Lecture 10

Problem set 1 + solutions

① Compute $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}$ in the free falling astronaut problem when $r<2M$ and show that the results are the same as for the $r>2M$ case.

Solution

We follow the same steps shown in the Lecture's slides, but now for $r<2M$ the vector ∂_t is spacelike and ∂_r timelike. Therefore, in the local frame of the astronaut, she should define her orthonormal basis to be: $\hat{e}_t = |g_{rr}|^{-1/2}\partial_r$ and $\hat{e}_r = |g_{tt}|^{-1/2}\partial_t$

Indeed $\hat{e}_t \cdot \hat{e}_t = (|g_{rr}|^{-1/2})^2 \partial_r \cdot \partial_r = |g_{rr}|^{-1} g_{rr} = -1$, $\hat{e}_r \cdot \hat{e}_r = |g_{tt}|^{-1} g_{tt} = +1$

so that $\hat{e}_\mu \cdot \hat{e}_\nu = \eta_{\hat{\mu}\hat{\nu}}$

Then, we should replace $\hat{r} \to t \qquad \hat{t} \to r$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = |g_{rr}|^{-1} |g_{tt}|^{-1} R_{trtr} = \left|1-\frac{2M}{r}\right|^{+1} \left|1-\frac{2M}{r}\right|^{-1} \left(-\frac{2M}{r^3}\right) = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = |g_{rr}|^{-1} |g_{\theta\theta}|^{-1} R_{\theta r\theta r} = \left|1-\frac{2M}{r}\right|^{+1} (r^2)^{-1} \left[-\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1}\right]$$

$$= \left[-\left(1-\frac{2M}{r}\right)\right] r^{-2} \left[-\frac{M}{r} \left(1-\frac{2M}{r}\right)^{-1}\right] = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = (\text{same}) = +\frac{2M}{r^3}$$

$$R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = |g_{rr}|^{-1} |g_{\phi\phi}|^{-1} R_{\phi r\phi r} = \left|1-\frac{2M}{r}\right| r^{-2} \sin^{-2}\theta \left[-\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1} \sin^2\theta\right]$$

$$= \left[-\left(1-\frac{2M}{r}\right)\right] \left[-\frac{M}{r^3}\left(1-\frac{2M}{r}\right)^{-1}\right] = +\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = |g_{tt}|^{-1} |g_{\theta\theta}|^{-1} R_{t\theta t\theta} = \left|1-\frac{2M}{r}\right|^{-1} r^{-2} \frac{M}{r}\left(1-\frac{2M}{r}\right)$$

$$= \left[-\left(1-\frac{2M}{r}\right)^{-1}\right] \frac{M}{r^3}\left(1-\frac{2M}{r}\right) = -\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = |g_{tt}|^{-1} |g_{\phi\phi}|^{-1} R_{t\phi t\phi} = \left|1-\frac{2M}{r}\right|^{-1} r^{-2} \sin^{-2}\theta \quad \frac{M}{r}\left(1-\frac{2M}{r}\right)\sin^2\theta$$

$$= \left[-\left(1-\frac{2M}{r}\right)\right] \frac{M}{r^3}\left(1-\frac{2M}{r}\right) = -\frac{M}{r^3}$$

We check that the results are the same as for $r>2M$.

(2) Consider the Killing vector field $\xi = \partial_t$ of the Schwarzschild geometry. Show that at the horizon $r = 2M$

$$D_\xi \xi^\mu = \xi^\nu \nabla_\nu \xi^\mu = \kappa \xi^\mu ,$$

and compute κ. κ is the "surface gravity" of the black hole (Use the Eddington-Finkelstein coordinates for the computation) ($\Gamma^\mu_{\nu\rho}$ for E-F coordinates can be found in the Mathematica notebook ρ Lecture 10)

. A stationary observer is hovering at $(t, r, \theta, \phi) = (t, R, \frac{\pi}{2}, 0)$, over a Schwarzschild black hole. Her 4-velocity is u^μ, and her acceleration $a^\mu = u^\nu \nabla_\nu u^\mu = \frac{Du^\mu}{d\tau}$. If $\xi = \partial_t$ is the timelike killing vector field, then:

- Compute a^μ and a_μ in the (t, r, θ, ϕ) and the (v, r, θ, ϕ) coordinate systems (E-F : Eddington-Finkelstein coordinates)
- Compute the scalar $a^2 = a_\mu a^\mu$ and show that a^μ is a spacelike vector for $r > 2M$
- Compute $V = (-\xi_\mu \xi^\mu)^{1/2}$, and show that $a \cdot V = \kappa = \begin{pmatrix}\text{surface} \\ \text{gravity}\end{pmatrix}$
- Compute $(\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu)$ using E-F coordinates, and show that, at $r = 2M$, $\kappa = -\frac{1}{2} (\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu)$
- Show that $\eta^\mu = \xi^\nu \nabla_\nu \xi^\mu$ is space like for $r > 2M$. Is this in contradiction with $\xi^\nu \nabla_\nu \xi^\mu = \kappa \xi^\mu$, since for $r > 2M$ ξ^μ is timelike?

