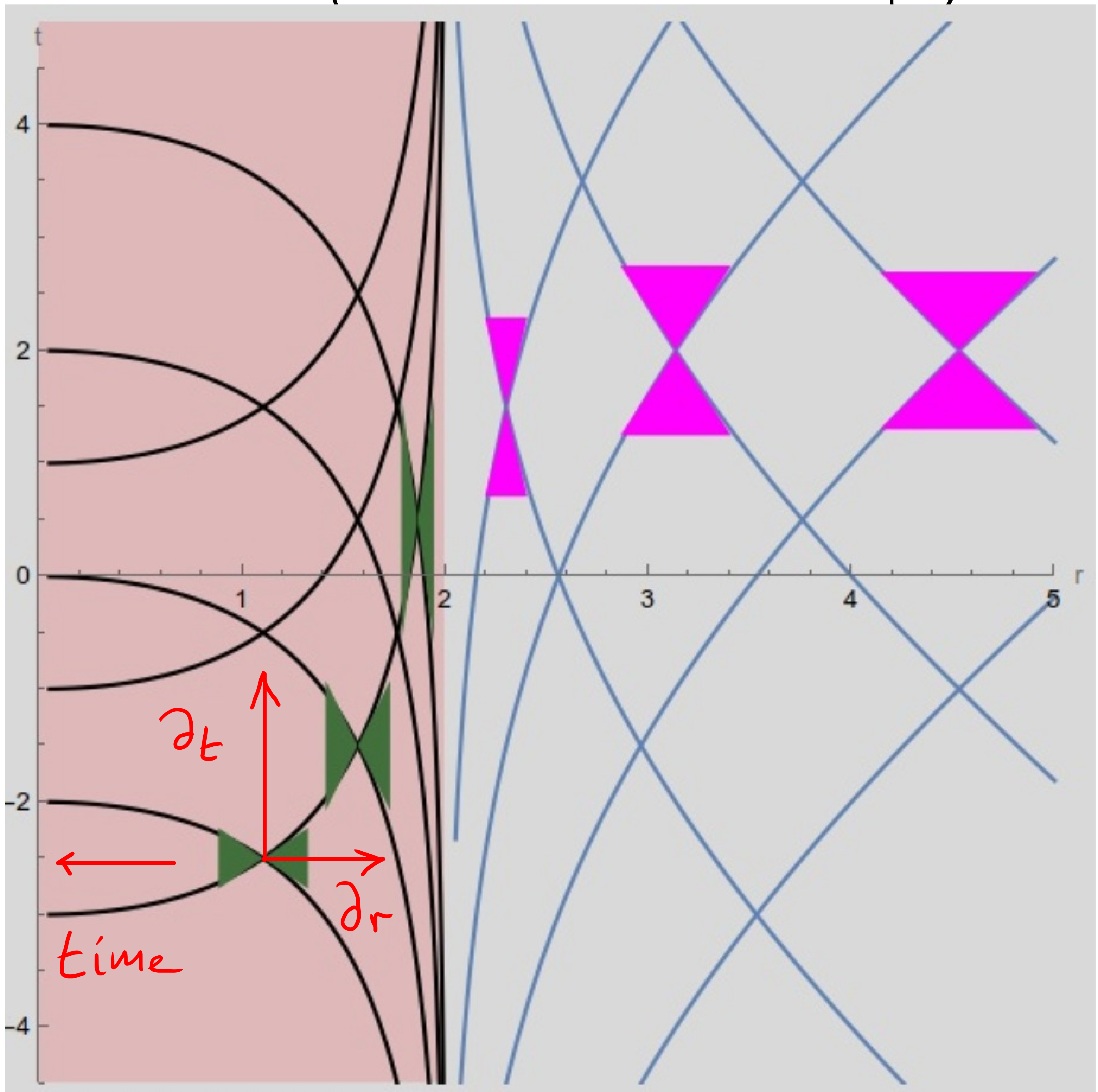

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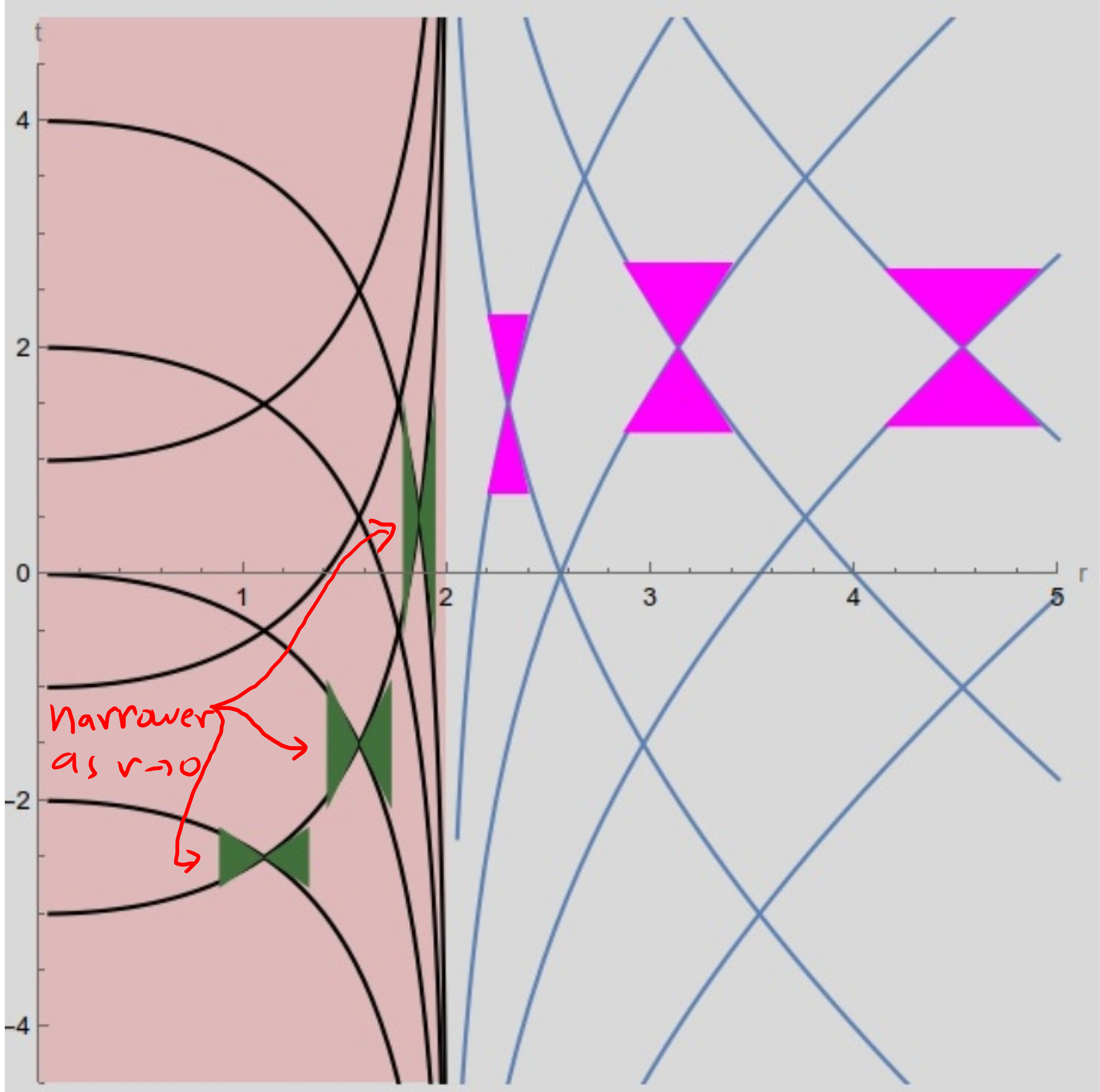

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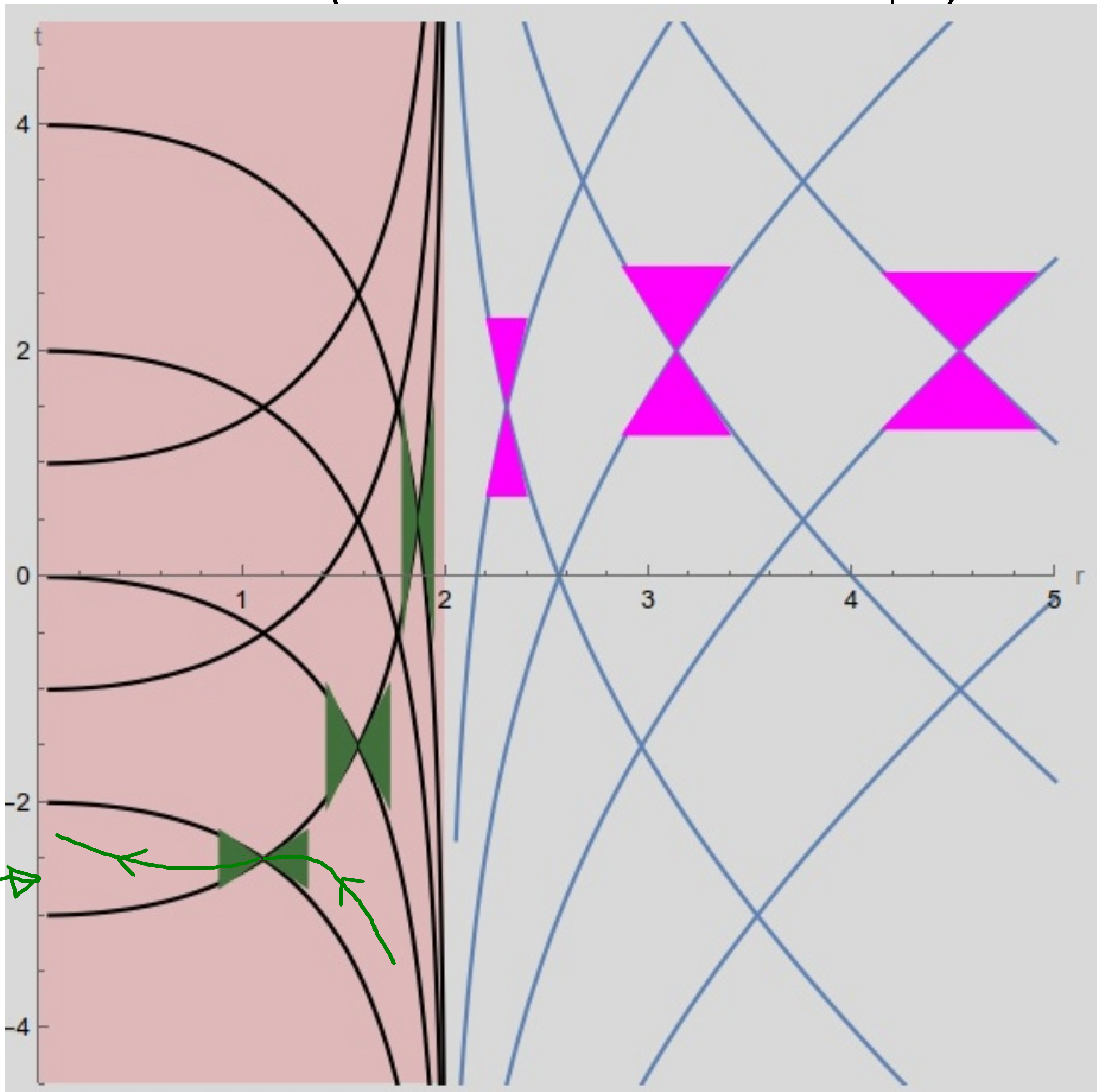

timelike curves must fall on $r = 0$

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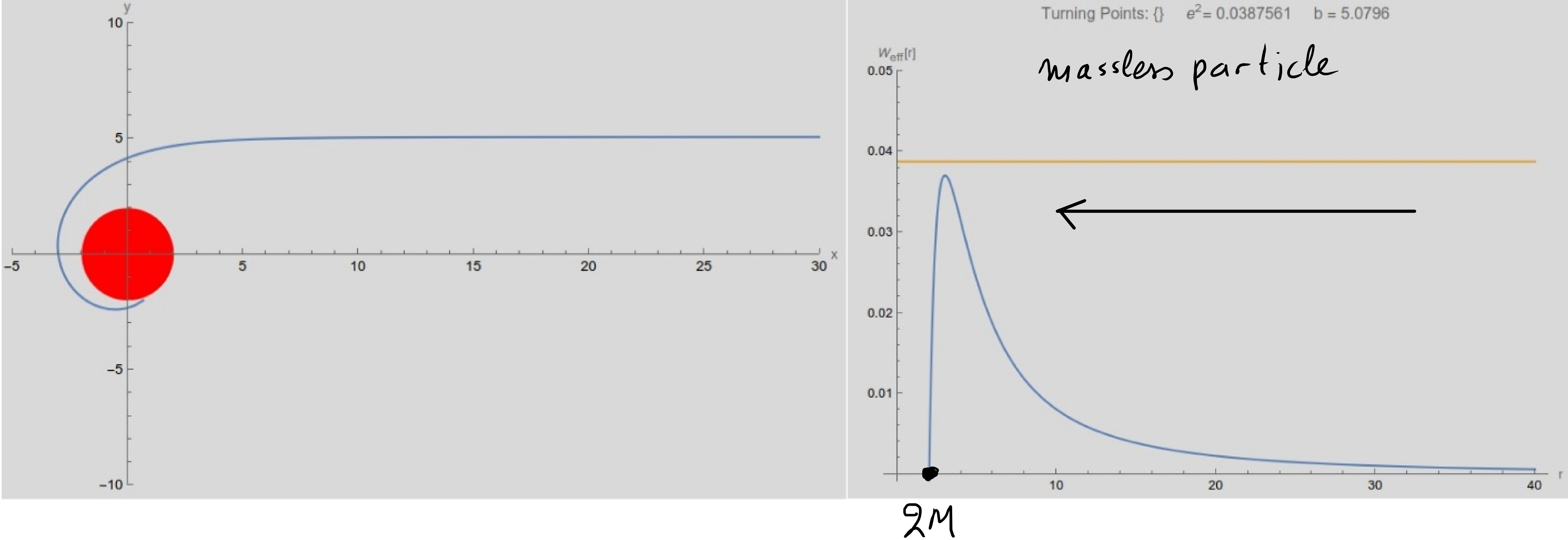
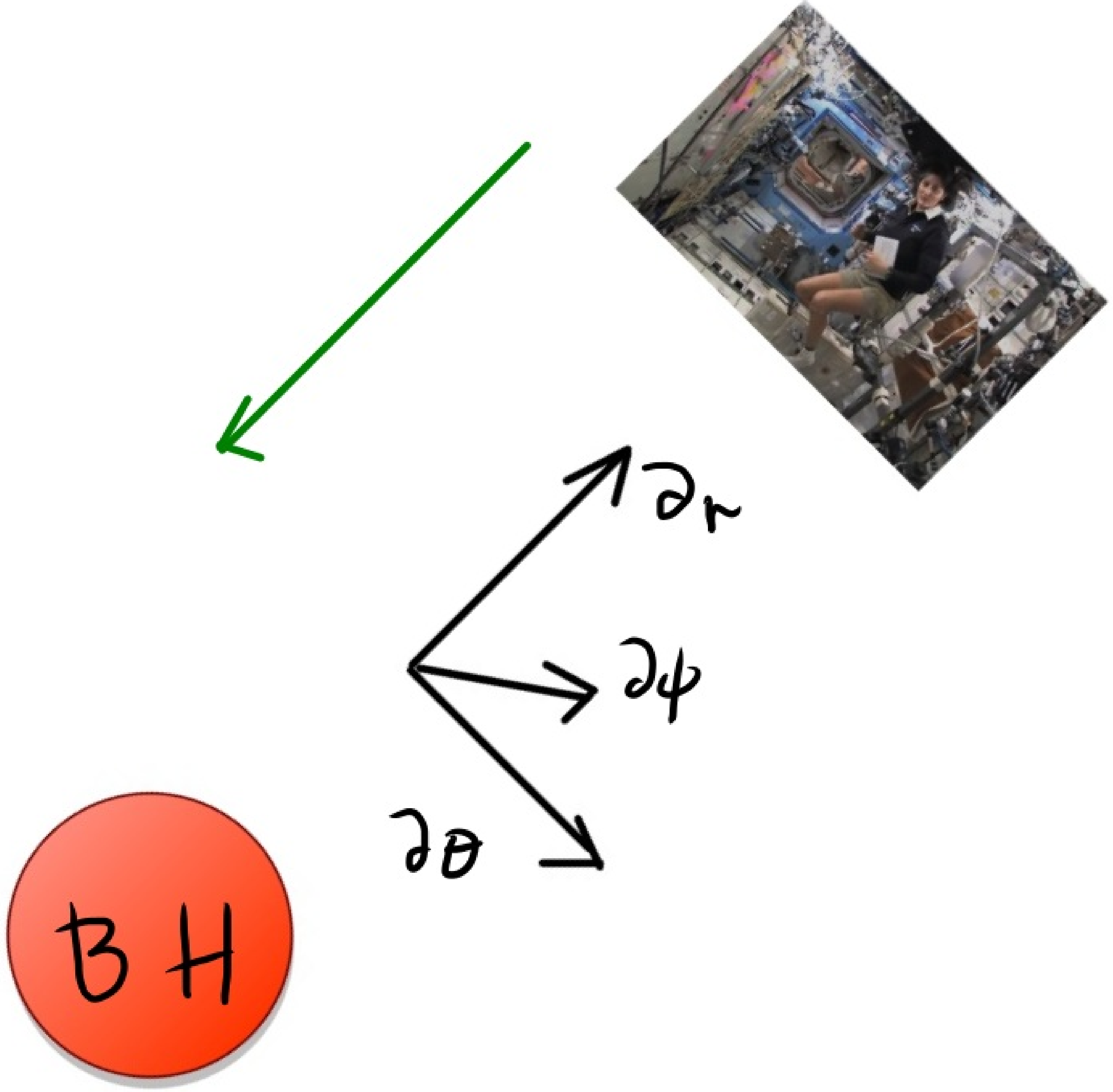


- we have seen particles (massless + massive) to get into the $r < 2M$ region in finite affine parameter

$\Rightarrow$ must be a problem of the coordinate's choice!

• How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?



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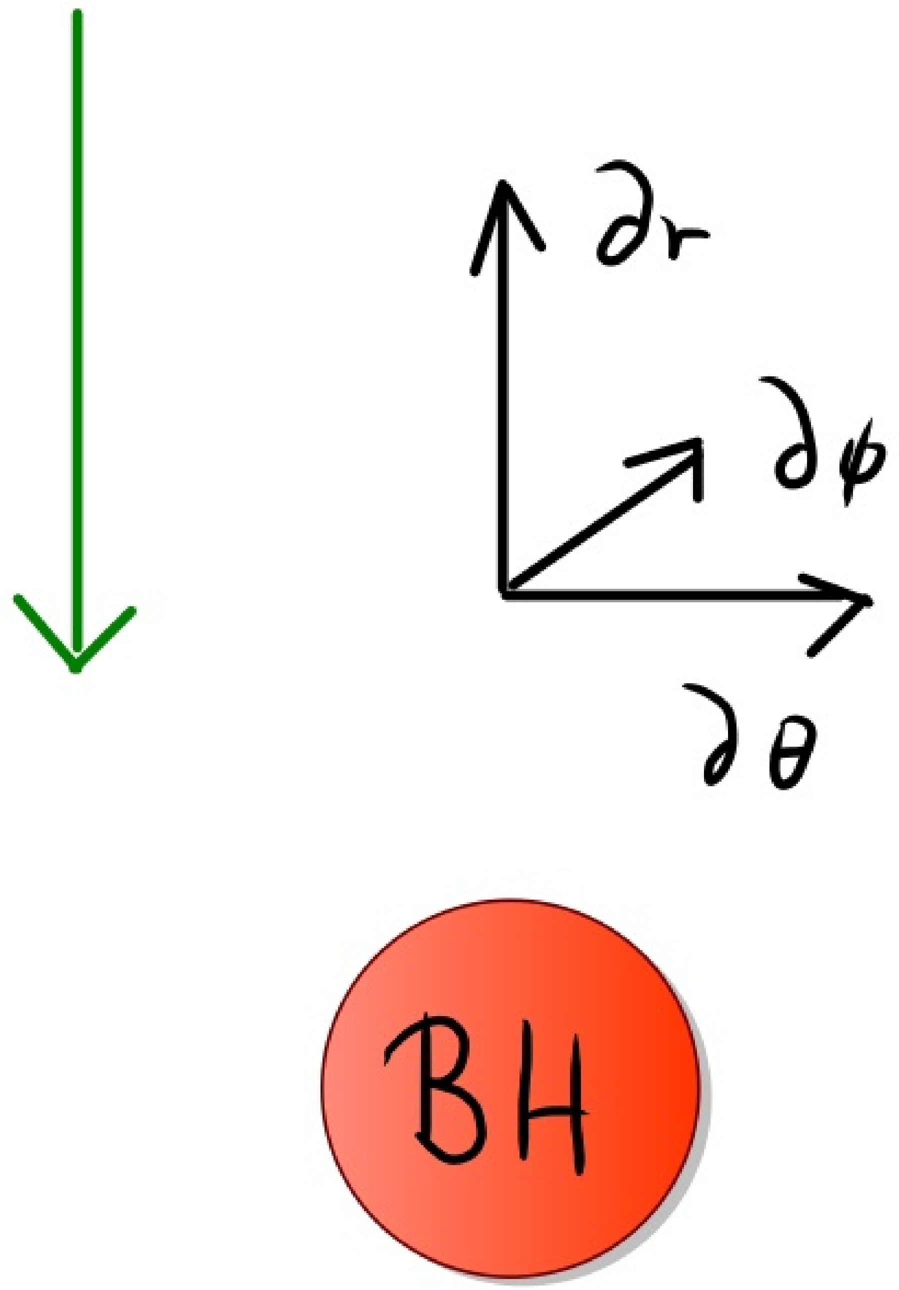

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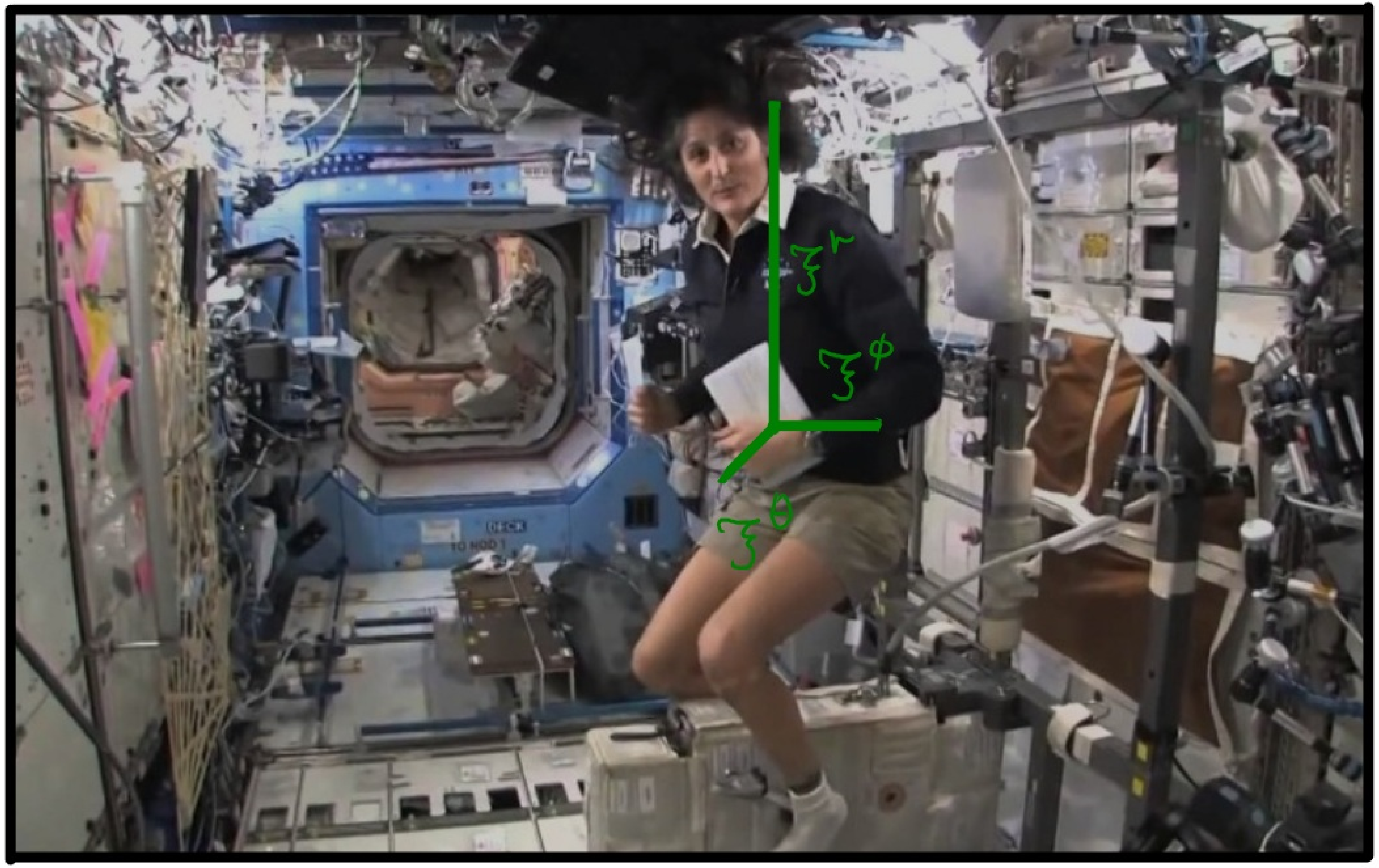

