The Schwarzschild Solution

Part I: Exterior Region $r > r_s$

The geometry outside a spherically symmetric star

- The metric in (t, r, θ, φ) coordinates:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\, d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

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  - spherically symmetric
  - asymptotically flat

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- The unique solution to Einstein's equation in the vacuum which is:

$\hookrightarrow R_{\mu\nu} = 0$

  - static
  - spherically symmetric
  - asymptotically flat $\rightarrow$ $r \to \infty \Rightarrow \left(1-\frac{2M}{r}\right) \to 1 \Rightarrow g_{\mu\nu} \to \eta_{\mu\nu}$

- $(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$

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$\Rightarrow$ ($\xi^\mu u_\mu$, $\eta^\mu u_\mu$ are conserved along geodesics w/ tangent vector u^μ)

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but: area of sphere: $A = 4\pi r^2$ (defines "r")

- Mass M:

For $r \gg 2M$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - 2\frac{M}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2 \qquad (1+x)^{-1} \approx 1 - x \quad x \ll 1$$

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* the parameter M is total mass of source of curvature
* there is no "particle" at $r=0$ of mass m. ($r=0$ not in spacetime)

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$$r > r_s \Rightarrow \begin{cases} \partial_t & \text{timelike} \\ \partial_r & \text{spacelike} \end{cases}$$

$$\left(\partial_t \cdot \partial_t = g_{00} = -\left(1 - \frac{r_s}{r}\right) < 0 \right)$$

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$\rightsquigarrow$ direction of "time"

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$$r > r_s \Rightarrow \begin{cases} \partial_t & \text{timelike} \\ \partial_r & \text{spacelike} \end{cases} \qquad r = r_s \begin{cases} \partial_t & \text{null} \\ g_{\mu\nu} & \text{singular} \end{cases}$$

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- timelike geodesics (= trajectories of freely falling massive particles) behave regularly across the horizon, which they cross in finite proper time
- curvature scalars have finite value on the horizon

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- r_s is irrelevant for ordinary stars, planets, objects in everyday life, where

$$R \gg r_s$$

$\hookrightarrow$ size of object

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$r_s =$ 8.8 mm for the Earth
2.95 km " " Sun
0.2 lyrs " " galaxy

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– $r = 0$ a true spacetime singularity:

- $K = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} = \dfrac{48M^2}{r^6} \rightarrow \infty$ (but regular at $r = 2M$)

↳ Kretschmann scalar

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  - $K = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} = \frac{48M^2}{r^6} \longrightarrow \infty$
  - timelike geodesics reach $r = 0$ at finite proper time

    $\rightarrow$ they end there: the end of time...

Units

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

SI:

$$G = 6.7 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$$

$$\kappa = \frac{8\pi G}{c^4} = 2.1 \times 10^{-43}\ N^{-1}$$

$$F = G\frac{Mm}{r^2} \qquad \Phi = \frac{GM}{r}$$

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Geometrized units

$$M(\text{in cm}) = \frac{G}{c^2} M(\text{in gr}) = 0.74 \times 10^{-28}\ \frac{\text{cm}}{\text{gr}}\ M(\text{in gr})$$

$$\Rightarrow \begin{cases} M_\odot = 1.5\ \text{km} \\ M_\oplus = 4.4\ \text{mm} \end{cases} \qquad (= 2 r_s)$$

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Restore metric units: $M \to \frac{GM}{c^2}$ $\quad t \to ct,\ \tau \to c\tau$

Quantity	SI dimension	Geometric dimension	Multiplication factor
Length	L	L	1
Time	T	L	c
Mass	M	L	$G\,c^{-2}$
Velocity	$\mathrm{L\,T^{-1}}$	1	c^{-1}
Angular velocity	$\mathrm{T^{-1}}$	$\mathrm{L^{-1}}$	c^{-1}
Acceleration	$\mathrm{L\,T^{-2}}$	$\mathrm{L^{-1}}$	c^{-2}
Energy	$\mathrm{M\,L^{2}\,T^{-2}}$	L	$G\,c^{-4}$
Energy density	$\mathrm{M\,L^{-1}\,T^{-2}}$	$\mathrm{L^{-2}}$	$G\,c^{-4}$
Angular momentum	$\mathrm{M\,L^{2}\,T^{-1}}$	$\mathrm{L^{2}}$	$G\,c^{-3}$
Force	$\mathrm{M\,L\,T^{-2}}$	1	$G\,c^{-4}$
Power	$\mathrm{M\,L^{2}\,T^{-3}}$	1	$G\,c^{-5}$
Pressure	$\mathrm{M\,L^{-1}\,T^{-2}}$	$\mathrm{L^{-2}}$	$G\,c^{-4}$
Density	$\mathrm{M\,L^{-3}}$	$\mathrm{L^{-2}}$	$G\,c^{-2}$
Electric charge	T I	L	$G^{1/2}\,c^{-2}\,\varepsilon_0^{-1/2}$
Electric potential	$\mathrm{M\,L^{2}\,T^{-3}\,I^{-1}}$	1	$G^{1/2}\,c^{-2}\,\varepsilon_0^{1/2}$
Electric field	$\mathrm{M\,L\,T^{-3}\,I^{-1}}$	$\mathrm{L^{-1}}$	$G^{1/2}\,c^{-2}\,\varepsilon_0^{1/2}$
Magnetic field	$\mathrm{M\,T^{-2}\,I^{-1}}$	$\mathrm{L^{-1}}$	$G^{1/2}\,c^{-1}\,\varepsilon_0^{1/2}$

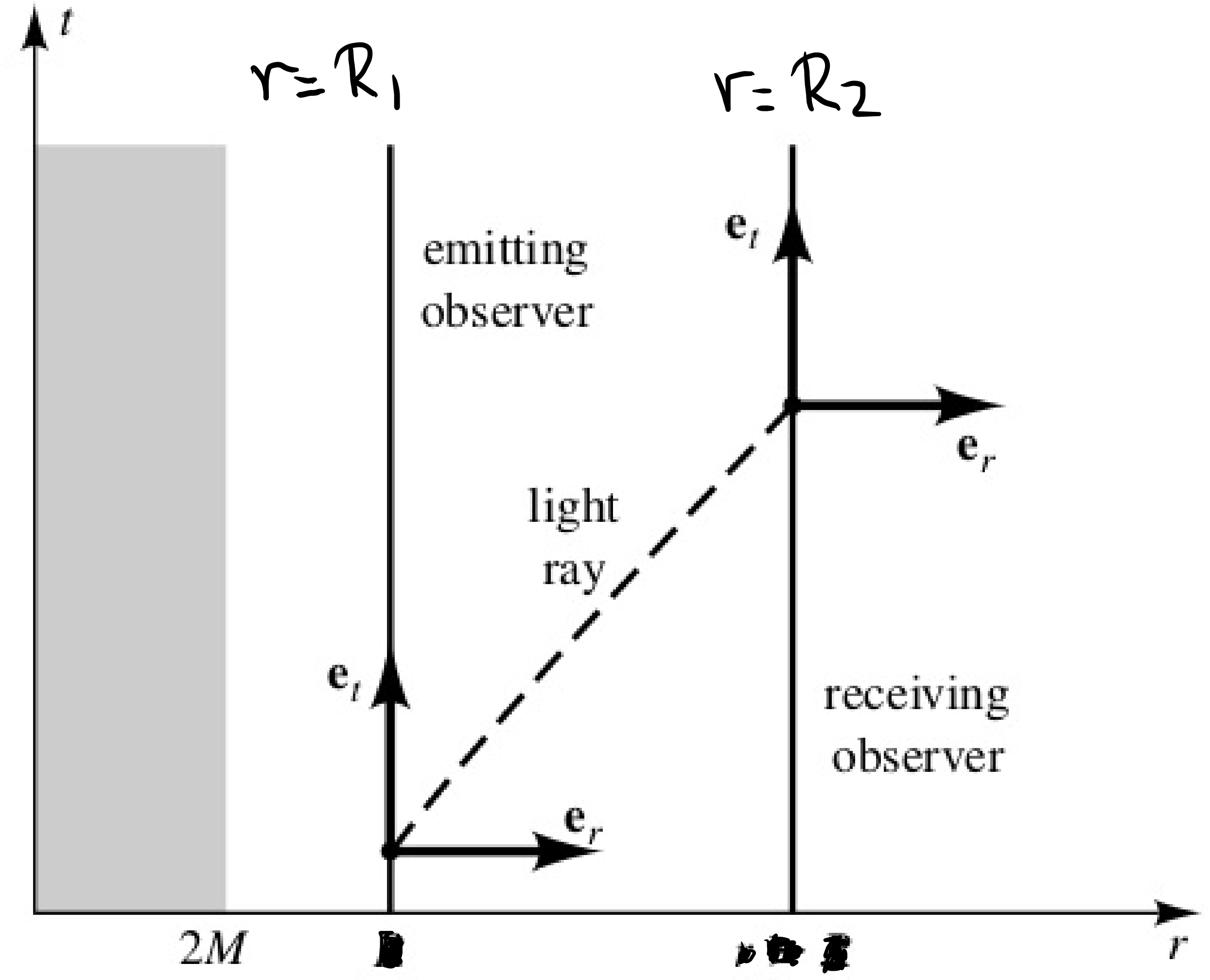
https://en.wikipedia.org/wiki/Geometrized_unit_system

Gravitational Redshift

- $\xi = \partial_t$ a Killing Vector field

  $p_\mu \xi^\mu$ conserved along geodesic w/ tangent vector p^μ

  $\xi^\mu = (1, 0, 0, 0)$



Hartle, Fig 9.1

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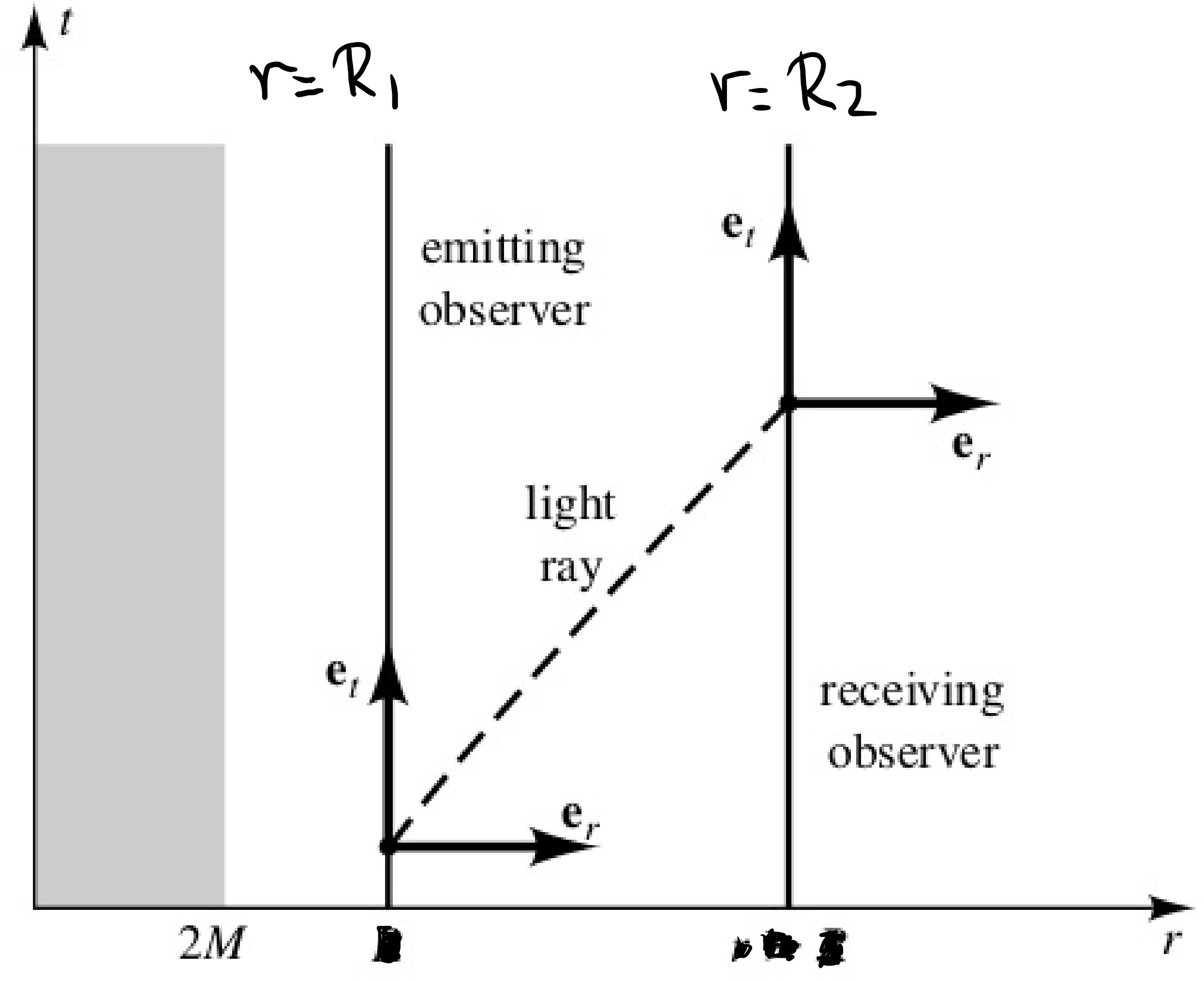

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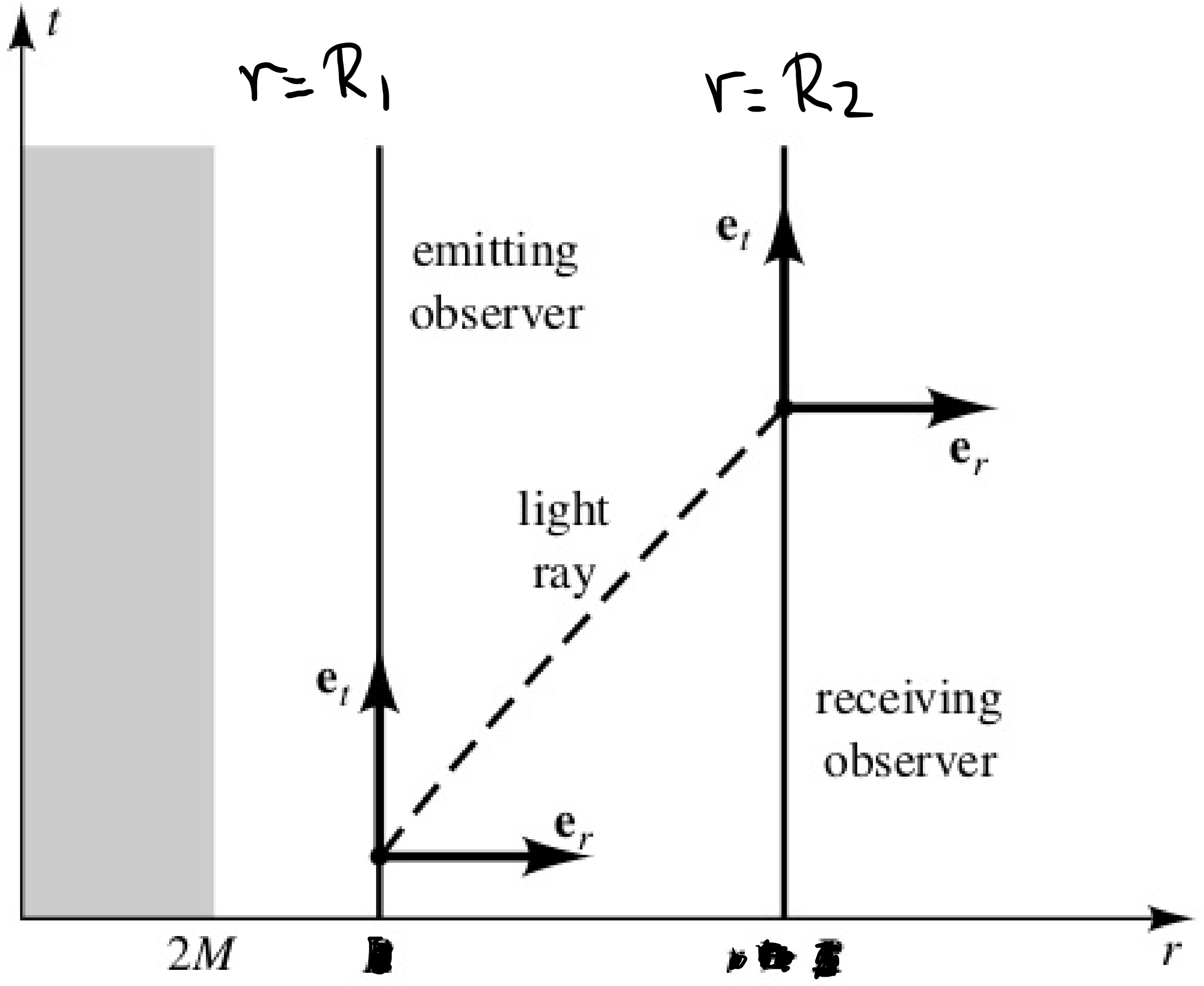

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- 4 velocity of observer at R_1: $u_1^\mu = (u_1^0, 0, 0, 0)$

  $u_1^\mu u_{1\mu} = -1 \Rightarrow g_{\mu\nu} u_1^\mu u_1^\nu = -1 \Rightarrow g_{00} u_1^0 u_1^0 = -1$

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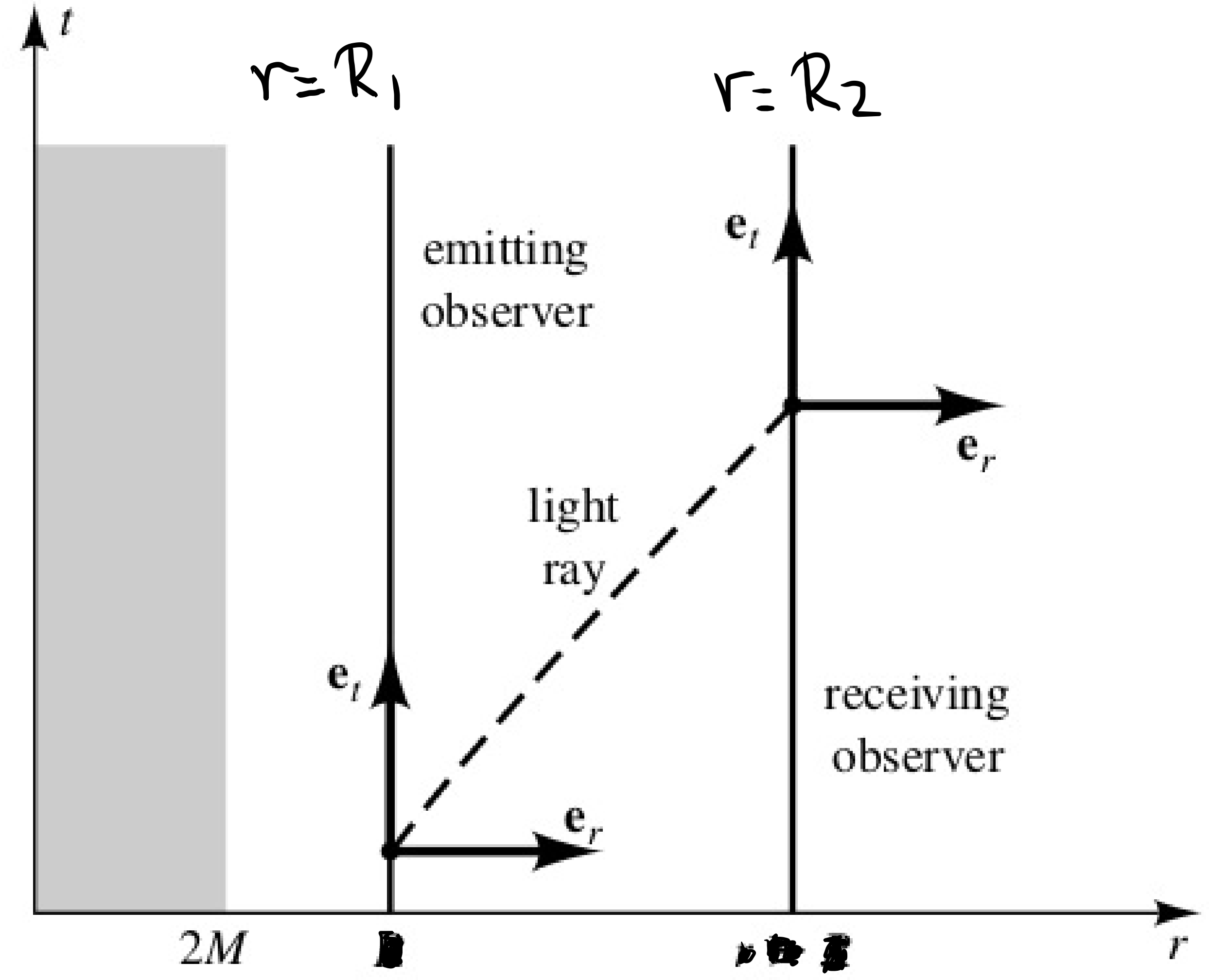