The KVF is $\xi = \partial_t$

We will work with E-F coordinates (v, r, θ, ϕ)

$$\partial_t = \frac{\partial}{\partial t} = \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = 1 \cdot \partial_v = \partial_v \quad , \text{ so } \quad \partial_t = [1, 0, 0, 0]$$

$$\xi^\nu \nabla_\nu \xi^\mu = \cancel{\frac{d}{d\tau}\xi^\mu}^{\,0} + \Gamma^\mu{}_{\nu\rho}\,\xi^\nu \xi^\rho = \Gamma^\mu{}_{00}\, 1 \cdot 1 = \Gamma^\mu{}_{00}$$

We look up the $\Gamma^\mu{}_{\nu\rho}$ of the E-F coordinates: (see notebook @ webpage - Lecture 10)

$$g_{\mu\nu} = \begin{pmatrix} -1 + \frac{2M}{r} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 - \frac{2M}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

Christoffel Symbols

$\Gamma^0{}_{0,0}$	$\frac{M}{r^2}$
$\Gamma^0{}_{2,2}$	$-r$
$\Gamma^0{}_{3,3}$	$-r\,\mathrm{Sin}[\theta]^2$
$\Gamma^1{}_{0,0}$	$\frac{M(-2M+r)}{r^3}$
$\Gamma^1{}_{1,0}$	$-\frac{M}{r^2}$
$\Gamma^1{}_{2,2}$	$2M - r$
$\Gamma^1{}_{3,3}$	$(2M - r)\,\mathrm{Sin}[\theta]^2$
$\Gamma^2{}_{2,1}$	$\frac{1}{r}$
$\Gamma^2{}_{3,3}$	$-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$
$\Gamma^3{}_{3,1}$	$\frac{1}{r}$
$\Gamma^3{}_{3,2}$	$\mathrm{Cot}[\theta]$

$\rightarrow$ only $\Gamma^0{}_{00} = \frac{M}{r^2}$ and $\Gamma^1{}_{00} = \frac{M}{r^2}\left(1 - 2\frac{M}{r}\right)$ are non zero

$$\rightarrow \xi^\nu \nabla_\nu \xi^\mu = \left[\frac{M}{r^2}, \frac{M}{r^2}\left(1 - 2\frac{M}{r}\right), 0, 0\right]$$

$$\Rightarrow \xi^\nu \nabla_\nu \xi^\mu \Big|_{r=2M} = \left[\frac{M}{(2M)^2}, 0, 0, 0\right] = \frac{1}{4M}[1, 0, 0, 0] = \frac{1}{4M}\,\xi^\mu \Rightarrow \kappa = \frac{1}{4M}$$

In (t, r, θ, ϕ) , $u^\mu = (u^0, 0, 0, 0)$

$$u^\mu u_\mu = -1 \Rightarrow g_{00} u^0 u^0 = -1 \Rightarrow -\left(1-\frac{2M}{r}\right)(u^0)^2 = -1 \Rightarrow u^0 = \left(1-\frac{2M}{r}\right)^{-1/2}$$

$$u^\mu = \left[\left(1-\frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right]$$

$$a^\mu = u^\nu \nabla_\nu u^\mu = u^0 \nabla_0 u^\mu = u^0 \cancel{\nabla_0} u^\mu + \Gamma^\mu{}_{0\lambda} u^0 u^\lambda = \Gamma^\mu{}_{00} u^0 u^0 = \Gamma^\mu{}_{00}\left(1-\frac{2M}{r}\right)^{-1}$$

Christoffel Symbols

$\Gamma^0{}_{1,0}$	$\frac{M}{r(-2M+r)}$
$\Gamma^1{}_{0,0}$	$\frac{M(-2M+r)}{r^3}$
$\Gamma^1{}_{1,1}$	$\frac{M}{2Mr-r^2}$
$\Gamma^1{}_{2,2}$	$2M-r$
$\Gamma^1{}_{3,3}$	$(2M-r)\,\mathrm{Sin}[\theta]^2$
$\Gamma^2{}_{2,1}$	$\frac{1}{r}$
$\Gamma^2{}_{3,3}$	$-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$
$\Gamma^3{}_{3,1}$	$\frac{1}{r}$
$\Gamma^3{}_{3,2}$	$\mathrm{Cot}[\theta]$

$\rightarrow$ only $\Gamma^1{}_{00} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right)$ is non zero

$$\Rightarrow a^\mu = \left[0, \frac{M}{r^2}, 0, 0\right]$$

$$\Rightarrow a^2 = a_\mu a^\mu = g_{\mu\nu} a^\mu a^\nu = g_{rr} a^r a^r = \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{M}{r^2}\right)^2$$

$$\Rightarrow a = \left(1-\frac{2M}{r}\right)^{-1/2} \frac{M}{r^2}$$

As $r \to 2M$, $a \to +\infty$. We notice that it corresponds to $a^{\hat{r}}$ of a stationary observer that we calculated in class (Hartle Box 12.2)