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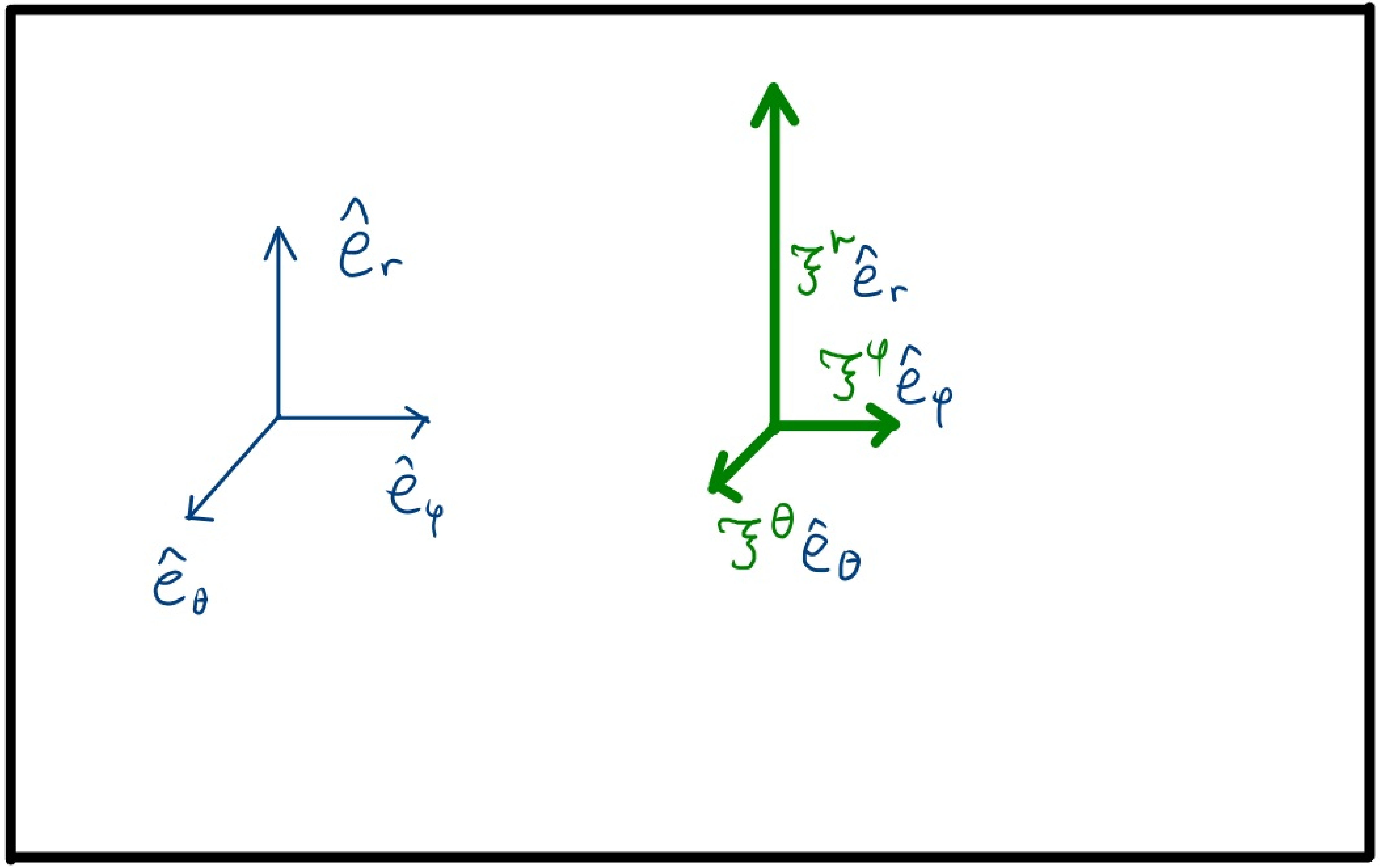

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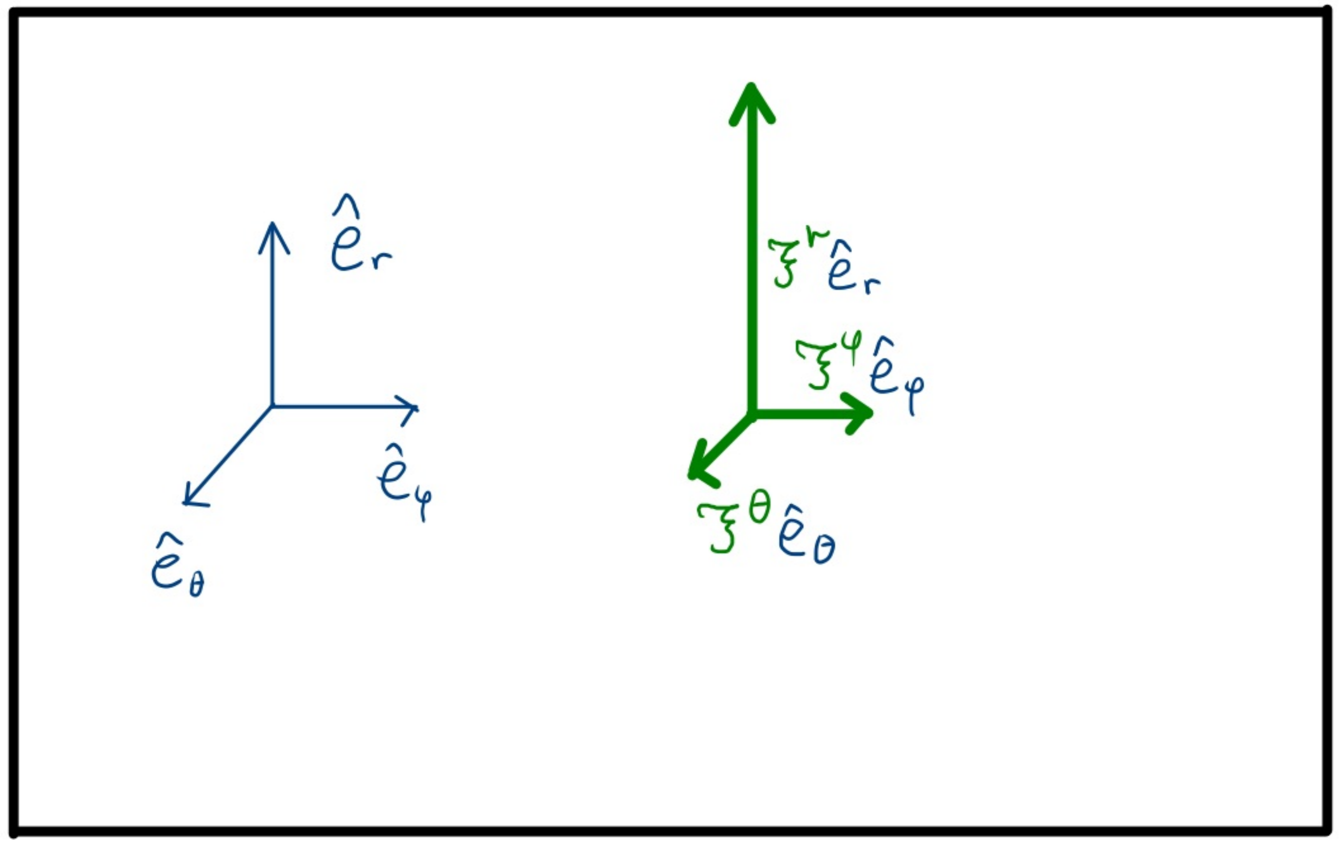

- Get into the ship:

  - orthonormal basis

$$\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$$

$$\hat{e}_t \cdot \hat{e}_t = \eta_{00} = -1$$

$$\hat{e}_r \cdot \hat{e}_r = \hat{e}_\theta \cdot \hat{e}_\theta = \hat{e}_\phi \cdot \hat{e}_\phi = \eta_{ii} = +1$$

$$\hat{e}_i \cdot \hat{e}_j = \hat{e}_t \cdot \hat{e}_o = 0$$

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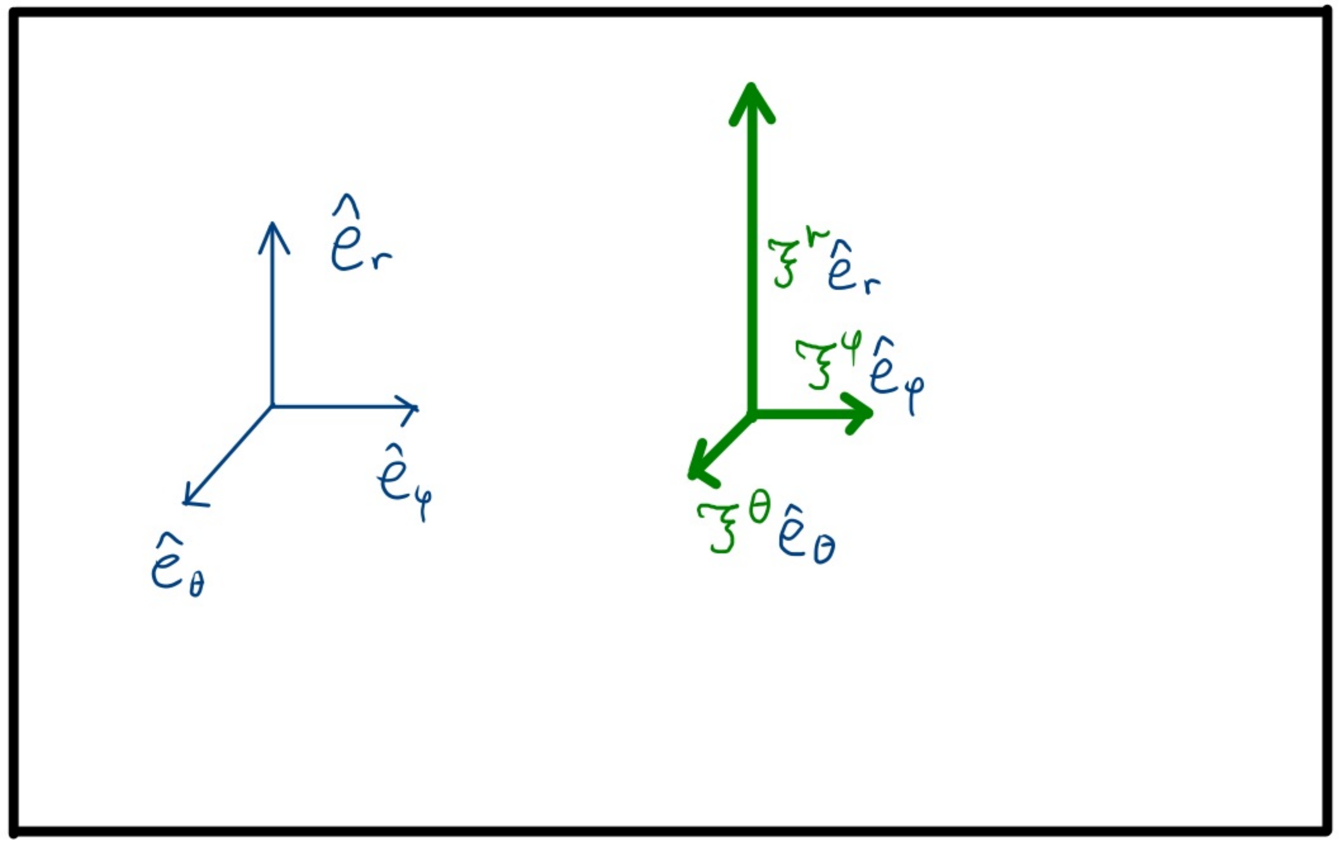

- Get into the ship:
  - orthonormal basis $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity

$$u^\mu = [u^t, 0, 0, 0]$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = g_{00} u^t u^t = -\left(1 - \frac{2M}{r}\right)(u^t)^2$$

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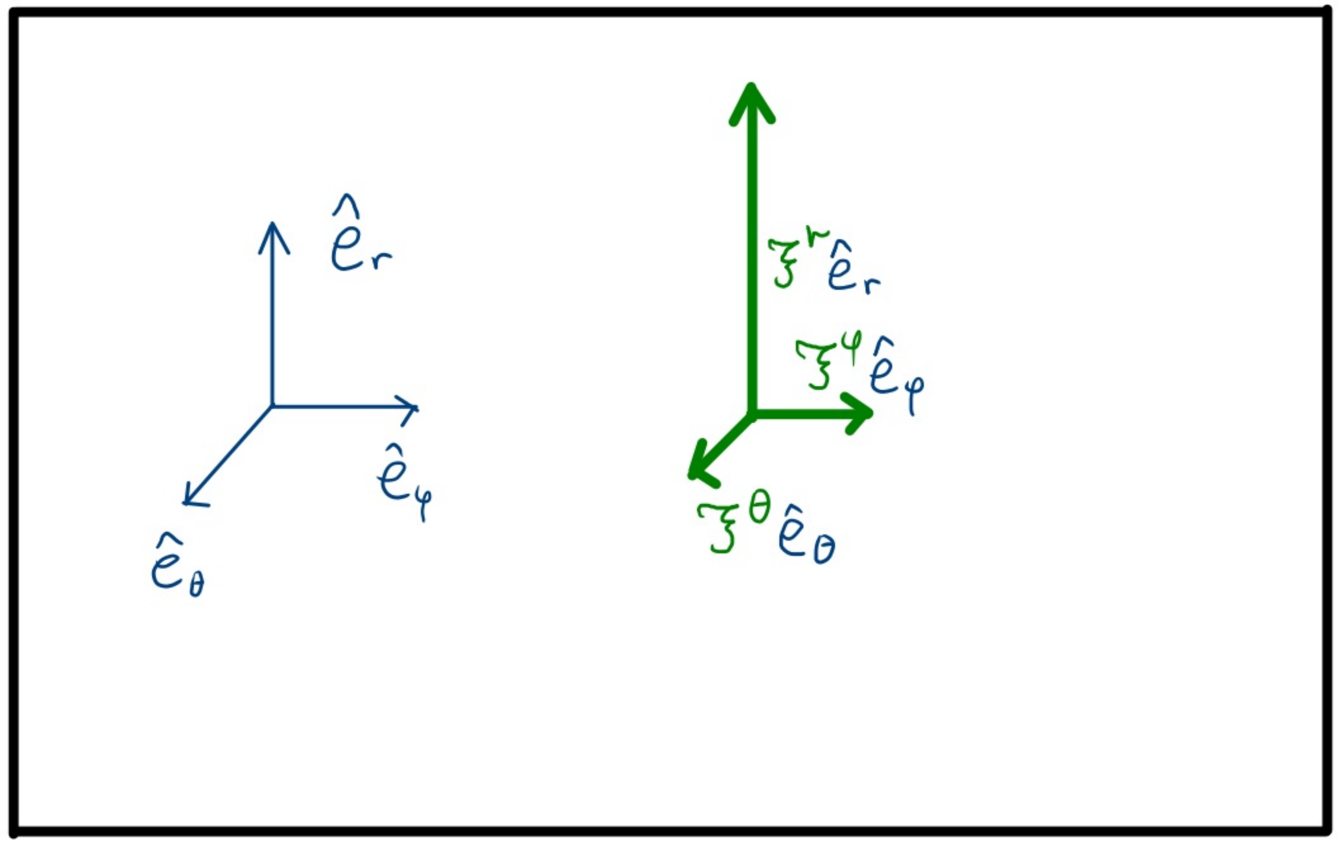

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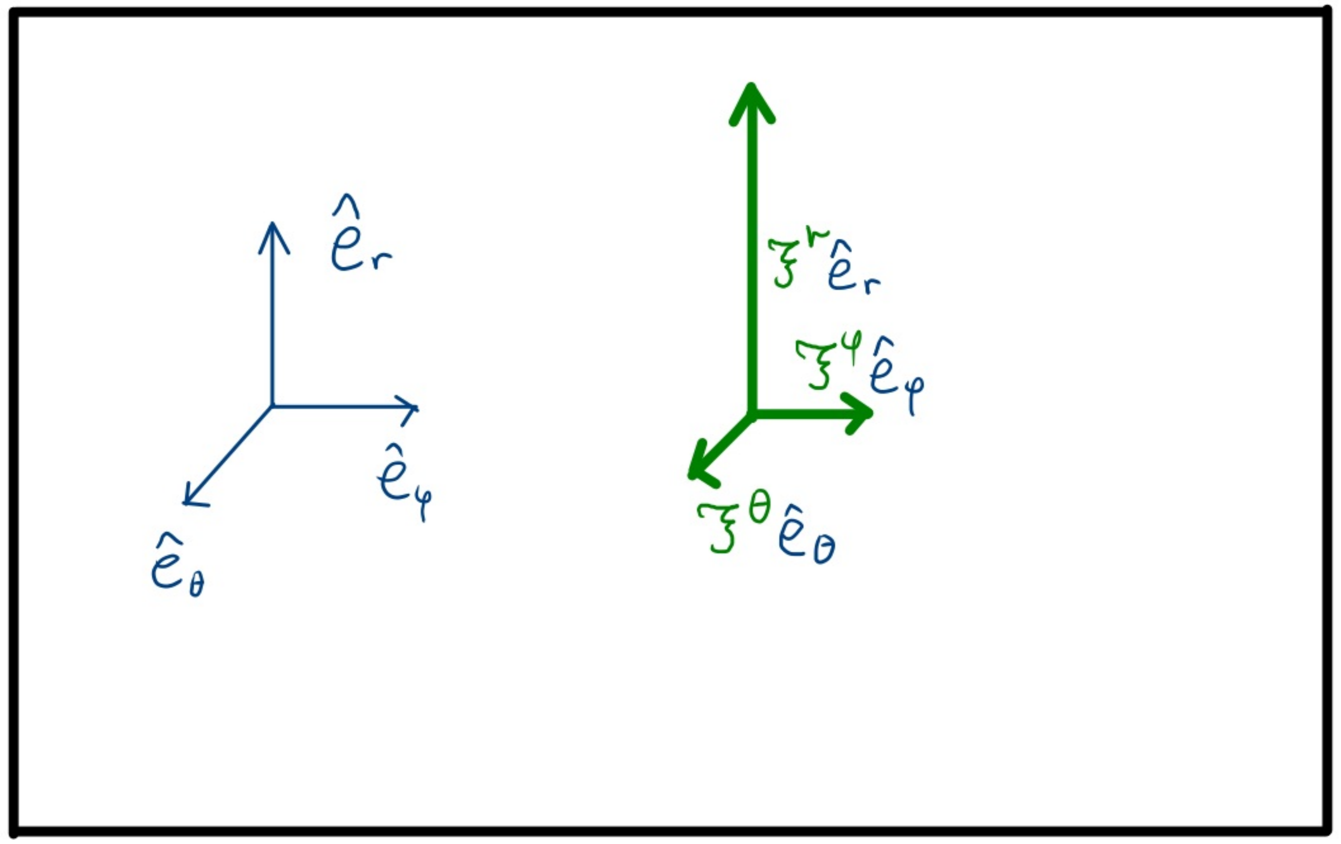

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  - 4-velocity

$$u^\mu = [u^t, 0, 0, 0] = \left[\left(1-\frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right] = \left(1-\frac{2M}{r}\right)^{-1/2} \partial_r$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = g_{00} u^t u^t = -\left(1-\frac{2M}{r}\right)(u^t)^2$$

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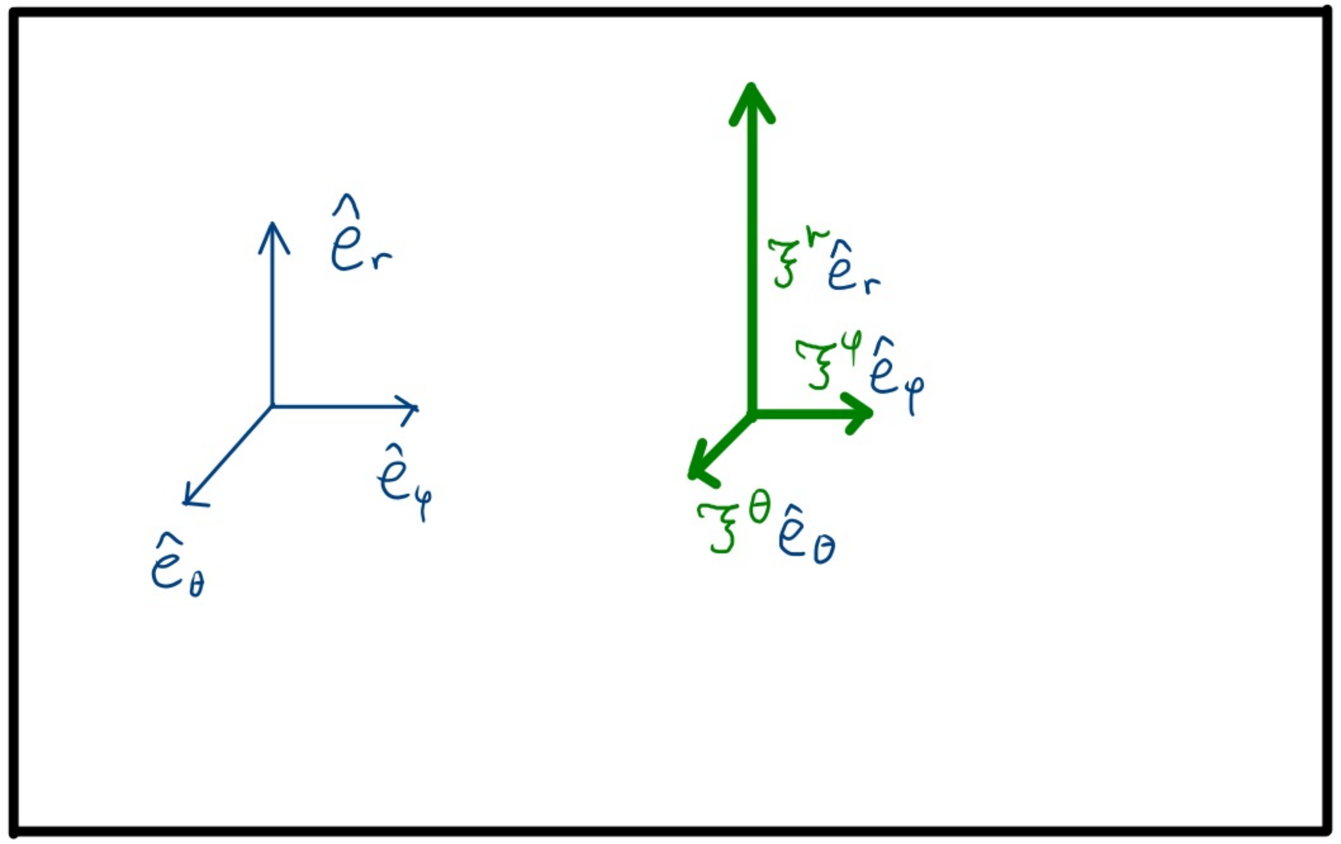

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$$\hat{e}_t = u = \left(1-\frac{2M}{r}\right)^{-1/2} \partial_r = \sqrt{|g_{tt}|}\, \partial_r \qquad r > 2M$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\, dt^2 + g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}\, d\phi^2$$

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- coordinate basis

$$e_t = \partial_t \quad \Rightarrow \quad e_t{}^{\mu} = [1, 0, 0, 0]$$

$$e_r = \partial_r \quad \Rightarrow \quad e_r{}^{\mu} = [0, 1, 0, 0]$$

$$e_\theta = \partial_\theta \quad \Rightarrow \quad e_\theta{}^{\mu} = [0, 0, 1, 0]$$

$$e_\phi = \partial_\phi \quad \Rightarrow \quad e_\phi{}^{\mu} = [0, 0, 0, 1]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu$$

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- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu \Rightarrow \hat{e}_\mu \cdot \hat{e}_\mu = \alpha^2\, e_\mu \cdot e_\mu$$

no summation if both
indices are downstairs
or upstairs

– Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu \Rightarrow \eta_{\mu\mu} = \alpha^2\, e_\mu \cdot e_\mu = \alpha^2\, g_{\mu\mu}$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu \Rightarrow \eta_{\mu\mu} = \alpha^2\, e_\mu \cdot e_\mu = \alpha^2\, g_{\mu\mu} \Rightarrow \alpha = |g_{\mu\mu}|^{-1/2}$$

same sign

– Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

no-summation!

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = [\,|g_{tt}|^{-1/2}, 0, 0, 0\,] = [\,|1 - \tfrac{2M}{r}|^{-1/2}, 0, 0, 0\,]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = [\,|g_{tt}|^{-1/2}, 0, 0, 0\,] = [\,|1 - \tfrac{2M}{r}|^{-1/2}, 0, 0, 0\,]$$

$$\hat{e}_r = [\,0, |g_{rr}|^{-1/2}, 0, 0\,] = [\,0, |1 - \tfrac{2M}{r}|^{1/2}, 0, 0\,]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis:

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = [\,|g_{tt}|^{-1/2}, 0, 0, 0\,] = [\,|1-\tfrac{2M}{r}|^{-1/2}, 0, 0, 0\,]$$

$$\hat{e}_r = [\,0, |g_{rr}|^{-1/2}, 0, 0\,] = [\,0, |1-\tfrac{2M}{r}|^{1/2}, 0, 0\,]$$

$$\hat{e}_\theta = [\,0, 0, |g_{\theta\theta}|^{-1/2}, 0\,] = [\,0, 0, r^{-1}, 0\,]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis:

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = [\,|g_{tt}|^{-1/2}, 0, 0, 0\,] = [\,|1-\tfrac{2M}{r}|^{-1/2}, 0, 0, 0\,]$$

$$\hat{e}_r = [\,0, |g_{rr}|^{-1/2}, 0, 0\,] = [\,0, |1-\tfrac{2M}{r}|^{1/2}, 0, 0\,]$$

$$\hat{e}_\theta = [\,0, 0, |g_{\theta\theta}|^{-1/2}, 0\,] = [\,0, 0, r^{-1}, 0\,]$$

$$\hat{e}_\phi = [\,0, 0, 0, |g_{\phi\phi}|^{-1/2}\,] = [\,0, 0, 0, r^{-1}\sin^{-1}\theta\,]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

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- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

vector: $v = v^\mu e_\mu = v^{\hat{\mu}}\, \hat{e}_\mu$

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$$|g_{\mu\mu}|^{1/2} v^\mu\, \hat{e}_\mu = v^{\hat{\mu}}\, \hat{e}_\mu$$

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vector: $v = v^\mu e_\mu = v^{\hat{\mu}}\,\hat{e}_\mu \Rightarrow$

$$|g_{\mu\mu}|^{1/2} v^\mu\,\hat{e}_\mu = v^{\hat{\mu}}\,\hat{e}_\mu \Rightarrow v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

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- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

$$v^{\hat{\mu}} = \underbrace{|g_{\mu\mu}|^{1/2}\, v^\mu}_{\text{no summation!}}$$

$$v^{\hat{t}} = |g_{tt}|^{1/2}\, v^t$$

$$v^{\hat{r}} = |g_{rr}|^{1/2}\, v^r$$

etc ...