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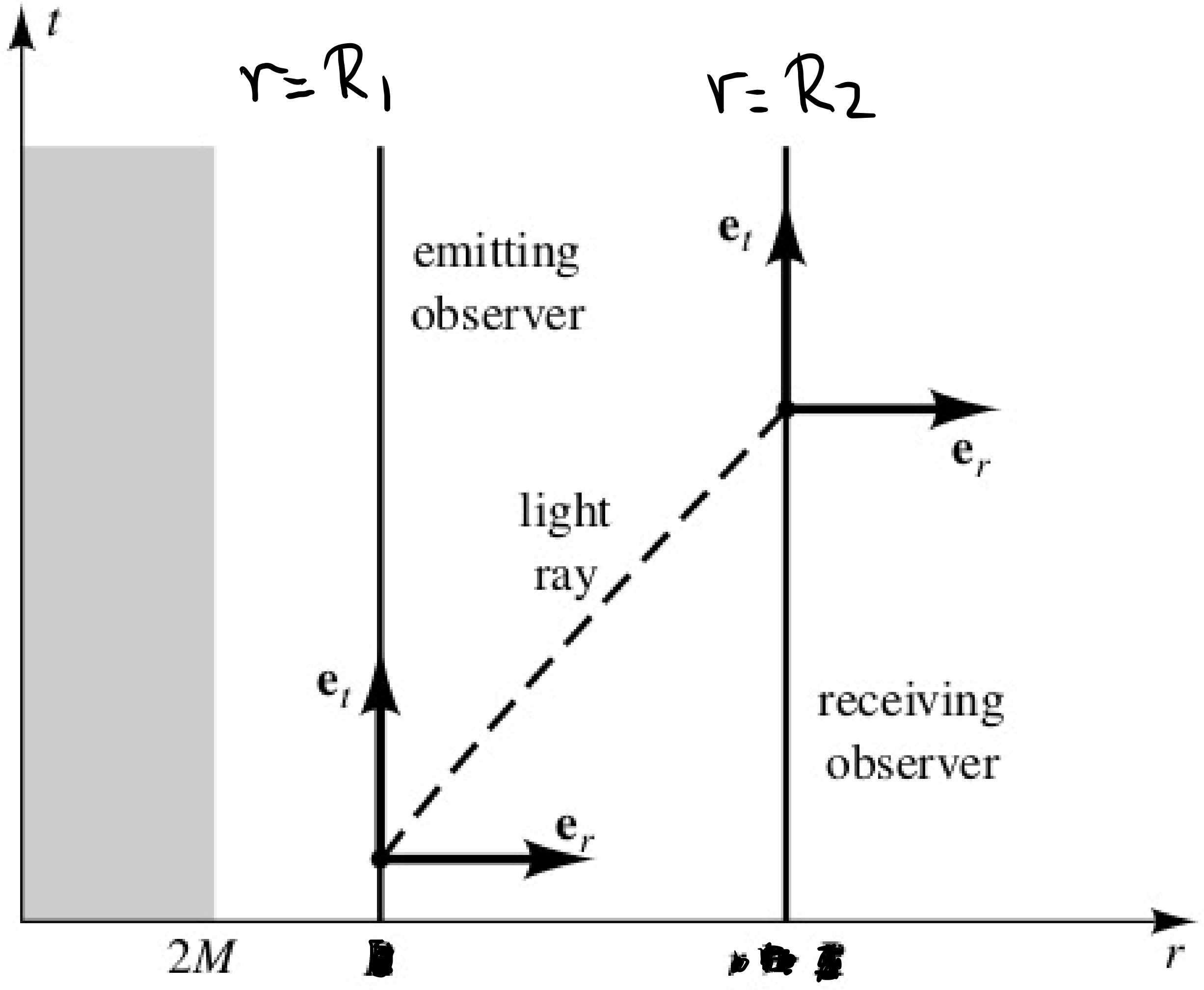

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  $u_1^0 = \left(1 - \frac{2M}{r}\right)^{-1/2}$

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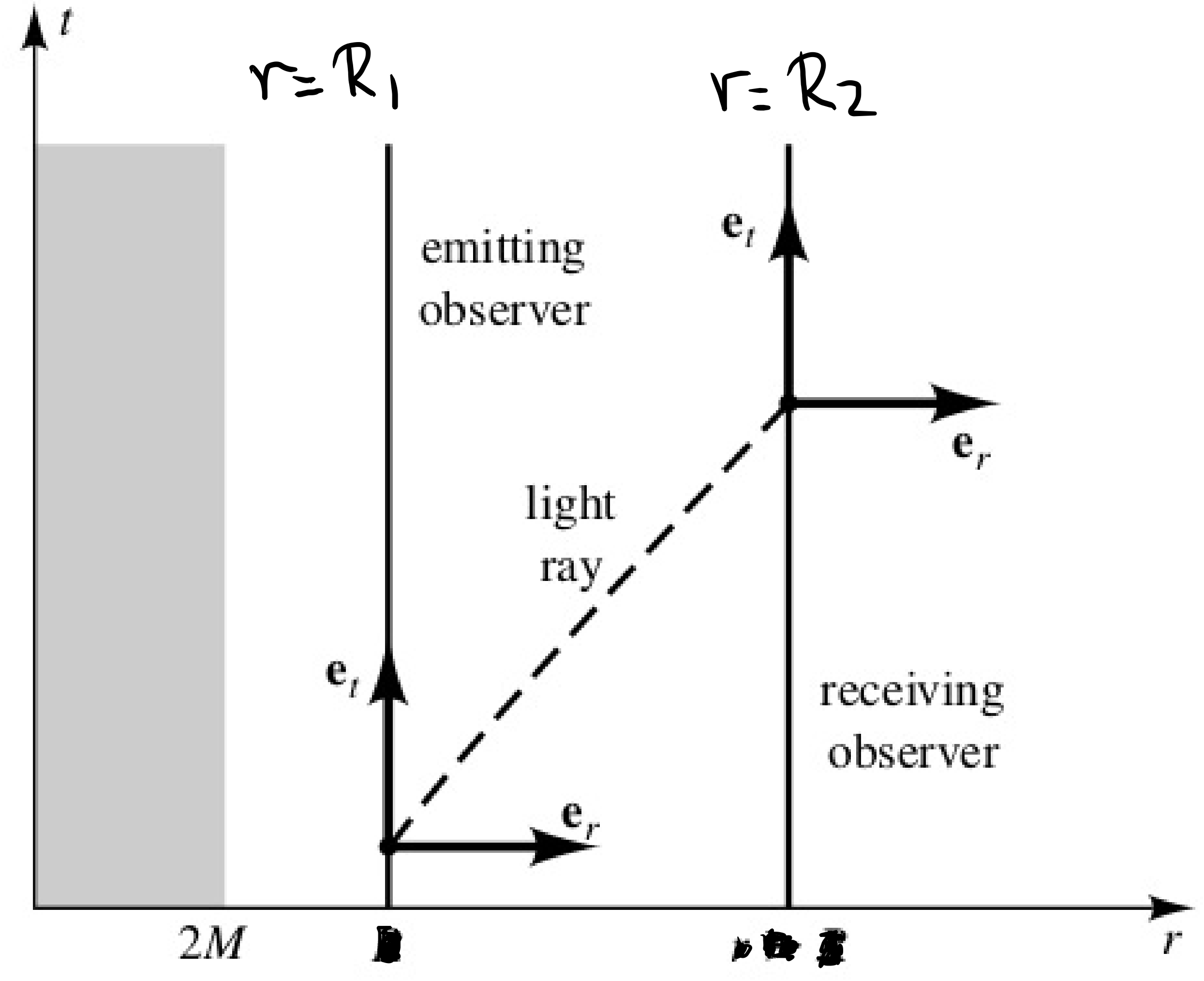

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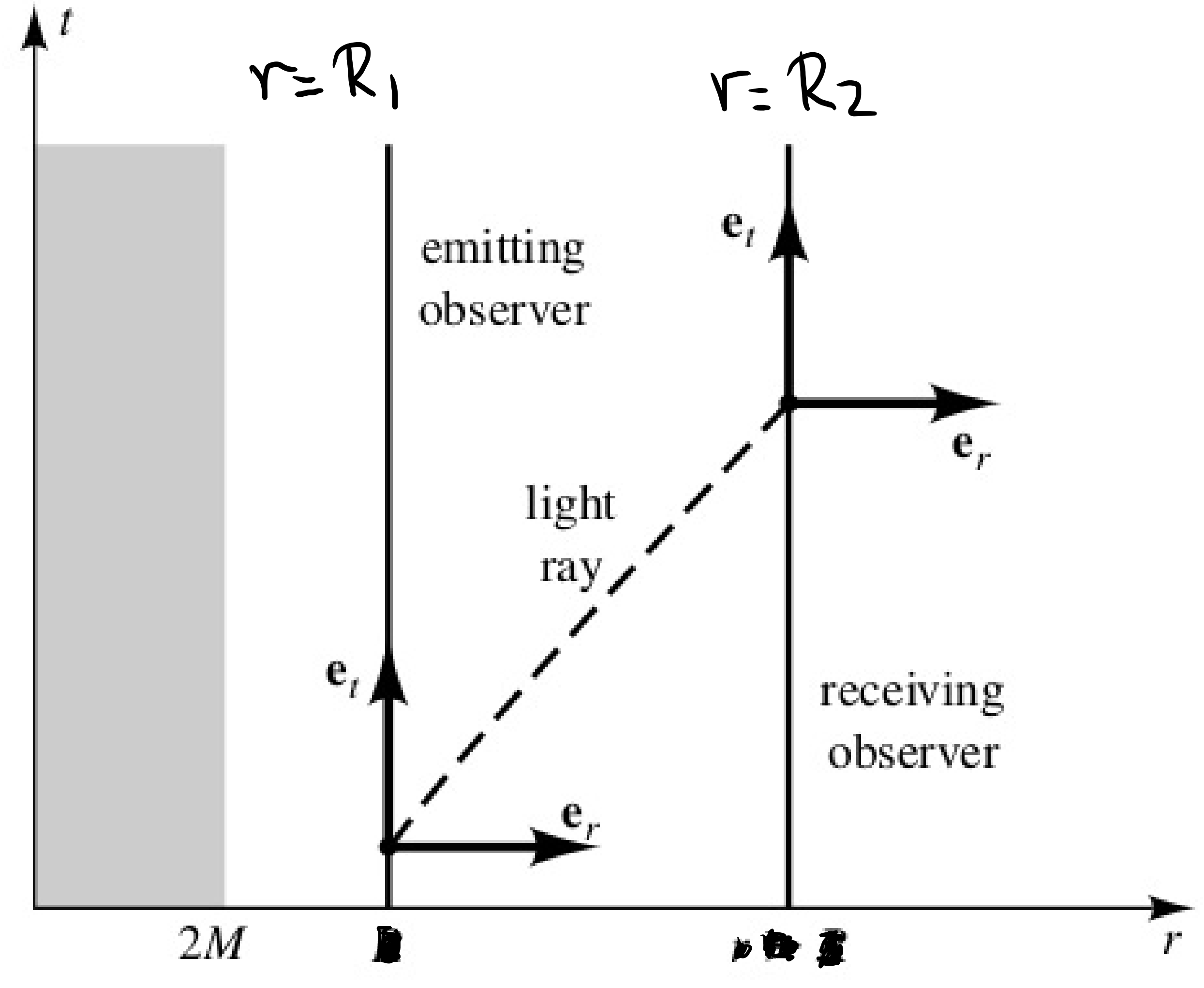

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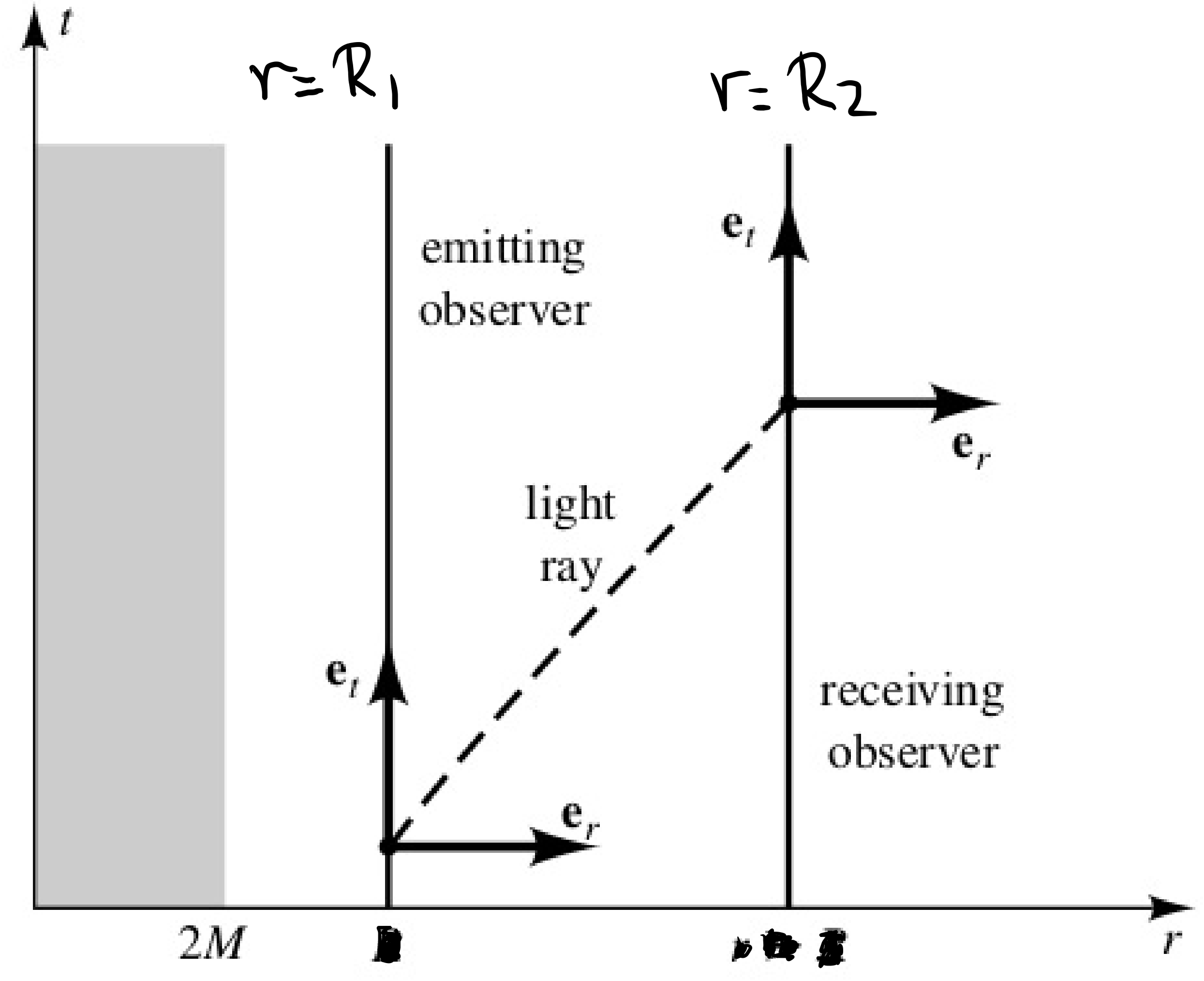
- " " R_2: $u_2^\mu = (u_2^0, 0, 0, 0) = \left(\left(1 - \frac{2M}{R_2}\right)^{-1/2}, 0, 0, 0 \right)$

$$\Rightarrow \quad u_1^\mu = \left(1-\frac{2M}{R_1}\right)^{-1/2}\xi^\mu \qquad u_2^\mu = \left(1-\frac{2M}{R_2}\right)^{-1/2}\xi^\mu$$



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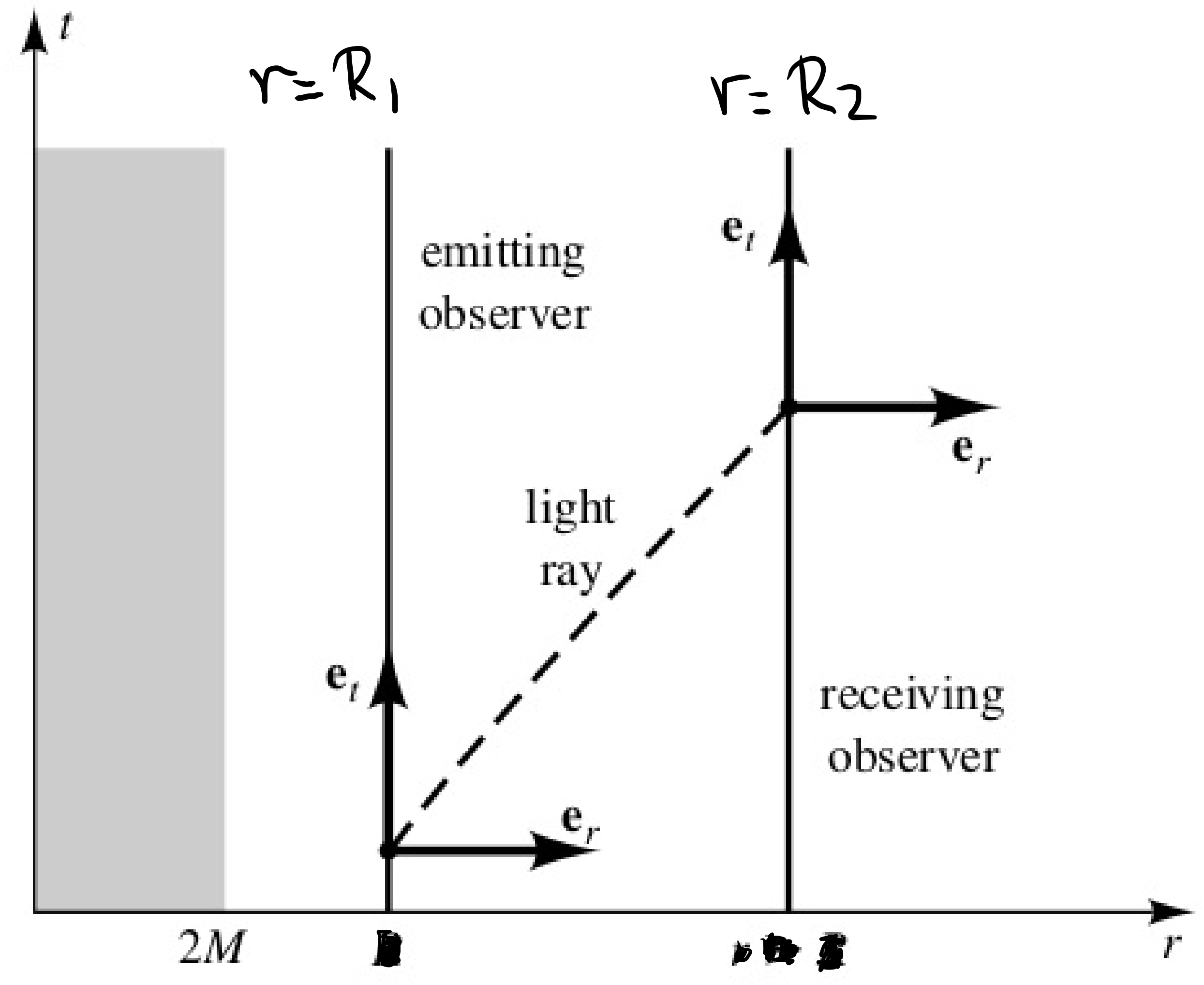
- " received " " " $r = R_2$

- 4 velocity of observer at R_1: $u_1^\mu = (u_1^0, 0, 0, 0) = \left(\left(1-\frac{2M}{R_1}\right)^{-1/2}, 0, 0, 0\right)$
- " " R_2: $u_2^\mu = (u_2^0, 0, 0, 0) = \left(\left(1-\frac{2M}{R_2}\right)^{-1/2}, 0, 0, 0\right)$

$$\Rightarrow \quad u_1^\mu = \left(1-\frac{2M}{R_1}\right)^{-1/2}\xi^\mu \qquad u_2^\mu = \left(1-\frac{2M}{R_2}\right)^{-1/2}\xi^\mu$$

Energies of photons, as measured by observers:

$$E_1 = -P_1^\mu\, u_{1\,\mu} = -\, P_1^\mu\, \xi_\mu \left(1-\frac{2M}{R_1}\right)^{-1/2}$$



Hartle, Fig 9.1

$$\Rightarrow \quad u_1^{\mu} = \left(1 - \frac{2M}{R_1}\right)^{-1/2} \xi^{\mu} \qquad u_2^{\mu} = \left(1 - \frac{2M}{R_2}\right)^{-1/2} \xi^{\mu}$$

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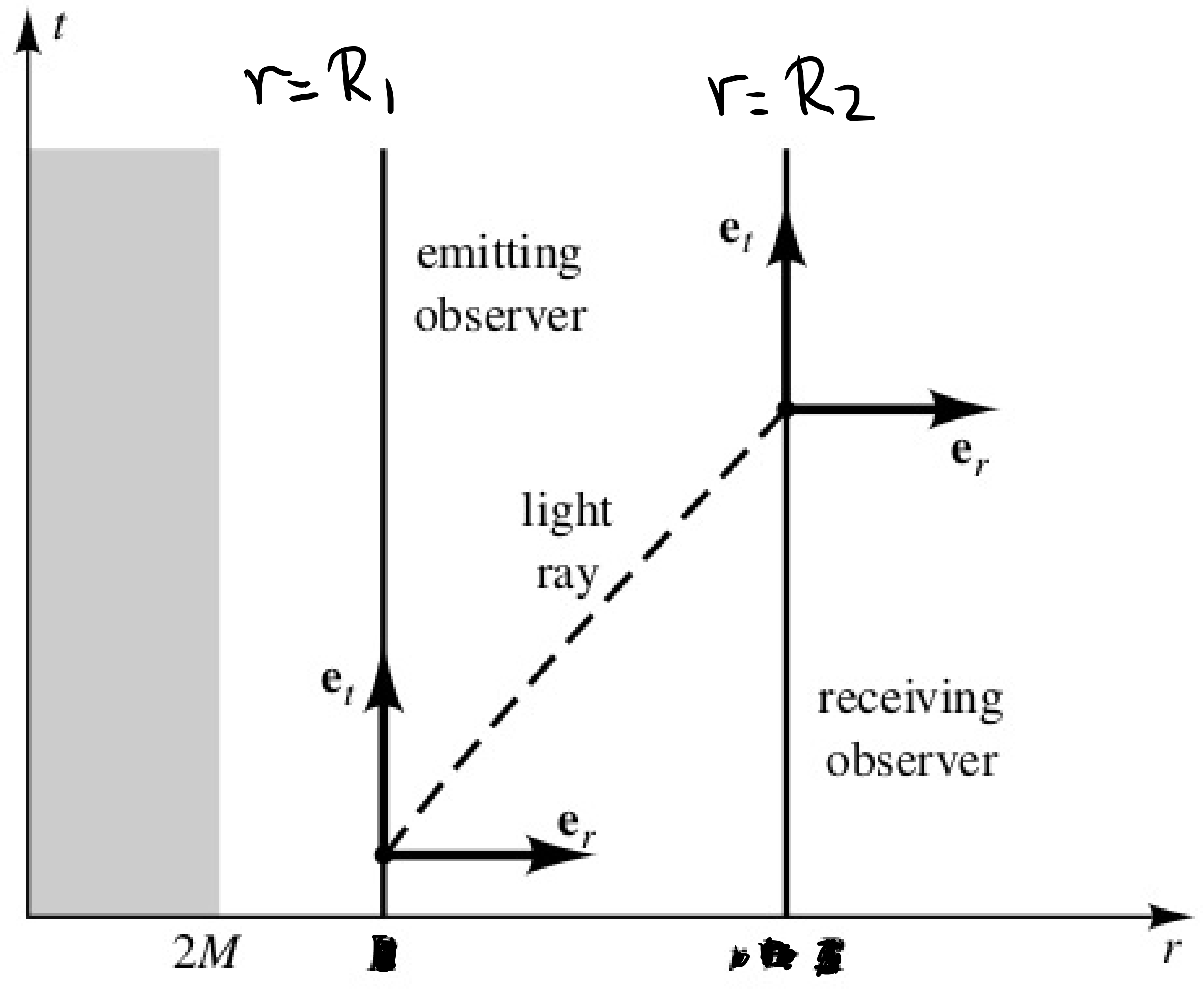