For E-F coordinates: $v = t + r + 2M \ln\left|\frac{r}{2M} - 1\right|$

$$g_{\mu\nu} = \begin{pmatrix} -1+\frac{2M}{r} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1-\frac{2M}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

$$\frac{\partial v}{\partial t} = 1 \qquad \frac{\partial v}{\partial r} = 1 + \frac{2M \frac{1}{2M}}{\frac{r}{2M} - 1} = \frac{\frac{r}{2M}}{\frac{r}{2M} - 1} = \frac{1}{1 - \frac{2M}{r}} = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$u^0 = u^v = \frac{\partial v}{\partial t} u^t = 1 \cdot \left(1 - \frac{2M}{r}\right)^{-1/2} \rightsquigarrow u^\mu = \left[\left(1 - \frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right]$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = g_{00} u^0 u^0 = -\left(1 - \frac{2M}{r}\right)\left(1 - \frac{2M}{r}\right) = -1$$

$$\left.\begin{aligned} a^0 &= a^v = \frac{\partial v}{\partial r} a^r = \left(1 - \frac{2M}{r}\right)^{-1} \cdot \frac{M}{r^2} \\ a^1 &= a^r = \text{(same)} \end{aligned}\right\} \Rightarrow a^\mu = \left[\left(1 - \frac{2M}{r}\right)^{-1} \frac{M}{r^2}, \frac{M}{r^2}, 0, 0\right]$$

$$a_0 = g_{0\mu} a^\mu = g_{00} a^0 + g_{01} a^1 = -\left(1 - \frac{2M}{r}\right)\left(1 - \frac{2M}{r}\right)^{-1} \frac{M}{r^2} + 1 \cdot \frac{M}{r^2} = 0$$

$$a_1 = g_{1\mu} a^\mu = g_{10} a^0 = 1 \left(1 - \frac{2M}{r}\right)^{-1} \frac{M}{r^2}$$

$$a^2 = a_\mu a^\mu = a_1 a^1 = \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{M}{r^2}\right)^2$$ (same as in S-coordinates)

$$a^2 = a_\mu a^\mu = \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{M}{r^2}\right)^2 > 0 \quad \text{for} \quad r > 2M \rightsquigarrow \text{spacelike}$$

$$a^2 = 0 \quad \text{at} \quad r = 2M$$

– In (t, r, θ, ϕ) $\quad \xi^\mu = [1, 0, 0, 0]$

$$\xi^\mu \xi_\mu = g_{\mu\nu} \xi^\mu \xi^\nu = g_{00} \xi^0 \xi^0 = -\left(1 - \frac{2M}{r}\right) \qquad (\text{timelike for } r > 2M)$$

$$V = (-\xi_\mu \xi^\mu)^{1/2} = \left|1 - \frac{2M}{r}\right|^{1/2}$$

$$a \cdot V = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \frac{M}{r^2} \cdot \left|1 - \frac{2M}{r}\right|^{1/2} = \frac{M}{r^2} \quad \text{for} \quad r > 2M$$

$$a \cdot V\Big|_{r=2M} = \frac{M}{(2M)^2} = \frac{1}{4M} = \kappa$$

- E-F coordinates:

$$g_{\mu\nu}=\begin{pmatrix} -1+\frac{2M}{r} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

$$g^{\mu\nu}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1-\frac{2M}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

Christoffel Symbols:

$\Gamma^{0}{}_{0,0}$	$\frac{M}{r^2}$
$\Gamma^{0}{}_{2,2}$	$-r$
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$\Gamma^{1}{}_{0,0}$	$\frac{M(-2M+r)}{r^3}$
$\Gamma^{1}{}_{1,0}$	$-\frac{M}{r^2}$
$\Gamma^{1}{}_{2,2}$	$2M-r$
$\Gamma^{1}{}_{3,3}$	$(2M-r)\,\mathrm{Sin}[\theta]^2$
$\Gamma^{2}{}_{2,1}$	$\frac{1}{r}$
$\Gamma^{2}{}_{3,3}$	$-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$
$\Gamma^{3}{}_{3,1}$	$\frac{1}{r}$
$\Gamma^{3}{}_{3,2}$	$\mathrm{Cot}[\theta]$

$$\xi^\mu = \partial_t = \partial_U$$
$$= [1, 0, 0, 0]$$

Compute $\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu$

show that at $r=2M$ $\kappa^2 = -\frac{1}{2}\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu$

- $\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu = g^{\mu\nu} g_{\rho\sigma} \nabla_\mu \xi^\rho \nabla_\nu \xi^\sigma$

- $\nabla_\mu \xi^\rho = \cancel{\partial_\mu \xi^\rho} + \Gamma^\rho{}_{\mu\lambda}\xi^\lambda$
$$= \Gamma^\rho{}_{\mu 0}\xi^0 = \Gamma^\rho{}_{\mu 0}$$

- $g^{\mu\nu} g_{\rho\sigma} \nabla_\mu \xi^\rho \nabla_\nu \xi^\sigma =$
$$g^{\mu\nu} g_{\rho\sigma} \Gamma^\rho{}_{\mu 0}\Gamma^\sigma{}_{\nu 0} =$$

$$= g_{\rho\sigma}\left(g^{01}\Gamma^\rho{}_{00}\Gamma^\sigma{}_{10} + g^{10}\Gamma^\rho{}_{10}\Gamma^\sigma{}_{00} + g^{11}\Gamma^\rho{}_{10}\Gamma^\sigma{}_{10}\right) =$$

$$= g_{\rho\sigma}\left(2 g^{01}\Gamma^\rho{}_{00}\Gamma^\sigma{}_{10} + g^{11}\Gamma^\rho{}_{10}\Gamma^\sigma{}_{10}\right) =$$

$$= 2 g_{00} g^{01}\Gamma^0{}_{00}\cancel{\Gamma^0{}_{10}} + 2 g_{01} g^{01}\Gamma^0{}_{00}\Gamma^1{}_{10} + 2 g_{10} g^{01}\Gamma^1{}_{00}\cancel{\Gamma^0{}_{10}}$$
$$g_{00} g^{11}\Gamma^0{}_{10}\cancel{\Gamma^0{}_{10}} \quad g_{01} g^{11}\cancel{\Gamma^0{}_{10}}\Gamma^1{}_{10} + g_{10} g^{11}\Gamma^1{}_{10}\cancel{\Gamma^0{}_{10}} = 2 g_{01} g^{01}\Gamma^0{}_{00}\Gamma^1{}_{10}$$