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2}\, v^\mu$$

dual basis: $\hat{e}^\mu = |g_{\mu\mu}|^{1/2} e^\mu = |g_{\mu\mu}|\, dx^\mu$

indeed $\hat{e}^\mu(\hat{e}_\nu) = \hat{e}^\mu(|g_{\nu\nu}|^{-1/2} e_\nu) = |g_{\mu\mu}|^{1/2} |g_{\nu\nu}|^{-1/2} e^\mu(e_\nu)$

$$= |g_{\mu\mu}|^{1/2} |g_{\nu\nu}|^{-1/2} \delta^\mu{}_\nu = |g_{\mu\mu}|^{1/2} |g_{\nu\nu}|^{-1/2} \delta^\mu{}_\nu = \delta^\mu{}_\nu$$

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- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$ $\quad \hat{e}^\mu = |g_{\mu\mu}|^{1/2} e^\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu$$

one forms: $\omega = \omega_\mu e^\mu = \omega_{\hat{\mu}} \hat{e}^\mu$

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$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2}\, v^\mu$$

one forms: $\omega = \omega_\mu e^\mu = \omega_{\hat{\mu}} \hat{e}^\mu =$

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$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu$$

one forms: $\omega = \omega_\mu e^\mu = \omega_{\hat{\mu}} \hat{e}^\mu =$

$$|g_{\mu\mu}|^{-1/2} \omega_\mu \hat{e}^\mu = \omega_{\hat{\mu}} \hat{e}^\mu \quad \Rightarrow \quad \omega_{\hat{\mu}} = |g_{\mu\mu}|^{-1/2} \omega_\mu$$

_Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\, dt^2 + g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}\, d\phi^2$$

- coordinate basis $\{ e_t, e_r, e_\theta, e_\phi \}$
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$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu \qquad \omega_{\hat{\mu}} = |g_{\mu\mu}|^{-1/2} \omega_\mu$$

Tensors: $R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}|^{1/2} |g_{\nu\nu}|^{-1/2} |g_{\rho\rho}|^{-1/2} |g_{\lambda\lambda}|^{-1/2} R^\mu{}_{\nu\rho\lambda}$

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$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

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Tensors: $R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}|^{1/2}\, |g_{\nu\nu}|^{-1/2}\, |g_{\rho\rho}|^{-1/2}\, |g_{\lambda\lambda}|^{-1/2}\, R^\mu{}_{\nu\rho\lambda}$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = \eta_{\hat{\mu}\hat{\sigma}}\, R^{\hat{\sigma}}{}_{\hat{\nu}\hat{\rho}\hat{\lambda}}$$

$$R_{0101} = R_{trtr} = -\frac{2M}{r^3}$$

$$R_{0202} = R_{t\theta t\theta} = \frac{M}{r}\left(1-\frac{2M}{r}\right)$$

$$R_{1212} = R_{r\theta r\theta} = -\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1}$$

$$R_{0303} = R_{t\phi t\phi} = \frac{M}{r}\left(1-\frac{2M}{r}\right)\sin^2\theta$$

$$R_{1313} = R_{r\phi r\phi} = -\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1}\sin^2\theta$$

$$R_{2323} = R_{\theta\phi\theta\phi} = 2M\, r \sin^2\theta$$

$$R_{0101} = R_{trtr} = -\frac{2M}{r^3}$$

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$$R_{2323} = R_{\theta\phi\theta\phi} = 2M\, r \sin^2\theta$$

Notice singular behavior as $r \to 2M$!

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = | g_{\mu\mu} \, g_{\nu\nu} \, g_{\rho\rho} \, g_{\lambda\lambda} |^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2}\, R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = |g_{tt}|^{-1}\, |g_{rr}|^{-1}\, R_{rtrt} = \left|1 - \frac{2M}{r}\right|^{-1} \left|1 - \frac{2M}{r}\right| \left(-\frac{2M}{r^3}\right)$$

$$= -\frac{2M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = \left| g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda} \right|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = |g_{tt}|^{-1} |g_{\theta\theta}|^{-1} R_{\theta t \theta t} = \left|1 - \frac{2M}{r}\right|^{-1} (r^2)^{-1} \frac{M(r-2M)}{r^2}$$

$$= \left(1 - \frac{2M}{r}\right)^{-1} \frac{1}{r^2} \frac{Mr}{r^2} \left(1 - \frac{2M}{r}\right)$$

$\swarrow$

Notice $\left|1 - \frac{2M}{r}\right|^{-1} = \left(1 - \frac{2M}{r}\right)^{-1}$

for $r > 2M$

$$R_{\hat\mu\hat\nu\hat\rho\hat\lambda} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat r\hat t\hat r\hat t} = -\frac{2M}{r^3}$$

$$R_{\hat\theta\hat t\hat\theta\hat t} = |g_{tt}|^{-1} |g_{\theta\theta}|^{-1} R_{\theta t\theta t} = \left|1-\frac{2M}{r}\right|^{-1} (r^2)^{-1} \frac{M(r-2M)}{r^2}$$

$$= \left(1-\frac{2M}{r}\right)^{-1} \frac{1}{r^2} \frac{Mr}{r^2}\left(1-\frac{2M}{r}\right)$$

$$= +\frac{M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = |g_{\theta\theta}|^{-1} |g_{\phi\phi}|^{-1} R_{\theta\phi\theta\phi} = r^{-2}\, r^{-2} \sin^{-2}\theta \;\, 2Mr \sin^2\theta$$

$$= +\frac{2M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = | g_{\mu\mu} \, g_{\nu\nu} \, g_{\rho\rho} \, g_{\lambda\lambda} |^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = +\frac{2M}{r^3}$$

$$R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = | g_{tt} |^{-1} | g_{\phi\phi} |^{-1} R_{\phi t \phi t} = \left(1 - \frac{2M}{r}\right)^{-1} r^{-2} \sin^{-2}\theta \, \frac{M(r-2M)}{r^2} \sin^2\theta$$

$$= +\frac{M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}} = +\frac{2M}{r^3}$$

$$R_{\hat{\varphi}\hat{t}\hat{\varphi}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = |g_{rr}|^{-1} |g_{\theta\theta}|^{-1} R_{r\theta r\theta} = \left(1 - \frac{2M}{r}\right) r^{-2} \frac{M}{2M-r} = -\frac{M}{r^3}$$

$$R_{\hat\mu\hat\nu\hat\rho\hat\lambda} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat r \hat t \hat r \hat t} = -\frac{2M}{r^3}$$

$$R_{\hat\theta \hat t \hat\theta \hat t} = +\frac{M}{r^3}$$

$$R_{\hat\theta\hat\phi\hat\theta\hat\phi} = +\frac{2M}{r^3}$$

$$R_{\hat\phi \hat t \hat\phi \hat t} = +\frac{M}{r^3}$$

$$R_{\hat r\hat\theta\hat r\hat\theta} = -\frac{M}{r^3}$$

$$R_{\hat r\hat\phi\hat r\hat\phi} = |g_{rr}|^{-1}|g_{\phi\phi}|^{-1} R_{r\phi r\phi} = \left(1-\frac{2M}{r}\right) r^{-2}\sin^{-2}\theta\, \frac{M\sin^2\theta}{2M-r} = -\frac{M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = +\frac{2M}{r^3}$$

$$R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -\frac{M}{r^3}$$

$$R_{\hat\mu\hat\nu\hat\rho\hat\lambda} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat r\hat t\hat r\hat t} = -\frac{2M}{r^3}$$

$$R_{\hat\theta\hat t\hat\theta\hat t} = +\frac{M}{r^3}$$

$$R_{\hat\theta\hat\phi\hat\theta\hat\phi} = +\frac{2M}{r^3}$$

$$R_{\hat\phi\hat t\hat\phi\hat t} = +\frac{M}{r^3}$$

$$R_{\hat r\hat\theta\hat r\hat\theta} = -\frac{M}{r^3}$$

$$R_{\hat r\hat\phi\hat r\hat\phi} = -\frac{M}{r^3}$$

For $r < 2M$ $\quad e_t = \partial_t$ spacelike

$e_r = \partial_r$ timelike

so

$$\hat e_t = |g_{rr}|^{-1/2} e_r$$

$$\hat e_r = |g_{tt}|^{-1/2} e_t \quad ,$$

and, e.g.

$$R_{\hat\theta\hat t\hat\theta\hat t} = |g_{\theta\theta}|^{-1} |g_{rr}|^{-1} R_{\theta r\theta r} \; !$$

<u>Exercise:</u> compute $R_{\hat\mu\hat\nu\hat\rho\hat\lambda}$, show they are the same!

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda}$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = +\frac{2M}{r^3}$$

$$R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -\frac{M}{r^3}$$

There is no singular behavior of $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}$ at $r = 2M$!

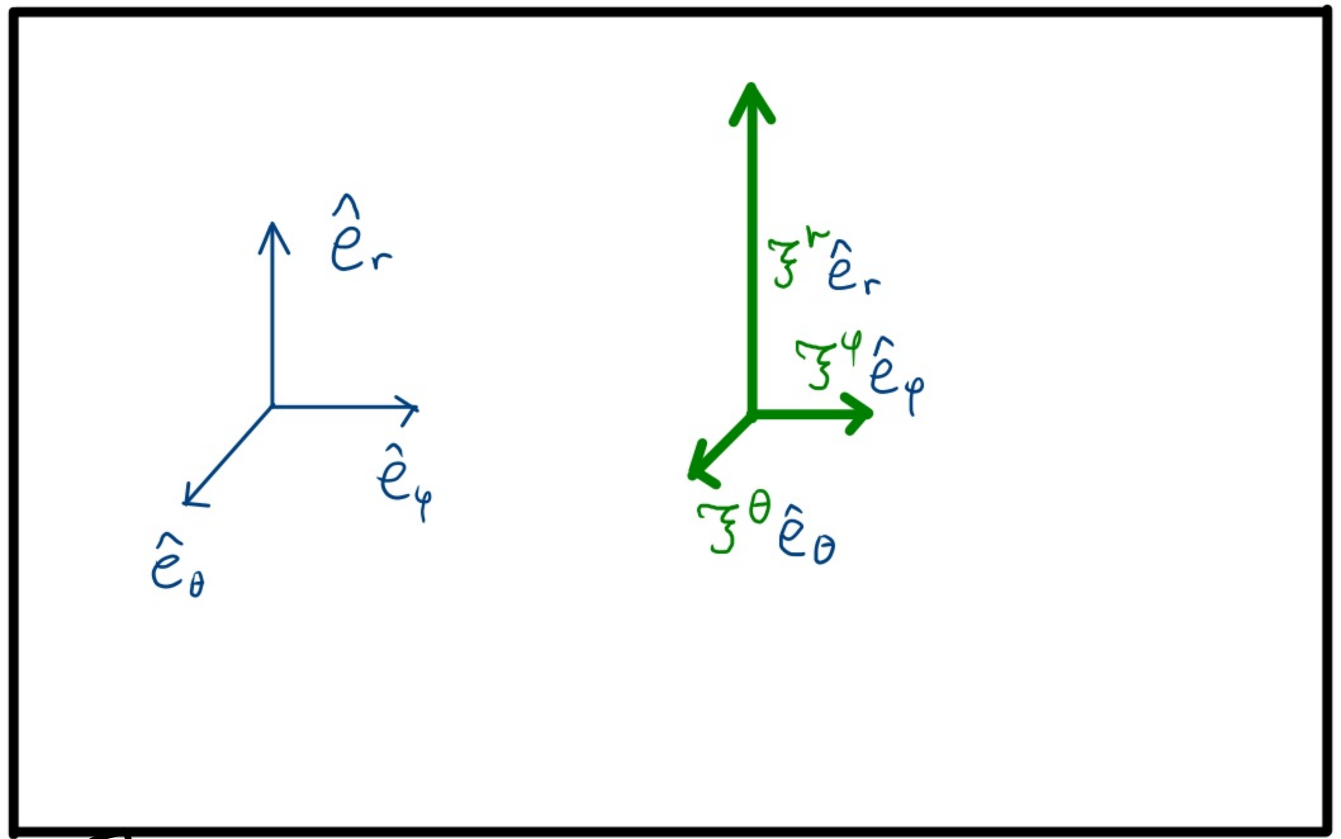
- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?

- Back to the ship:

  - orthonormal basis $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$

  - 4-velocity $u^{\hat{\mu}} = \hat{e}_t{}^{\hat{\mu}} = [1,0,0,0]$

  - Relative accelerations given by geodesic deviation equation:



$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\rho}} \xi^{\hat{\sigma}}$$

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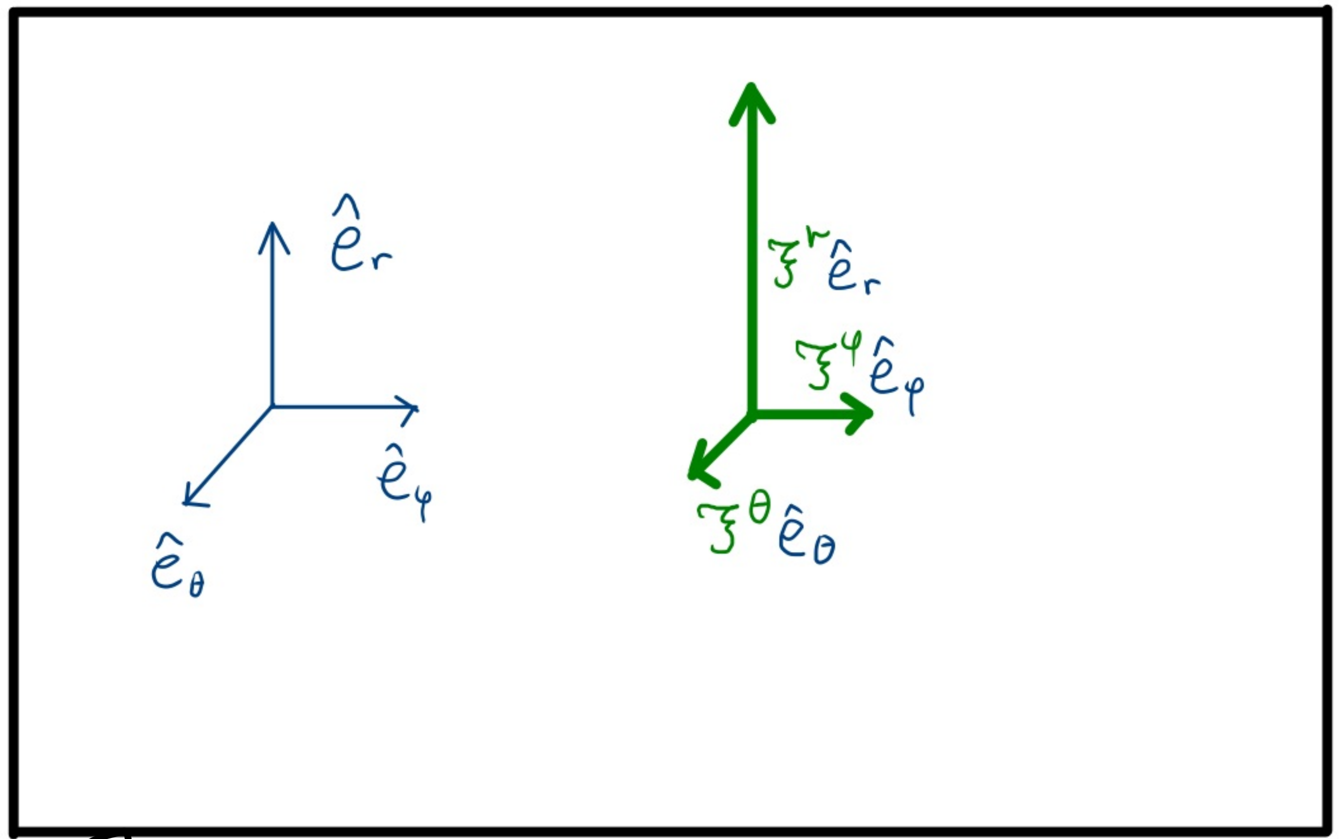
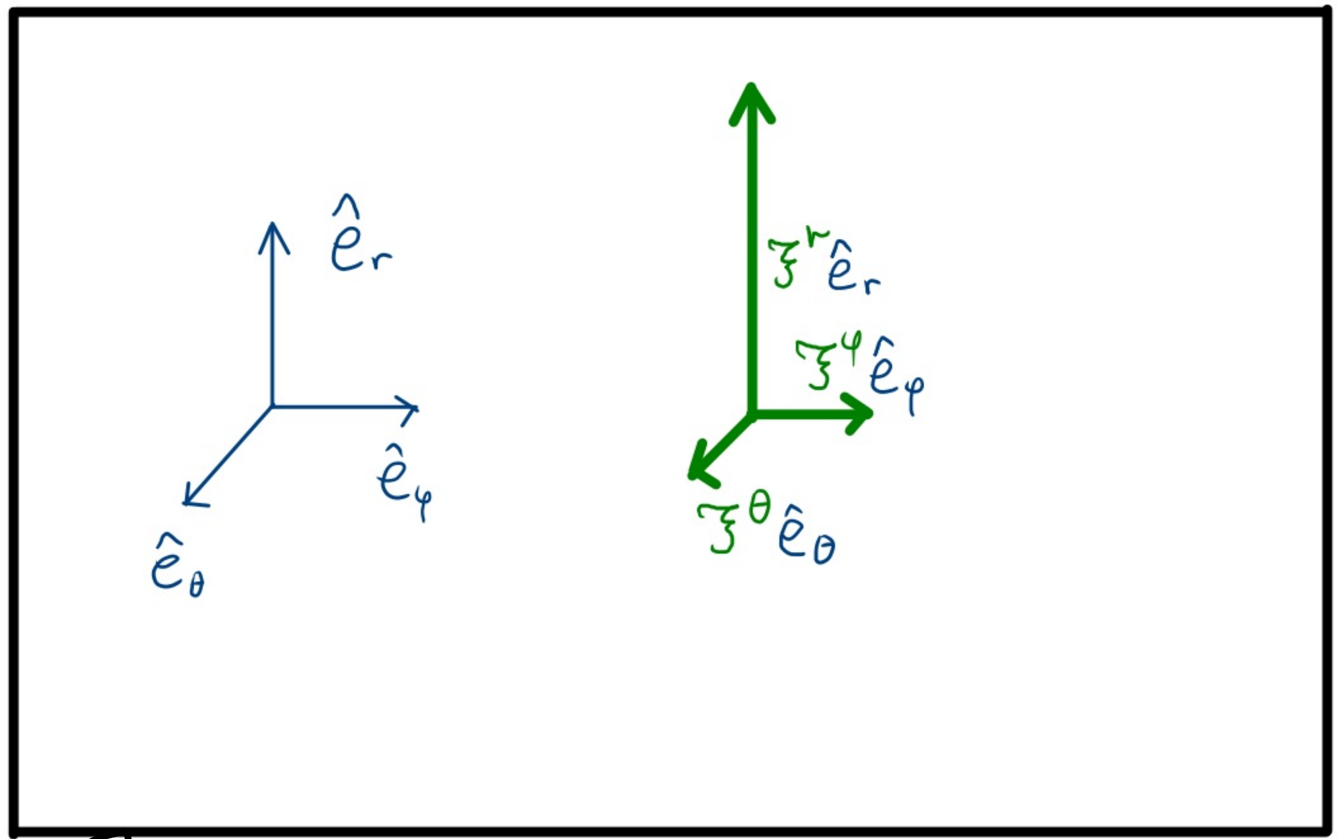


  - orthonormal basis $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity $\quad u^{\hat{\mu}} = \hat{e}_t{}^{\hat{\mu}} = [1,0,0,0]$
  - Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\sigma}}\, u^{\hat{\nu}}\, u^{\hat{\rho}}\, \xi^{\hat{\sigma}}$$

only $\hat{\nu} = \hat{\rho} = \hat{t}$ survive

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?



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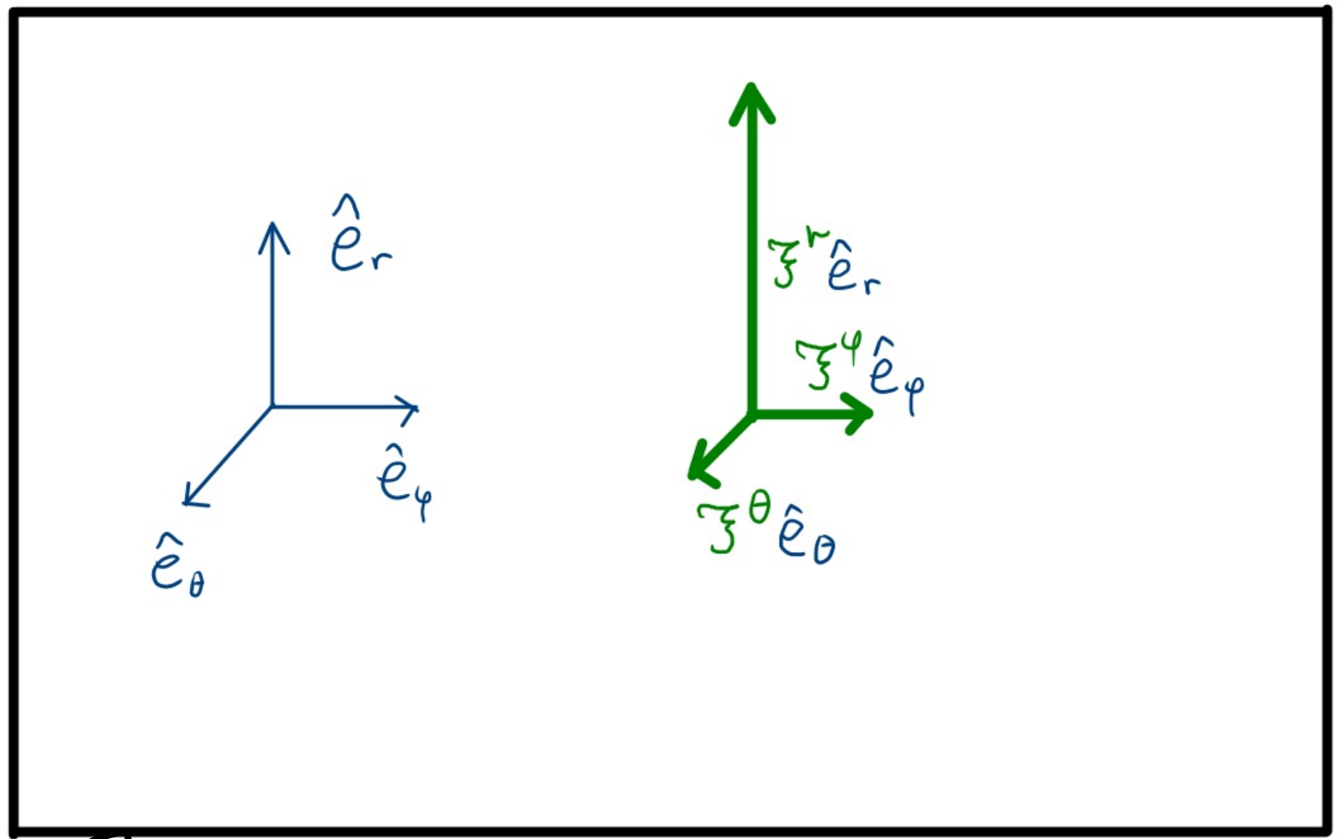
  - orthonormal basis $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity $u^{\hat{\mu}} = \hat{e}_t{}^{\hat{\mu}} = [1,0,0,0]$
  - Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\rho}} \xi^{\hat{\sigma}} = R^{\hat{\mu}}{}_{\hat{t}\hat{t}\hat{\sigma}}\, \xi^{\hat{\sigma}}$$

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?

- Back to the ship:



  - orthonormal basis $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity $\quad u^{\hat{\mu}} = \hat{e}_t{}^{\hat{\mu}} = [1,0,0,0]$
  - Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\rho}} \xi^{\hat{\sigma}} = \underbrace{R^{\hat{\mu}}{}_{\hat{t}\hat{t}\hat{\sigma}}\, \xi^{\hat{\sigma}}}_{\text{sum over } \hat{\sigma}}$$

$$\frac{D^2 \xi^t}{d\tau^2} = -R^{\hat{t}}{}_{\hat{t}\hat{\sigma}\hat{t}}\, \xi^{\hat{\sigma}} = 0$$

$$-R_{\hat{r}\hat{t}\hat{r}\hat{t}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = \frac{M}{r^3}$$

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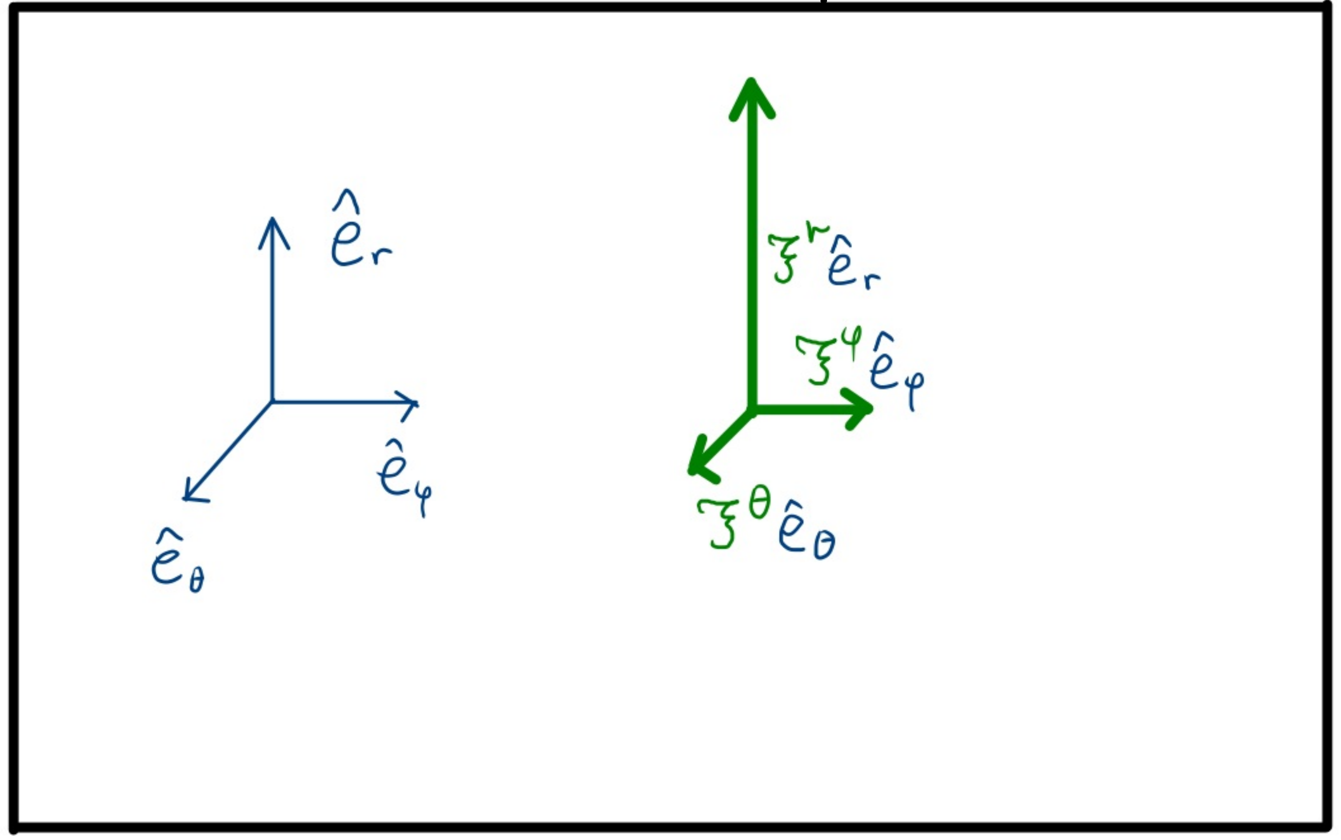

– Relative accelerations given by geodesic deviation equation:

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antisymmetric change

$$\frac{D^2 \xi^{t}}{d\tau^2} = -R^{\hat{t}}{}_{\hat{t}\hat{\sigma}\hat{t}}\, \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{r}}}{d\tau^2} = -R^{\hat{r}}{}_{\hat{t}\hat{\sigma}\hat{t}}\, \xi^{\hat{\sigma}} = -R_{\hat{r}\hat{t}\hat{r}\hat{t}}\, \xi^{\hat{r}}$$

lower index with $\eta_{\hat{\mu}\hat{\nu}}$

$$-R_{\hat{r}\hat{t}\hat{r}\hat{t}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

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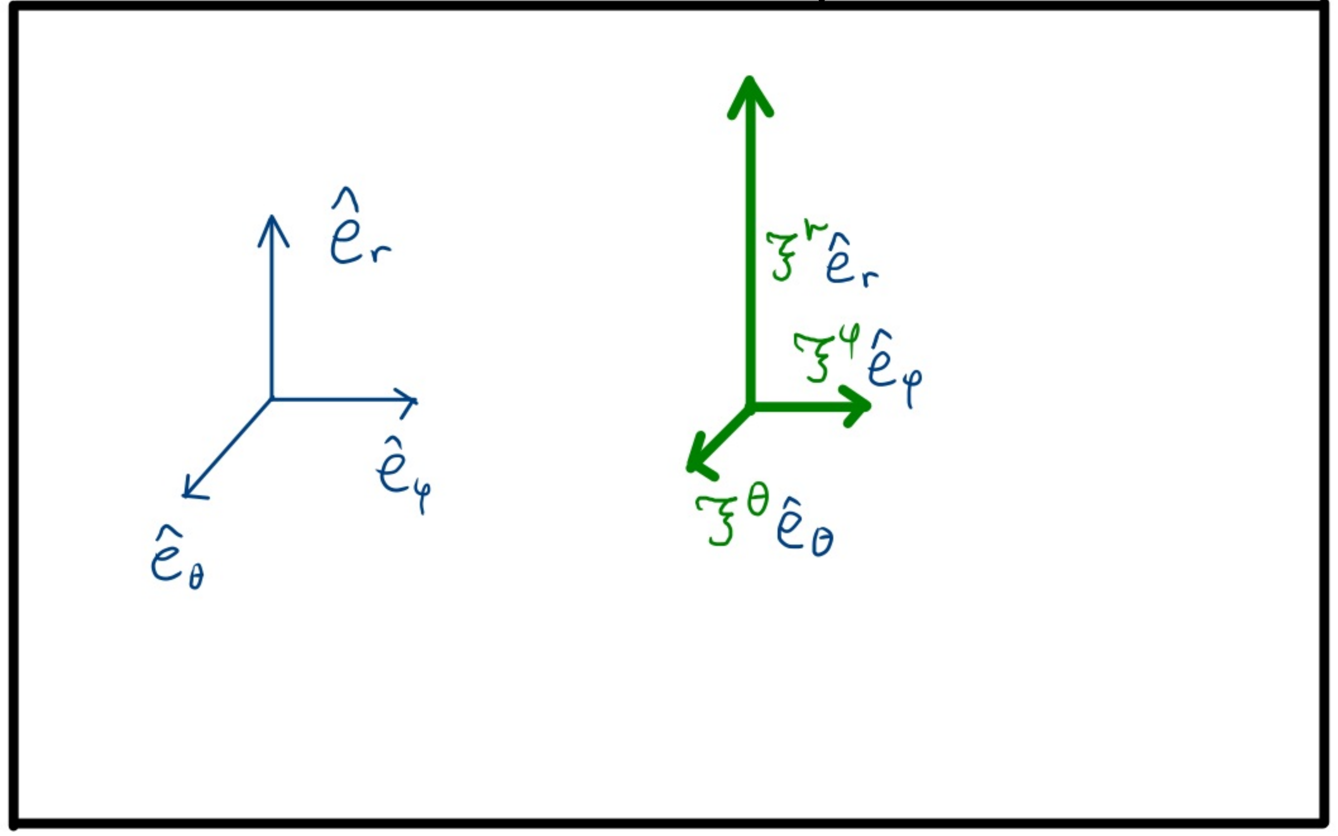

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\rho}} \xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{t}\hat{\sigma}\hat{t}}\, \xi^{\hat{\sigma}}$$

$$\frac{D^2 \xi^{t}}{d\tau^2} = -R^{\hat{t}}{}_{\hat{t}\hat{\sigma}\hat{t}}\,\xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{r}}}{d\tau^2} = -R^{\hat{r}}{}_{\hat{t}\hat{\sigma}\hat{t}}\,\xi^{\hat{\sigma}} = -R_{\hat{r}\hat{t}\hat{r}\hat{t}}\,\xi^{\hat{r}}$$

$$= -\left(-\frac{2M}{r^3}\right)\xi^{\hat{r}} = +\frac{2M}{r^3}\,\xi^{\hat{r}}$$

$$-R_{\hat{r}\hat{t}\hat{r}\hat{t}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = \frac{M}{r^3}$$

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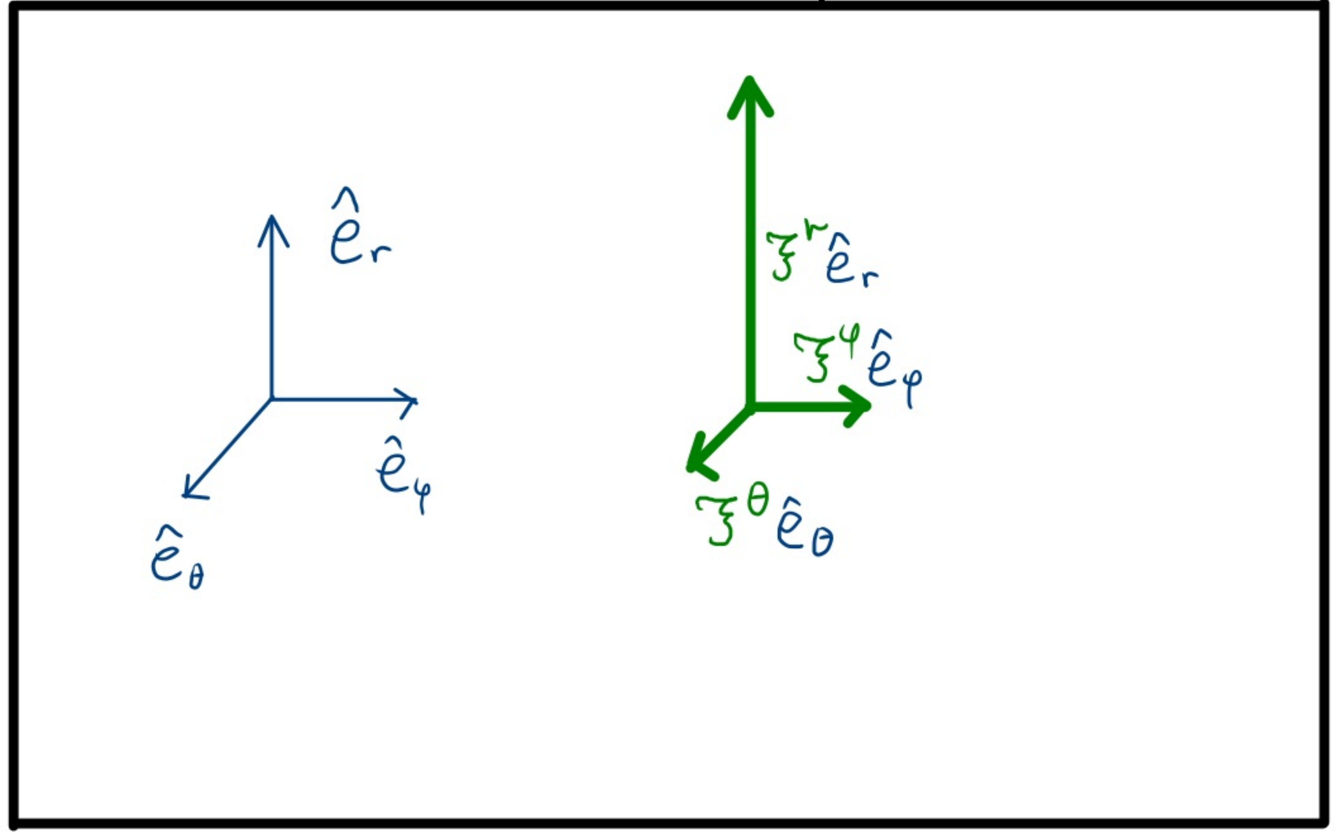

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\sigma}}\,u^{\hat{\nu}} u^{\hat{\rho}}\,\xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{t}\hat{\sigma}\hat{t}}\,\xi^{\hat{\sigma}}$$

$$\frac{D^2 \xi^{t}}{d\tau^2} = - R^{\hat t}{}_{\hat t \hat\sigma \hat t}\, \xi^{\hat\sigma} = 0$$

$$\frac{D^2 \xi^{\hat r}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat r}$$

↳ relative acceleration of head w.r.t. waist

$$- R_{\hat r \hat t \hat r \hat t} = R_{\hat\theta \hat\phi \hat\theta \hat\phi} = \frac{2M}{r^3}$$

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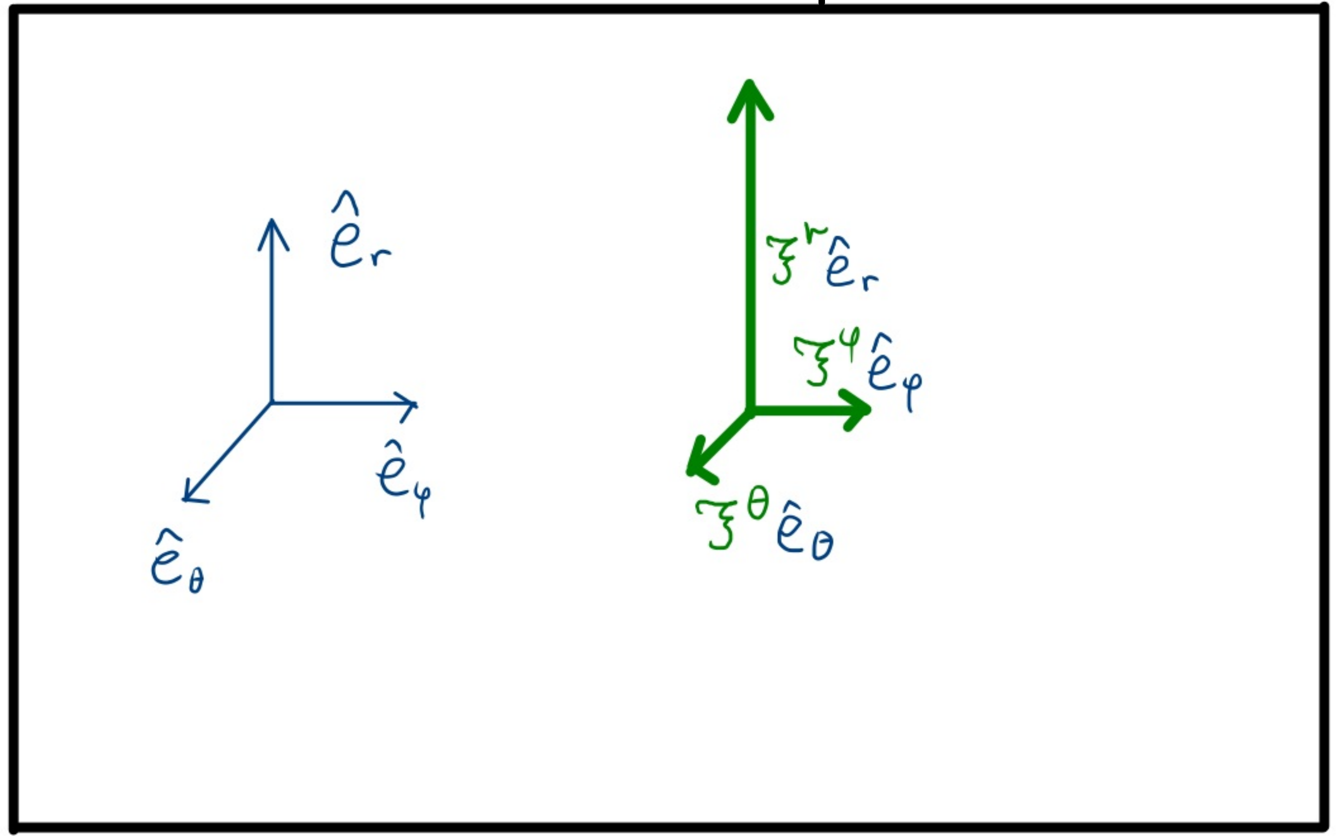

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$$\frac{D^2\xi^{t}}{d\tau^2} = -R^{\hat{t}}{}_{\hat{t}\hat{\sigma}\hat{t}}\,\xi^{\hat{\sigma}} = 0$$

$$\frac{D^2\xi^{\hat{r}}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat{r}}$$

↳ relative acceleration of head w.r.t. waist

→ there is no $\left(1-\frac{2M}{r}\right)$

$$-R_{\hat{r}\hat{t}\hat{r}\hat{t}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = \frac{M}{r^3}$$

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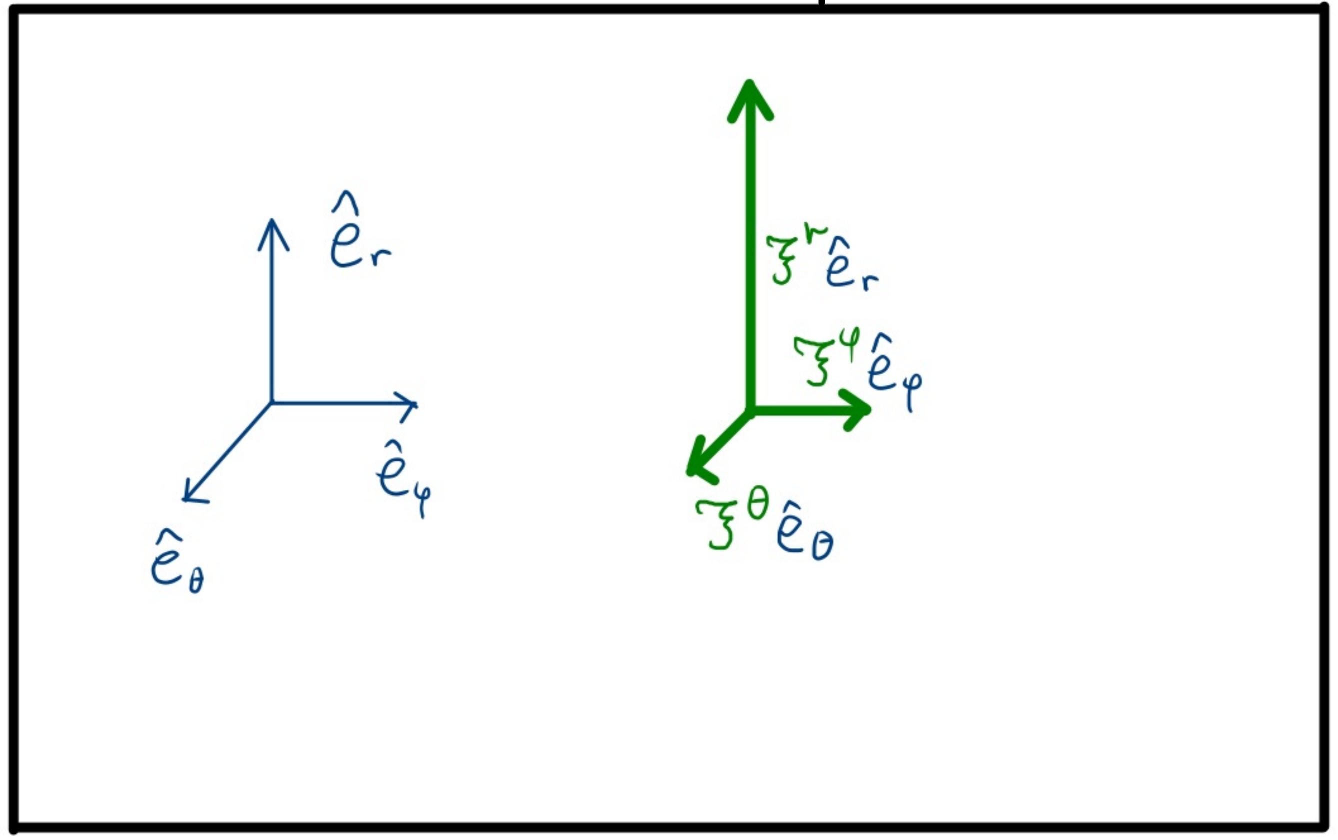

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$$\frac{D^2\xi^t}{d\tau^2} = -R^{\hat t}{}_{\hat t\hat\sigma\hat t}\,\xi^{\hat\sigma} = 0$$

$$\frac{D^2\xi^{\hat r}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat r}$$

↳ relative acceleration of head w.r.t. waist

→ there is no $\left(1-\frac{2M}{r}\right)$

→ blows up as $r \to 0$

$$-R_{\hat r\hat t\hat r\hat t} = R_{\hat\theta\hat\phi\hat\theta\hat\phi} = \frac{2M}{r^3}$$

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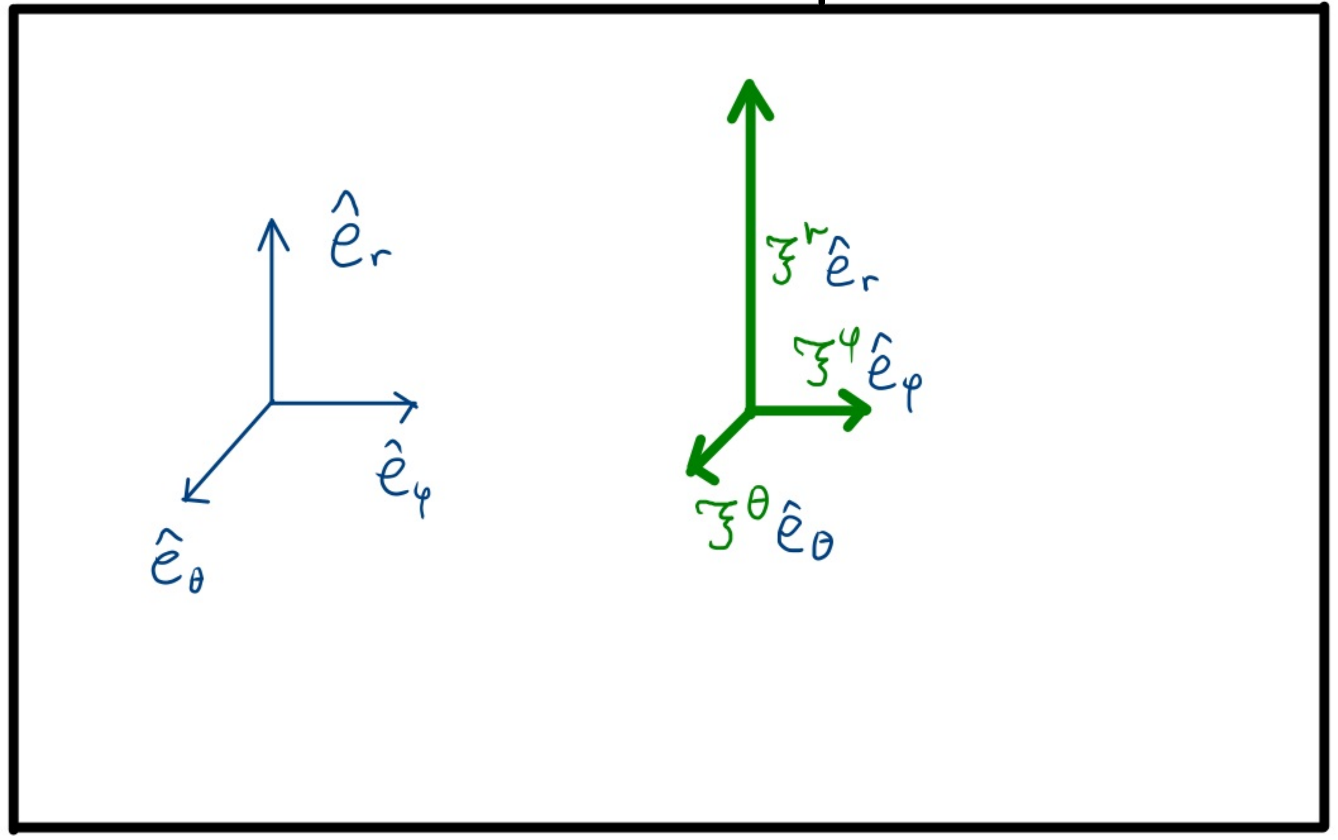

– Relative accelerations given by geodesic deviation equation:

$$\frac{D^2\xi^{\hat\mu}}{d\tau^2} = R^{\hat\mu}{}_{\hat\nu\hat\rho\hat\sigma}\,u^{\hat\nu}u^{\hat\rho}\,\xi^{\hat\sigma} = -R^{\hat\mu}{}_{\hat t\hat\sigma\hat t}\,\xi^{\hat\sigma}$$

$$\frac{D^2\xi^t}{d\tau^2} = -R^{\hat t}{}_{\hat t\hat\sigma\hat t}\,\xi^{\hat\sigma} = 0$$

$$\frac{D^2\xi^{\hat r}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat r}$$

↳ relative acceleration of head w.r.t. waist

→ there is no $\left(1-\frac{2M}{r}\right)$

→ blows up as $r \to 0$

→ head moves away from feet

$$-R_{\hat r\hat t\hat r\hat t} = R_{\hat\theta\hat\phi\hat\theta\hat\phi} = \frac{2M}{r^3}$$

$$R_{\hat\theta\hat t\hat\theta\hat t} = R_{\hat\phi\hat t\hat\phi\hat t} = \frac{M}{r^3}$$

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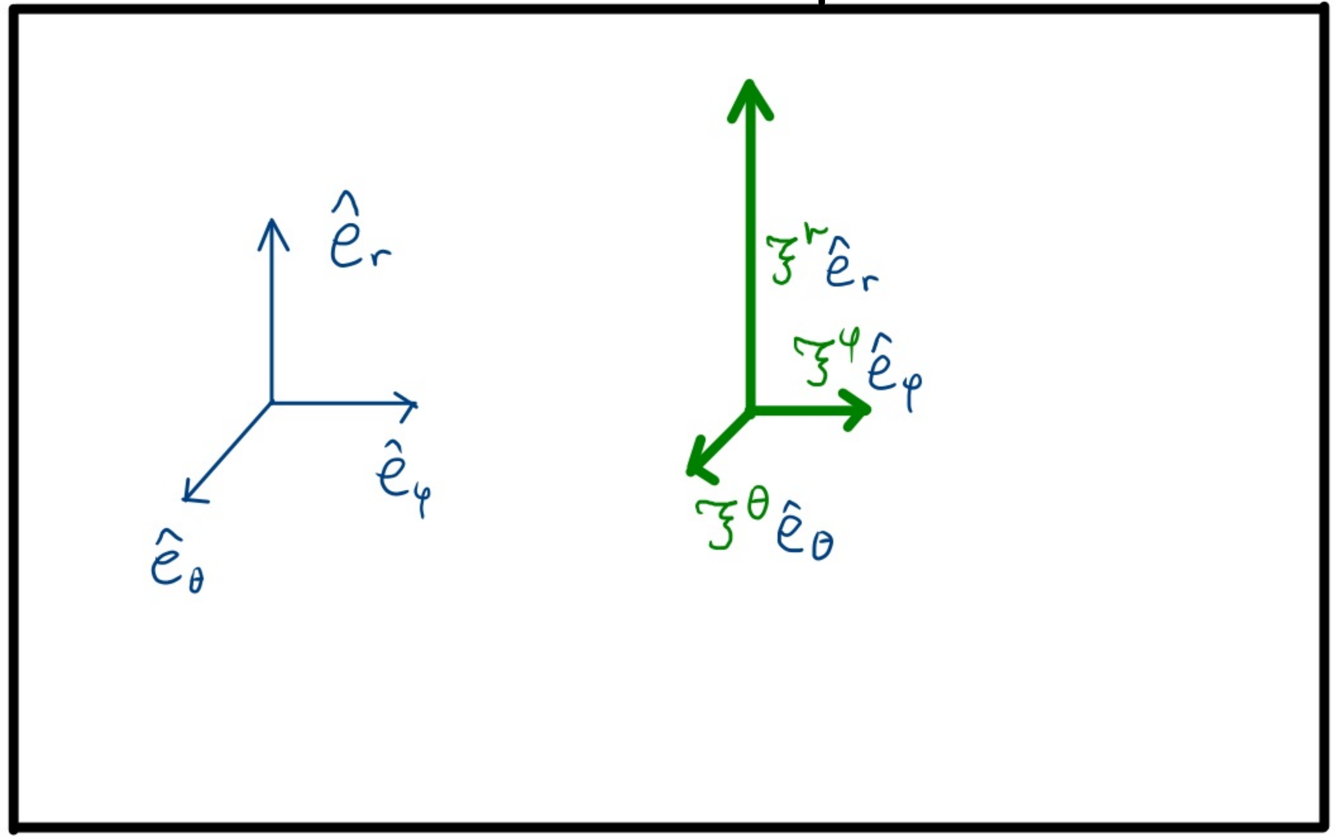

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$$\frac{D^2 \xi^t}{d\tau^2} = -R^{\hat t}{}_{\hat t \hat\sigma \hat t}\, \xi^{\hat\sigma} = 0$$

$$\frac{D^2 \xi^{\hat r}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat r}$$

$$\frac{D^2 \xi^{\theta}}{d\tau^2} = -R^{\hat\theta}{}_{\hat t \hat\sigma \hat t}\, \xi^{\hat\sigma} = -R_{\hat\theta \hat t \hat\theta \hat t}\, \xi^{\hat\theta}$$

$$= -\frac{M}{r^3}\, \xi^{\hat\theta}$$

$$-R_{\hat r \hat t \hat r \hat t} = R_{\hat\theta \hat\phi \hat\theta \hat\phi} = \frac{2M}{r^3}$$