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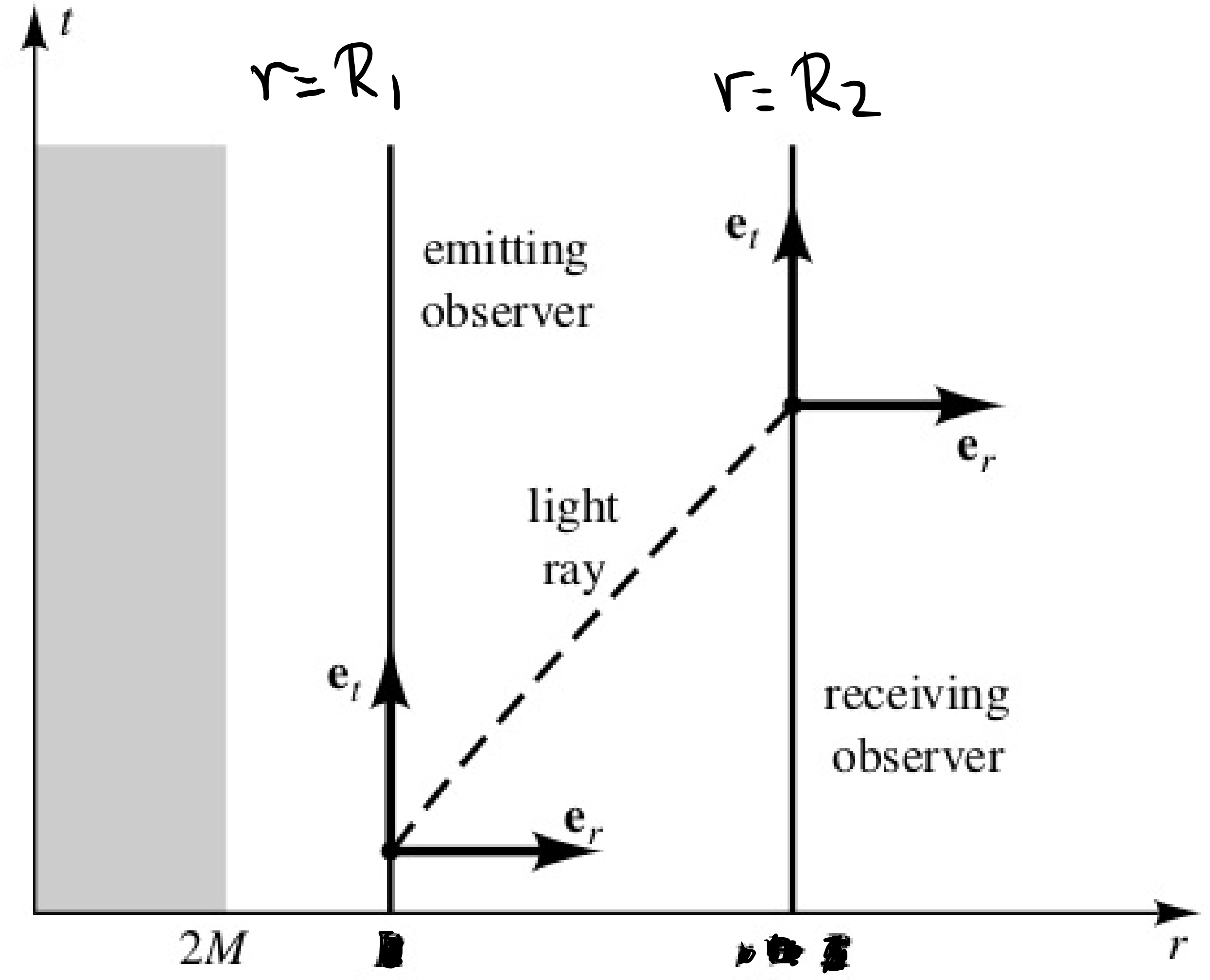
$$\Rightarrow \quad \frac{E_1}{E_2} = \frac{-P_1^{\mu}\,\xi_{\mu} \left(1 - \frac{2M}{R_1}\right)^{-1/2}}{-P_2^{\mu}\,\xi_{\mu} \left(1 - \frac{2M}{R_1}\right)^{-1/2}}$$



Hartle , Fig 9.1

$$\Rightarrow \quad u_1^{\mu} = \left(1-\frac{2M}{R_1}\right)^{-1/2}\xi^{\mu} \qquad u_2^{\mu} = \left(1-\frac{2M}{R_2}\right)^{-1/2}\xi^{\mu}$$

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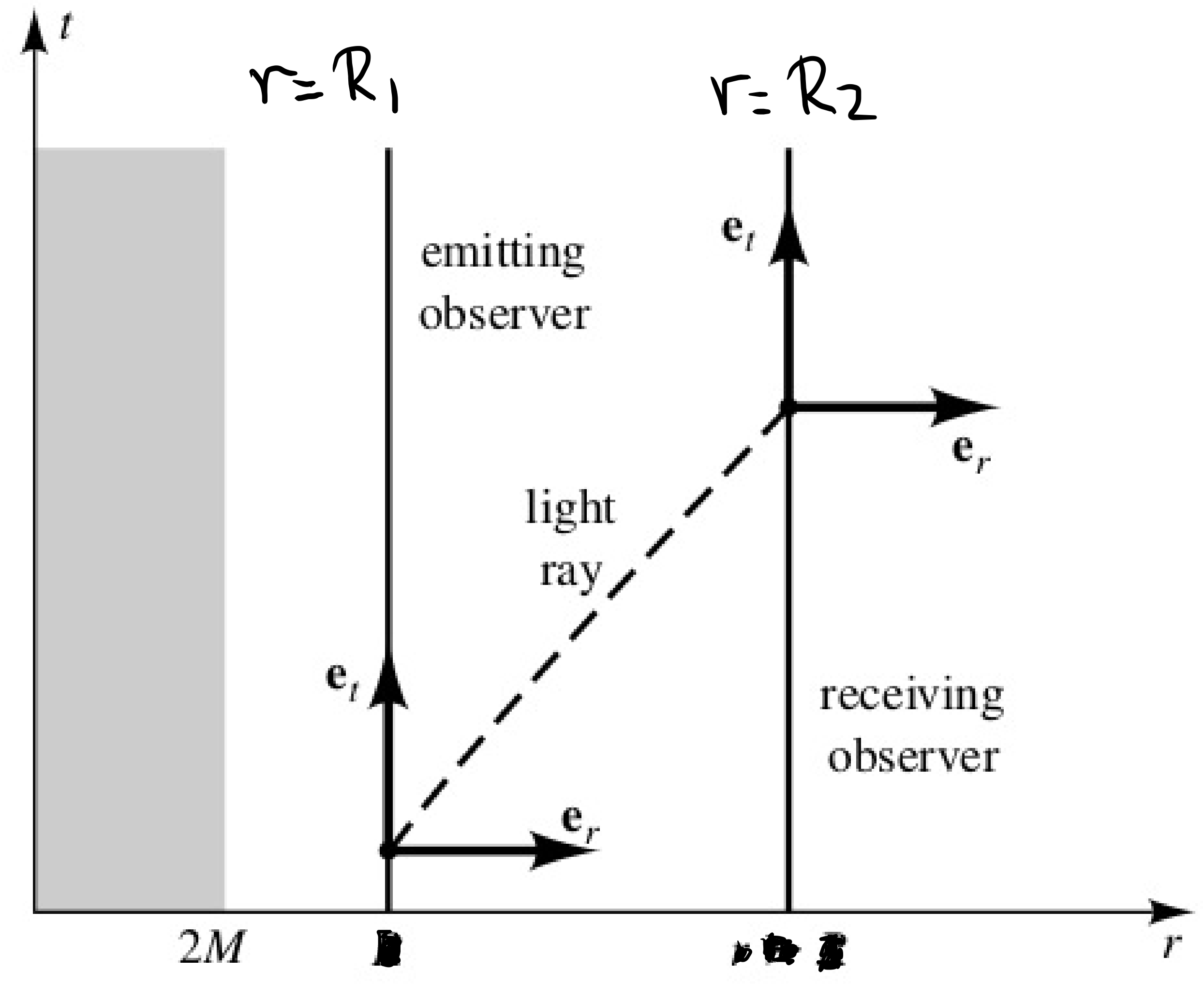
$$E_2 = -p_2^{\mu}\, u_{2\,\mu} = -\, p_2^{\mu}\,\xi_{\mu}\left(1-\frac{2M}{R_2}\right)^{-1/2}$$

conserved!

$$\Rightarrow \frac{E_1}{E_2} = \frac{-\cancel{p_1^{\mu}}\,\xi_{\mu}\left(1-\frac{2M}{R_1}\right)^{-1/2}}{-\cancel{p_2^{\mu}}\,\xi_{\mu}\left(1-\frac{2M}{R_1}\right)^{-1/2}} \Rightarrow$$

$$E_2 = \frac{\left(1-\frac{2M}{R_1}\right)^{1/2}}{\left(1-\frac{2M}{R_2}\right)^{1/2}}\, E_1$$



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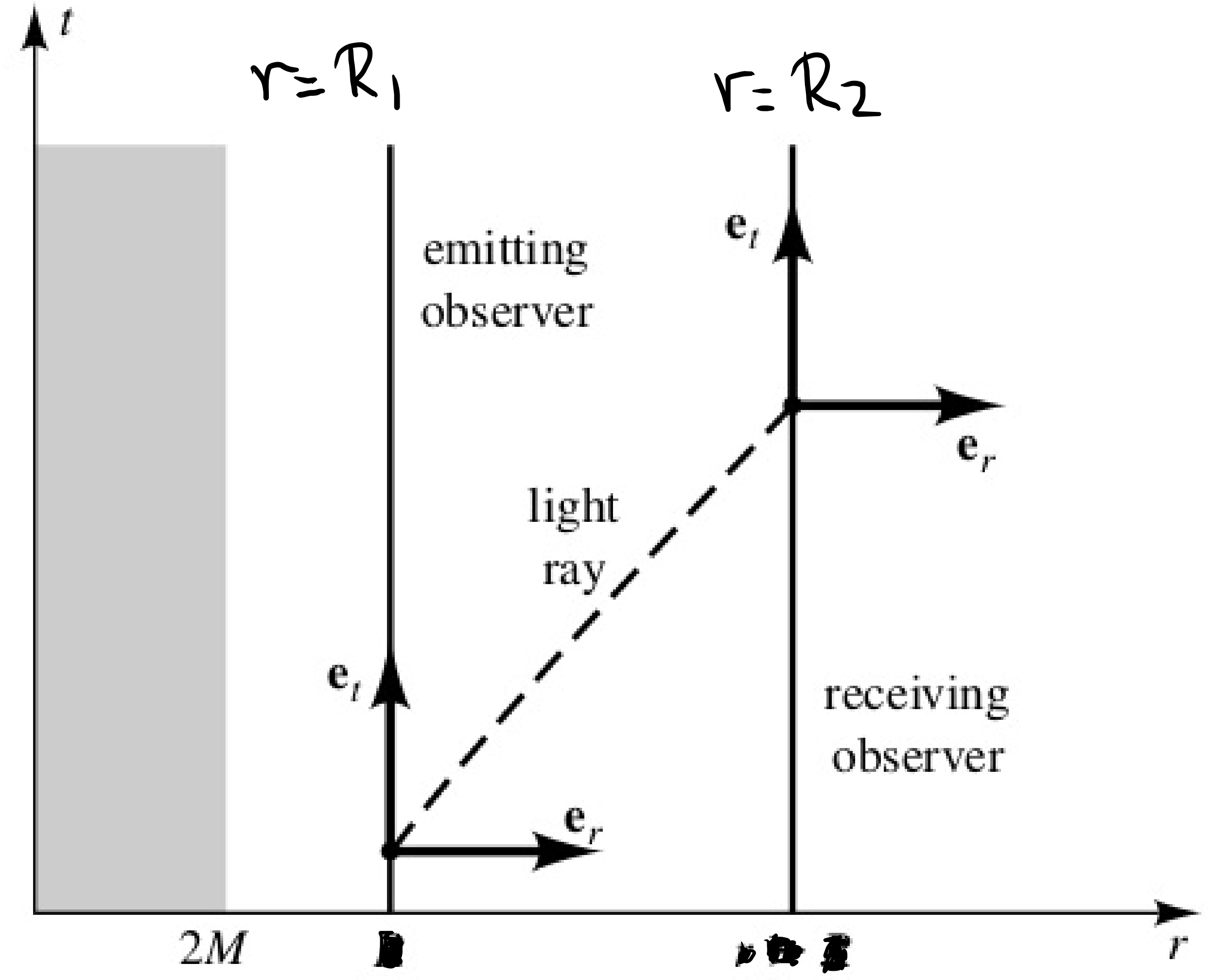

Hartle, Fig 9.1

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$$\Rightarrow \frac{E_1}{E_2} = \frac{-\cancel{P_1^\mu \xi_\mu} \left(1-\frac{2M}{R_1}\right)^{-1/2}}{-\cancel{P_2^\mu \xi_\mu} \left(1-\frac{2M}{R_1}\right)^{-1/2}} \Rightarrow$$

$$E_2 = \frac{\left(1-\frac{2M}{R_1}\right)^{1/2}}{\left(1-\frac{2M}{R_2}\right)^{1/2}} E_1$$

If $R_2 = \infty$

$$E_\infty = \left(1-\frac{2M}{R}\right)^{1/2} E(R)$$

$$\Rightarrow \quad u_1^\mu = \left(1-\frac{2M}{R_1}\right)^{-1/2}\xi^\mu \qquad u_2^\mu = \left(1-\frac{2M}{R_2}\right)^{-1/2}\xi^\mu$$

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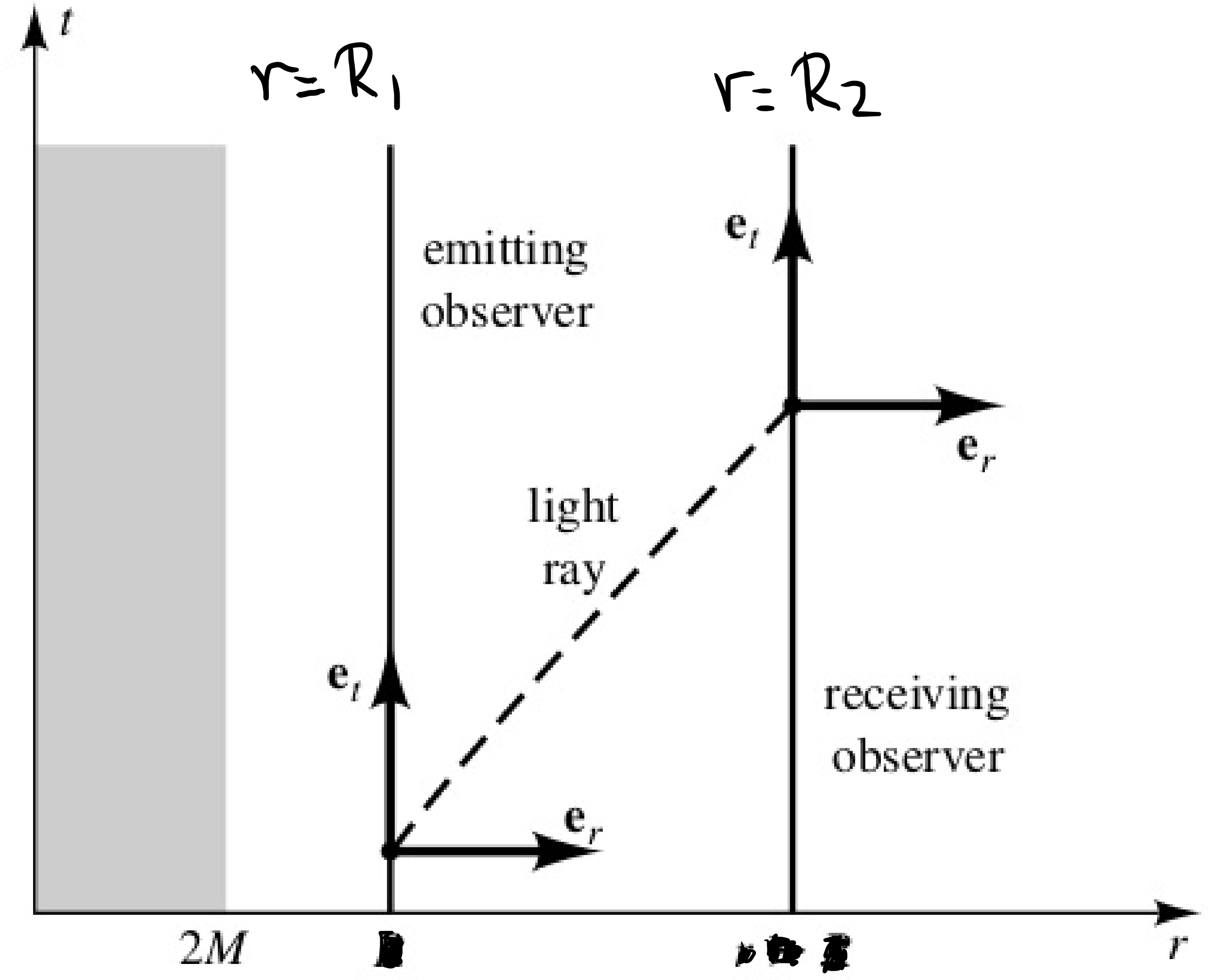

Hartle, Fig 9.1

conserved!

$$\Rightarrow \frac{E_1}{E_2} = \frac{-\cancel{P_1^\mu} \xi_\mu \left(1-\frac{2M}{R_1}\right)^{-1/2}}{-\cancel{P_2^\mu} \xi_\mu \left(1-\frac{2M}{R_1}\right)^{-1/2}} \Rightarrow$$

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If $R_2 = \infty$ $\qquad E = \hbar\omega$

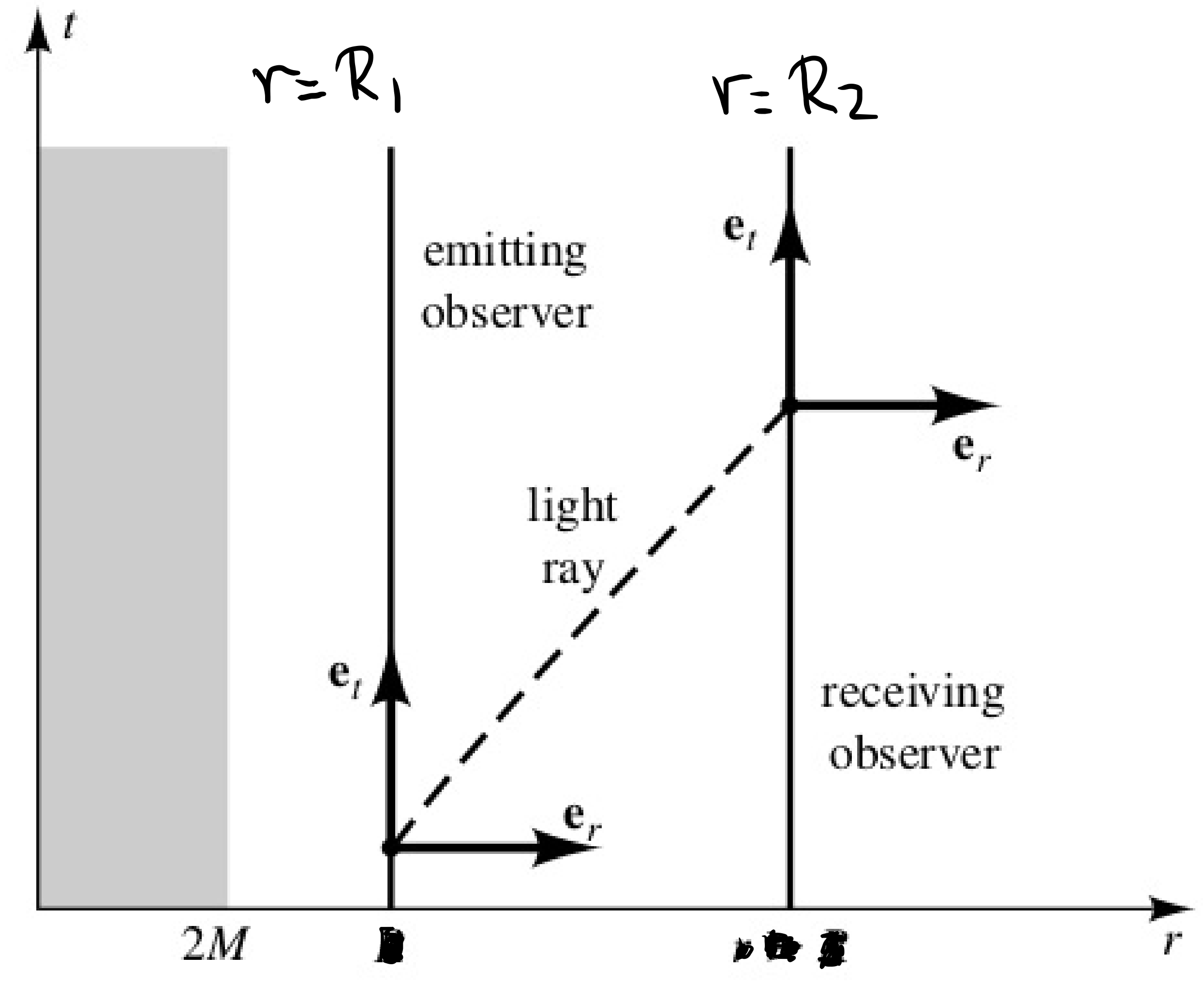
$$\omega_\infty = \left(1-\frac{2M}{R}\right)^{1/2}\omega(R)$$