$$= 2\cdot 1\cdot 1\left(\frac{M}{r^2}\right)\left(-\frac{M}{r^2}\right) = -2\left(\frac{M}{r^2}\right)^2$$

$$\Rightarrow -\frac{1}{2}\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu = -\frac{1}{2}\left[-2\left(\frac{M}{r^2}\right)^2\right] = \left(\frac{M}{r^2}\right)^2$$

for $r=2M$ $\quad = \left[\frac{M}{(2M)^2}\right]^2 = \left(\frac{1}{4M}\right)^2 = k^2$

— In previous slides we computed $\eta^\mu = \xi^\nu \nabla_\nu \xi^\mu = \left[\frac{M}{r^2}, \left(1-\frac{2M}{r}\right)\frac{M}{r^2}, 0, 0\right]$ in E-F coordinates

$$\eta^\mu \eta_\mu = g_{\mu\nu}\eta^\mu\eta^\nu = g_{00}\eta^0\eta^0 + 2g_{01}\eta^0\eta^1$$

$$= -\left(1-\frac{2M}{r}\right)\left(\frac{M}{r^2}\right)^2 + 2\cdot 1\cdot\frac{M}{r^2}\left(1-\frac{2M}{r}\right)\frac{M}{r^2} = +\left(1-\frac{2M}{r}\right)\frac{M}{r^2} > 0 \quad \text{for } r>2M$$

so η^μ is spacelike for $r>2M$.

We have also shown that $\xi^\mu\xi_\mu = -\left(1-\frac{2M}{r}\right) < 0$ for $r>2M$, timelike

At $r=2M$, η^μ and ξ^μ are null, and the equation $\xi^\nu\nabla_\nu\xi^\mu = k\xi^\mu$ is consistent

$\xi^\nu\nabla_\nu\xi^\mu \neq k\xi^\mu$ for $r>2M$, as can be seen from $\eta^\mu = \left[\frac{M}{r^2}, \left(1-\frac{2M}{r}\right)\cdot\frac{M}{r^2}, 0, 0\right]$, $\xi^\mu = [1,0,0,0]$

(3) A particle of rest mass m is initially at rest relative to a stationary observer O_1 at $r=\infty$ from a Schwarzschild black hole. The particle is left to fall freely towards the center of the black hole. Another stationary observer O_2 is hovering above the black hole at $(t,r,\theta,\phi)=(t,R,\frac{\pi}{2},0)$, $R=$ const $>2M$.

- What is the speed of the particle as measured by O_2, when the particle goes through her lab?
- The particle is converted to radiation at O_2, which is sent back to $r=\infty$. What is the total energy of that radiation, as measured by stationary detectors at $r=\infty$?
- Suppose that when the particle reaches O_2, it is brought to rest in the O_2's lab, and the excess energy is converted to radiation, which is sent back to $r=\infty$. What is the energy of the radiation as measured by stationary detectors at $r=\infty$?

The particle has $u^\mu=\left(\frac{dt}{d\tau},\frac{dr}{d\tau},0,0\right)=(u^0,u^1,0,0)$ in (t,r,θ,ϕ) coordinates $(r>2M)$

The quantity $e=\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}$ is conserved $\Rightarrow \frac{dt}{d\tau}=e\left(1-\frac{2M}{r}\right)^{-1}$

As $r\to\infty$, the particle is at rest relative do the stationary observer. $\frac{dt}{d\tau}=1$, $\left(1-\frac{2M}{r}\right)\to 1$

so $e=1$ (1)

$$u^\mu u_\mu=-1 \Rightarrow g_{00}(u^0)^2+g_{11}(u^1)^2=-1 \Rightarrow -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2+\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2=-1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)e^2\left(1-\frac{2M}{r}\right)^{-2}+\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2=-1 \Rightarrow \left(\frac{dr}{d\tau}\right)^2=(e^2-1)+\frac{2M}{r}=\frac{2M}{r}$$

$$\Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

so $$u^\mu = (u^0, u^r, 0, 0) = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right) = \left(e\left(1-\frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right) = \left(\left(1-\frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

– O_2 measures the speed of the particle in her orthonormal frame $\{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3\}$

$$u^{\hat{0}} = |g_{00}|^{1/2} u^0 = \left(1-\frac{2M}{R}\right)^{1/2}\left(1-\frac{2M}{R}\right)^{-1} = \left(1-\frac{2M}{R}\right)^{-1/2}$$

$$u^{\hat{1}} = |g_{11}|^{1/2} u^1 = \left(1-\frac{2M}{R}\right)^{-1/2}\left(-\left(\frac{2M}{R}\right)^{1/2}\right) = -\left(1-\frac{2M}{R}\right)^{-1/2}\left(\frac{2M}{R}\right)^{1/2}$$