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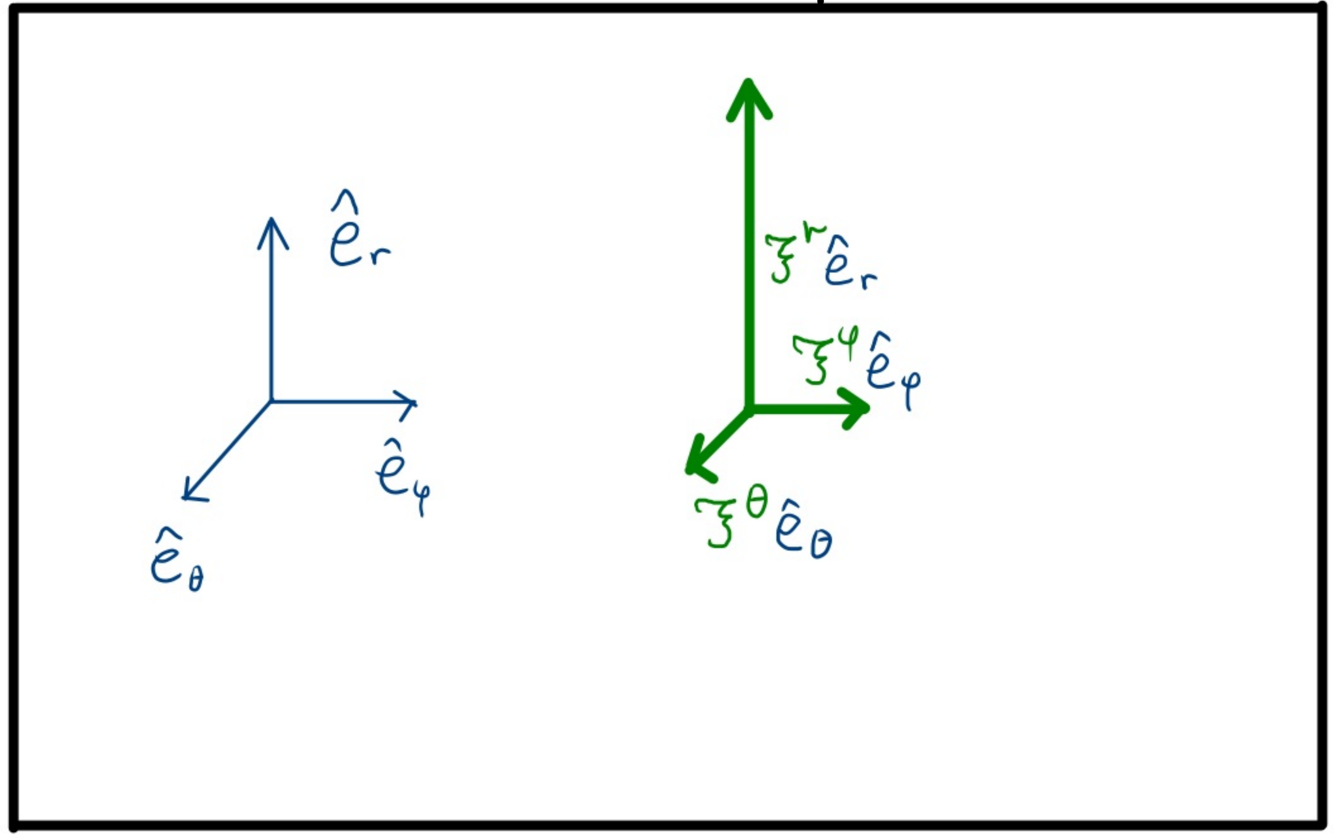

– Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat\mu}}{d\tau^2} = R^{\hat\mu}{}_{\hat\nu \hat\rho \hat\sigma}\, u^{\hat\nu} u^{\hat\rho} \xi^{\hat\sigma} = -R^{\hat\mu}{}_{\hat t \hat\sigma \hat t}\, \xi^{\hat\sigma}$$

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$$\frac{D^2 \xi^{\hat{r}}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat{r}}$$

$$\frac{D^2 \xi^{\theta}}{d\tau^2} = -\frac{M}{r^3}\, \xi^{\hat{\theta}}$$

$$-R_{\hat{r}\hat{t}\hat{r}\hat{t}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = \frac{M}{r^3}$$

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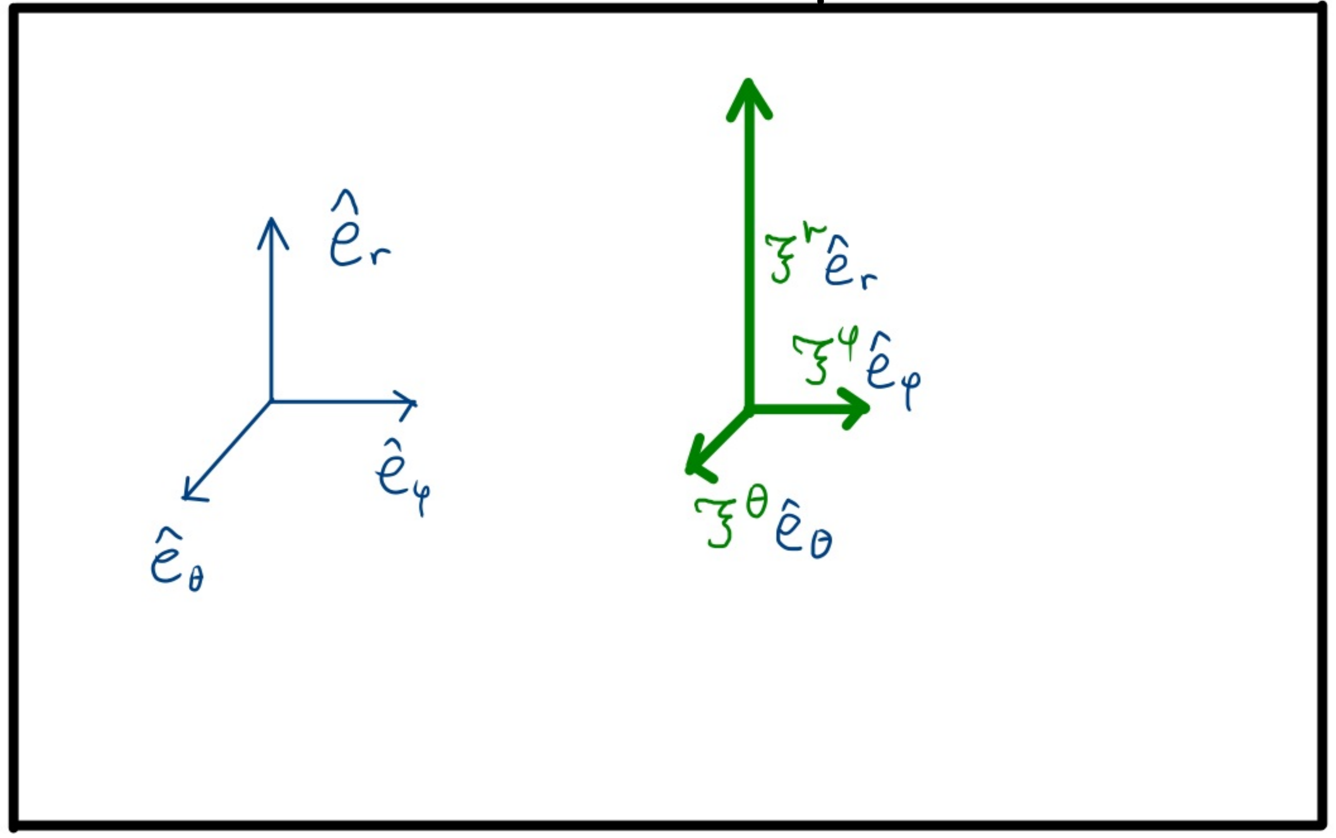

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$$\frac{D^2\xi^{\hat{r}}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat{r}}$$

$$\frac{D^2\xi^{\theta}}{d\tau^2} = -\frac{M}{r^3}\,\xi^{\hat{\theta}}$$

→ nothing wrong at $r=2M$

→ left + right hand approach squeezed as $r \to 0$

$$-R_{\hat{r}\hat{t}\hat{r}\hat{t}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = \frac{M}{r^3}$$

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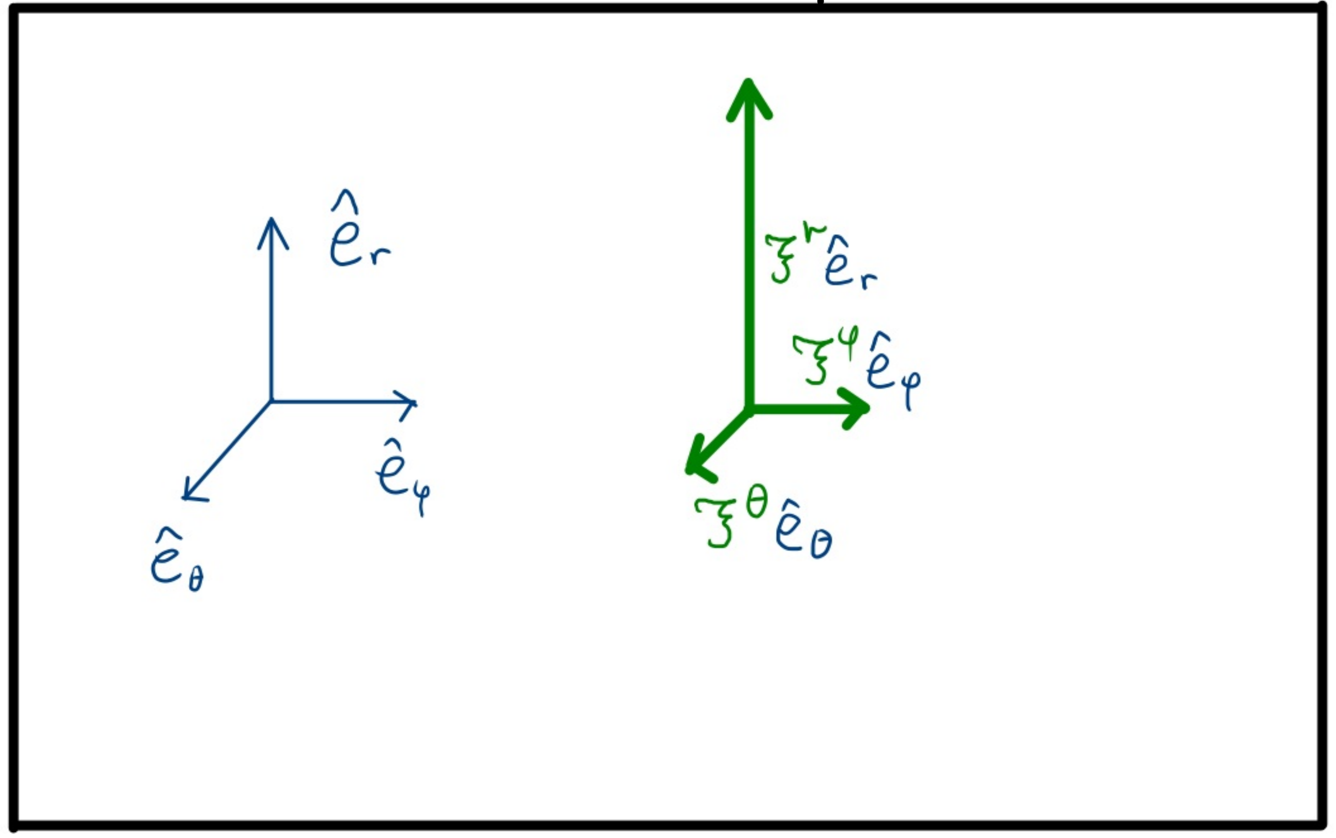

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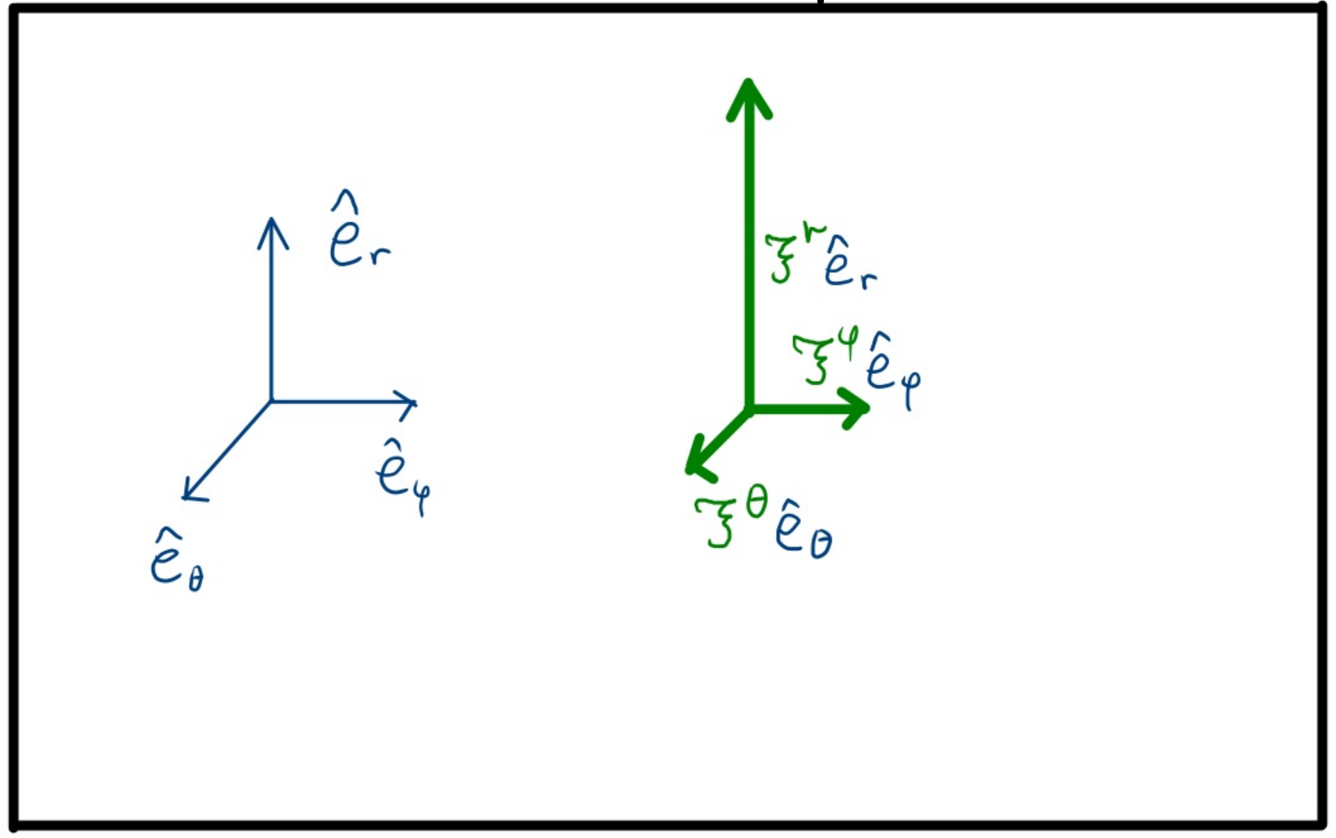

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$$\frac{D^2\xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\sigma}}\,u^{\hat{\nu}}u^{\hat{\rho}}\xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{t}\hat{\sigma}\hat{t}}\,\xi^{\hat{\sigma}}$$

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$$\frac{D^2\xi^{\hat r}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat r}$$

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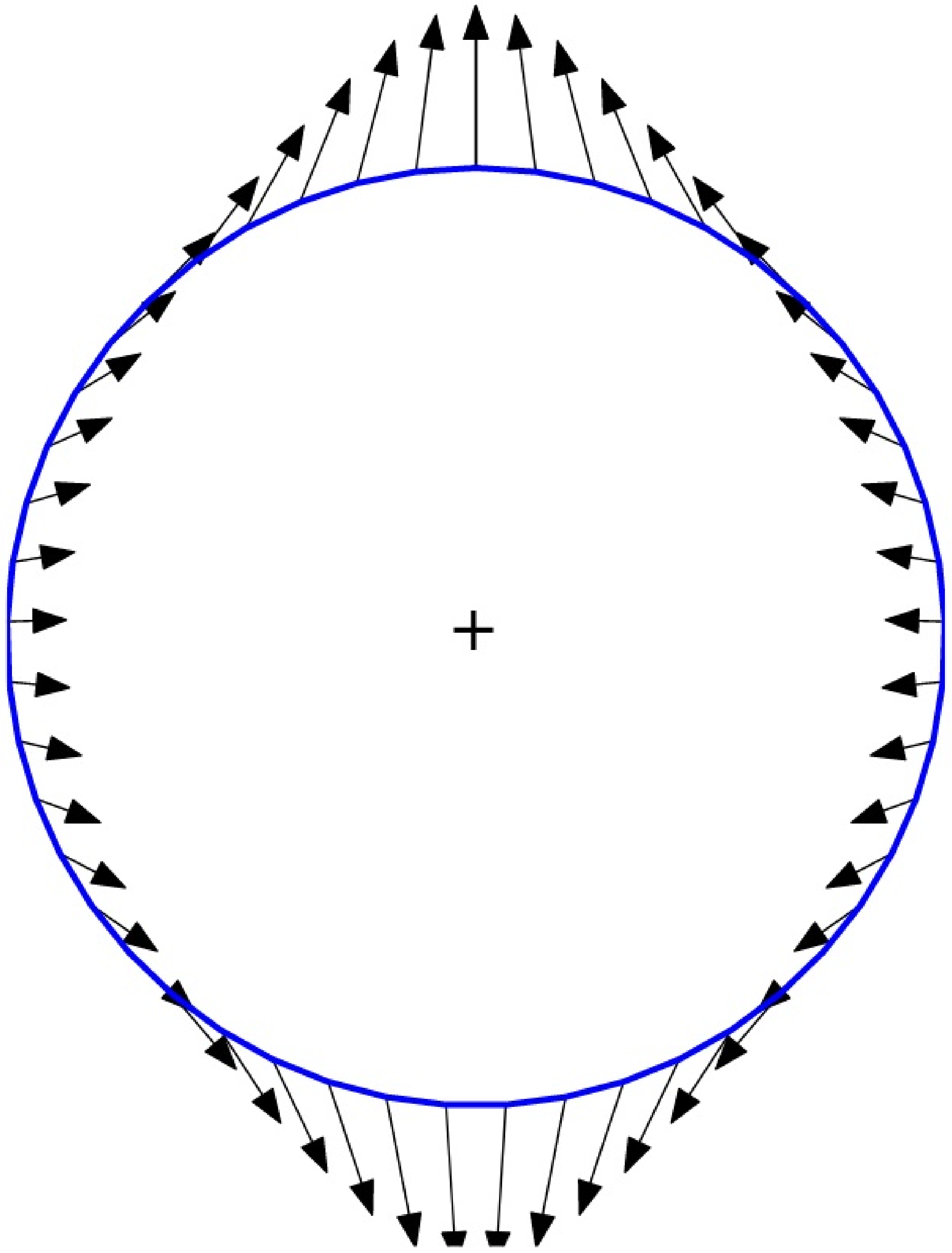
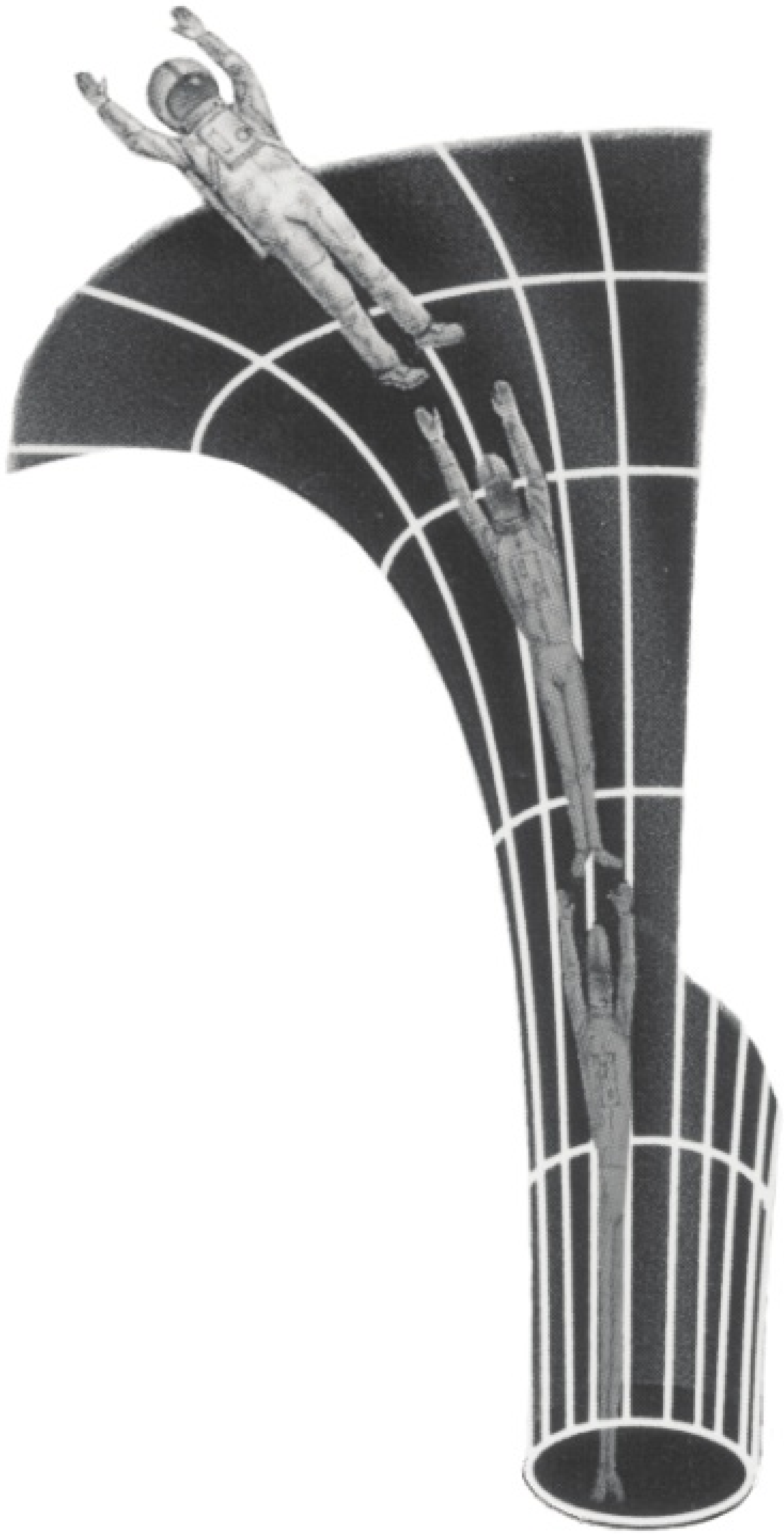
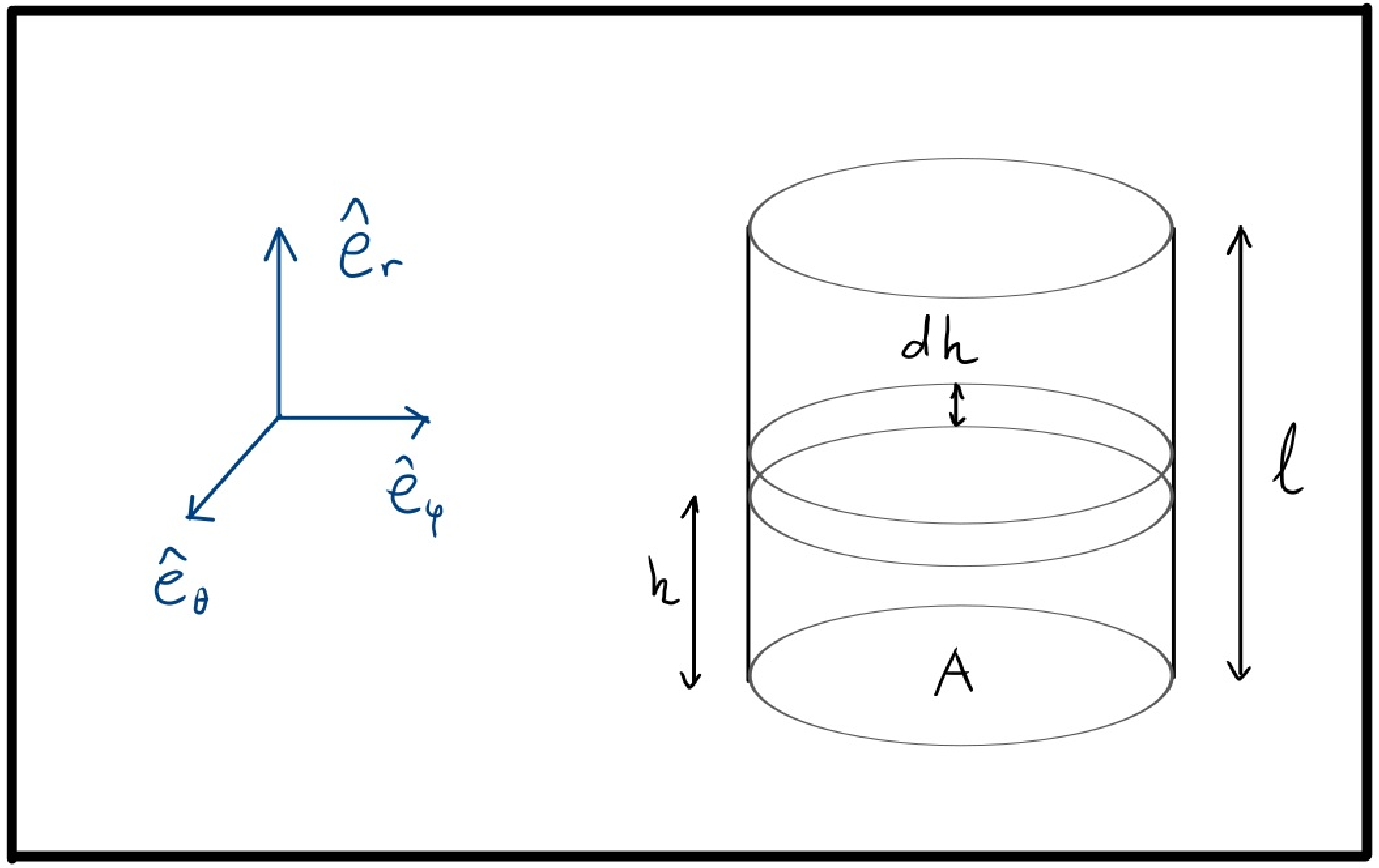




spaghettification ...

$$\frac{D^2 \xi^{\hat{r}}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat{r}}$$