For $R, R_1, R_2 \gg 2M$

$$\omega_\infty \approx \left(1 - \frac{M}{R}\right)\omega(R) = \left(1 + \Phi(R)\right)\omega(R)$$

$$\omega_2 \approx \left(1 - \frac{M}{R_1}\right)\left(1 + \frac{M}{R_2}\right)\omega_1$$

$$(1-x)^{-\frac{1}{2}} \approx 1 - \left(-\tfrac{1}{2}\right)x = 1 + \tfrac{1}{2}x$$



Hartle, Fig 9.1

$$\omega_2 = \frac{\left(1 - \frac{2M}{R_1}\right)^{1/2}}{\left(1 - \frac{2M}{R_2}\right)^{1/2}}\,\omega_1$$

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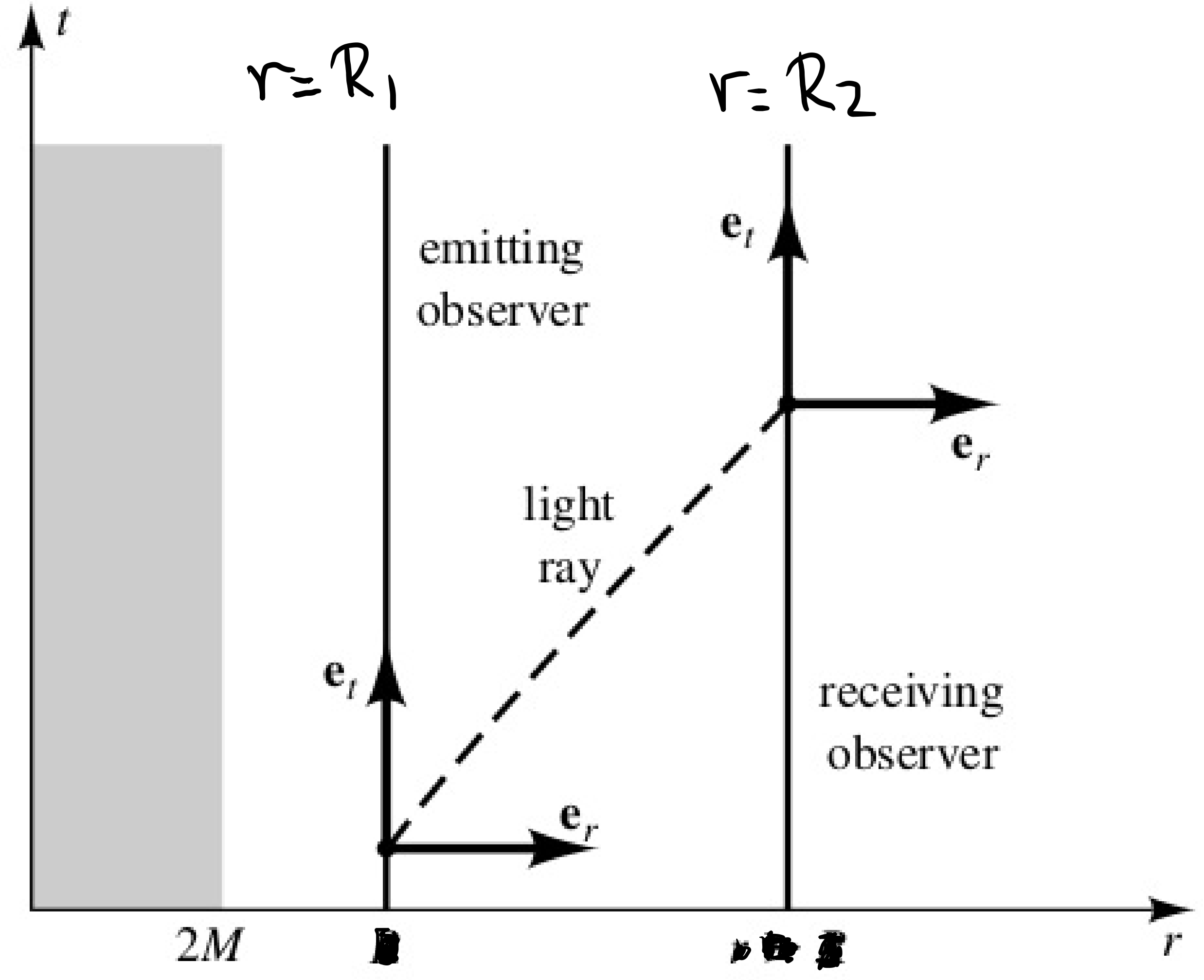
$$\approx \left(1 - \frac{M}{R_1} + \frac{M}{R_2}\right)\omega_1$$

$$= \left(1 + \Phi(R_1) - \Phi(R_2)\right)\omega_1$$



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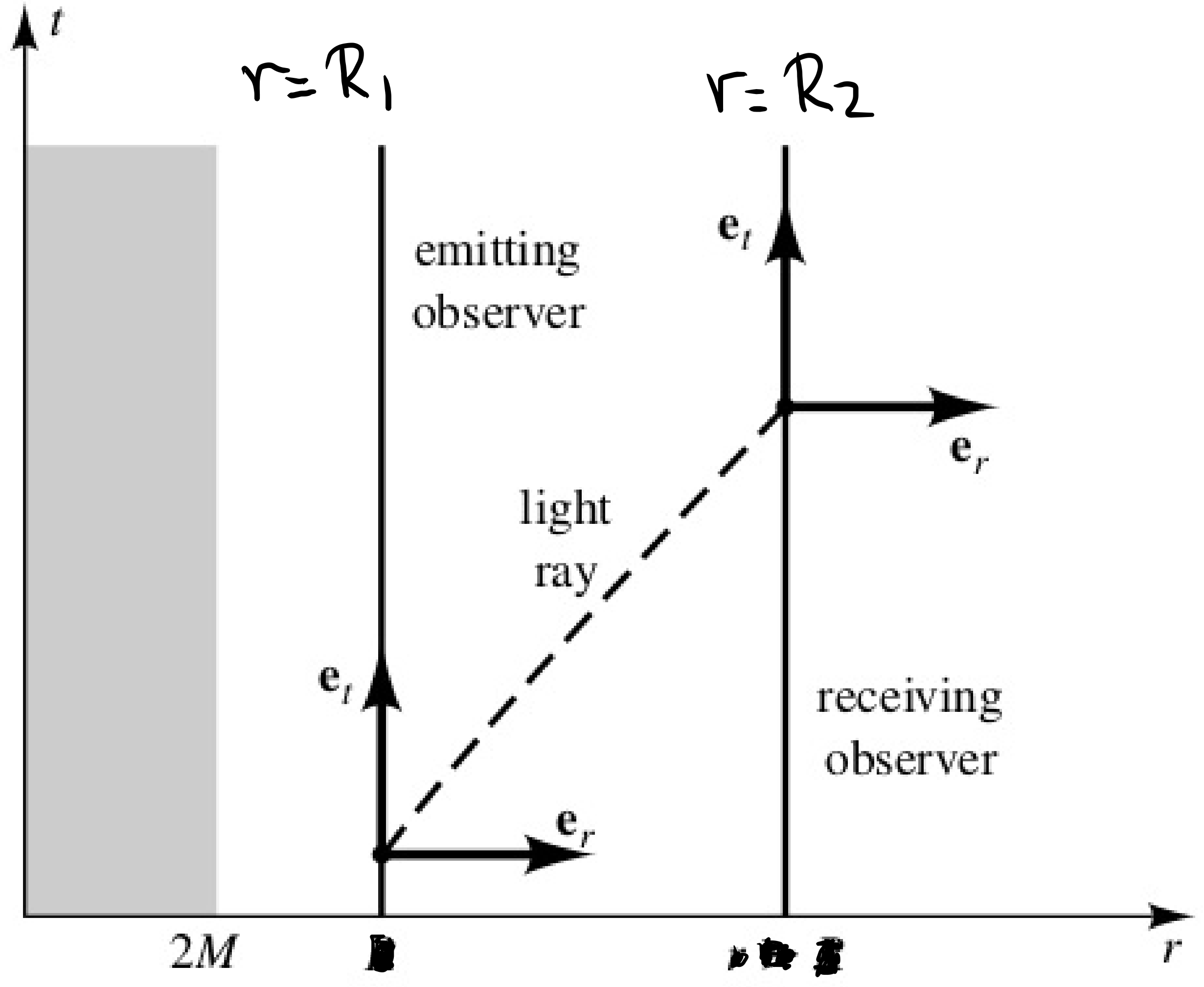
If $R = 2M$, $\omega(R) < \infty$ $\Big\} \Rightarrow \omega_\infty = 0$



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If $R_2 = \infty$

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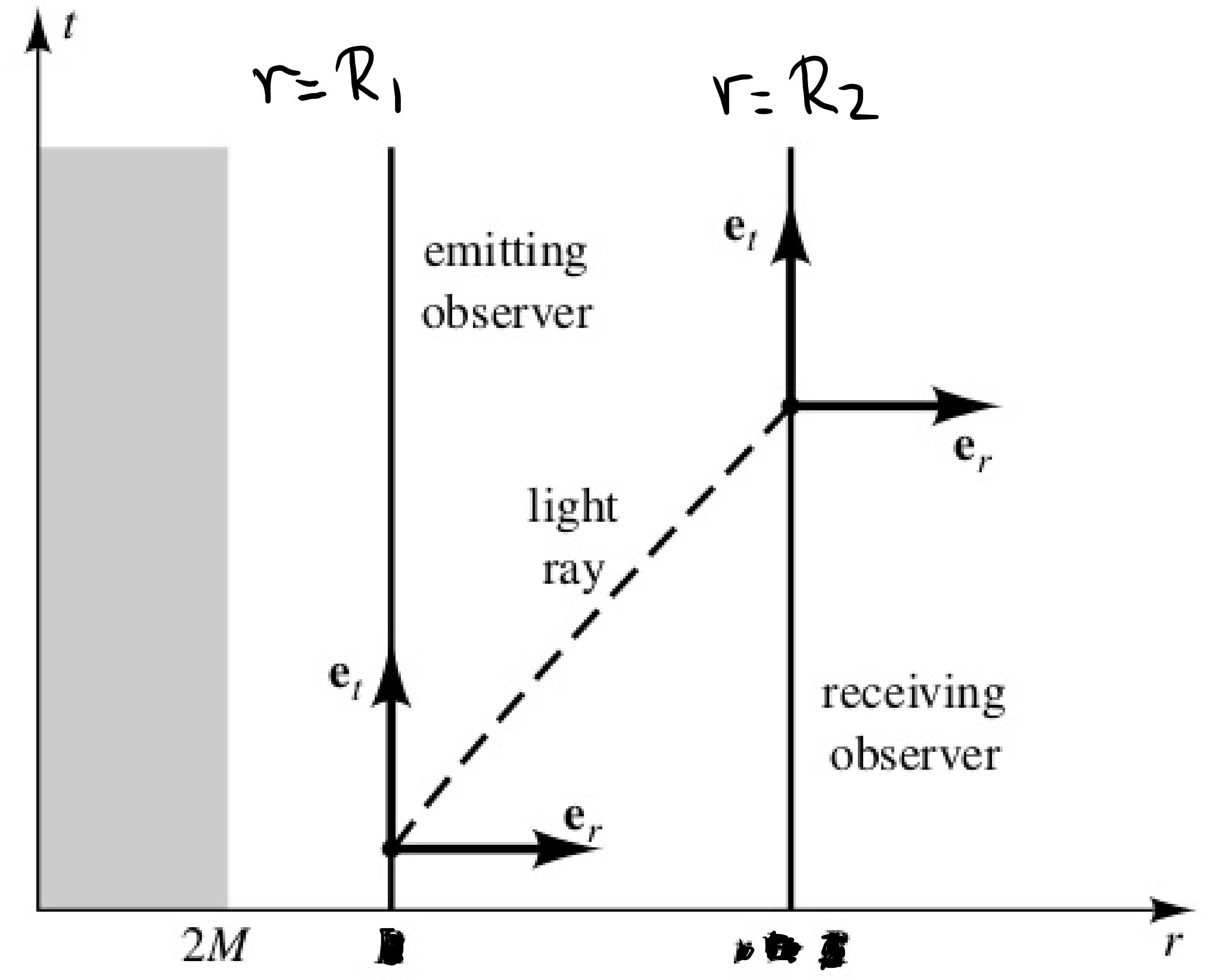
$$\omega_\infty = \left(1 - \frac{2M}{R}\right)^{1/2} \omega(R)$$

$$\left.\begin{array}{l} \text{If } R = 2M \\ \omega(R) < \infty \end{array}\right\} \Rightarrow \omega_\infty = 0$$

$\Rightarrow$ • infinite redshift $\dfrac{\omega - \omega_\infty}{\omega_\infty}$

• cannot "see" objects as they approach $R \to 2M$

their light signals are redshifted to zero



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Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu$ conserved

$$\xi^\mu = (1, 0, 0, 0) \qquad u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau} \right)$$

4-velocity of freely falling particle
u^μ: tangent to its timelike geodesic
world line

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$$(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$$

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$$-\xi^\mu u_\mu = -g_{\mu\nu}\xi^\mu u^\nu = -g_{00}\xi^0 u^0 = +\left(1 - \frac{2M}{r}\right) \cdot 1 \cdot \frac{dt}{d\tau}$$

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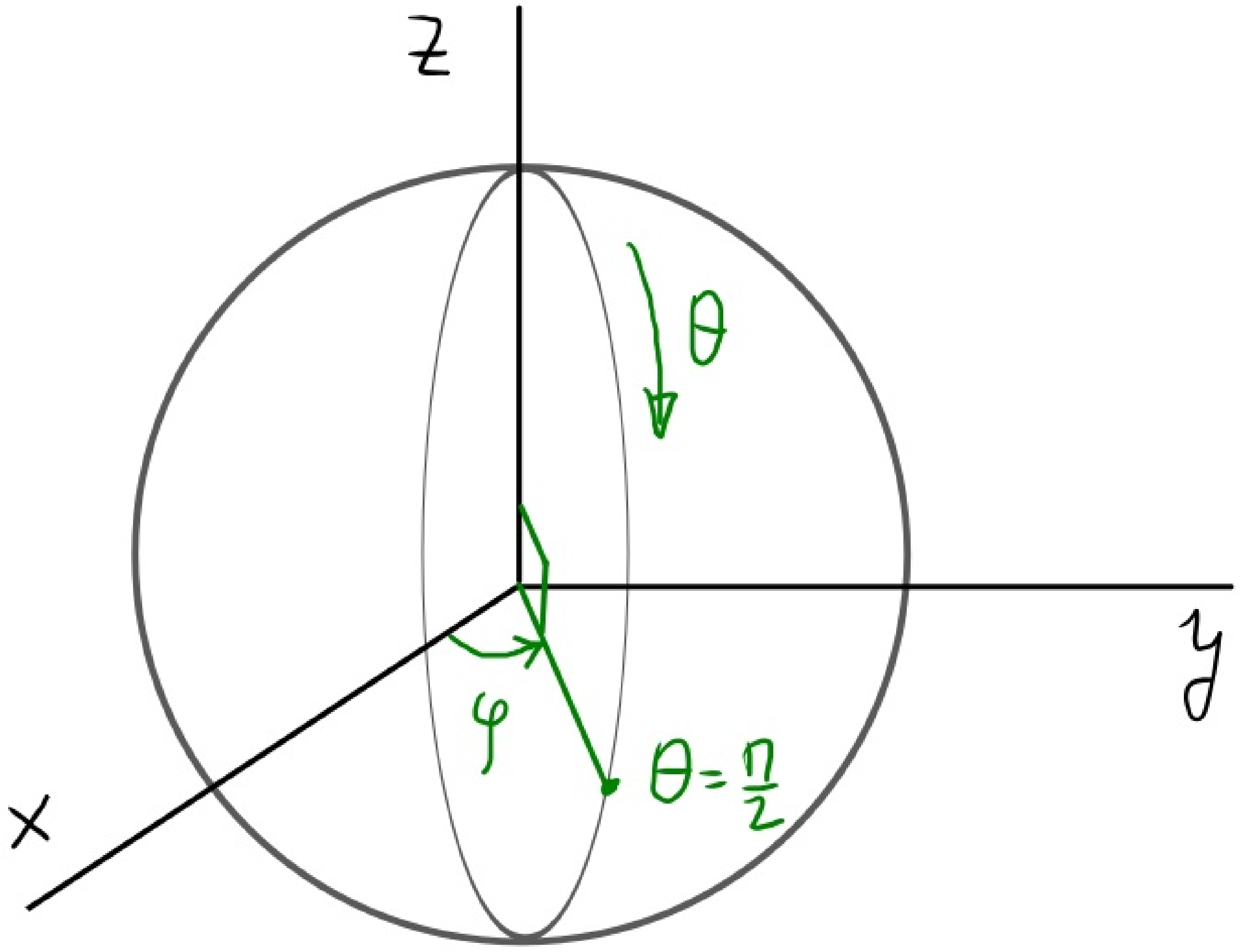
l conserved $\Rightarrow$ motion on a plane!

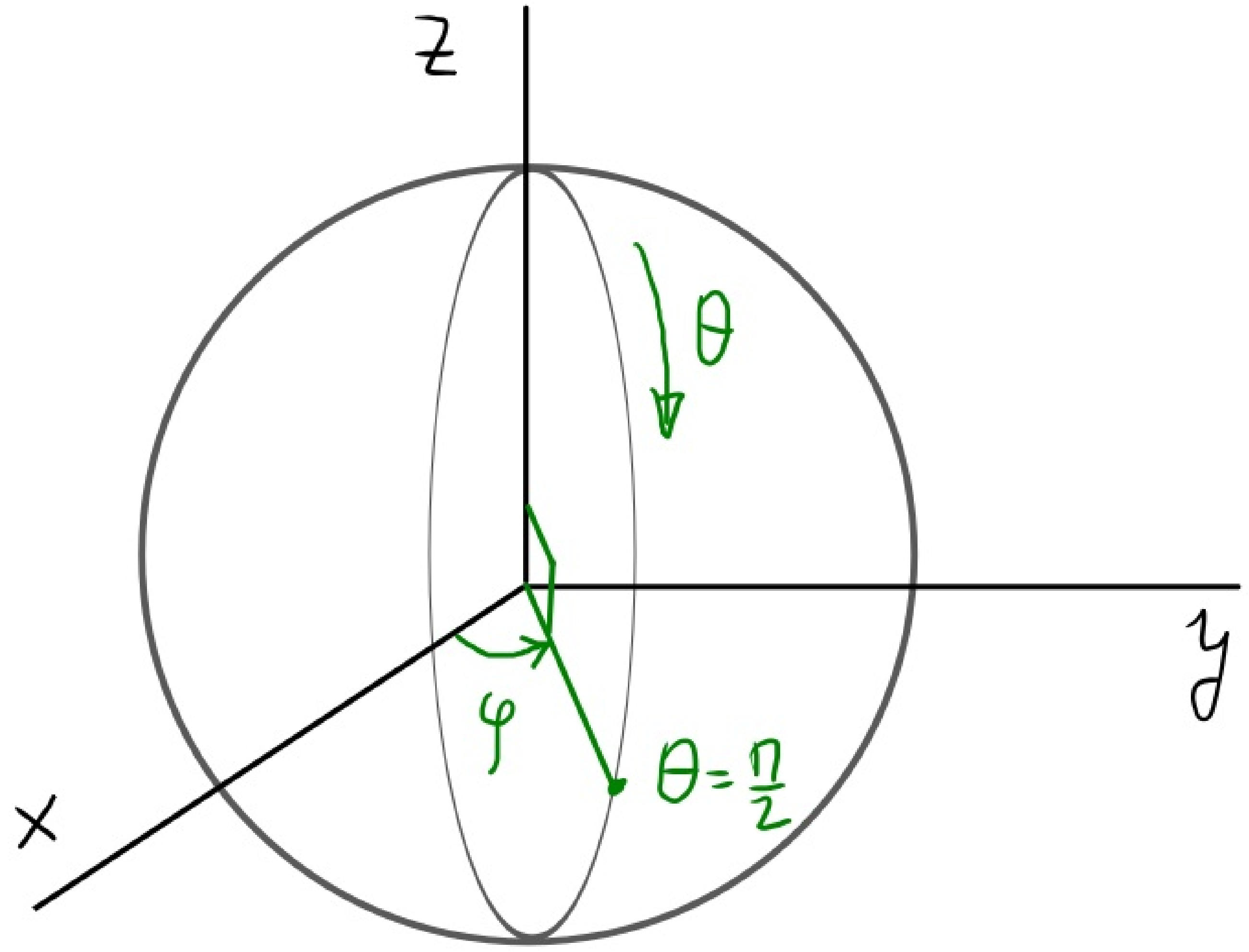
$\ell = r^2 \sin^2\theta \, \frac{d\varphi}{d\tau}$ conserved $\Rightarrow$ motion on a plane