But $u^{\hat{\mu}} = (\gamma, \gamma\vec{v}) = \left[\left(1-\frac{2M}{R}\right)^{-1/2}, -\left(1-\frac{2M}{R}\right)^{-1/2}\left(\frac{2M}{R}\right)^{1/2}, 0, 0\right]$, so $\gamma = \left(1-\frac{2M}{r}\right)^{1/2}$

$$v^1 = -\left(\frac{2M}{R}\right)^{1/2}$$

Can also be derived from $E = -p_\mu u^\mu{}_{O_2} \Rightarrow \gamma = \frac{E}{m} = -u_\mu u^\mu{}_{O_2}$

$u^\mu_{O_2} = \left[\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$ (as we have shown for a stationary observer)

$$-u_\mu u^\mu_{O_2} = -g_{00} u^0 u^0_{O_2} = \left(1-\frac{2M}{R}\right)\left(1-\frac{2M}{R}\right)^{-1/2}\left(1-\frac{2M}{R}\right)^{-1} = \left(1-\frac{2M}{R}\right)^{-1/2}$$

$$\Rightarrow \gamma = \left(1-\frac{2M}{R}\right)^{-1/2} \Rightarrow \gamma^2 = \left(1-\frac{2M}{R}\right)^{-1} \Rightarrow v^2 = 1 - \frac{1}{\gamma^2} = 1 - \left(1-\frac{2M}{R}\right) = \frac{2M}{R}$$

$$\Rightarrow v = \pm \left(\frac{2M}{R}\right)^{1/2}$$

— When the particle reaches O_2's lab, its energy is measured to be:

$$E = -P_\mu u^\mu_{O_2} = m\left(1-\frac{2M}{R}\right)^{-1/2}$$

This energy is converted to radiation: $E_R = E$

and received by observer at $r=\infty$ redshifted

$$E_\infty = E_R\left(1-\frac{2M}{R}\right)^{1/2} = m\left(1-\frac{2M}{R}\right)^{-1/2}\left(1-\frac{2M}{R}\right)^{1/2} = m$$

So the detector will detect energy equal to the energy of the particle before it left the asymptotically flat region

- The particle reaches O_2 with energy $E_R = m\left(1-\frac{2M}{R}\right)^{-1/2}$

Its kinetic energy is converted to radiation:

$$T = E_R - m = m\left[\left(1-\frac{2M}{R}\right)^{-1/2} - 1\right]$$

This radiation arrives at $r=\infty$ with redshifted energy

$$E_{rad} = T\left(1-\frac{2M}{R}\right)^{1/2} = m - m\left(1-\frac{2M}{R}\right)^{1/2}$$

The $r=\infty$ stationary observer may interpret this as

$$\underbrace{m}_{\text{initial energy}} = \underbrace{E_{rad}}_{\text{radiated energy}} + \underbrace{m\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}}_{\text{energy left to the particle}}$$

(can in principle be received as radiation by annihilating the particle @ O_2)

4) Rindler Coordinates: Consider the (t,r,θ,φ) coordinates of the Schwarzschild metric, and make the transformation to the (t,ξ,θ,φ) coordinates, where

$$r-2M=\frac{\xi^2}{8M}.$$

Show that the metric in the new coordinates is given by

$$ds^2=-\frac{k^2\xi^2}{k^2\xi^2+1}dt^2+(k^2\xi^2+1)\,d\xi^2+\frac{1}{4k^2}(k^2\xi^2+1)^2d\Omega^2 \quad \text{, where}$$

$$k=\frac{1}{4M}, \quad d\Omega^2=d\theta^2+\sin^2\theta\,d\varphi^2$$

- Show that for $r\approx 2M$

$$ds^2\approx -k^2\xi^2dt^2+d\xi^2+\frac{1}{4k^2}d\Omega^2$$

- Consider the coordinate transformation

$$T=\xi\sinh(kt)$$

$$X=\xi\cosh(kt)$$

and show that the approximate metric above is flat

$$r-2M=\frac{\xi^2}{8M} \qquad k=\frac{1}{4M} \qquad \Rightarrow \qquad r=\frac{1}{2k}(k^2\xi^2+1) \quad , \quad dr=k\xi\,d\xi \Rightarrow dr^2=k^2\xi^2d\xi^2$$

$$1-\frac{2M}{r}=\frac{k\xi^2}{2}\frac{1}{r}=\frac{k^2\xi^2}{k^2\xi^2+1}$$

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2d\Omega^2=-\frac{k^2\xi^2}{k^2\xi^2+1}dt^2+\frac{k\xi^2+1}{k^2\xi^2}\left(k^2\xi^2d\xi^2\right)+\left(\frac{1}{2k}(k^2\xi^2+1)\right)^2d\Omega^2$$

$$=-\frac{k^2\xi^2}{k^2\xi^2+1}dt^2+(k^2\xi^2+1)\,d\xi^2+\frac{1}{4k^2}(k^2\xi^2+1)^2d\Omega^2$$

For $r \approx 2M \Rightarrow k^2\xi^2 + 1 = 2kr = \frac{r}{2M} \approx 1$

$$\Rightarrow ds^2 \approx -k^2\xi^2\, dt^2 + d\xi^2 + \frac{1}{4k^2} d\Omega^2$$