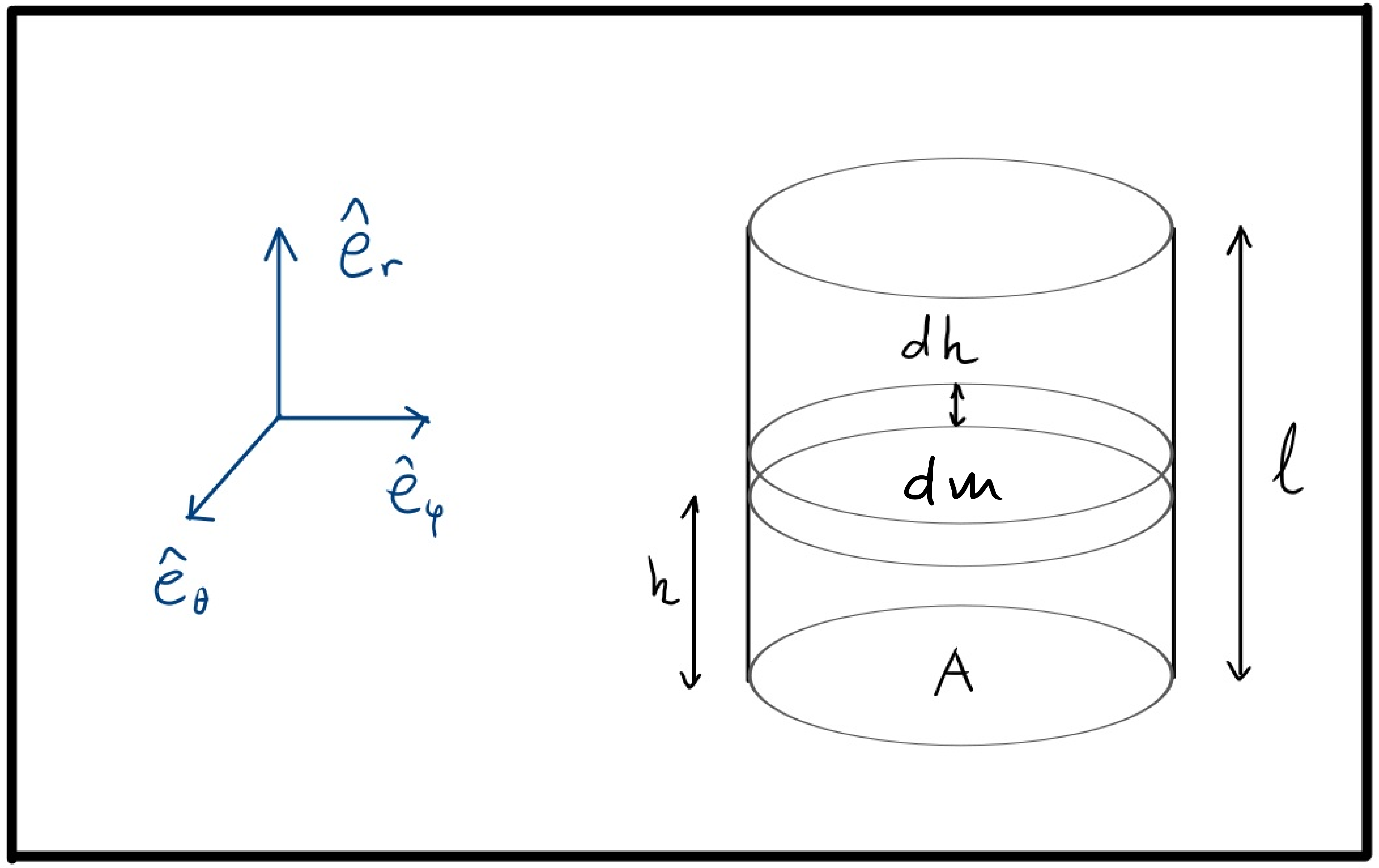


Relative acceleration of disks at distance h:

$$a = \frac{2M}{r^3} \cdot h$$

$$\frac{D^2 \xi^{r}}{d\tau^2} \qquad\qquad \xi^{\hat{r}}$$

$$\frac{D^2 \xi^{\hat{r}}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat{r}}$$



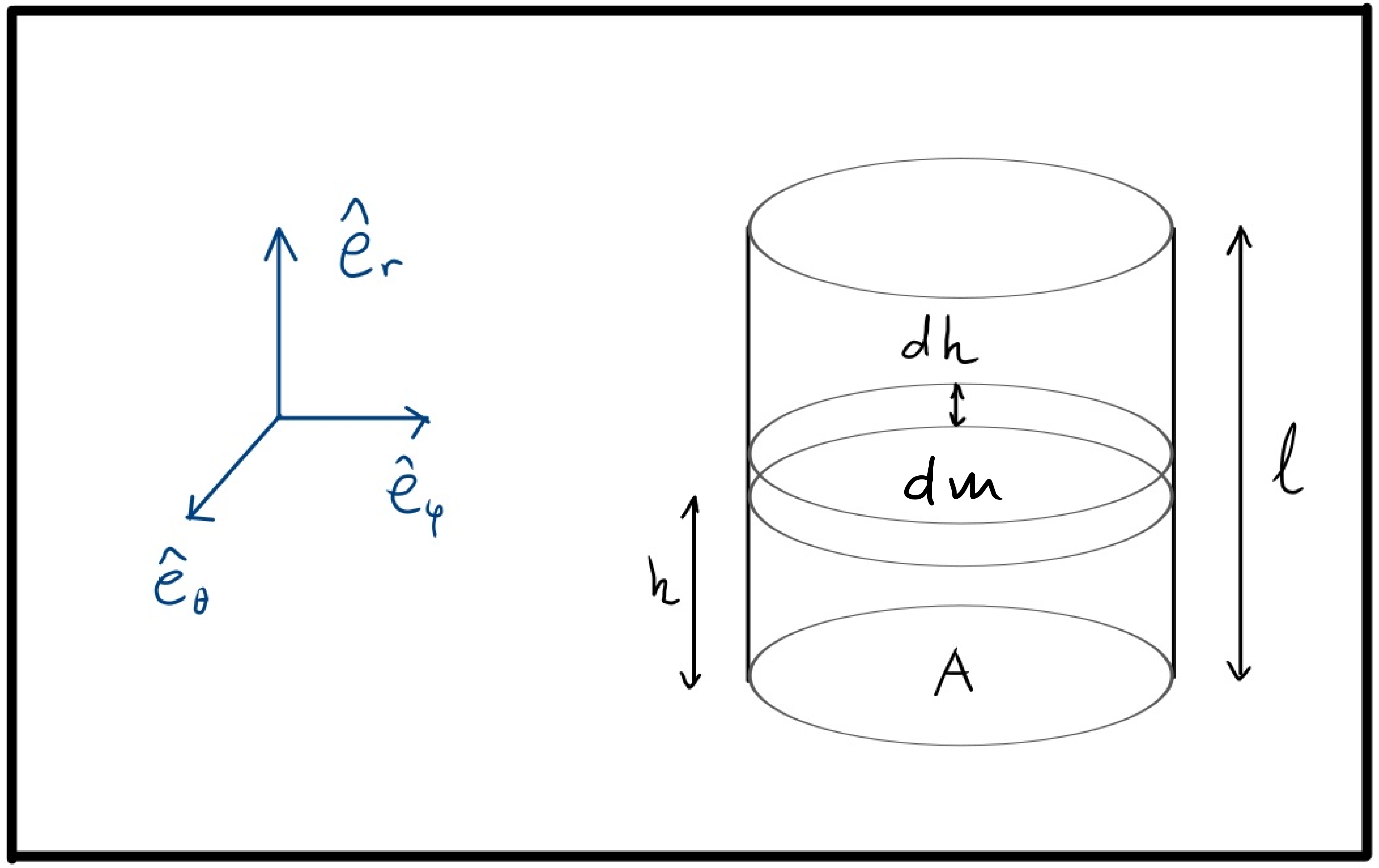
Relative acceleration of disks at distance h:

$$a = \frac{2M}{r^3} \cdot h$$

Astronaut of height ℓ:

$$\left(\begin{matrix}\text{Pressure on} \\ \text{waist}\end{matrix}\right) = \frac{1}{A}\int_0^{\ell/2} dF = \frac{1}{A}\int_0^{\ell/2} a\, dm$$

$$\frac{D^2 \xi^{\hat{r}}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat{r}}$$



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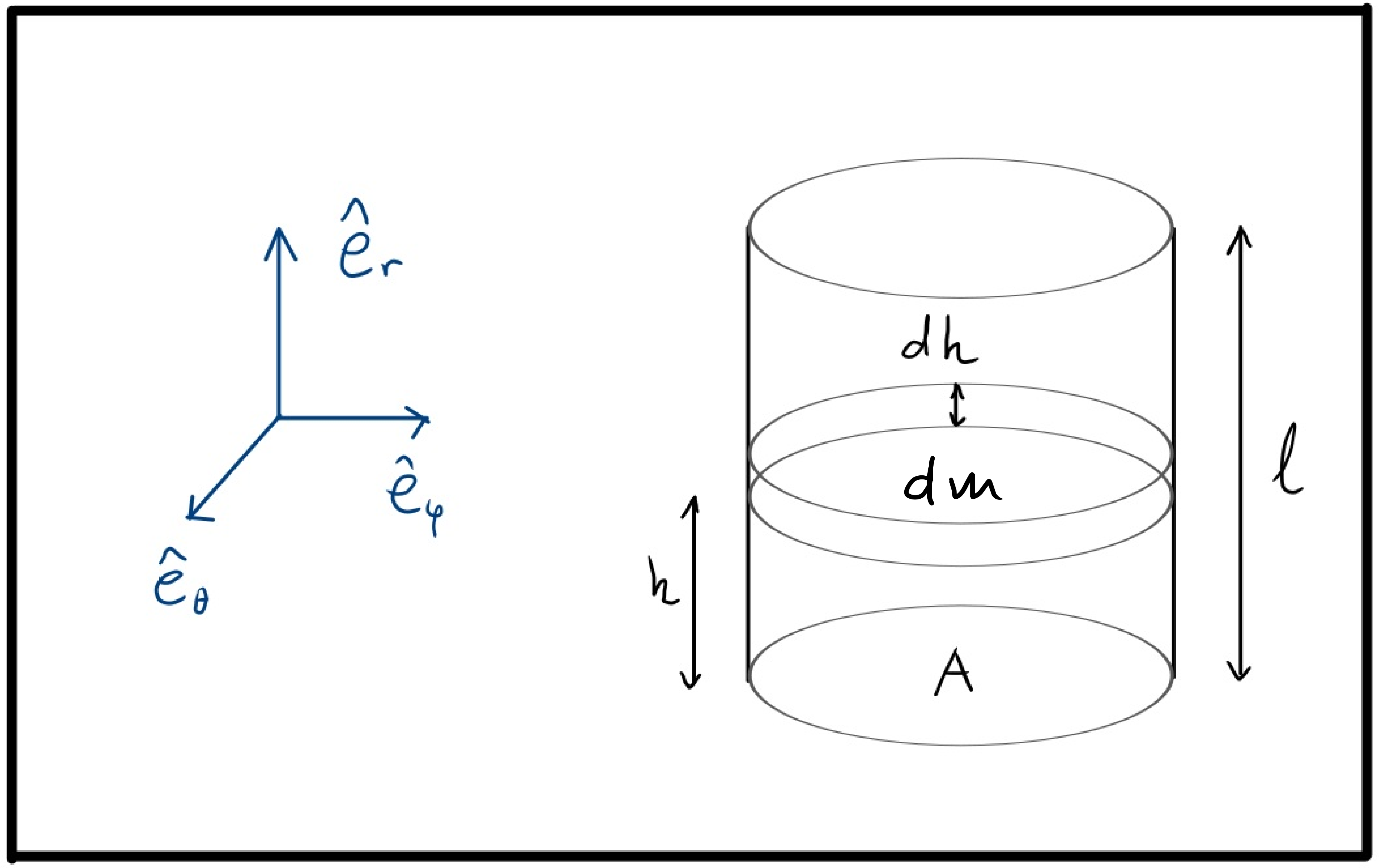
$$\frac{D^2 \xi^{\hat{r}}}{d\tau^2} = \frac{2M}{r^3} \xi^{\hat{r}}$$



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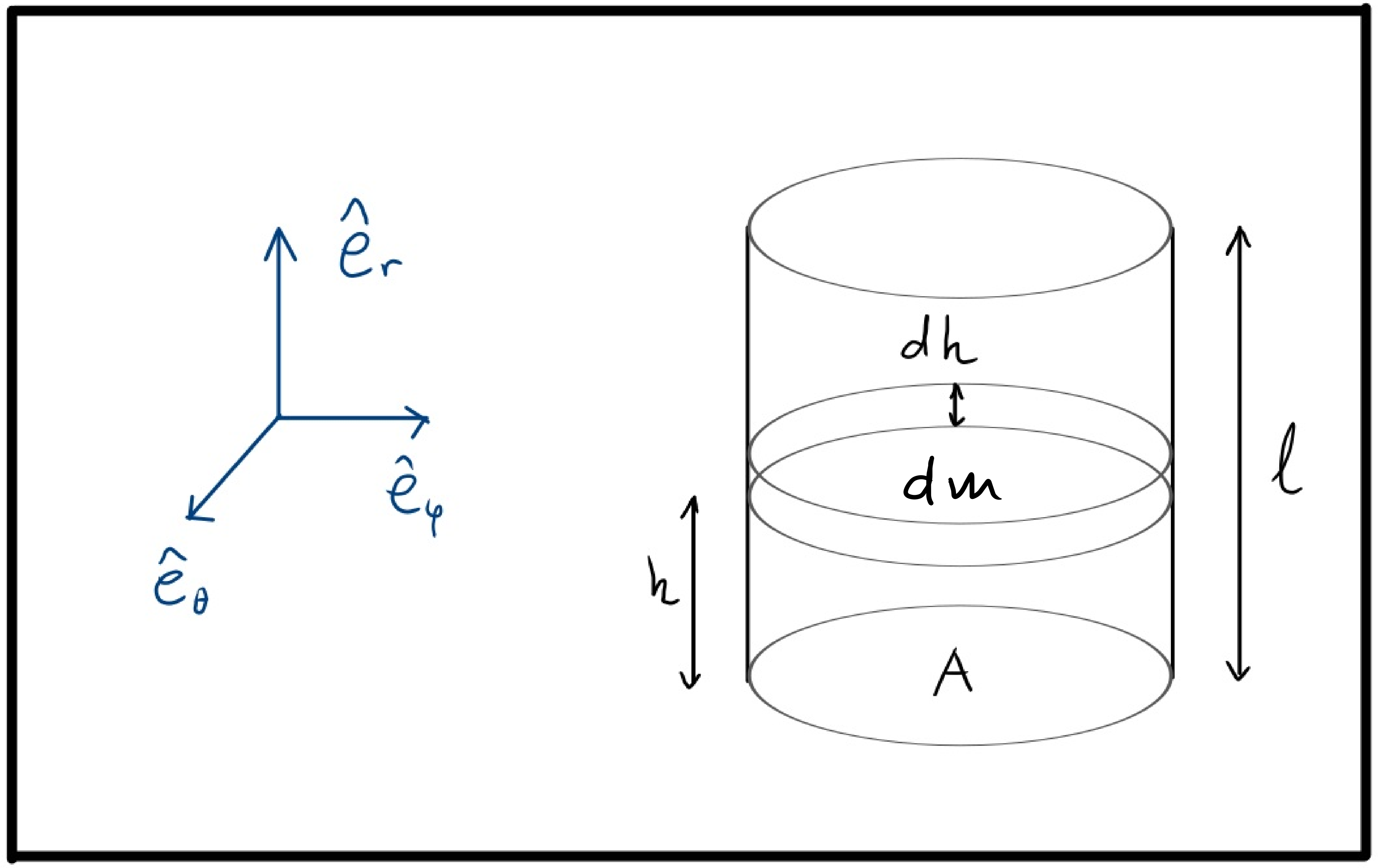
$$= \frac{1}{4}\frac{mM\ell}{Ar^3}$$

$m = 75\,\mathrm{kg} \quad A = (0.2\,\mathrm{m})^2 \quad \ell = 1.8\,\mathrm{m}$

$\Rightarrow$

$$(\text{Pressure}) \approx 1.1 \times 10^9 \, \frac{(M/M_\odot)}{(r/1\mathrm{km})^3} \ \text{Atm}$$

Misner § 32.6



$$\left(\begin{array}{c}\text{Pressure on}\\ \text{waist}\end{array}\right) = \frac{1}{A}\int_0^{\ell/2} dF = \frac{1}{A}\int_0^{\ell/2} a\,dm = \frac{1}{A}\int_0^{\ell/2} \underbrace{\left(\frac{2M}{r^3}h\right)}_{a}\underbrace{\left(\frac{m}{\ell A}\right)}_{\rho}\underbrace{(A\,dh)}_{dV}$$

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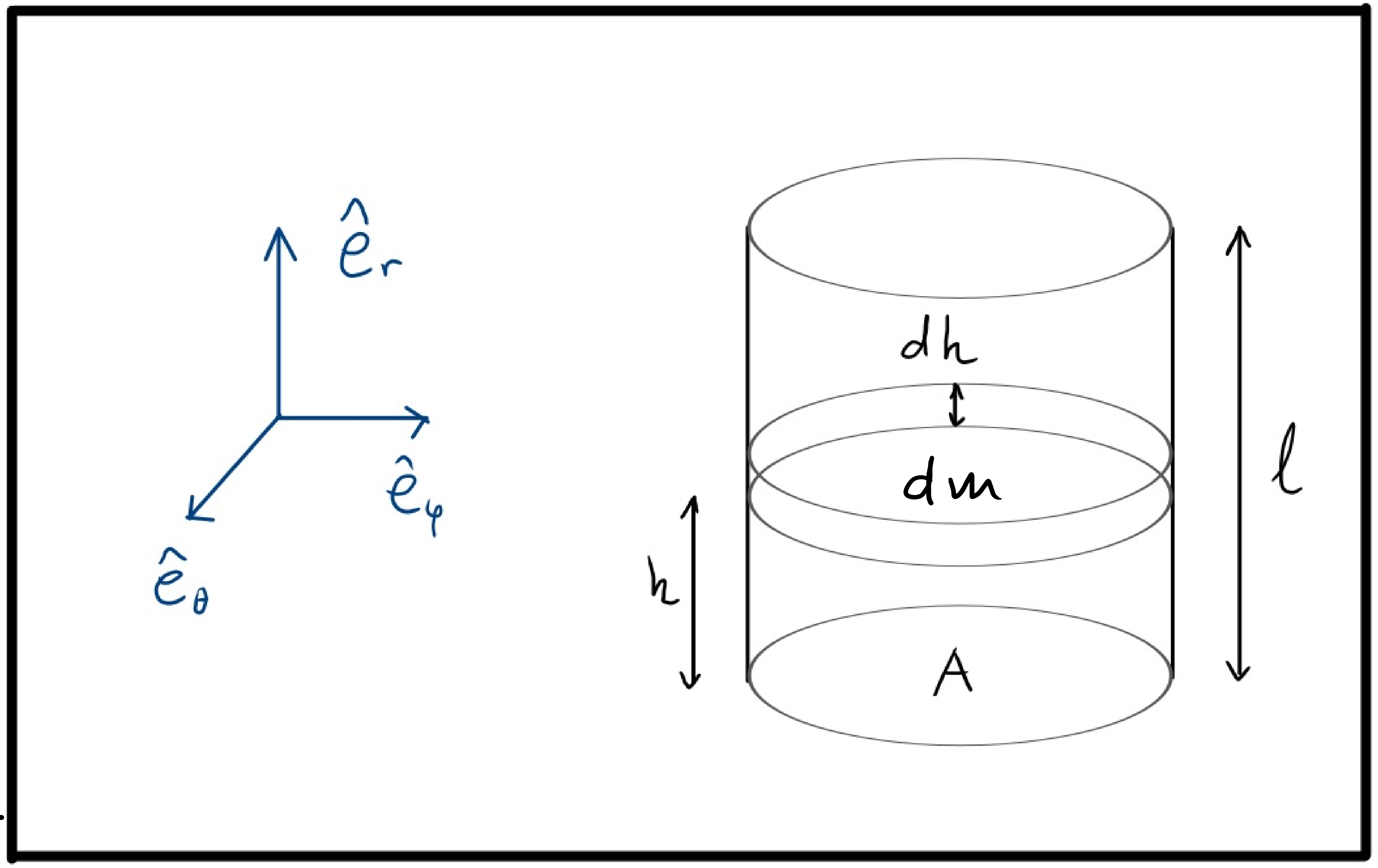


For a stellar black hole

$M \approx M_\odot \approx 1.5\ \mathrm{km} \qquad r_s \approx 3.0\ \mathrm{km}$

so at $r = r_s$ $\quad (\text{pressure}) \approx 10^9 \frac{1}{3^3}\ \mathrm{Atm} \approx 10^7\ \mathrm{Atm}$

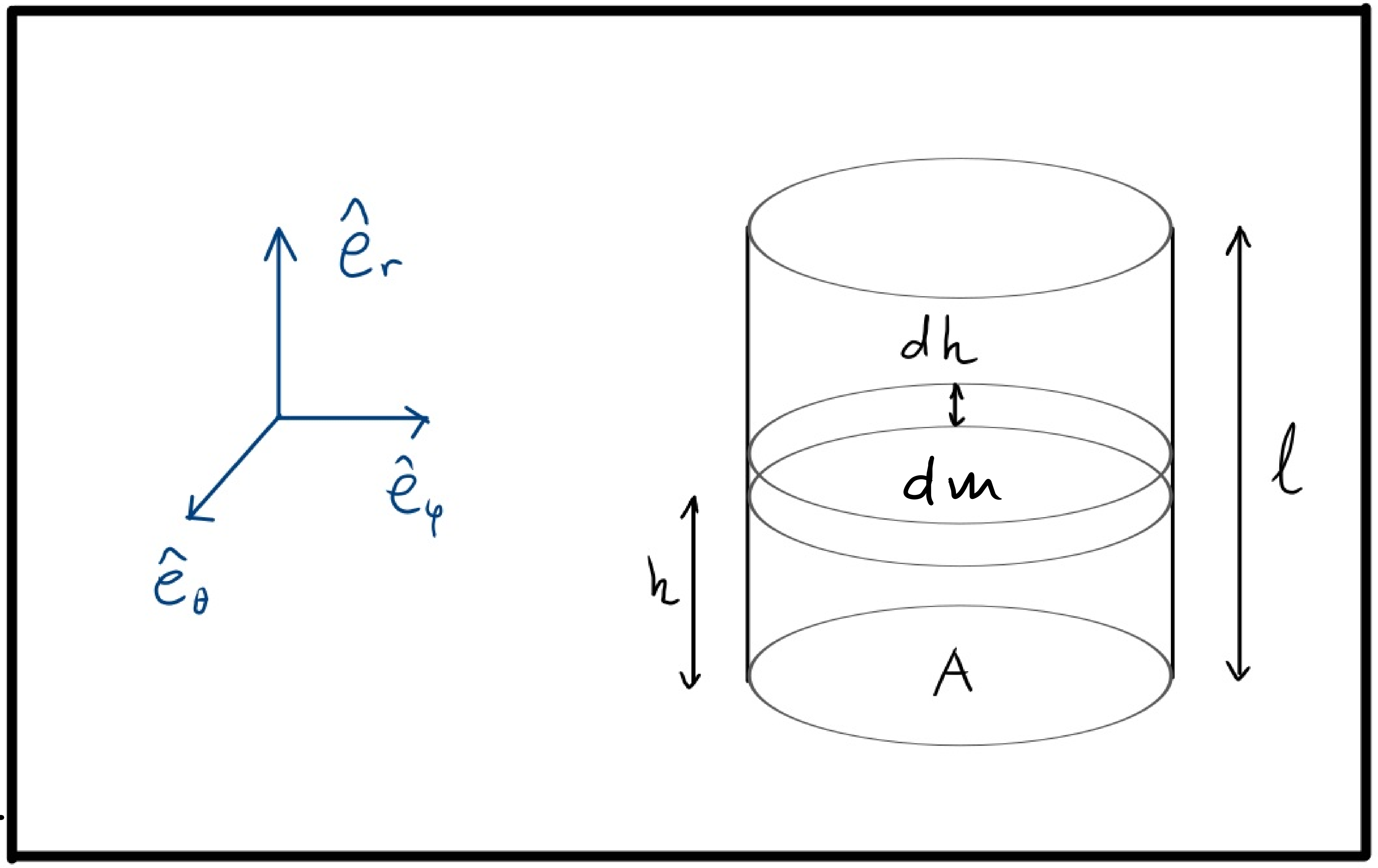
Human body may withstand $\approx 10^2\ \mathrm{Atm}$

$m = 75\,kg \quad A = (0.2\,m)^2 \quad \ell = 1.8\,m$

$\Rightarrow$

$$(\text{pressure}) \approx 1.1 \times 10^9 \frac{(M/M_\odot)}{(r/1\,km)^3}\ \text{Atm}$$

Misner § 32.6



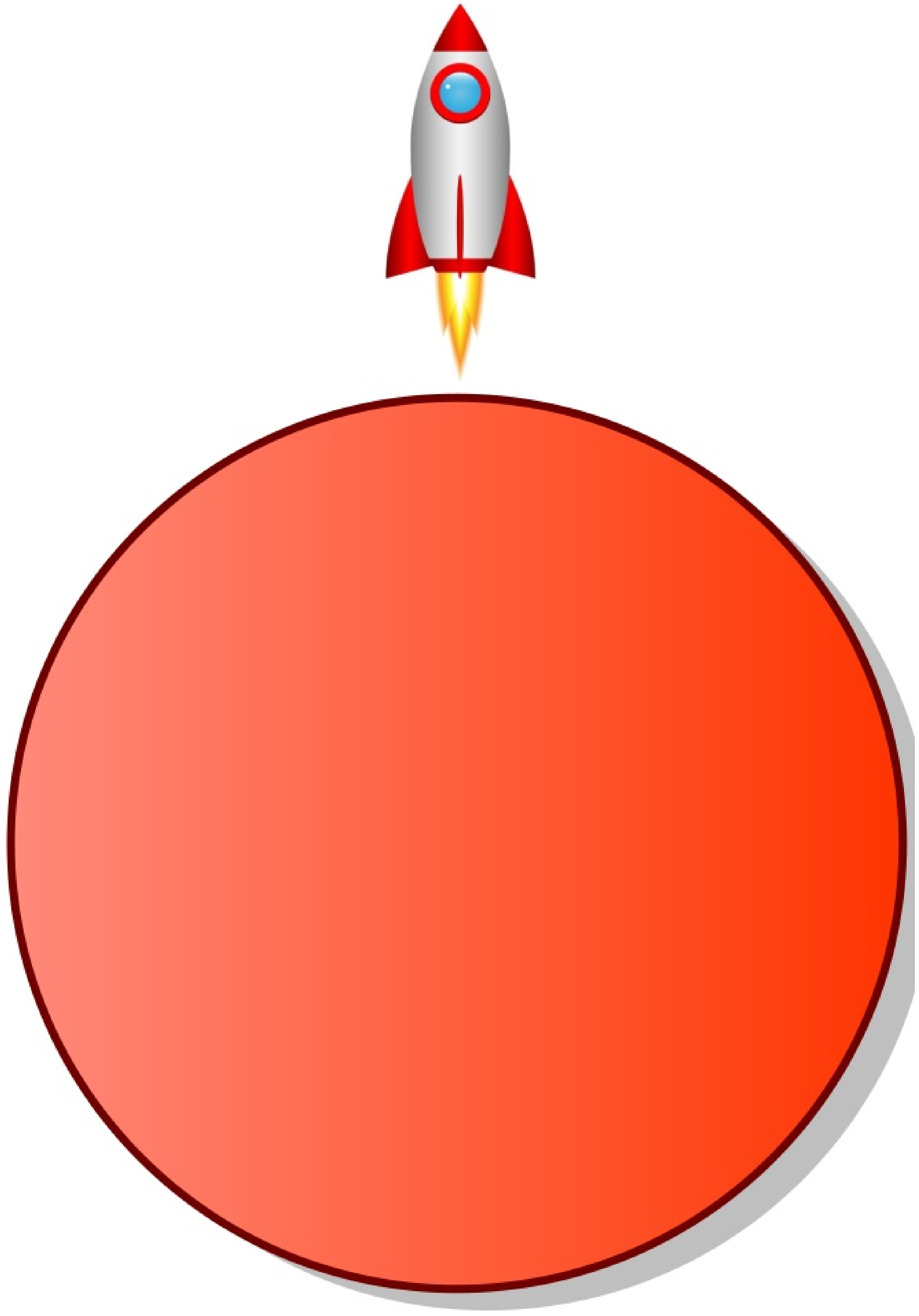
For supermassive black hole

$M \approx 10^9 M_\odot \approx 10^9\,km \qquad r_s \approx 10^9\,km$

so at $r = r_s$ $\quad (\text{pressure}) \approx 10^9 \dfrac{10^9}{(10^9)^3}\ \text{Atm} \approx 10^{18-27}\ \text{Atm} \lesssim 10^{-9}\ \text{Atm}$