- consider $u^\mu = (u^0, \vec{u})$
- orient coordinate system so that $u^\phi = \frac{d\phi}{d\tau} = 0$ at some instant of time τ_0

$\Rightarrow \ell = 0 \quad$ at τ_0

$\ell = r^2 \sin^2\theta \, \frac{d\varphi}{d\tau}$ conserved $\Rightarrow$ motion on a plane



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$$\Rightarrow \ell = 0 \quad \forall \tau$$

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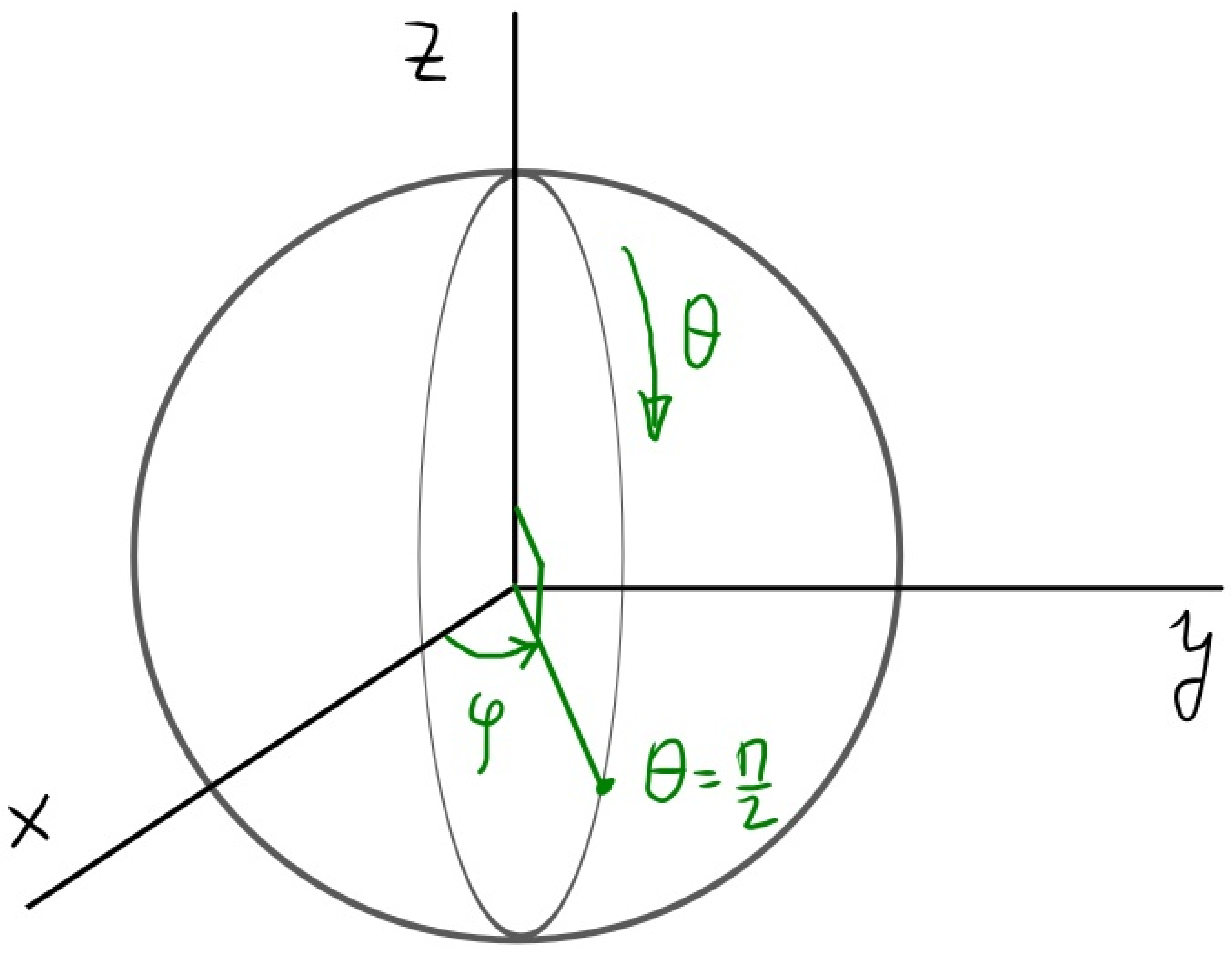

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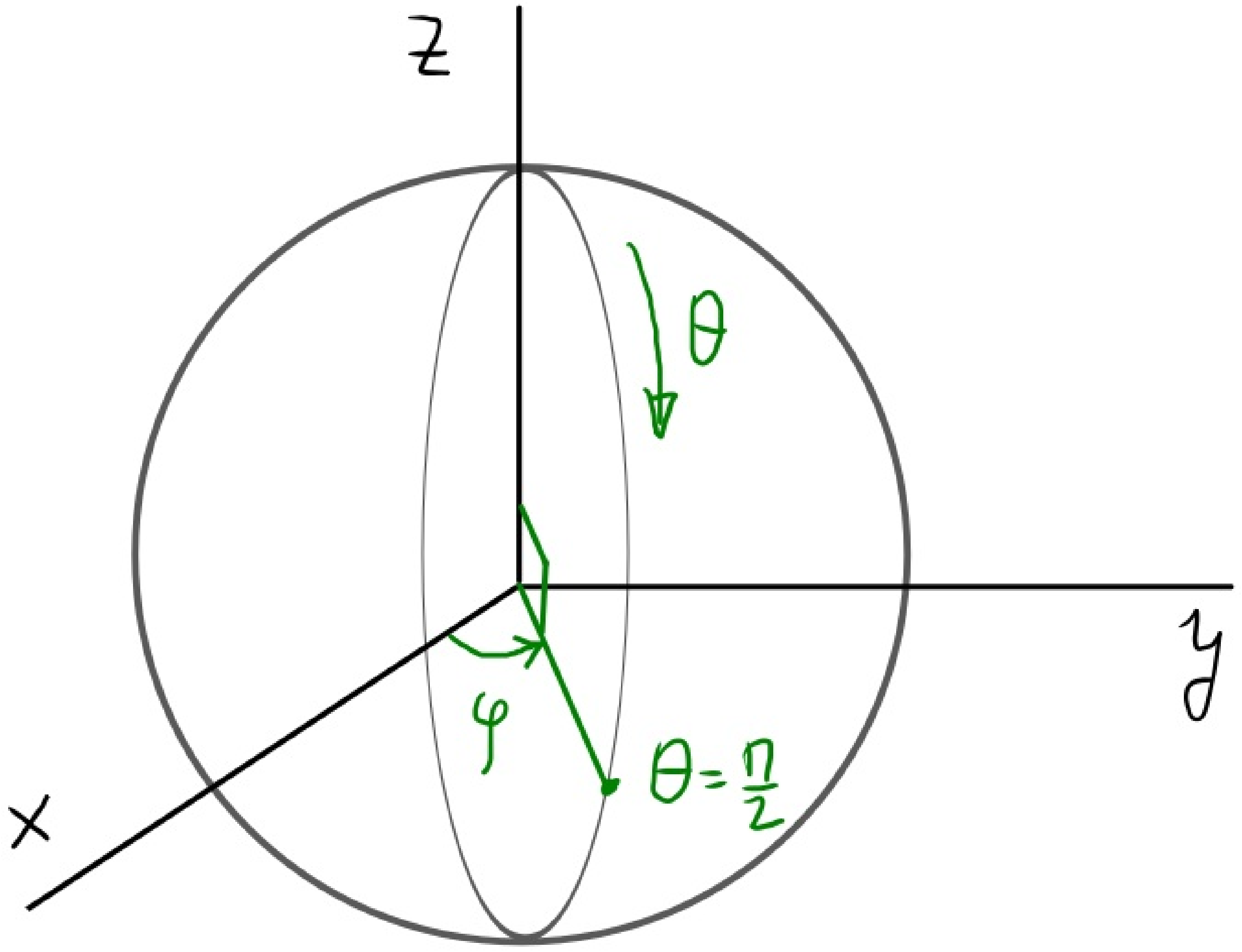
$\Rightarrow \ell = 0 \quad \forall \tau$

$\Rightarrow \frac{d\phi}{d\tau} = 0 \Rightarrow \phi =$ const $\Rightarrow$ stays on $\phi =$ const plane

$\ell = r^2 \sin^2\theta \dfrac{d\varphi}{d\tau}$ conserved $\Rightarrow$ motion on a plane



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$$\Rightarrow \ell = 0 \quad \text{at } \tau_0$$

$$\Rightarrow \ell = 0 \quad \forall \ \tau$$

$$\Rightarrow \frac{d\phi}{d\tau} = 0 \Rightarrow \phi = \text{const} \Rightarrow \text{stays on } \phi = \text{const plane}$$

- now reorient coordinates, so that the plane of motion is $\theta = \frac{\pi}{2}$

$$\Rightarrow u^\theta = \frac{d\theta}{d\tau} = 0$$

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$
- $\eta = \partial_\varphi$ spacelike " " ": $\ell = \eta^\mu u_\mu = r^2 \sin^2\theta \frac{d\varphi}{d\tau}$

ℓ conserved $\Rightarrow$ motion on $\theta = \frac{\pi}{2}$ plane $\Rightarrow \begin{cases} \sin\theta = 1 \\ u^\theta = \frac{d\theta}{d\tau} = 0 \end{cases}$

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$\Rightarrow u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right) \quad (g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\right)$

$\hookrightarrow \sin\theta = 1$

$$u^\mu u_\mu = -1 \Rightarrow g_{\mu\nu} u^\mu u^\nu = -1$$

$$e = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} = \text{const} \qquad l = r^2 \frac{d\varphi}{d\tau} = \text{const}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right) \qquad (g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\right)$$

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$$-\left(1-\frac{2M}{r}\right)(u^0)^2 + \left(1-\frac{2M}{r}\right)^{-1}(u^1)^2 + r^2 (u^3)^2 = -1$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = \text{const} \qquad l = r^2 \frac{d\varphi}{d\tau} = \text{const}$$

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$$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = \text{const} \qquad l = r^2 \frac{d\varphi}{d\tau} = \text{const}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right) \qquad (g_{\mu\nu}) = \operatorname{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\right)$$

$$u^\mu u_\mu = -1 \Rightarrow g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)(u^0)^2 + \left(1-\frac{2M}{r}\right)^{-1}(u^1)^2 + r^2 (u^3)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)^{-1} e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2 \frac{\ell^2}{r^4} = -1$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \Rightarrow \frac{dt}{d\tau} = \left(1-\frac{2M}{r}\right)^{-1} e, \quad \ell = r^2 \frac{d\varphi}{d\tau} \Rightarrow \frac{d\varphi}{d\tau} = \frac{\ell}{r^2}$$

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$$- e^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\frac{\ell^2}{r^2} = -\left(1-\frac{2M}{r}\right)$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \Rightarrow \frac{dt}{d\tau} = \left(1-\frac{2M}{r}\right)^{-1} e, \quad \ell = r^2 \frac{d\varphi}{d\tau} \Rightarrow \frac{d\varphi}{d\tau} = \frac{\ell}{r^2}$$

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$$-e^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\frac{\ell^2}{r^2} = -\left(1-\frac{2M}{r}\right) \Rightarrow$$

$$-e^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right) = 0$$

$$U^\mu U_\mu = -1 \Rightarrow g_{\mu\nu} U^\mu U^\nu = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)(U^0)^2 + \left(1-\frac{2M}{r}\right)^{-1}(U^1)^2 + r^2 (U^3)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1 \Rightarrow$$

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$$- e^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right) = 0 \Rightarrow$$

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right) - 1\right]$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

$$\underbrace{\frac{e^2-1}{2}}_{\mathcal{E}} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{\frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)-1\right]}_{V_{eff}(r)} = 0 \quad \Rightarrow$$

$$\underbrace{\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)}$$

analyze radial motion as we do in Newtonian theory

$$\underbrace{\frac{e^2-1}{2}}_{\mathcal{E}} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{\frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)-1\right]}_{V_{eff}(r)} = 0 \quad \Rightarrow$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

"effective potential energy"

"energy" (conserved)

"kinetic energy"

$$\underbrace{\frac{e^2-1}{2}}_{\mathcal{E}} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{\frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)-1\right]}_{V_{eff}(r)} = 0 \quad \Rightarrow$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

$$V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

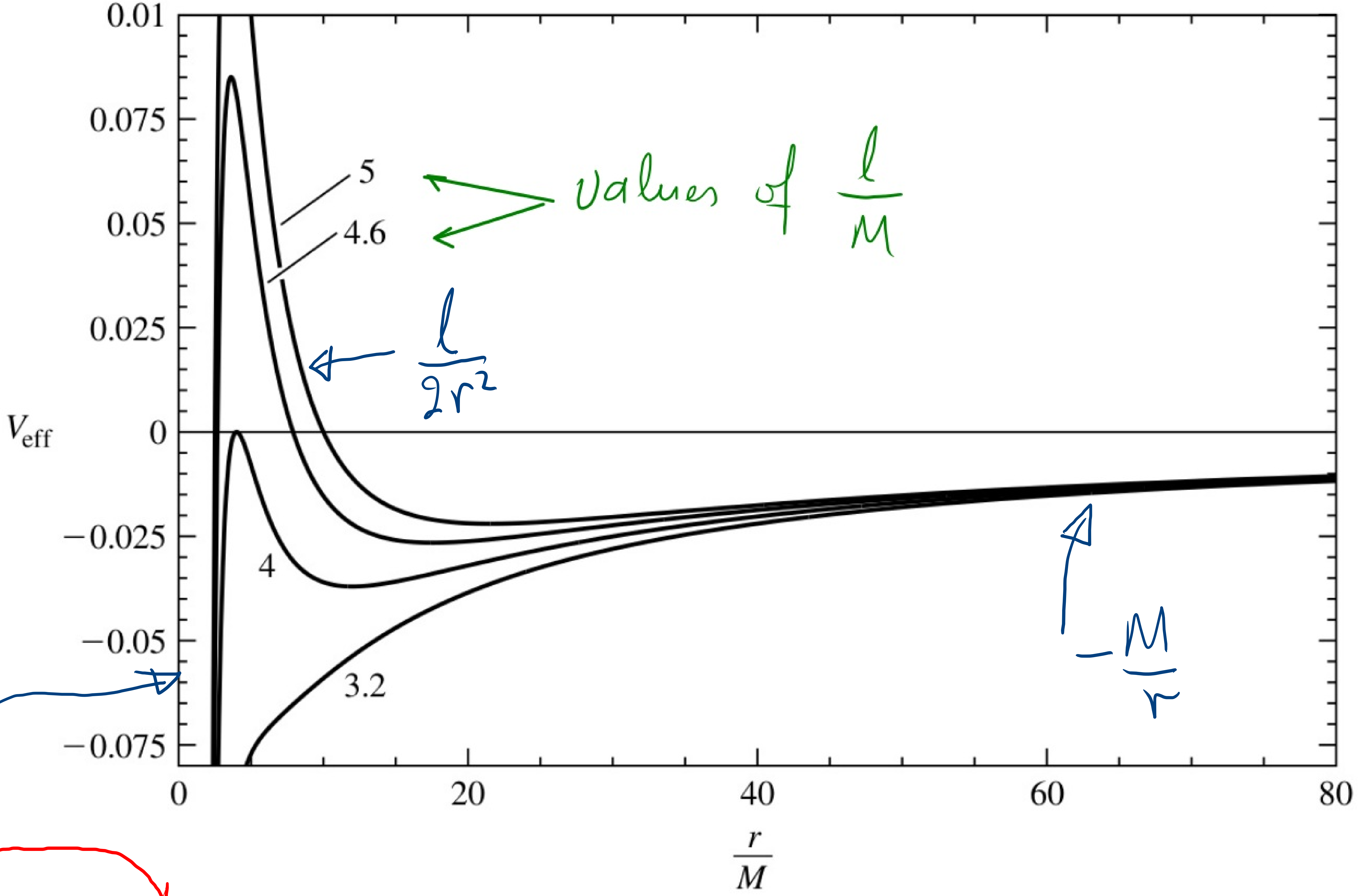
attractive Newtonian potential energy

angular momentum repulsion

general relativity term (attractive, dominant for small r)

$$\underbrace{\frac{e^2-1}{2}}_{\mathcal{E}} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{\frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)-1\right]}_{V_{eff}(r)} = 0 \Rightarrow$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$



$V(\infty) = 0$

$-\frac{M\ell^2}{r^3}$

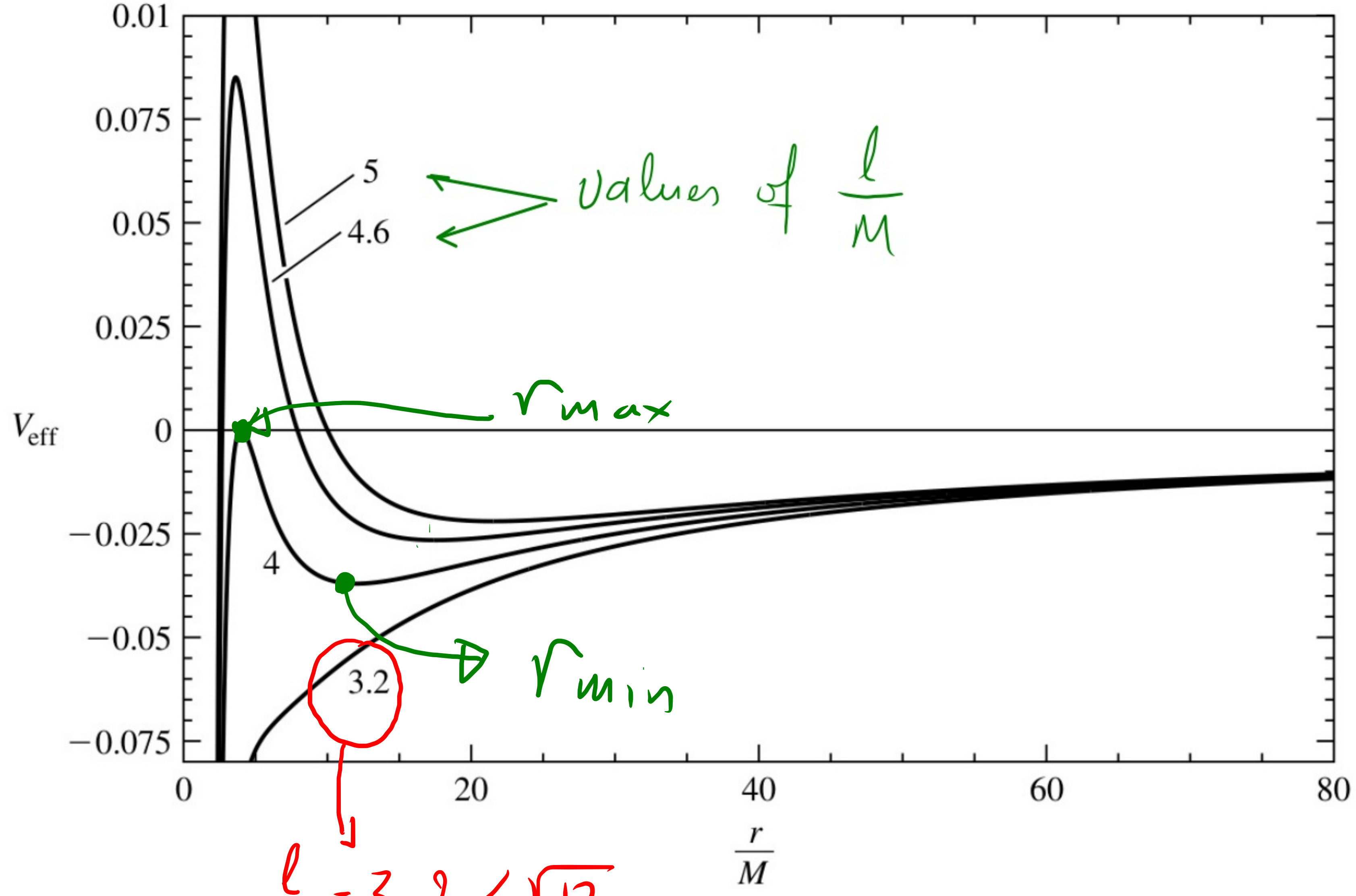
$V(2M) = -\frac{1}{2}$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

$$\frac{dV_{eff}(r)}{dr} = 0 \Rightarrow$$

$$+\frac{M}{r^2} - \frac{\ell^2}{r^3} + \frac{3\,\ell^2 M}{r^4} = 0 \Rightarrow$$

$$r_{min,max} = \frac{\ell^2}{2M}\left[1 \underset{r_{max}}{\overset{r_{min}}{\pm}} \sqrt{1-12\left(\frac{M}{\ell}\right)^2}\right]$$



$$\frac{\ell}{M} = 3.2 < \sqrt{12}$$

$$\text{for } 1-12\left(\frac{M}{\ell}\right)^2 \geq 0 \Rightarrow \ell \geq \sqrt{12}\,M \Rightarrow \frac{\ell}{M} \geq \sqrt{12} \approx 3.464$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