$T = \xi \sinh kt \Rightarrow dT = k\xi \cosh kt\, dt + \sinh kt\, d\xi$

$X = \xi \cosh kt \Rightarrow dX = k\xi \sinh kt\, dt + \cosh kt\, d\xi$

$$\Rightarrow dT^2 = k^2\xi^2 \cosh^2 kt\, dt^2 + \sinh^2 kt\, d\xi^2 + 2k\xi \cosh kt \sinh kt\, dt\, d\xi$$

$$dX^2 = k^2\xi^2 \sinh^2 kt\, dt^2 + \cosh^2 kt\, d\xi^2 + 2k\xi \sinh kt \cosh kt\, dt\, d\xi$$

$$\Rightarrow -dT^2 + dX^2 = -k^2\xi^2(\cosh^2 kt - \sinh^2 kt)\, dt^2 + (-\sinh^2 kt + \cosh^2 kt)\, d\xi^2$$

$$= -k^2\xi^2 dt^2 + d\xi^2$$

$\Rightarrow ds^2 \approx -dT^2 + dX^2 + \frac{1}{4k^2} d\Omega^2$ which a flat metric

- Orbital speed: Consider a hovering stationary observer O_1 at $(t, r, \theta, \phi) = (t, R, \frac{\pi}{2}, 0)$, $R = \text{const} > 2M$, over a Schwarzschild black hole.
A particle falls freely, moving on a $r = R$ circular trajectory, and goes through the lab of O_1.
Calculate the speed $v(R)$ of the particle, as measured by O_1.

The observer O_1 is stationary with 4-velocity

$$u_R^\mu = \left[\left(1 - \frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$$

The particle is falling freely with 4-velocity

$$u^\mu = \left[\frac{dt}{d\tau}, 0, 0, \frac{d\phi}{d\tau}\right]$$, and $e = \left(1 - \frac{2M}{R}\right)\frac{dt}{d\tau}$, $\ell = R^2 \frac{d\phi}{d\tau}$ are conserved, so

$$u^\mu = \left[e\left(1 - \frac{2M}{R}\right)^{-1}, 0, 0, \frac{\ell}{r^2}\right] , \quad u_\mu = [-e, 0, 0, \ell]$$

$$V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

$$V_{eff}'(r) = +\frac{M}{r^2} - \frac{\ell^2}{r^3} + \frac{3M\ell^2}{r^4}$$

For circular orbits $V_{eff}'(R) = 0 \Rightarrow M - \frac{\ell^2}{R} + \frac{3M\ell^2}{R^2} = 0 \Rightarrow \ell^2 = \frac{MR}{1 - \frac{3M}{R}}$

$$\frac{e^2 - 1}{2} = V_{eff}(R) \Rightarrow e^2 = \left(1 - \frac{2M}{R}\right)\left(1 + \frac{\ell^2}{R^2}\right) = \left(1 - \frac{2M}{R}\right)\left(1 + \frac{1}{R^2}\frac{MR}{1 - \frac{3M}{R}}\right)$$

$$= \left(1 - \frac{2M}{R}\right)\frac{1 - \frac{3M}{R} + \frac{M}{R}}{1 - \frac{3M}{R}} = \left(1 - \frac{2M}{R}\right)^2\left(1 - \frac{3M}{R}\right)^{-1}$$

$$\Rightarrow e = \left(1 - \frac{2M}{R}\right)\left(1 - \frac{3M}{R}\right)^{-1/2}$$

The speed is calculated from: $-E = p_\mu u^\mu_R = m\, u_\mu u^\mu_R \Rightarrow -\frac{E}{m} = -\gamma = u_\mu u^\mu_R \Rightarrow$

$$\gamma = e\left(1-\frac{2M}{R}\right)^{-1/2} = \left(1-\frac{2M}{R}\right)\left(1-\frac{3M}{R}\right)^{-1/2}\left(1-\frac{2M}{R}\right)^{-1} = \left(1-\frac{2M}{R}\right)^{1/2}\left(1-\frac{3M}{R}\right)^{-1/2}$$

But $\gamma = \dfrac{1}{\sqrt{1-v^2}} \Rightarrow v^2 = 1 - \dfrac{1}{\gamma^2} = 1 - \dfrac{1-\frac{3M}{R}}{1-\frac{2M}{R}} = \dfrac{1-\frac{2M}{R} - 1 + \frac{3M}{R}}{1-\frac{2M}{R}} = \dfrac{M/R}{1-\frac{2M}{R}}$