Human body may withstand $\approx 10^2$ Atm

- Freely falling observer sees nothing special crossing the horizon



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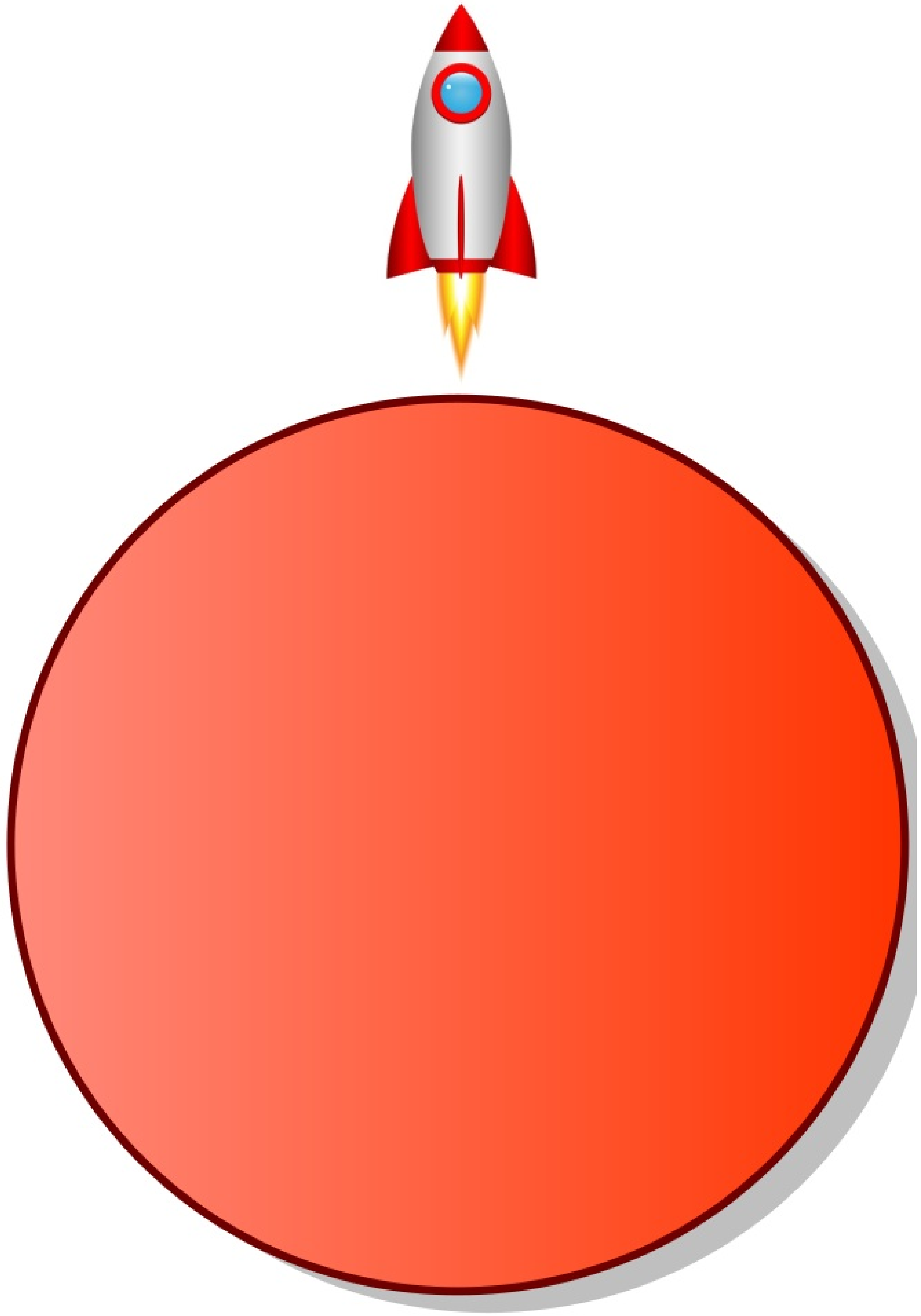
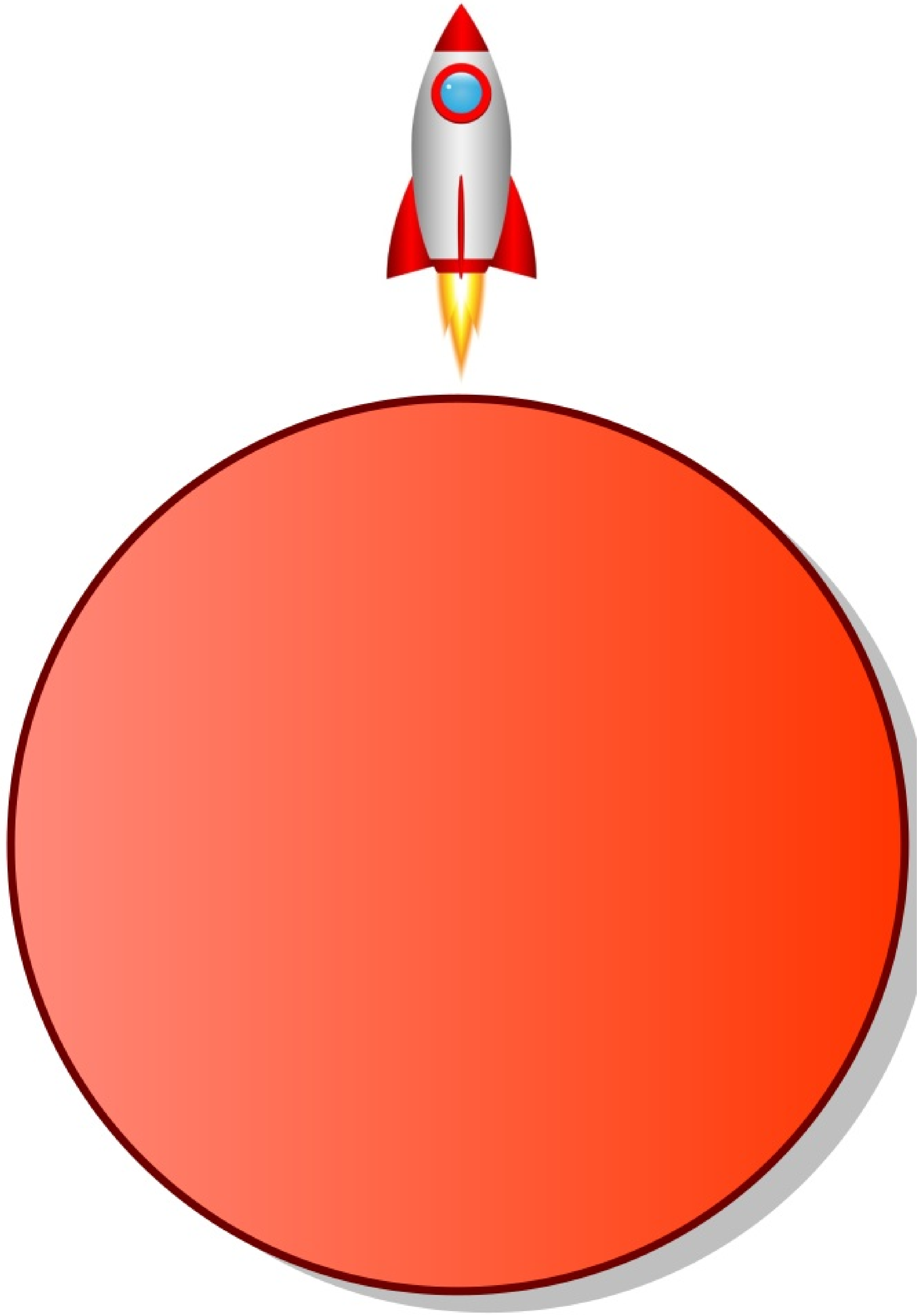


- Freely falling observer sees nothing special crossing the horizon
- But an (accelerated) observer struggles infinitely hard to remain stationary infinitesimally close to the horizon
- Indeed, 4-force per unit mass needed is:

$$f^\mu = \frac{Du^\mu}{d\tau} = u^\nu \nabla_\nu u^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho$$



$$\Gamma^0{}_{10} = \frac{M}{r(r-2M)} \qquad \Gamma^1{}_{00} = \frac{M(r-2M)}{r^3} \qquad \Gamma^1{}_{11} = \frac{M}{2Mr - r^2}$$

$$\Gamma^1{}_{22} = 2M - r \qquad \Gamma^1{}_{33} = (2M-r)\sin^2\theta \qquad \Gamma^2{}_{21} = \frac{1}{r}$$

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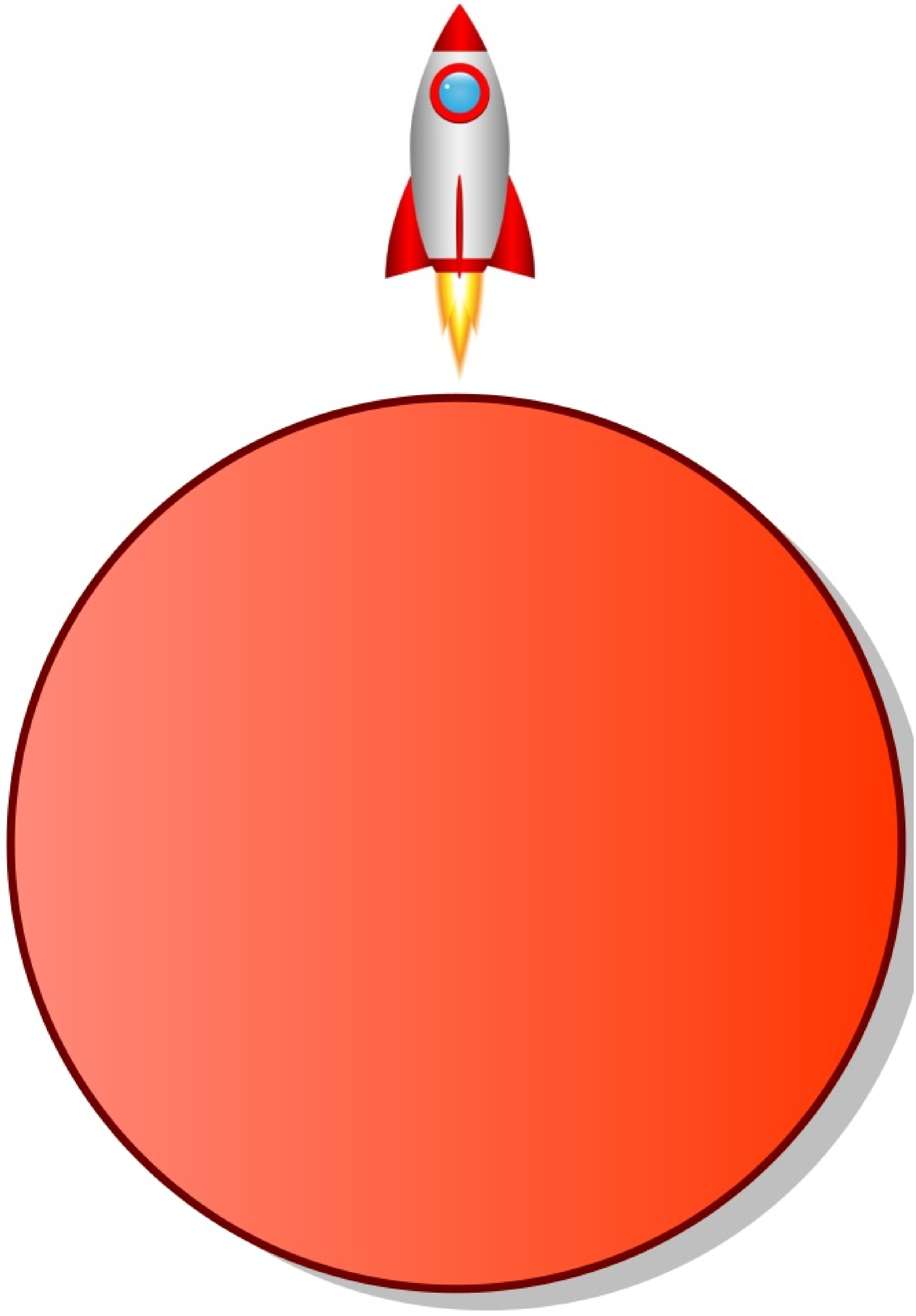

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$$u^\mu = [u^0, 0, 0, 0]$$

$$u_\mu u^\mu = -1 \Rightarrow -\left(1 - \frac{2M}{r}\right)(u^0)^2 = -1 \Rightarrow u^0 = \left(1 - \frac{2M}{r}\right)^{-1/2}$$

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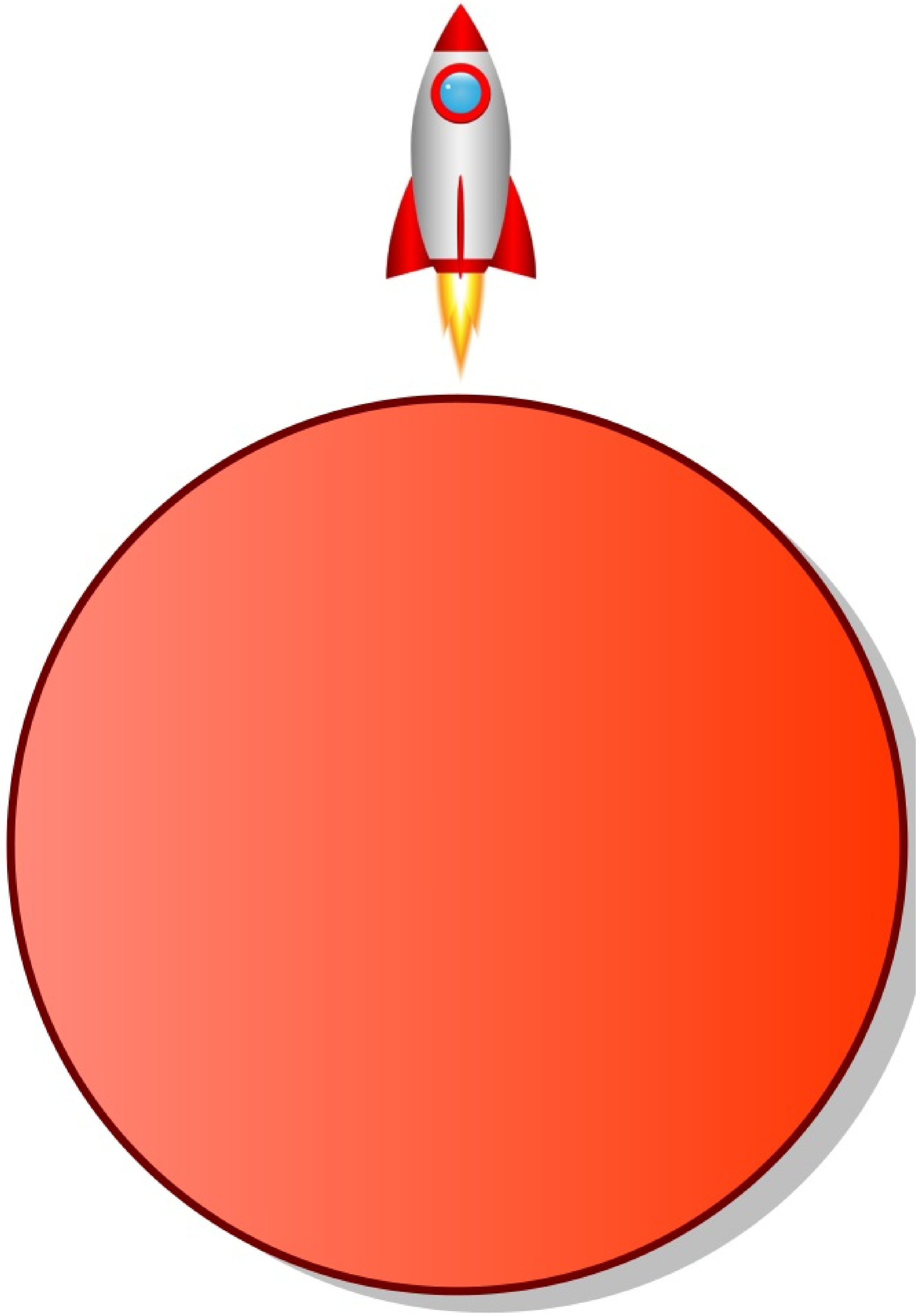

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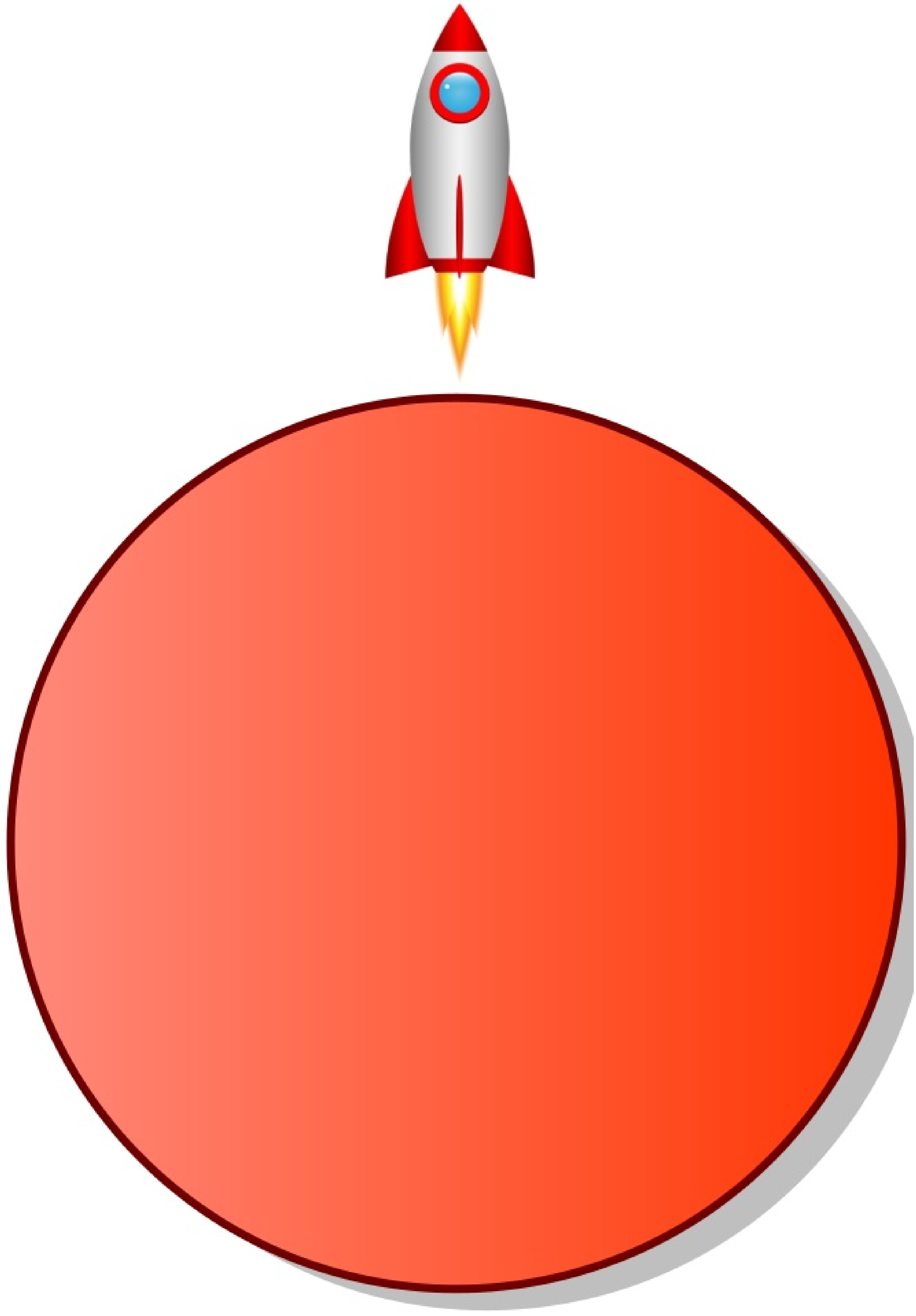

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$$u^\mu = \left[\left(1-\frac{2M}{R}\right)^{-1/2},\ 0,\ 0,\ 0\right]$$

Since she stays at fixed $r = R$

$$\frac{du^\mu}{d\tau} = 0$$

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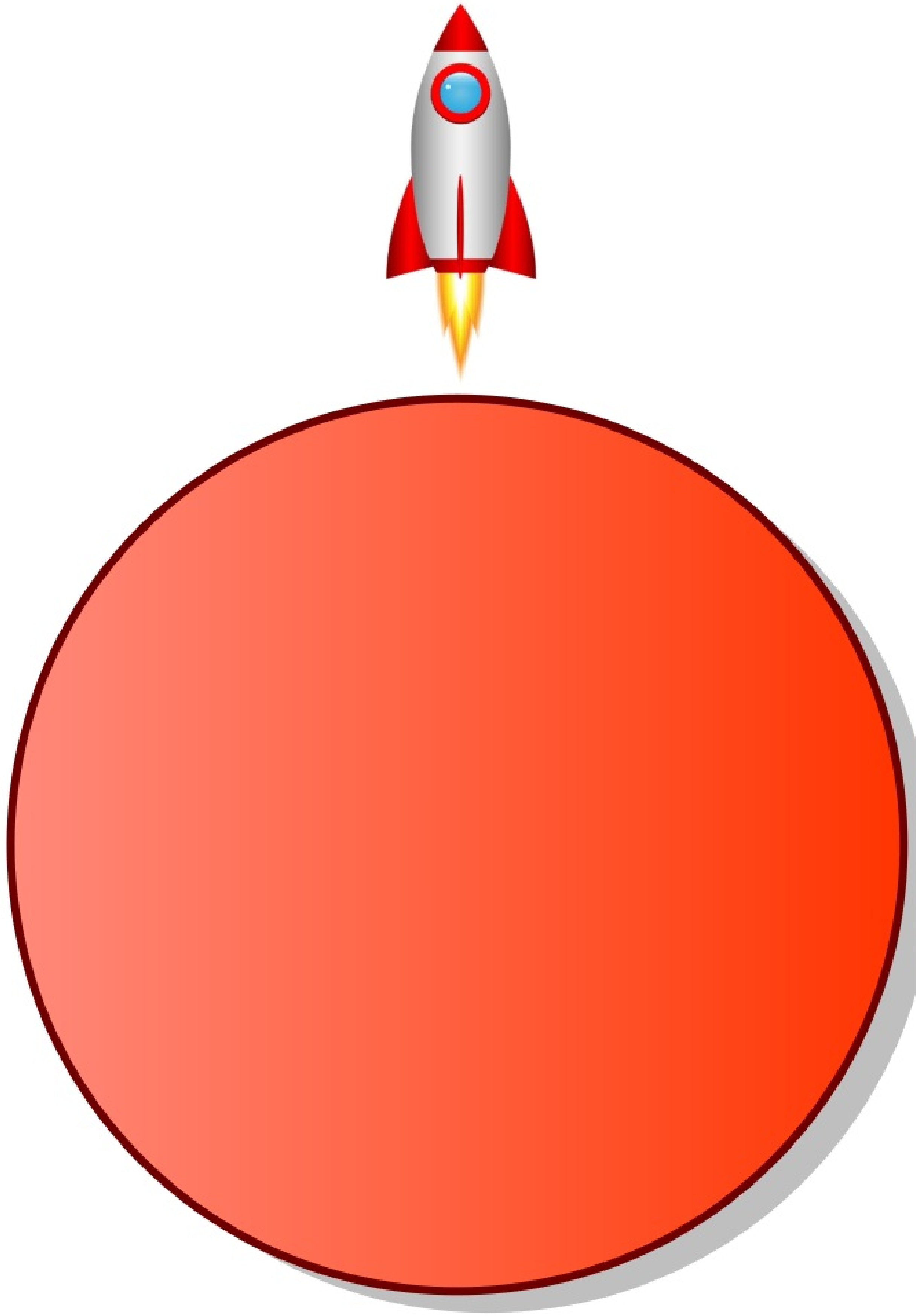

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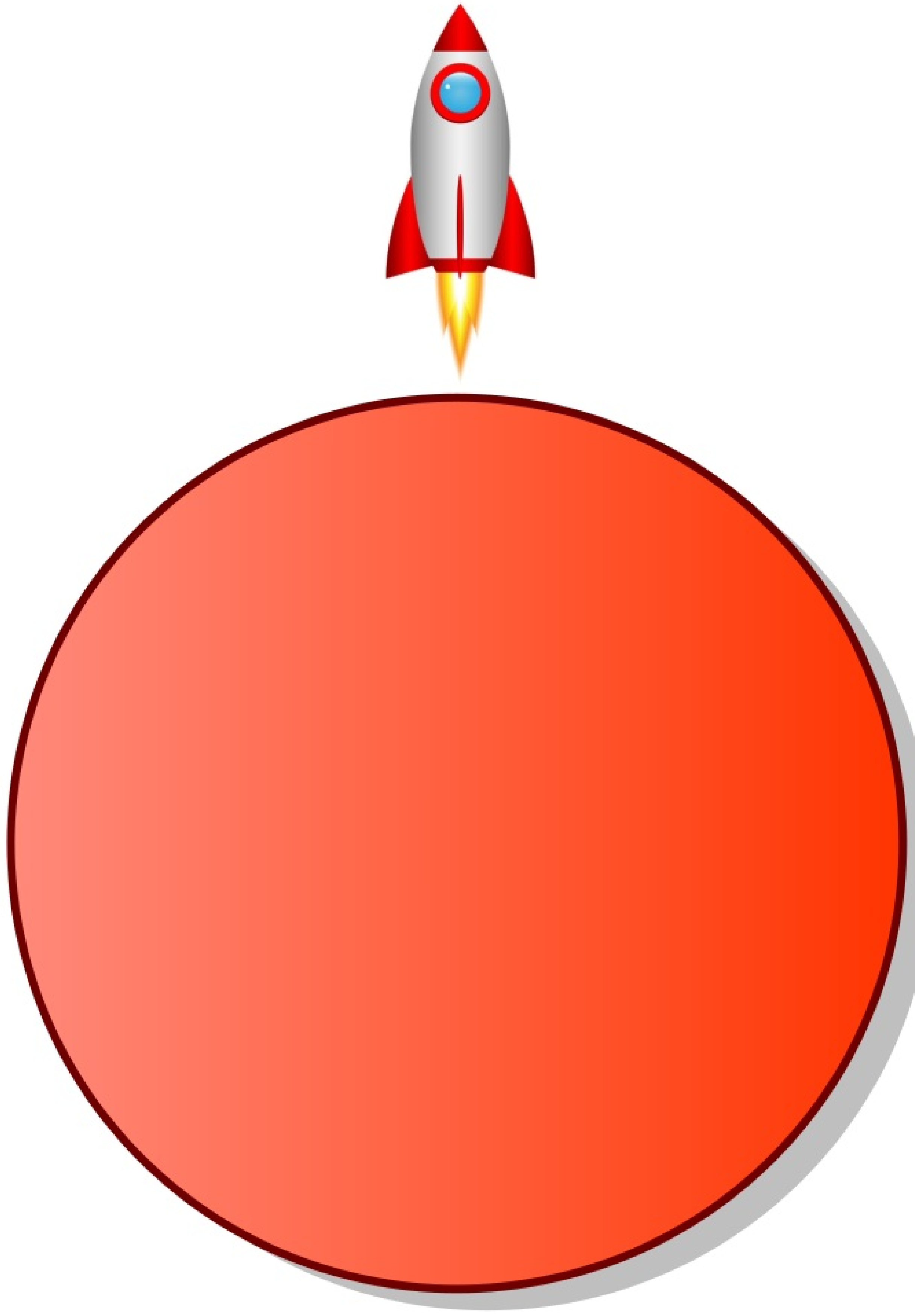