$$V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

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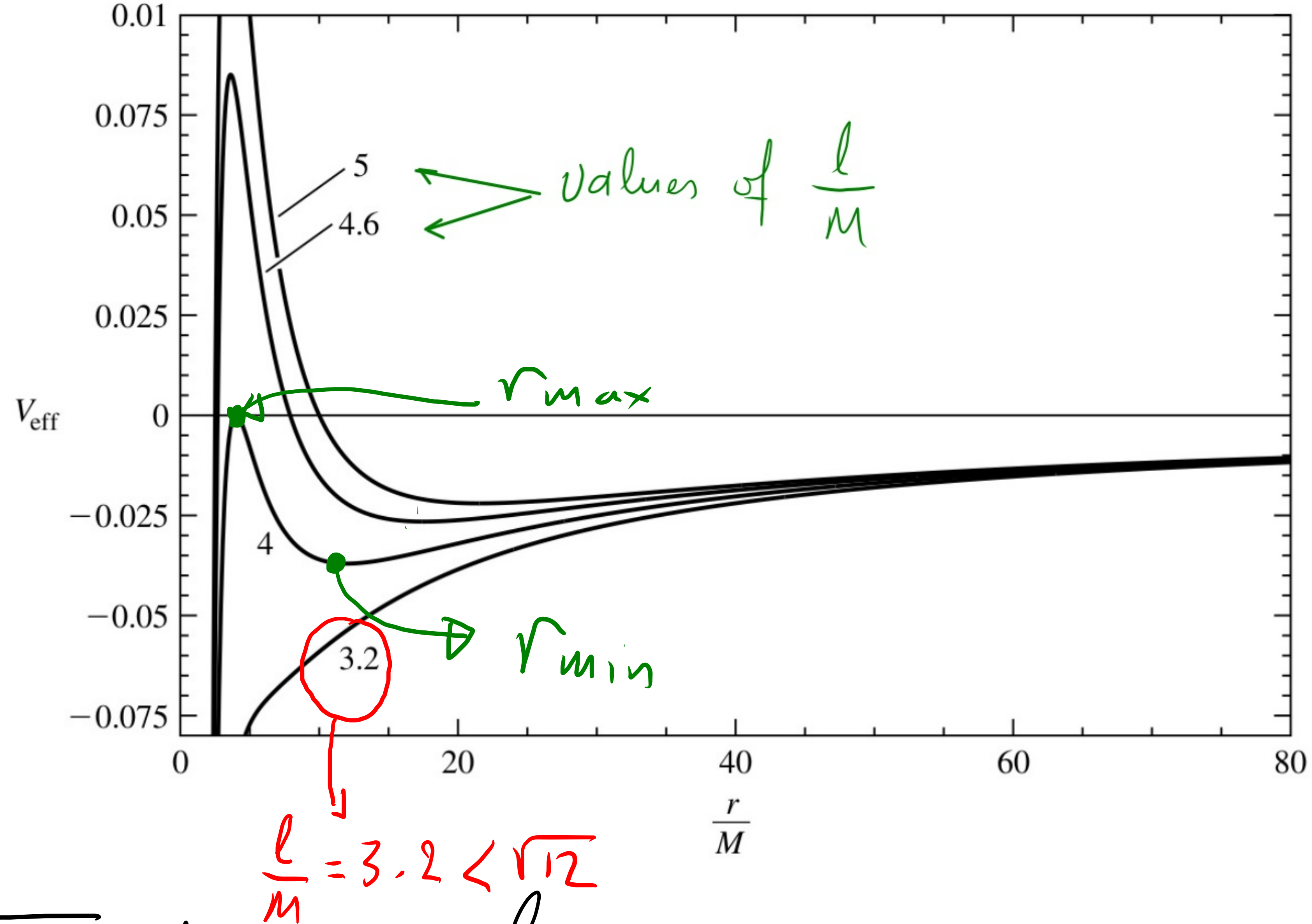


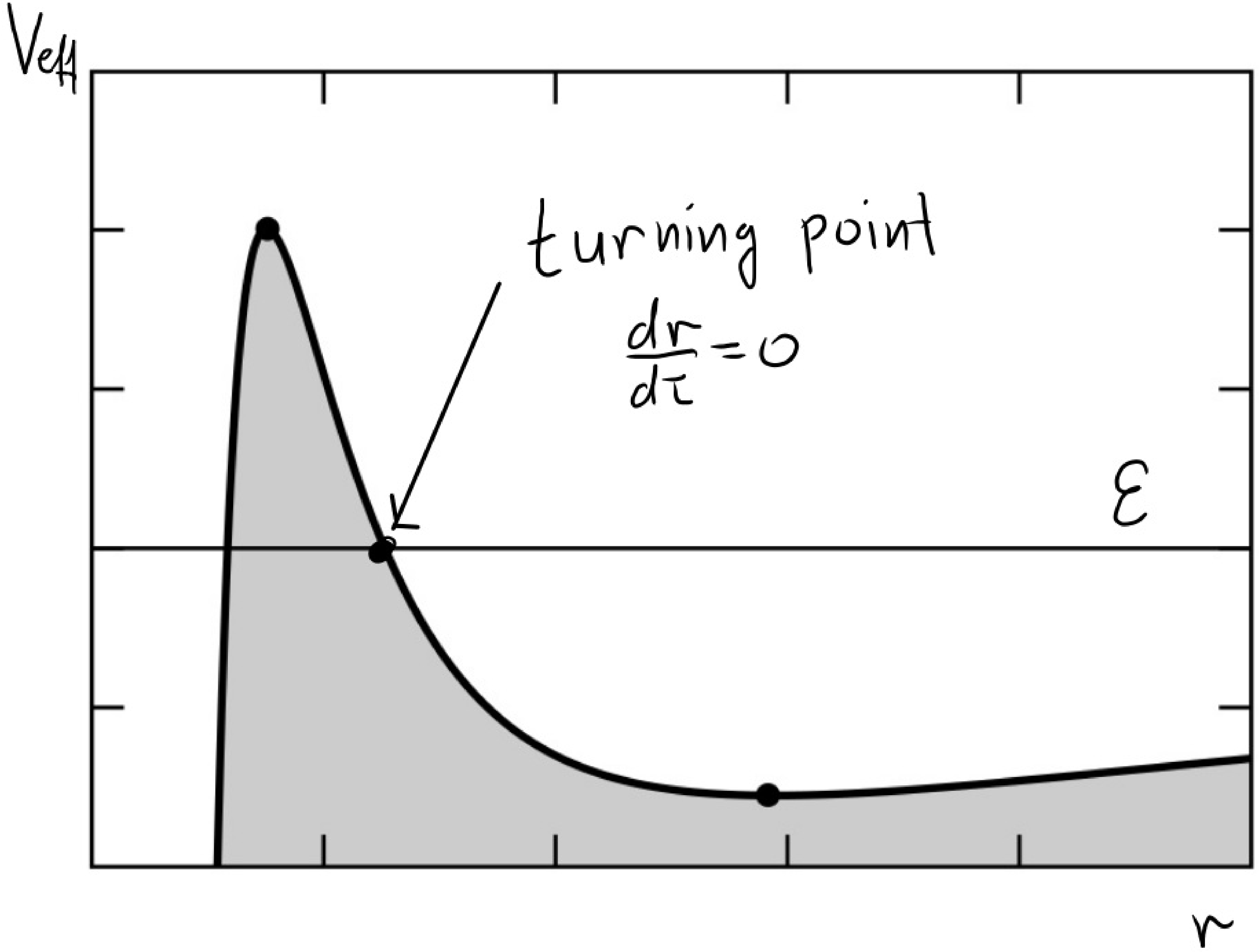
for $1 - 12\left(\frac{M}{\ell}\right)^2 \geq 0 \Rightarrow \ell \geq \sqrt{12}\,M \Rightarrow \frac{\ell}{M} \geq \sqrt{12} \approx 3.464$

$$V_{eff}(r_{max}) = \frac{2\left(8 - \left(\frac{\ell}{m}\right)^2 + \left(\frac{\ell}{m}\right)\sqrt{\left(\frac{\ell}{m}\right)^2 - 12}\right)}{\frac{\ell}{m}\left(\frac{\ell}{m} - \sqrt{\left(\frac{\ell}{m}\right)^2 - 12}\right)}$$

Radial motion:

Fix $\mathcal{E}$, determine turning points $\frac{dr}{d\tau} = 0$

$\Leftrightarrow \ \mathcal{E} = V_{eff}(r)$



Hartle, Fig 9.4

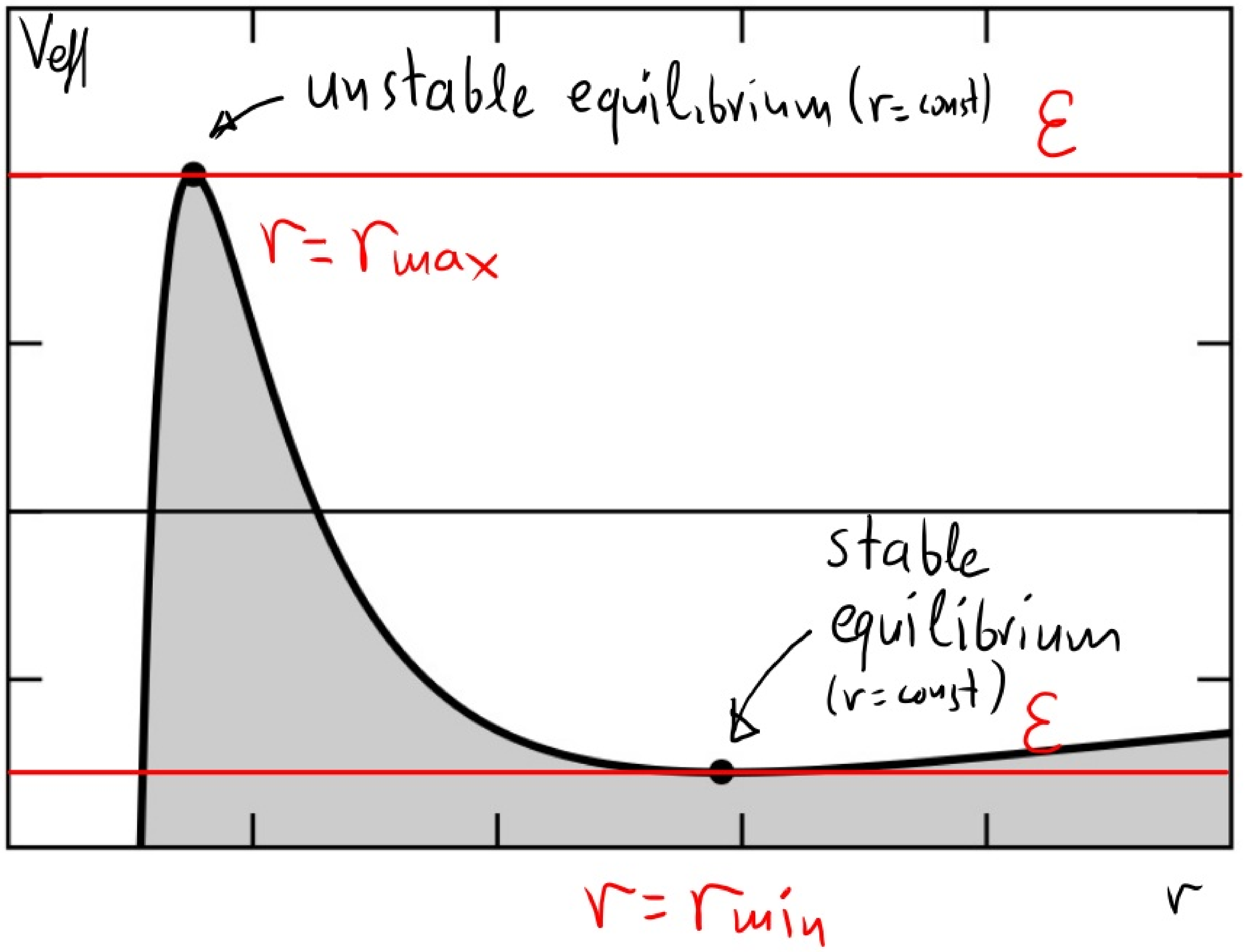
Radial motion:

Fix $\mathcal{E}$, determine turning points $\frac{dr}{d\tau} = 0$

$\Leftrightarrow \quad \mathcal{E} = V_{eff}(r)$

"Equilibrium" points:

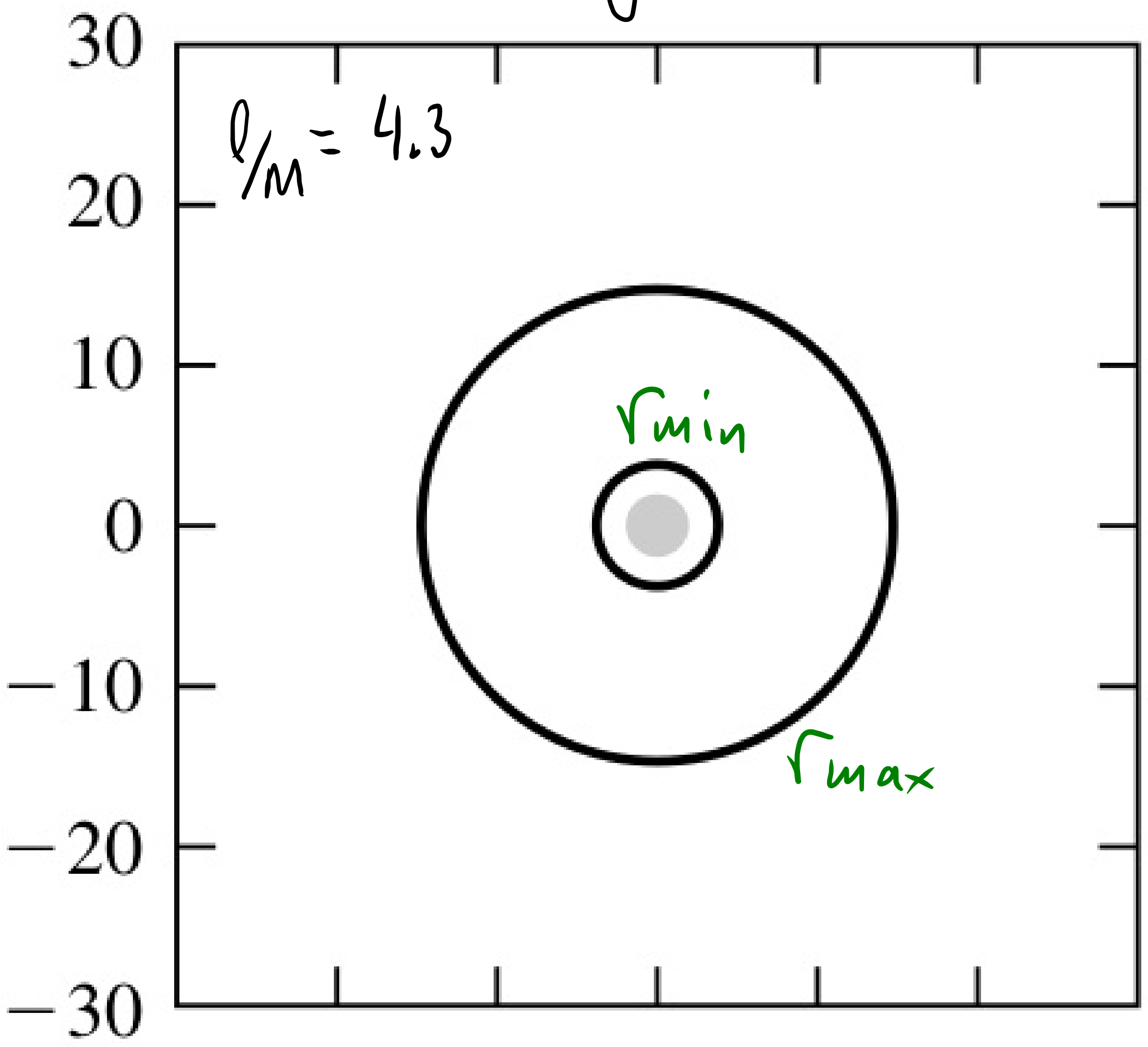
$r = \text{const} \Rightarrow$ circular orbits