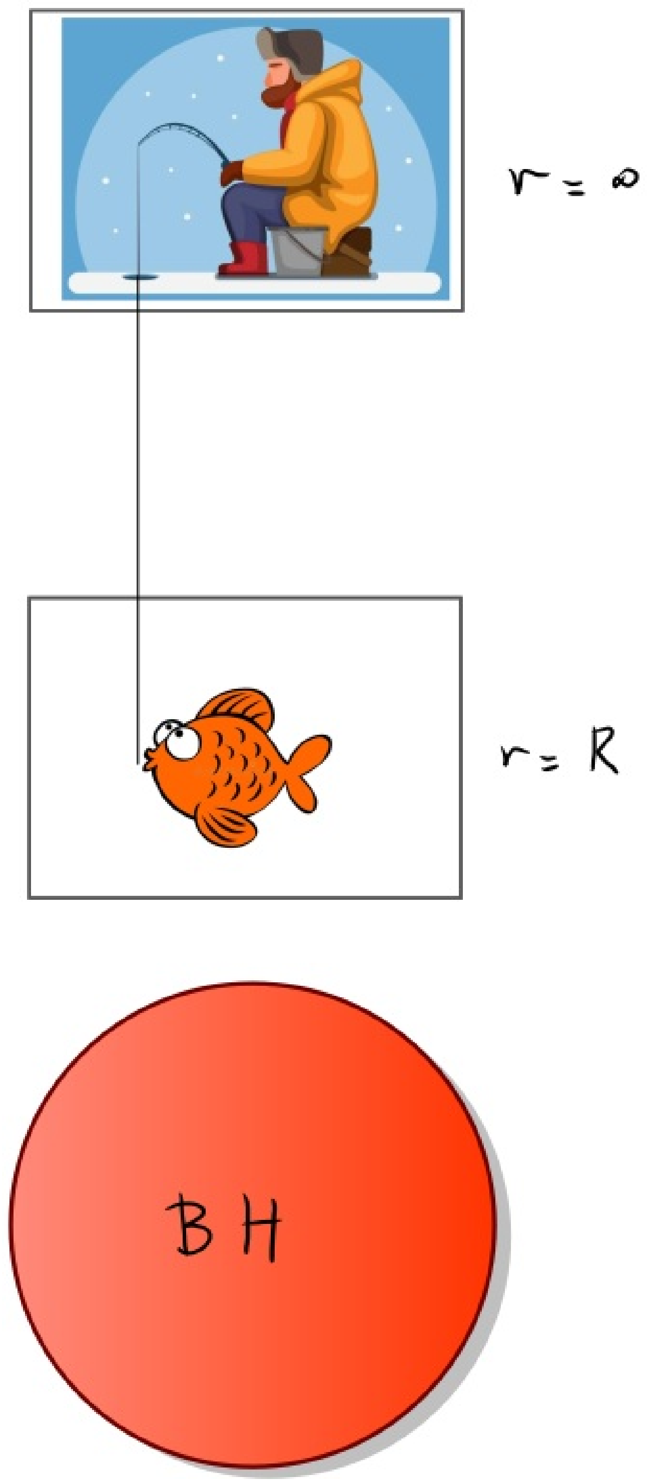
$$\Rightarrow v = \left(\frac{M}{R}\right)^{1/2}\left(1-\frac{2M}{R}\right)^{-1/2}$$

A note on surface gravity

We have seen that for a hovering observer in (t, r, θ, ϕ) coordinates

$$u^\mu = \left[\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right] \qquad a^\mu = \left[0, \frac{M}{r^2}, 0, 0\right]$$

$$a = \sqrt{a_\mu a^\mu} = \left(1-\frac{2M}{R}\right)^{-\frac{1}{2}} \frac{M}{R^2} \xrightarrow{R \to 2M} \infty$$



If an observer at $r=\infty$ holds an object at $r=R$ via a massless, inextensible string, then he has to exert a force $\propto a_\infty$

Indeed, moving "up" the string by a small $\Delta \ell$

$$\frac{1}{m} \Delta E_\infty = a_\infty \Delta \ell$$

Then the string at $r=R$ will exert a force per unit mass a_R such that

$$\frac{1}{m} \Delta E_R = a_R \Delta \ell$$

A note on surface gravity

But, as we have seen in the problems with the observer that annihilates the particle, or stops it and sends the kinetic energy via radiation, conservation of energy requires that

$$\Delta E_\infty = \Delta E \left(1-\frac{2M}{R}\right)^{1/2} \quad \text{is redshifted} \, , \Rightarrow$$

$$a_\infty = a_R \left(1-\frac{2M}{R}\right)^{1/2} = \left(1-\frac{2M}{R}\right)^{-\frac{1}{2}} \frac{M}{R^2} \left(1-\frac{2M}{R}\right)^{1/2} = \frac{M}{R^2}$$

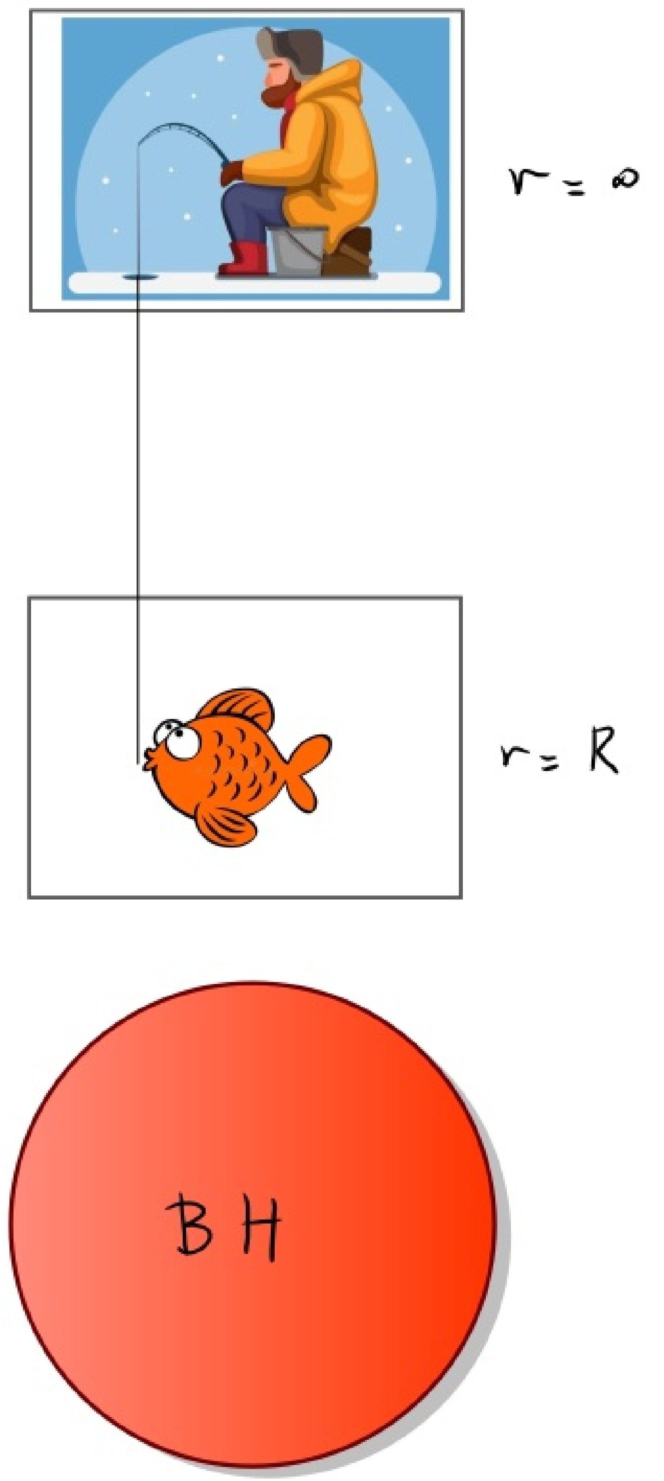
If an observer at $r=\infty$ holds an object at $r=R$ via a massless, inextensible string, then he has to exert a force $\propto a_\infty$