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Since she stays at fixed $r = R$

$$\frac{du^\mu}{d\tau} = 0 \quad \Rightarrow$$

all other components $= 0$

$$\frac{Du^\mu}{d\tau} = 0 + \Gamma^\mu{}_{00}\, u^0 u^0$$

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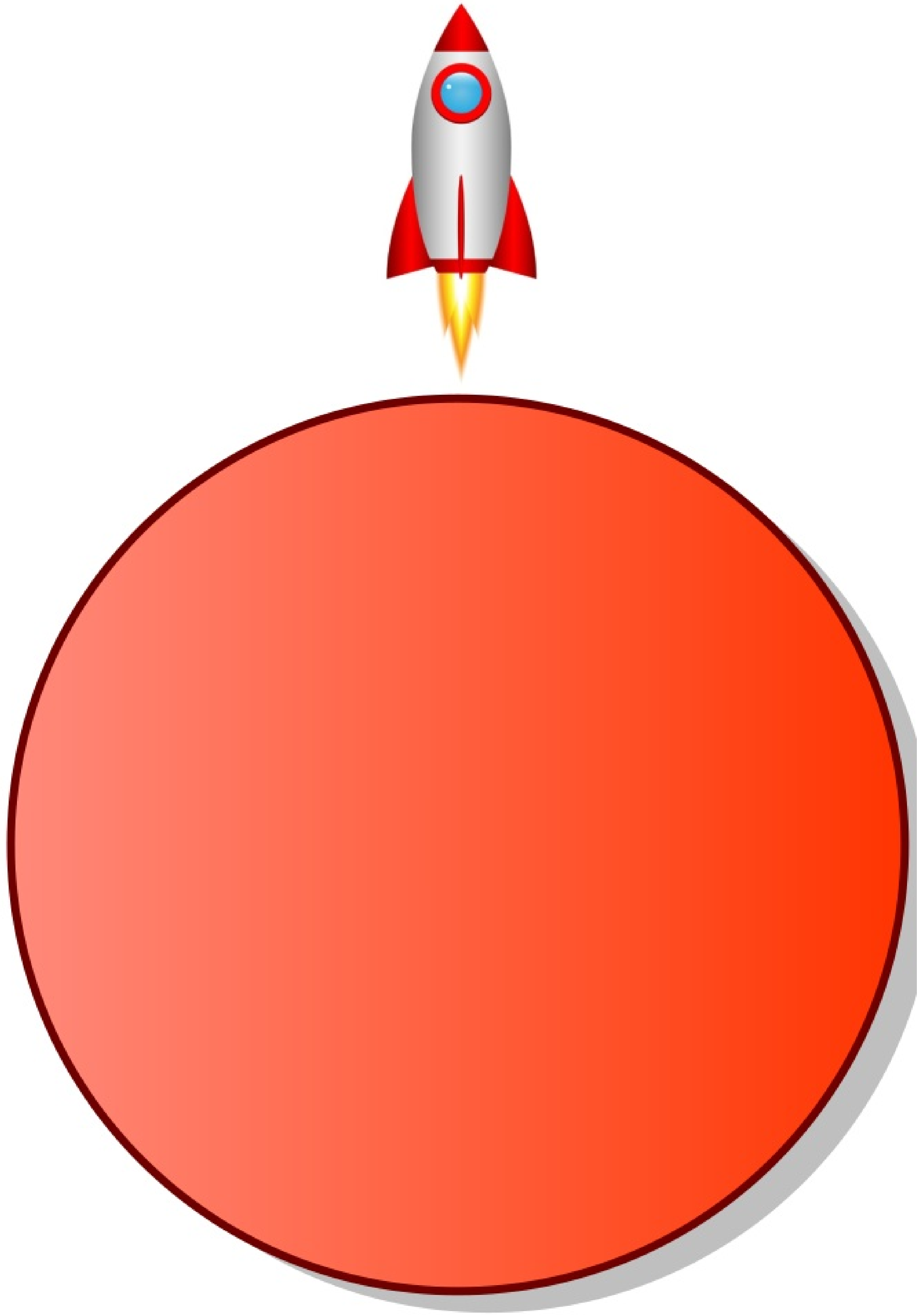

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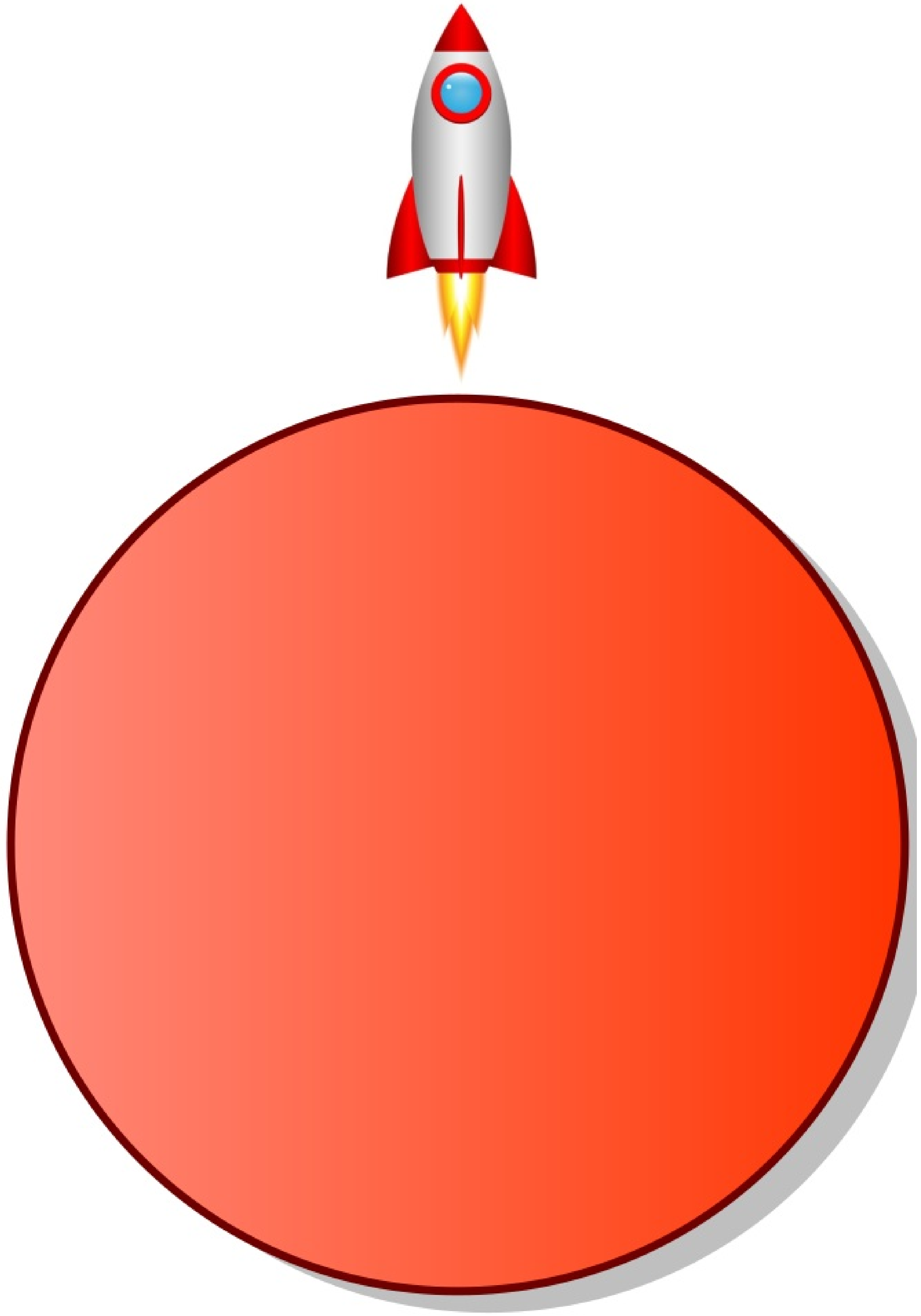

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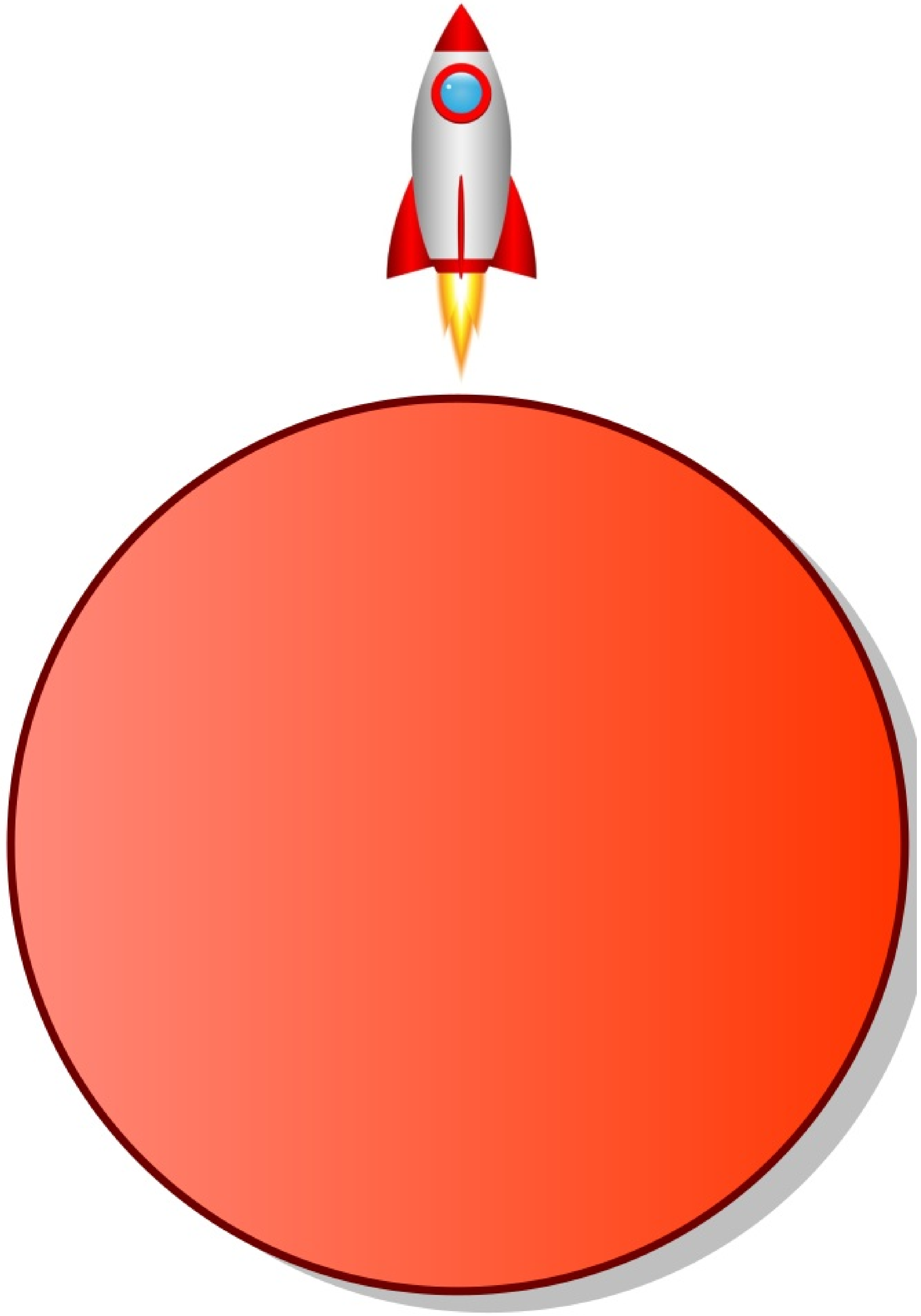

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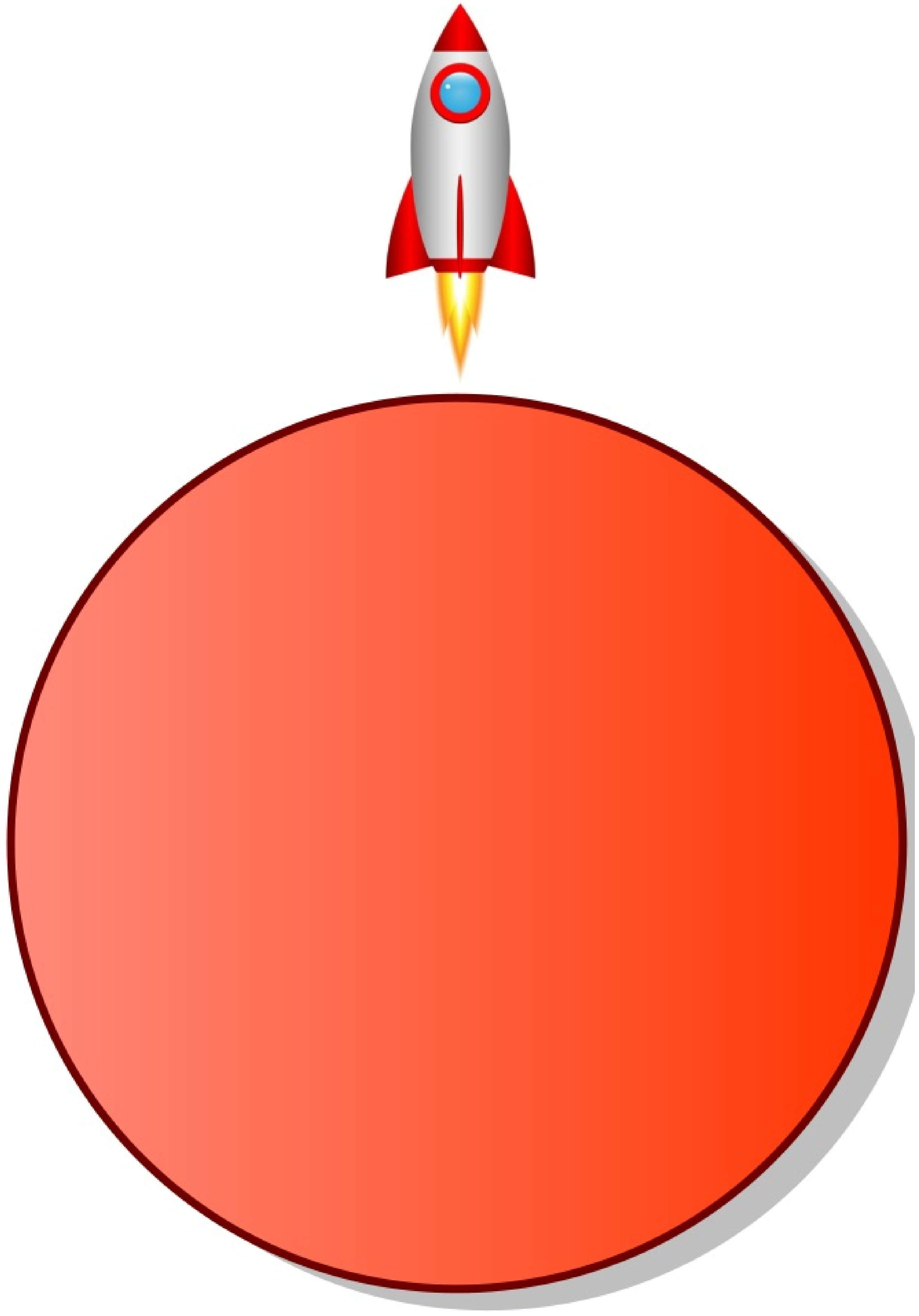
$$f^r = \frac{M}{r^2} \qquad\qquad f^\mu = \left[0, \frac{M}{R^2}, 0, 0\right]$$

But the observer measures force in her local inertial frame $\{\hat{e}_0, \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$

$$\Rightarrow f^{\hat{r}} = |g_{rr}|^{1/2} f^r$$

$$= \left(1 - \frac{2M}{R}\right)^{-1/2} \frac{M}{R^2}$$



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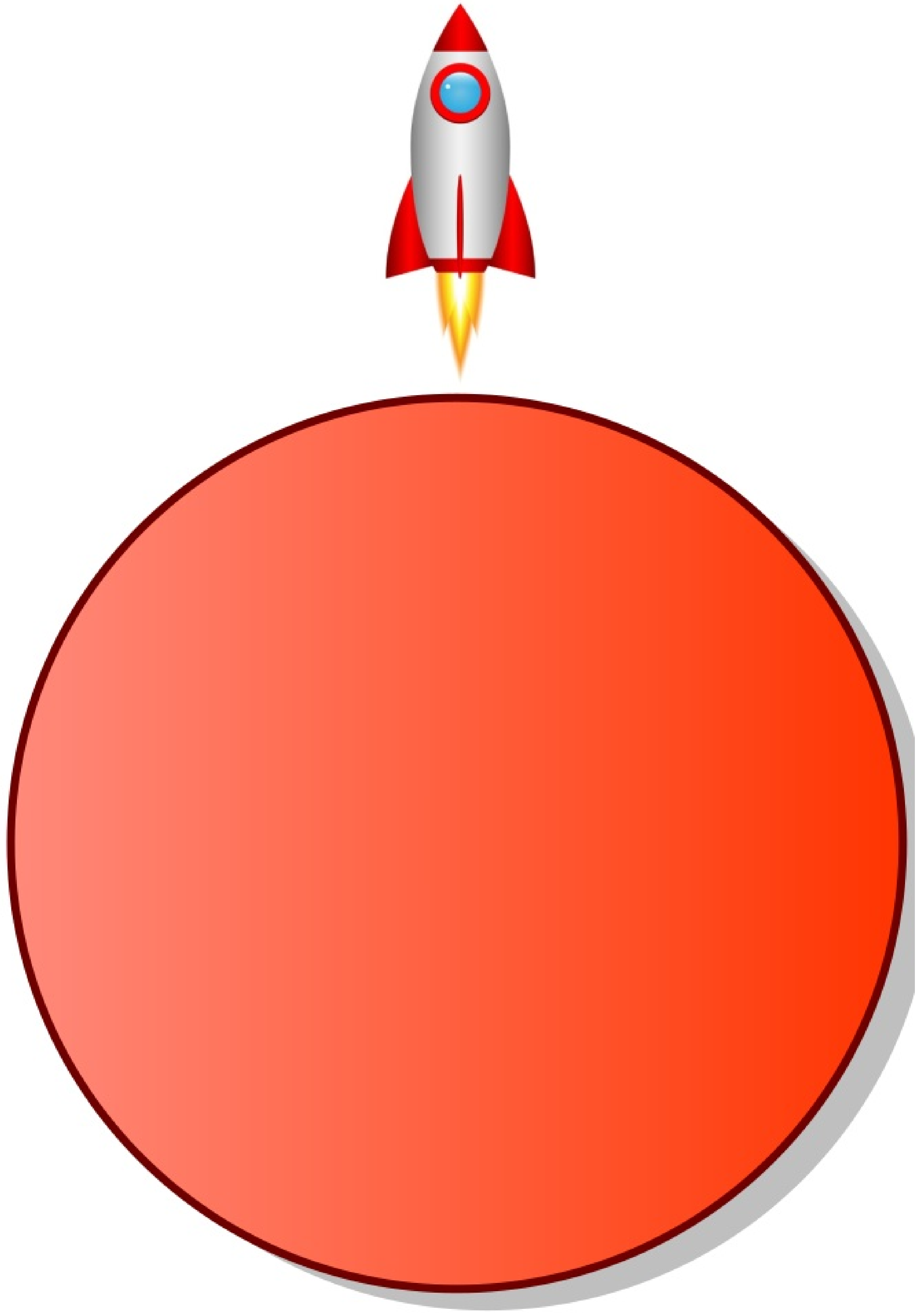

$$\Rightarrow f^{\hat{r}} = |g_{rr}|^{1/2} f^r$$

$$= \left(1 - \frac{2M}{R}\right)^{-1/2} \frac{M}{R^2}$$

Blows up as $R \rightarrow 2M$!

$$f^\mu = \left[0, \frac{M}{R^2}, 0, 0\right]$$

- Although it takes an infinite t for an observer to fall radially to $r=2M$, her proper time is finite. Even worse, after a finite τ, she will crush on $r=0$ (her time ends! - think about it carefully ...)

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- Consider an observer that starts from rest, relative to a stationary observer at $r = 10M$. How much time will elapse on the observer's clock before hitting the singularity?

Hartle, problem 5, ch. 12

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity:

$$\left.\begin{aligned} U^\mu &= [U^0, 0, 0, 0] \\ U^\mu U_\mu &= -1 \end{aligned}\right\} \Rightarrow U^\mu = \left[\left(1 - \frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$$

- Consider an observer that starts from rest, relative to a stationary observer at $r = 10M$. How much time will elapse on the observer's clock before hitting the singularity?

Hartle, problem 5, ch. 12

- $R = 10M$
- $l = 0$ (radial motion)
- stationary's observer 4-velocity: $U^\mu = \left[\frac{\sqrt{5}}{2}, 0, 0, 0\right]$

$$\left.\begin{array}{l} U^\mu = [U^0, 0, 0, 0] \\ U^\mu U_\mu = -1 \end{array}\right\} \Rightarrow U^\mu = \left[\left(1 - \frac{2M}{10M}\right)^{-1/2}, 0, 0, 0\right] = \left[\left(1 - \frac{2}{10}\right)^{-1/2}, 0, 0, 0\right]$$

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- $e = \left(1 - \frac{2M}{R}\right) \cdot u^0 = \left(1 - \frac{2M}{R}\right)\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{5}}$

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- $\frac{e^2 - 1}{2} = \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r}$

$\hookrightarrow V_{eff}(r)$ for $\ell = 0$

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$$\Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r} - \frac{1}{5}\right)^{1/2}$$

$\hookrightarrow$ r decreases

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity: $U^\mu = [\sqrt{5}, 0, 0, 0]$
- falling observer initially at rest w.r.t. U^μ, so has $u^\mu = U^\mu$

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$$\Rightarrow \quad \frac{\tau}{M} = 5\sqrt{5}\,\pi$$

- $e = \left(1 - \frac{2M}{R}\right) \cdot u^0 = \left(1 - \frac{2M}{R}\right)\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{5}}$

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Eddington - Finkelstein Coordinates

$$(t, r, \theta, \phi) \rightarrow (v, r, \theta, \phi)$$

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

Eddington - Finkelstein Coordinates

$$(t, r, \theta, \phi) \to (v, r, \theta, \phi)$$

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- regular at $r = 2M$, show explicitly that it is not true singularity

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$$dt = dv - dr - 2M \frac{\frac{dr}{2M}}{\frac{r}{2M} - 1}$$

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$$dt = dv - dr - 2M \frac{\frac{dr}{2M}}{\frac{r}{2M} - 1} = dv - dr\left[1 + \frac{1}{\frac{r}{2M} - 1}\right]$$

$$= dv - dr \frac{\cancel{\frac{r}{2M}}}{\cancel{\frac{r}{2M}}\left(1 - \frac{2M}{r}\right)} = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr$$

Eddington - Finkelstein Coordinates

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

$$dt = dv - \left(1-\frac{2M}{r}\right)^{-1} dr \quad \Rightarrow dt^2 = dv^2 + \left(1-\frac{2M}{r}\right)^{-2} dr^2 - 2\left(1-\frac{2M}{r}\right)^{-1} dv\,dr$$

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$$= -\left(1 - \frac{2M}{r}\right)\left(dv^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr\right) + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

$$= -\left(1 - \frac{2M}{r}\right) dv^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + 2\,dv\,dr + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

Eddington - Finkelstein Coordinates

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

$$dt = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr \quad \Rightarrow dt^2 = dv^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr$$

$$-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 =$$

$$= -\left(1 - \frac{2M}{r}\right)\left(dv^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr\right) + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

$$= -\left(1 - \frac{2M}{r}\right) dv^2 - \cancel{\left(1 - \frac{2M}{r}\right)^{-1} dr^2} + 2\,dv\,dr + \cancel{\left(1 - \frac{2M}{r}\right)^{-1} dr^2}$$

$$= -\left(1 - \frac{2M}{r}\right) dv^2 + 2\,dv\,dr$$

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- $r \to \infty \Rightarrow ds^2$ flat, $t = v - r$

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$$\left(r \to \infty \quad \frac{r}{2M} \gg \ln\frac{r}{2M} \right)$$

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- $r \to \infty \Rightarrow ds^2$ flat, $t = v - r$
- $r \to 0$ singularity
- $g_{\mu\nu}$ non-diagonal

$$(g_{\mu\nu}) = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad (g^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \left(1-\frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}$$

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$d\theta = d\varphi = 0$

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$$\Rightarrow \begin{cases} dv = 0 & (\alpha) \\ -\left(1 - \frac{2M}{r}\right) dv + 2dr = 0 & (\beta) \end{cases}$$

$$r = 2M \qquad (dr = 0) \qquad (\gamma)$$

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$$r = 2M \quad (dr = 0) \qquad (\gamma)$$

$$\Rightarrow \begin{cases} v = \text{const} & (\alpha) \\ v = 2\left(r + 2M \ln\left|\frac{r}{2M} - 1\right|\right) + c & (\beta) \\ r = 2M & (\gamma) \end{cases}$$