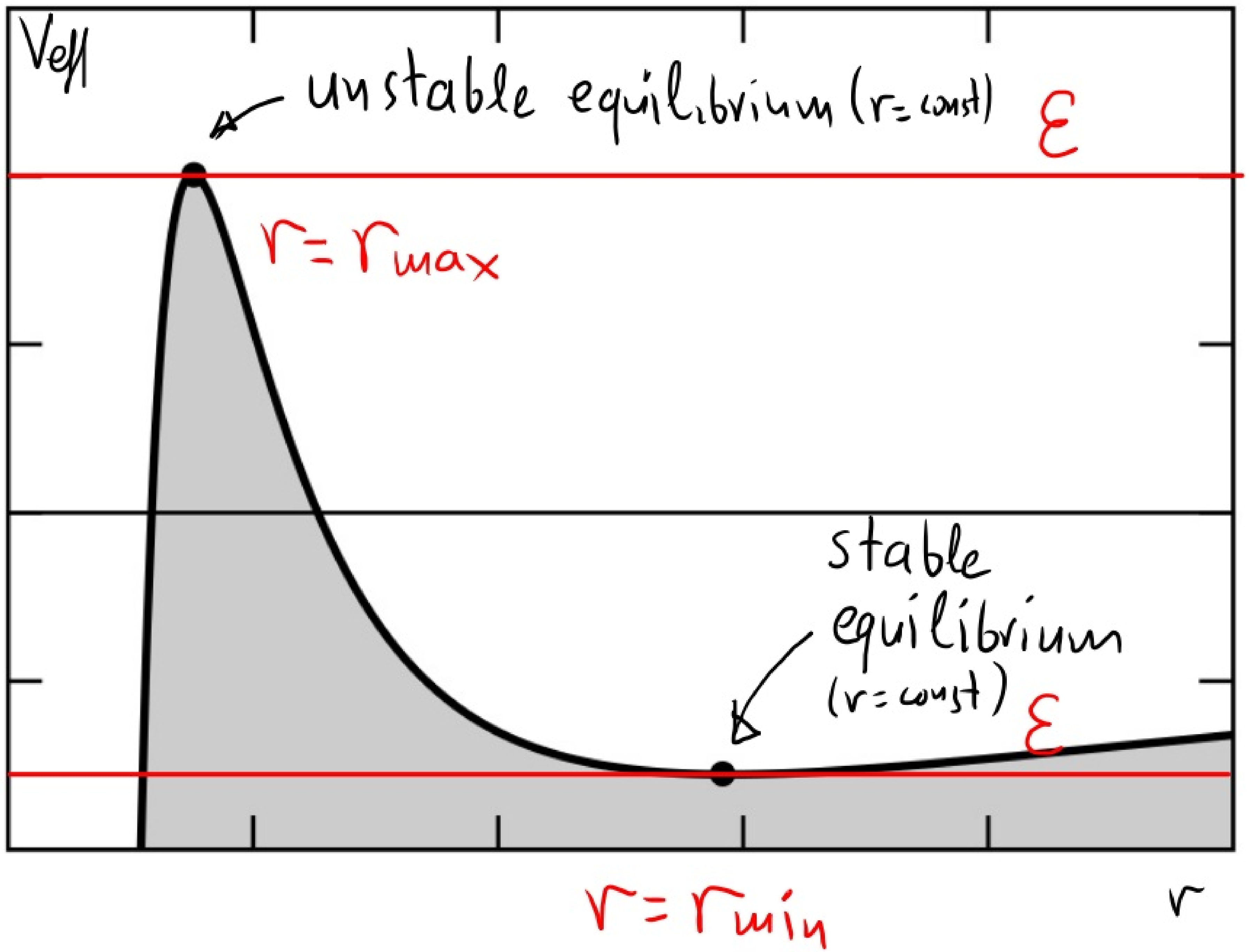


when $\mathcal{E} = V(r_{max})$ – unstable circular orbits

$\mathcal{E} = V(r_{min})$ – stable circular orbits

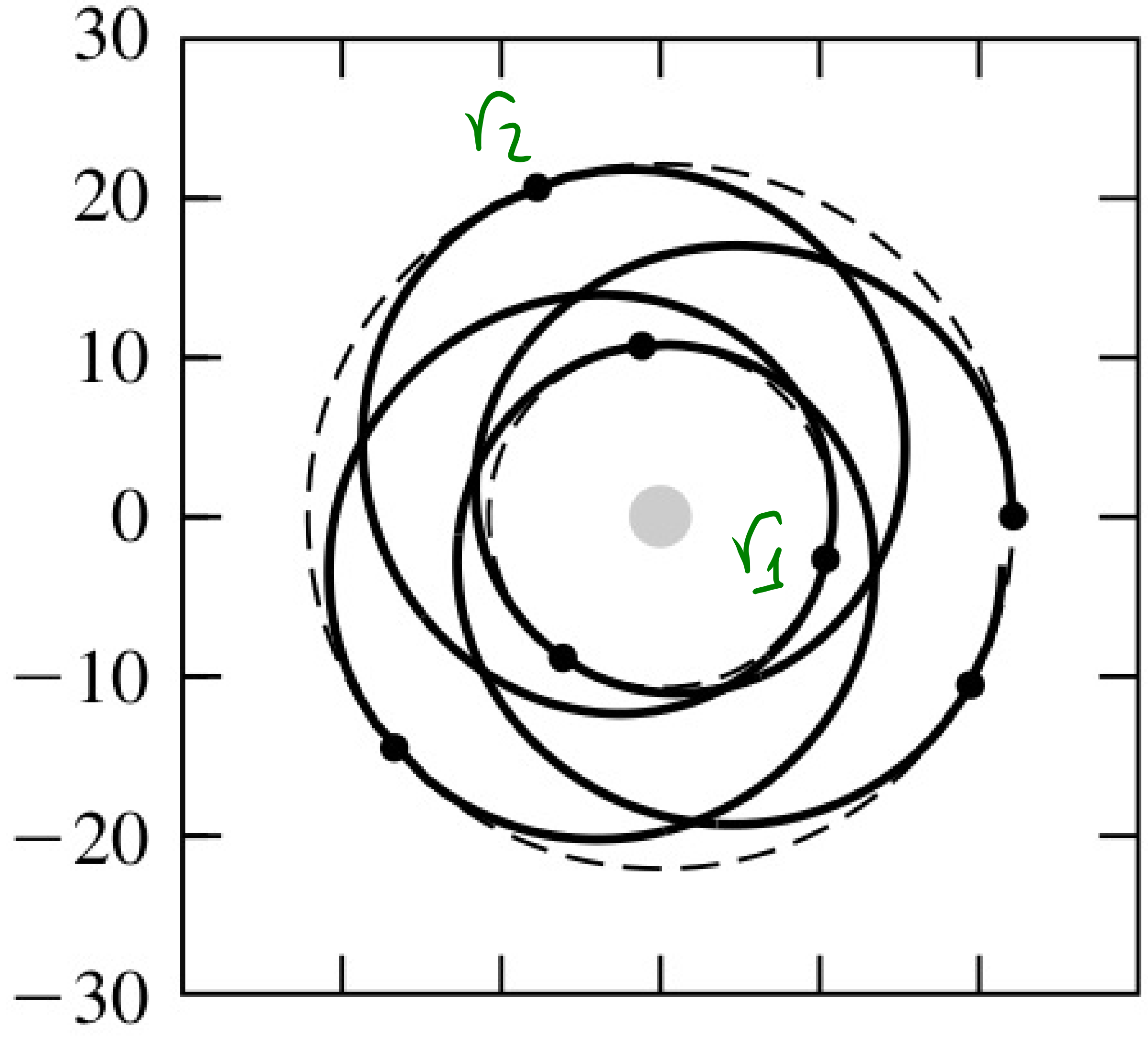
Hartle, Fig 9.4



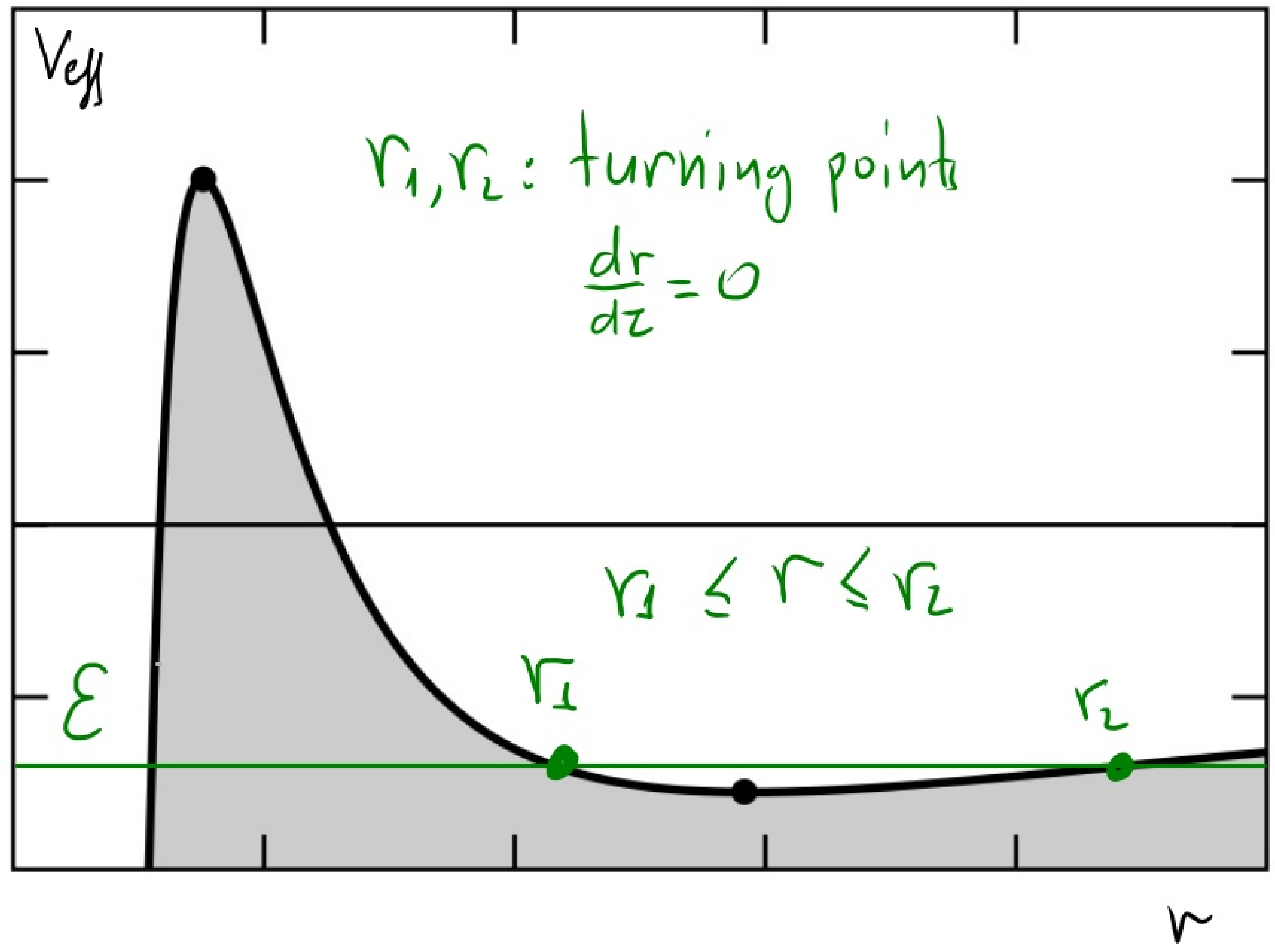


when $\mathcal{E} = V(r_{max})$ – unstable circular orbits

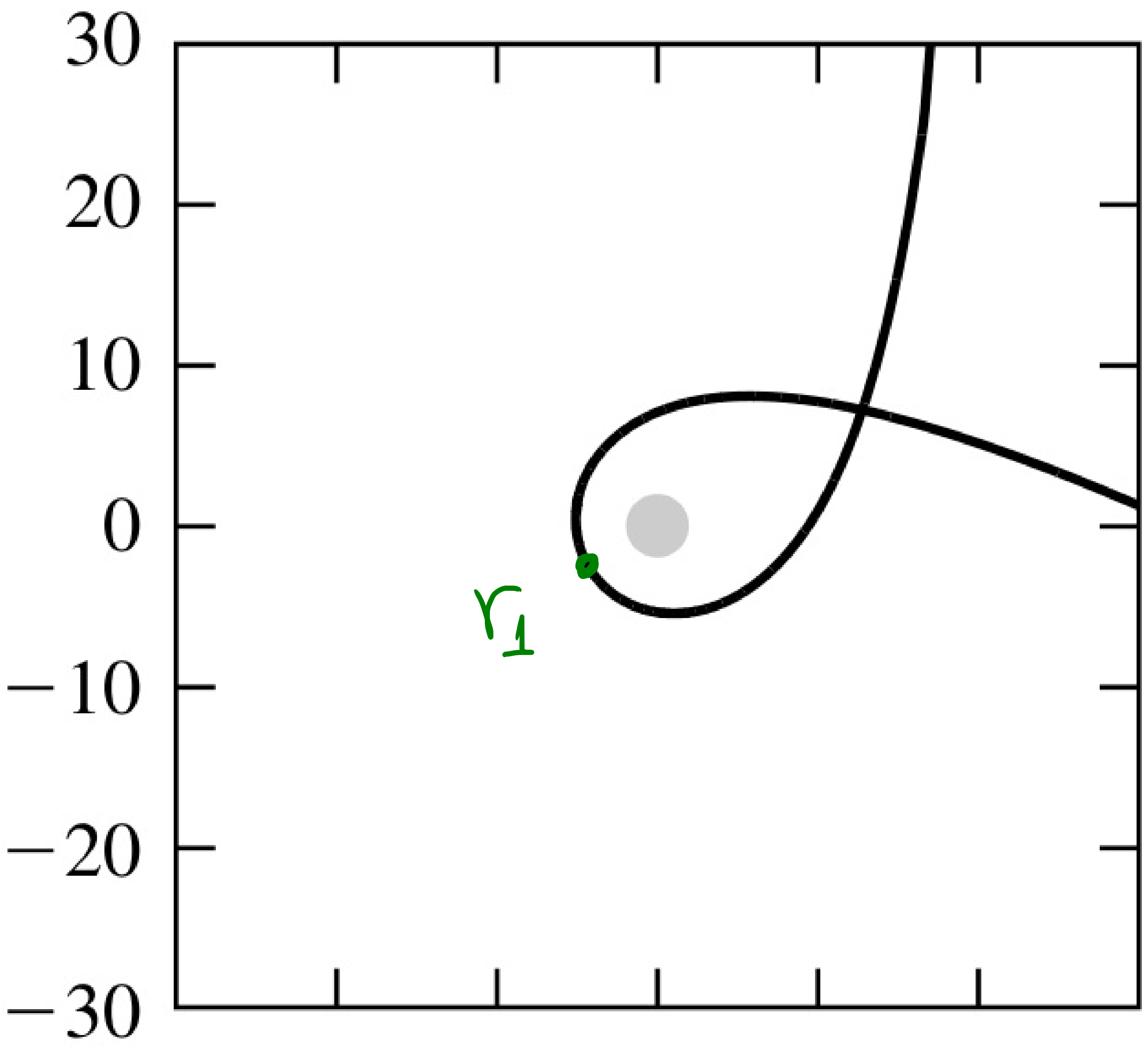
$\mathcal{E} = V(r_{min})$ – stable circular orbits

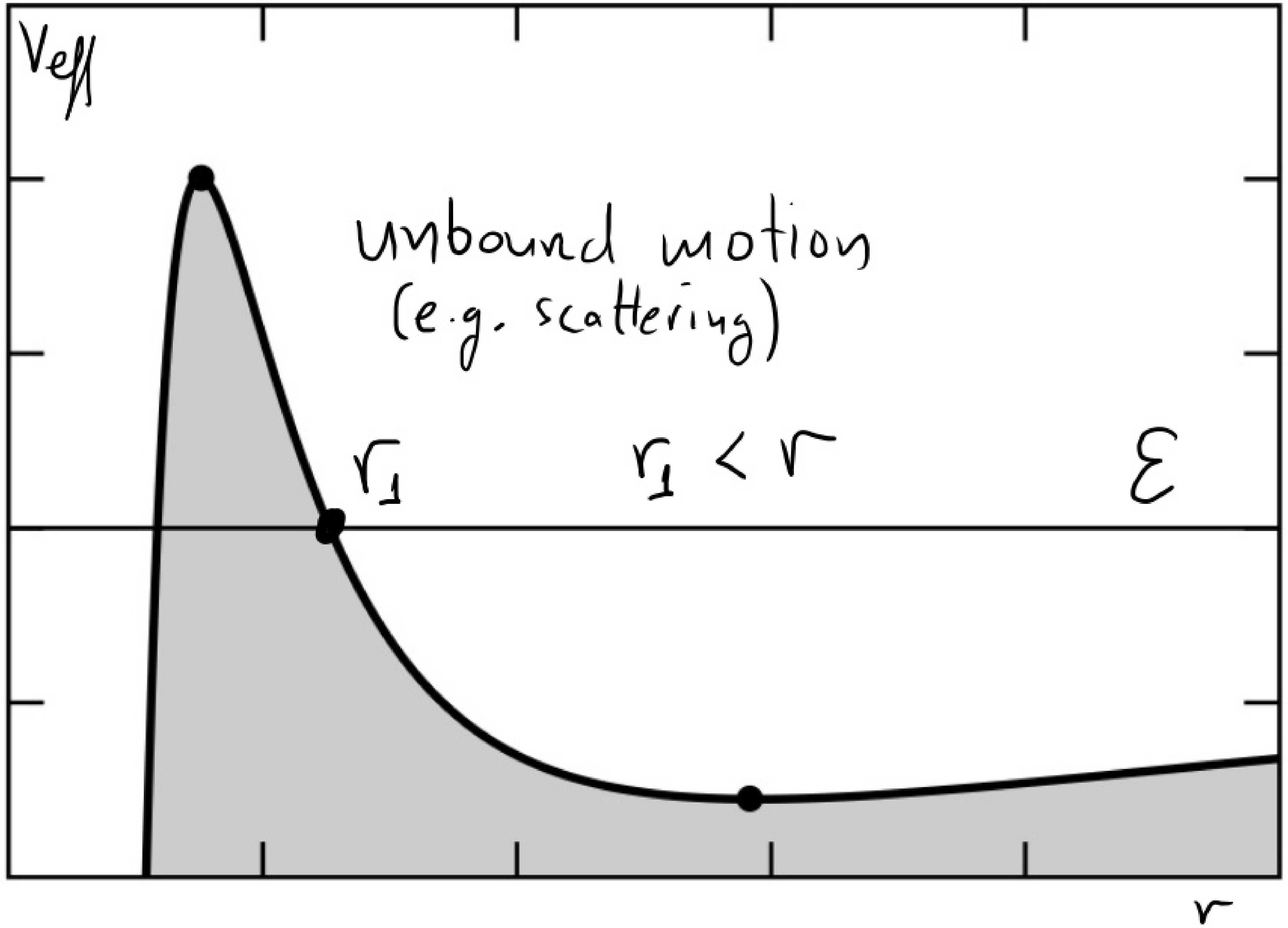


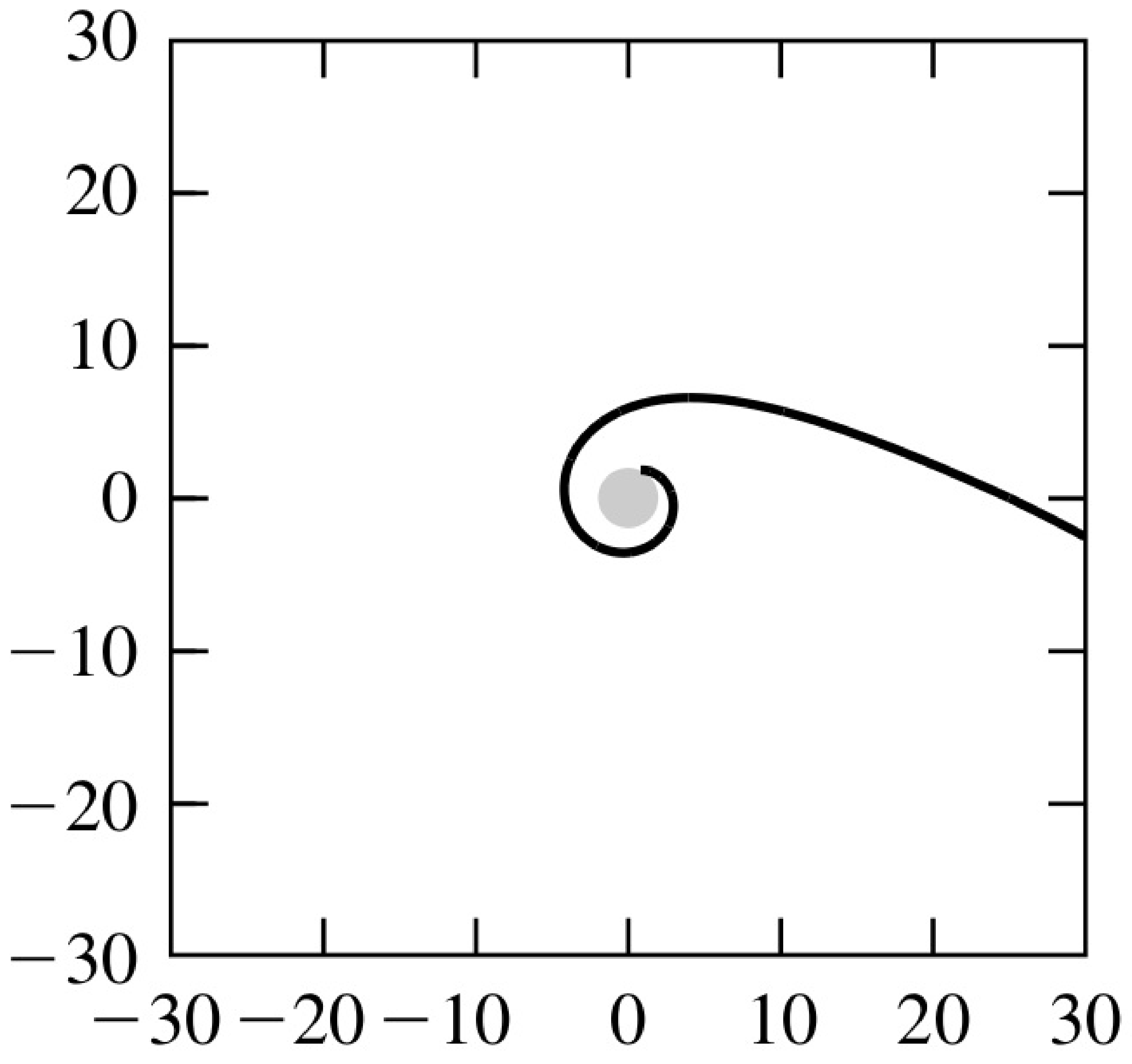
non-periodic ...