Indeed, moving "up" the string by a small Δl

$$\frac{1}{m} \Delta E_\infty = a_\infty \Delta l$$

Then the string at $r=R$ will exert a force per unit mass a_R such that

$$\frac{1}{m} \Delta E_R = a_R \Delta l$$



A note on surface gravity

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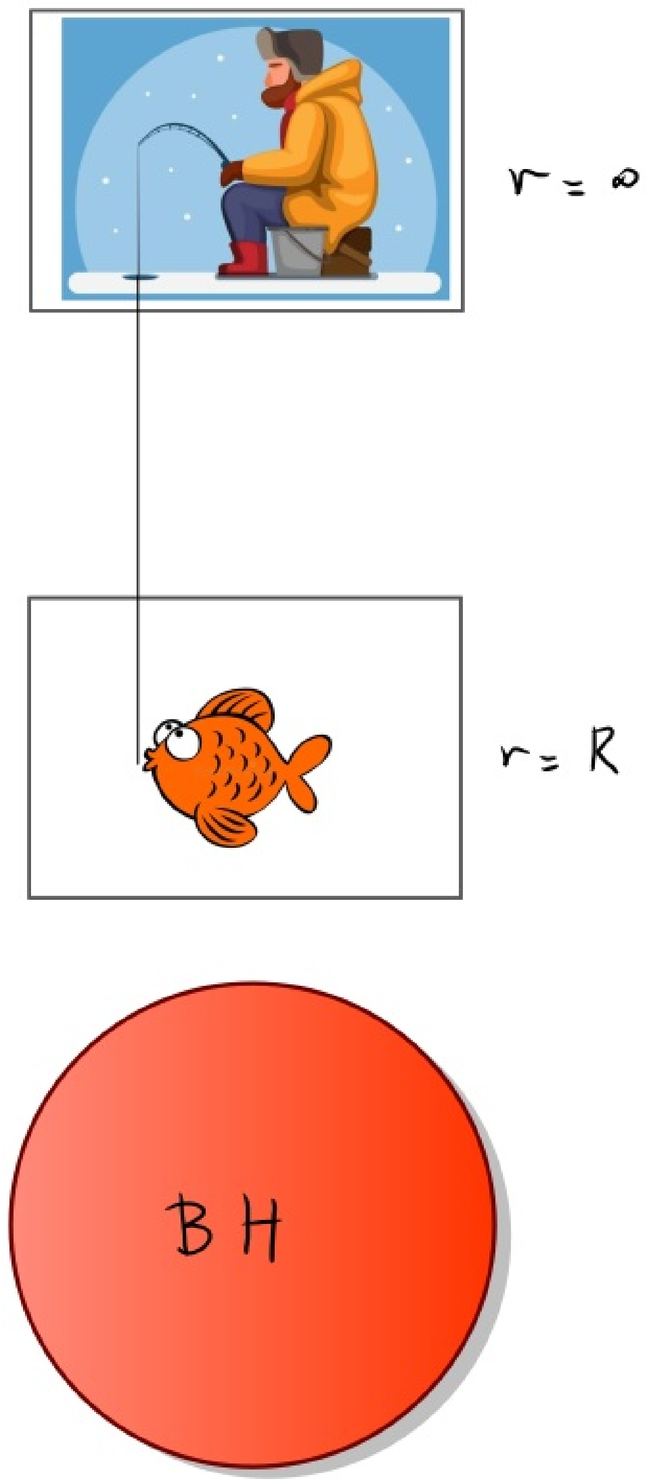
So, when $R \to 2M$, $\quad a_\infty = \dfrac{M}{(2M)^2} = \dfrac{1}{4M} = \kappa$, and

the surface gravity is the force per unit mass that a stationary observer at $r=\infty$ will have to exert to hold the object at the horizon.

Hence the term "surface gravity" used in astrophysics to denote the acceleration of (Newtonian) gravity on the surface of a star/planet.



(of course there is no real inextensible string that can do the job : $a_{R\to 2M} = \infty$, and soon a real string will extend + break)