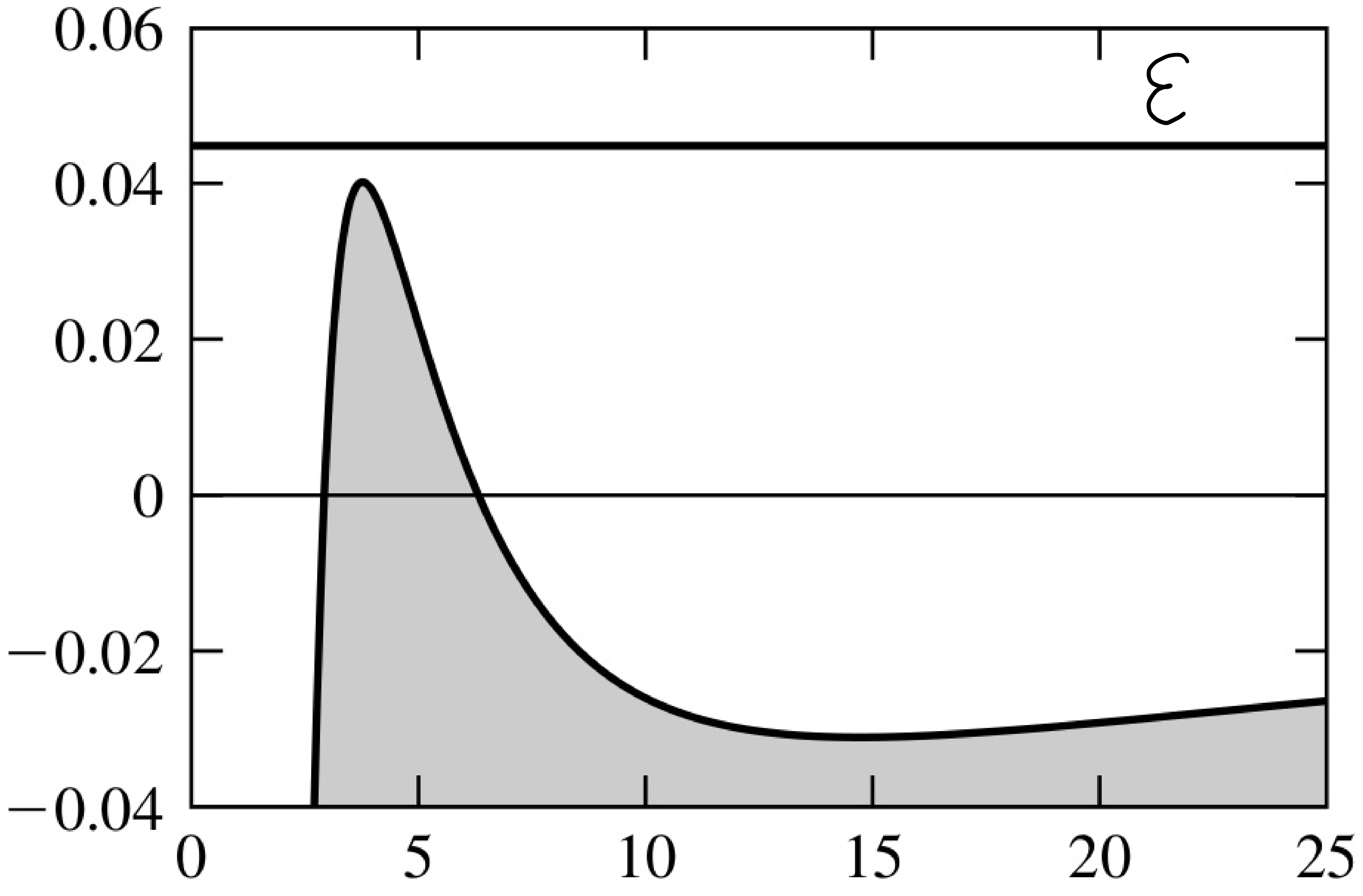


Bound motion

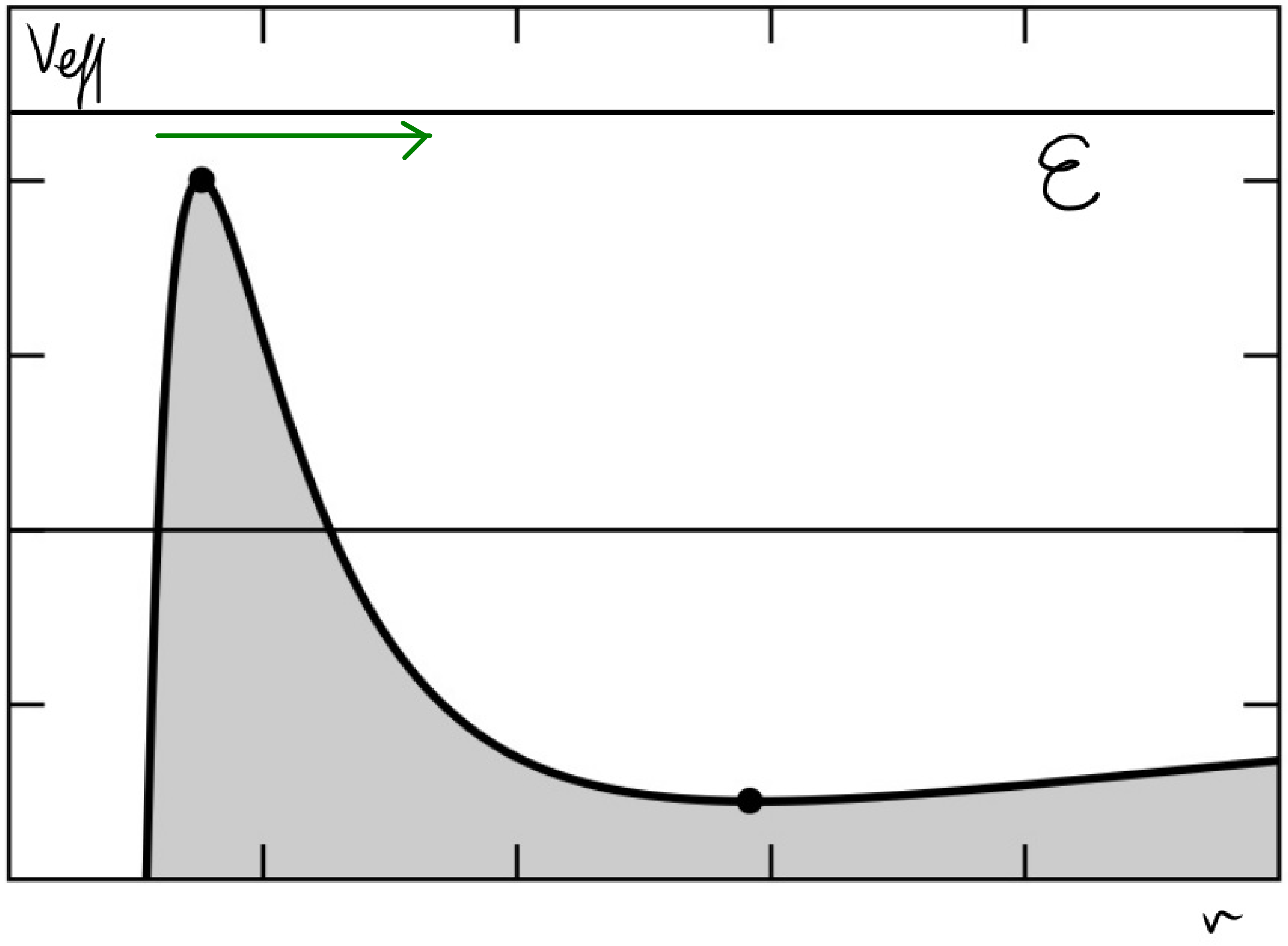

30
20
10
0
−10
−20
−30
r_1

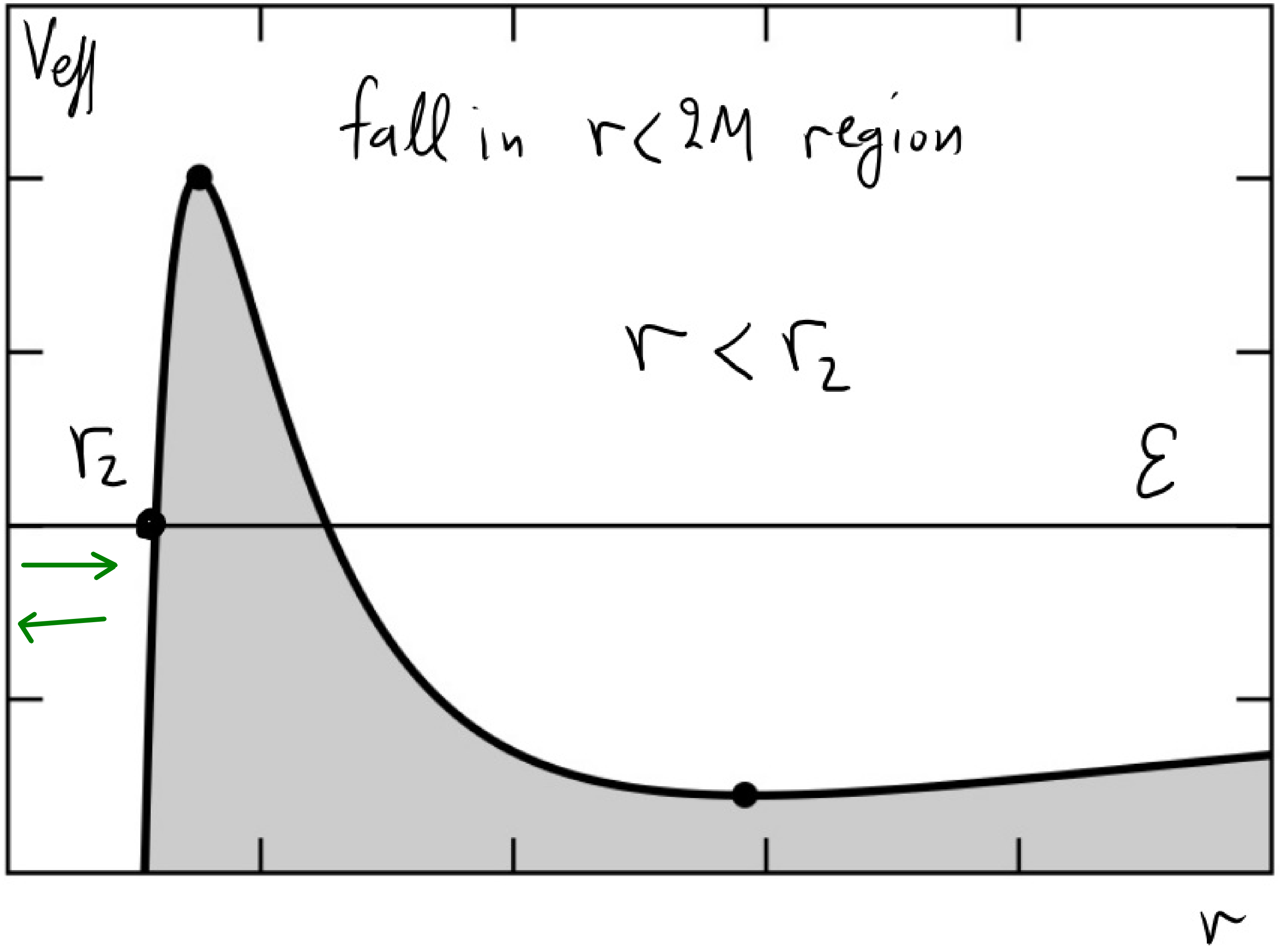

V_{eff}
unbound motion
(e.g. scattering)
r_1
$r_1 < r$
$\mathcal{E}$
r





Fall into the black hole!





Escape from the black hole

Fall into the black hole, cannot escape!

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1-\frac{2M}{r}\right)^{-1} \underset{r\gg 2M}{\approx} e \Rightarrow e = 1$$

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1 - \frac{2M}{r}\right)^{-1} \underset{r \gg 2M}{\approx} e \Rightarrow e = 1 \Rightarrow \mathcal{E} = \frac{e^2 - 1}{2} = 0$$

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$$l = 0 < \sqrt{12}\, M \Rightarrow \frac{d\varphi}{d\tau} = 0$$

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1 - \frac{2M}{r}\right)^{-1} \underset{r \gg 2M}{\approx} e \Rightarrow e = 1 \Rightarrow \mathcal{E} = \frac{e^2 - 1}{2} = 0$$

$$\ell = 0 < \sqrt{12}\, M \Rightarrow \frac{d\varphi}{d\tau} = 0$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0$$

$\hookrightarrow$ $\ell = 0$, so $V_{eff}(r) = -\frac{M}{r}$

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1 - \frac{2M}{r}\right)^{-1} \underset{r \gg 2M}{\approx} e \Rightarrow e = 1 \Rightarrow \mathcal{E} = \frac{e^2 - 1}{2} = 0$$

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$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

inward motion

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1 - \frac{2M}{r}\right)^{-1} \underset{r \gg 2M}{\approx} e \Rightarrow e = 1 \Rightarrow \mathcal{E} = \frac{e^2 - 1}{2} = 0$$

$$l = 0 < \sqrt{12}\,M \Rightarrow \frac{d\varphi}{d\tau} = 0$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \qquad \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}$$

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1-\frac{2M}{r}\right)^{-1} \underset{r \gg 2M}{\approx} e \Rightarrow e = 1 \Rightarrow \mathcal{E} = \frac{e^2-1}{2} = 0$$

$$l = 0 < \sqrt{12}\,M \Rightarrow \frac{d\varphi}{d\tau} = 0$$

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$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \qquad \Rightarrow \frac{dt}{d\tau} = \left(1-\frac{2M}{r}\right)^{-1}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1-\frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow r^{1/2}\, dr = -(2M)^{1/2}\, d\tau$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1 - \frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \quad \Rightarrow \quad r^{1/2}\, dr = -(2M)^{1/2}\, d\tau$$

$$\Rightarrow \quad \int_0^r r'^{1/2}\, dr' = -\int_{\tau_*}^{\tau} (2M)^{1/2}\, d\tau' \qquad , \quad r(\tau_*) = 0$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \quad \Rightarrow \quad \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \quad \Rightarrow \quad \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1 - \frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

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$$\Rightarrow \int_0^r r'^{1/2}\, dr' = -\int_{\tau_*}^{\tau} (2M)^{1/2}\, d\tau' \quad , \quad r(\tau_*) = 0$$

$$\Rightarrow \frac{r^{3/2}}{\frac{3}{2}} = -(2M)^{1/2}(\tau - \tau_*)$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1 - \frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow r^{1/2}\,dr = -(2M)^{1/2}\,d\tau$$

$$\Rightarrow \int_0^r r'^{1/2}\,dr' = -\int_{\tau_*}^{\tau} (2M)^{1/2}\,d\tau' \quad , \quad r(\tau_*) = 0$$

$$\Rightarrow \frac{r^{3/2}}{\frac{3}{2}} = -(2M)^{1/2}(\tau - \tau_*)$$

$$\Rightarrow r(\tau) = \left(3/2\right)^{2/3} (2M)^{1/3} (\tau_* - \tau)^{2/3}$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \qquad \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1 - \frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow r(\tau) = \left(3/2\right)^{2/3}\left(2M\right)^{1/3}\left(\tau_* - \tau\right)^{2/3}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{dt/d\tau}{dr/d\tau} = \frac{dt}{dr} = -\left(\frac{2M}{r}\right)^{-1/2}\left(1-\frac{2M}{r}\right)^{-1}$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \qquad (1)$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \Rightarrow \frac{dt}{d\tau} = \left(1-\frac{2M}{r}\right)^{-1} \qquad (2)$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1-\frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

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- From $r(\tau)$ $\Rightarrow$ $\begin{cases} \text{takes finite } \tau \text{ to cross } r = 2M \\ \quad\text{"} \quad\quad \text{"} \quad\quad \text{"} \quad \text{to fall on } r = 0 \end{cases}$

Shape of bound orbits

We want $r = r(\phi)$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \Rightarrow \left(\frac{dr}{d\tau}\right) = \pm\left[2\left(\mathcal{E} - V_{eff}\right)\right]^{1/2}$$

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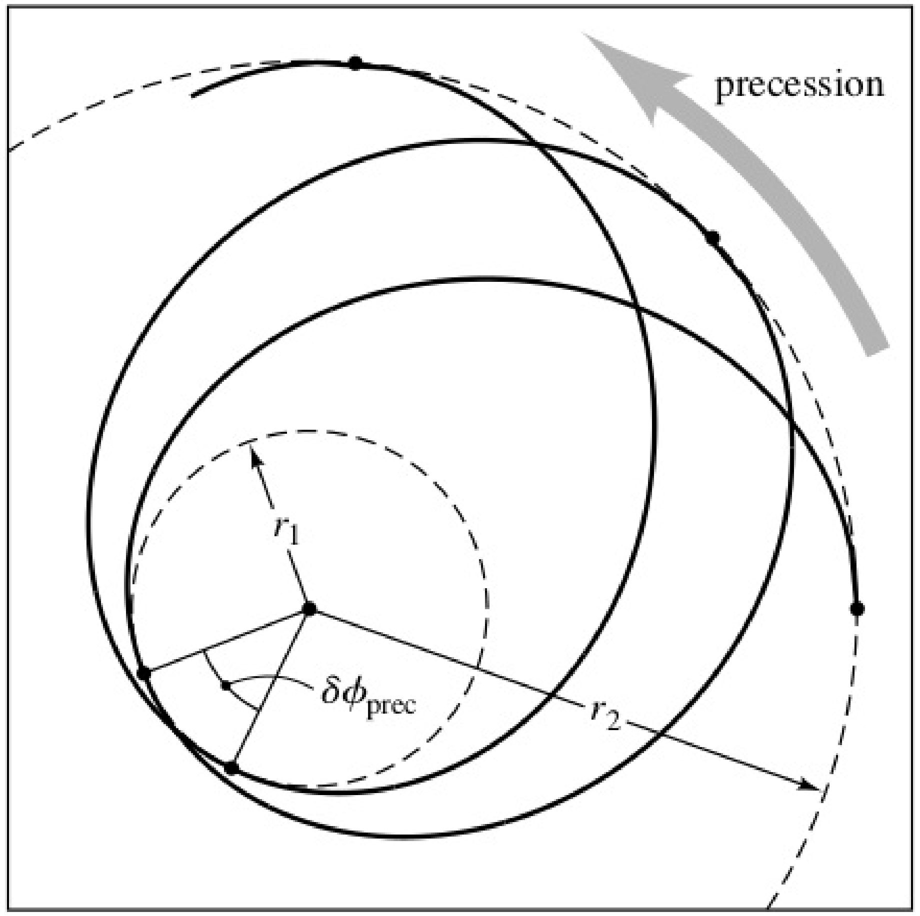
$$\Delta\phi = 2\left[\int_{r_1}^{r_2} dr\, \frac{\ell}{r^2}\left[e^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)\right]^{-1/2}\right]$$

$$\left.\frac{dr}{d\tau}\right|_{r_1} = \left.\frac{dr}{d\tau}\right|_{r_2} = 0$$



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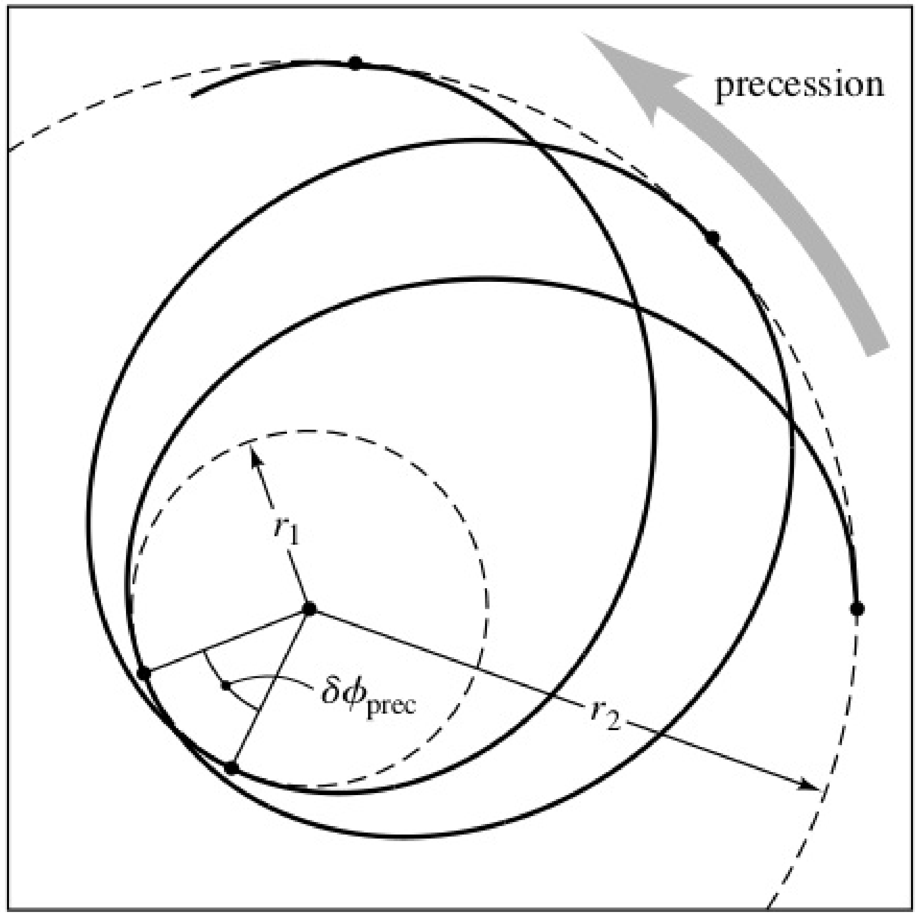
when $\frac{2M\ell^2}{r^3}$ term neglected $\Rightarrow \Delta\phi = 2\pi$ (no precession)



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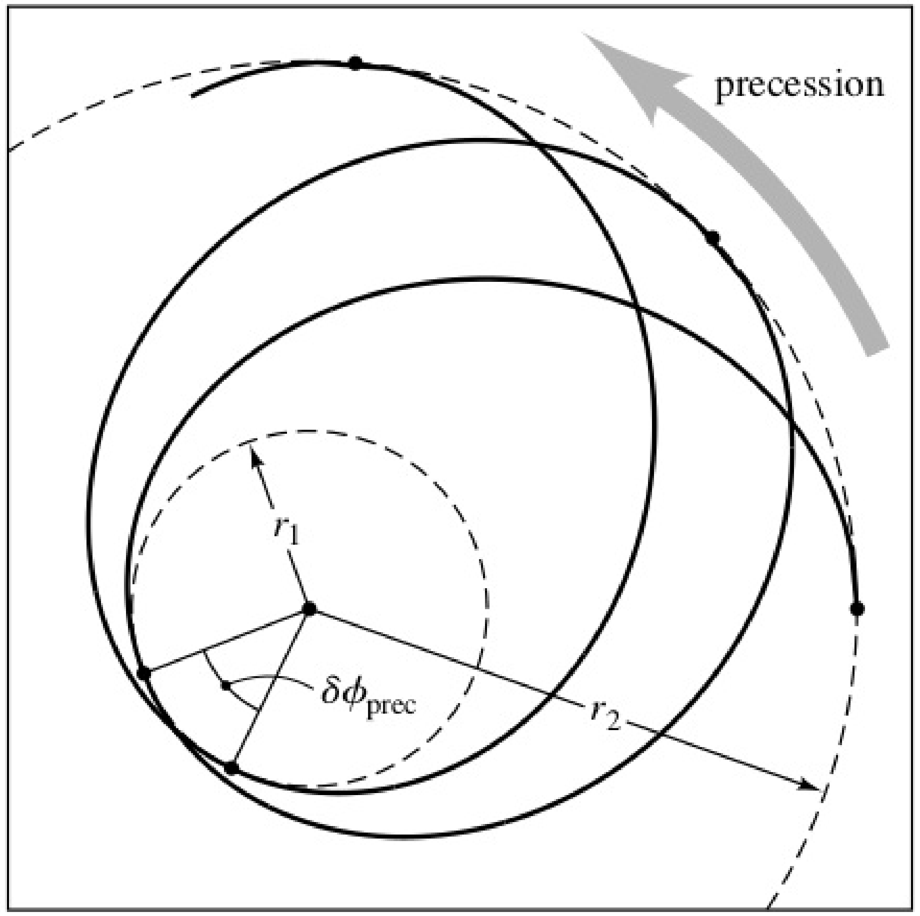

when $\frac{2 \cdot M \ell^2}{r^3}$ term is small, then

$$\delta\phi \approx 6\pi\left(\frac{M}{\ell}\right)^2$$

For Mercury $\sim 43''$/century (detectable)

Stable Circular Orbits

$$r = \frac{\ell^2}{2M}\left[1 + \left(1 - 12\left(\frac{M}{\ell}\right)^2\right)^{1/2}\right]$$

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smallest when $\frac{\ell}{M} = \sqrt{12}$ $\Rightarrow$ $r_{ISCO} = \frac{(\sqrt{12}\,M)^2}{2M} = 6M$

Innermost Stable Circular Orbit

Stable Circular Orbits

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Compute $\frac{\ell}{e}$ as function of M, r:

$$\frac{dV_{eff}}{dr} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2} + \frac{3M\ell^2}{r^3} = 0$$

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$$= \left(1 - \frac{2M}{r}\right)^2 \frac{\frac{r}{M}}{\frac{r^2}{\ell^2}}$$

$$\frac{r}{M} - 3 = \frac{r^2}{\ell^2}$$

$$\Uparrow$$

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$$= \left(1-\frac{2M}{r}\right)^2 \frac{\frac{r}{M}}{\frac{r^2}{\ell^2}} = \left(1-\frac{2M}{r}\right)^2 \frac{\ell^2}{Mr}$$

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$$\Rightarrow \Omega^2 = \frac{M}{r^3}$$ Kepler's law in GR!

Four velocity $u^\mu = \left(\frac{dt}{d\tau}, \cancel{\frac{dr}{d\tau}}, \cancel{\frac{d\theta}{d\tau}}, \frac{d\varphi}{d\tau} \right)$

$$= \left(\frac{dt}{d\tau}, 0, 0, \frac{d\varphi}{dt}\frac{dt}{d\tau} \right)$$

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$$\Rightarrow \quad u^\mu = \left(\left(1 - \frac{3M}{r}\right)^{-1/2}, 0, 0, \left(1 - \frac{3M}{r}\right)^{-1/2} \frac{M}{r^3} \right)$$

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