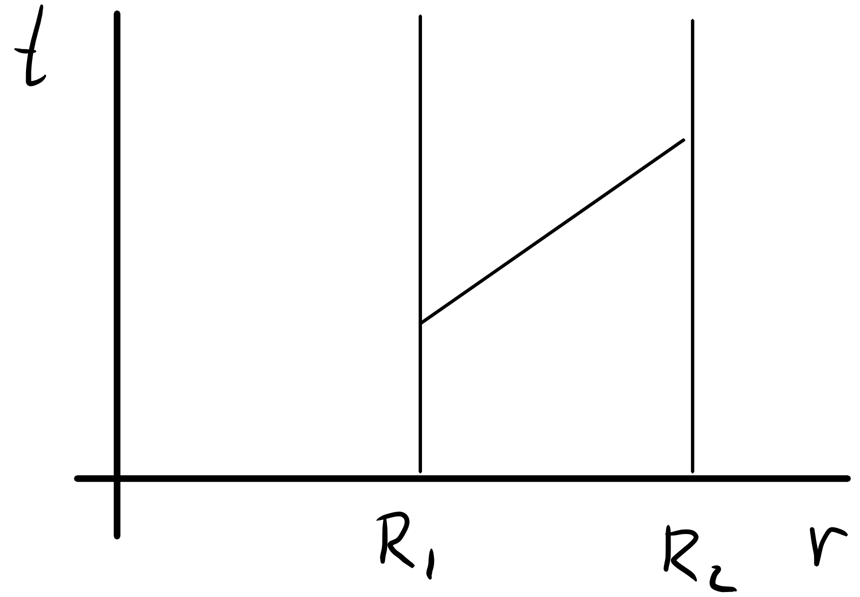
① Consider the metric (not a solution to Einstein's Equations)

$$ds^2 = \left(1-\frac{2M}{r}\right)\left[-dt^2+dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\varphi^2\right]$$

- Calculate the redshift of photons for stationary observers at $r=R_1$ and $r=R_2$
- Go through the analysis of the radial motion of a freely falling particle (both massive & massless case), like we did in class for the Schwarzschild metric. In particular find the respective conserved quantities and the conditions that lead to bound/unbound trajectories and the qualitative properties of their shape.

- Anne and George are twins living on the surface of the "earth" at $r=R_1$.
  Anne takes a lift along the radial direction on a (non geodesic) trajectory $r=vt+c_1$ to $r=R_2>R_1$, and then she immediately comes back along $r=-vt+c_2$ to meet with George again
  - calculate c_1 and c_2
  - who is going to be older when they meet again, and by how much?
  - Calulate Anne's 4-velocity and write down the condition $u^\mu u_\mu=-1$ as a relation between $\left(\frac{dt}{d\tau}\right)^2, \left(\frac{dr}{d\tau}\right)^2$ and r.

KVF = Killing Vector Field



- Redshift:

$\partial_t \equiv \xi$ is a KVF. If $p^\mu = \frac{dx^\mu}{d\lambda}$ is the 4-momentum of a photon propagating freely, then

$p_\mu \xi^\mu$ is conserved. $\xi^\mu = (1,0,0,0)$

The stationary observers have 4-velocities $u^\mu = (u^0,0,0,0)$

$$u_\mu u^\mu = -1 \Rightarrow g_{00}(u^0)^2 = -1 \Rightarrow -\left(1-\frac{2M}{r}\right)(u^0)^2 = -1 \Rightarrow u_0 = \left(1-\frac{2M}{r}\right)^{-1/2} \Rightarrow u^\mu = \left[\left(1-\frac{2M}{r}\right)^{-1/2},0,0,0\right] = \left(1-\frac{2M}{r}\right)^{-1/2}\xi^\mu$$

Then $E_1 = -p_\mu u^\mu|_{R_1} = -\left(1-2\frac{M}{R_1}\right)^{-1/2} p_\mu \xi^\mu|_{R_1}$

$E_2 = -p_\mu u^\mu|_{R_2} = -\left(1-\frac{2M}{R_2}\right) \quad p_\mu \xi^\mu|_{R_2}$

$$\Rightarrow \frac{E_1}{E_2} = \frac{\left(1-\frac{2M}{R_1}\right)^{-1/2}}{\left(1-\frac{2M}{R_2}\right)^{-1/2}} \frac{\cancel{p_\mu \xi^\mu}|_{R_1}}{\cancel{p_\mu \xi^\mu}|_{R_2}} \Rightarrow \frac{\omega_1}{\omega_2} = \frac{\left(1-\frac{2M}{R_1}\right)^{-1/2}}{\left(1-\frac{2M}{R_2}\right)^{-1/2}}$$

Massive particle: $\quad u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$

$\partial_t, \partial_\phi$ are KVFs

$\xi = \partial_t \quad \eta = \partial_\phi$

$$\Rightarrow \quad e = -u_\mu \xi^\mu = \left(1-\frac{2M}{R}\right)\frac{dt}{d\tau}$$

$$l = u_\mu \eta^\mu = \left(1-\frac{2M}{R}\right) r^2 \sin^2\theta \frac{d\phi}{d\tau}$$

are conserved quantities

using the same arguments as in the lecture (Hartle §9.3), l conserved $\Rightarrow$ motion on plane

So we choose $\theta = \frac{\pi}{2} \Rightarrow \frac{d\theta}{d\tau} = 0$, $\sin\theta = 1$, so

$$\frac{dt}{d\tau} = e\left(1-\frac{2M}{r}\right)^{-1} \qquad \frac{d\phi}{d\tau} = \left(1-\frac{2M}{r}\right)\frac{l}{r^2}$$

$$u_\mu u^\mu = -1 \Rightarrow g_{00}(u^0)^2 + g_{11}(u^1)^2 + g_{33}(u^3)^2 = -1$$

assume $r > 2M$

$$\Rightarrow \left(1-\frac{2M}{r}\right)\left[-\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right] = -1$$

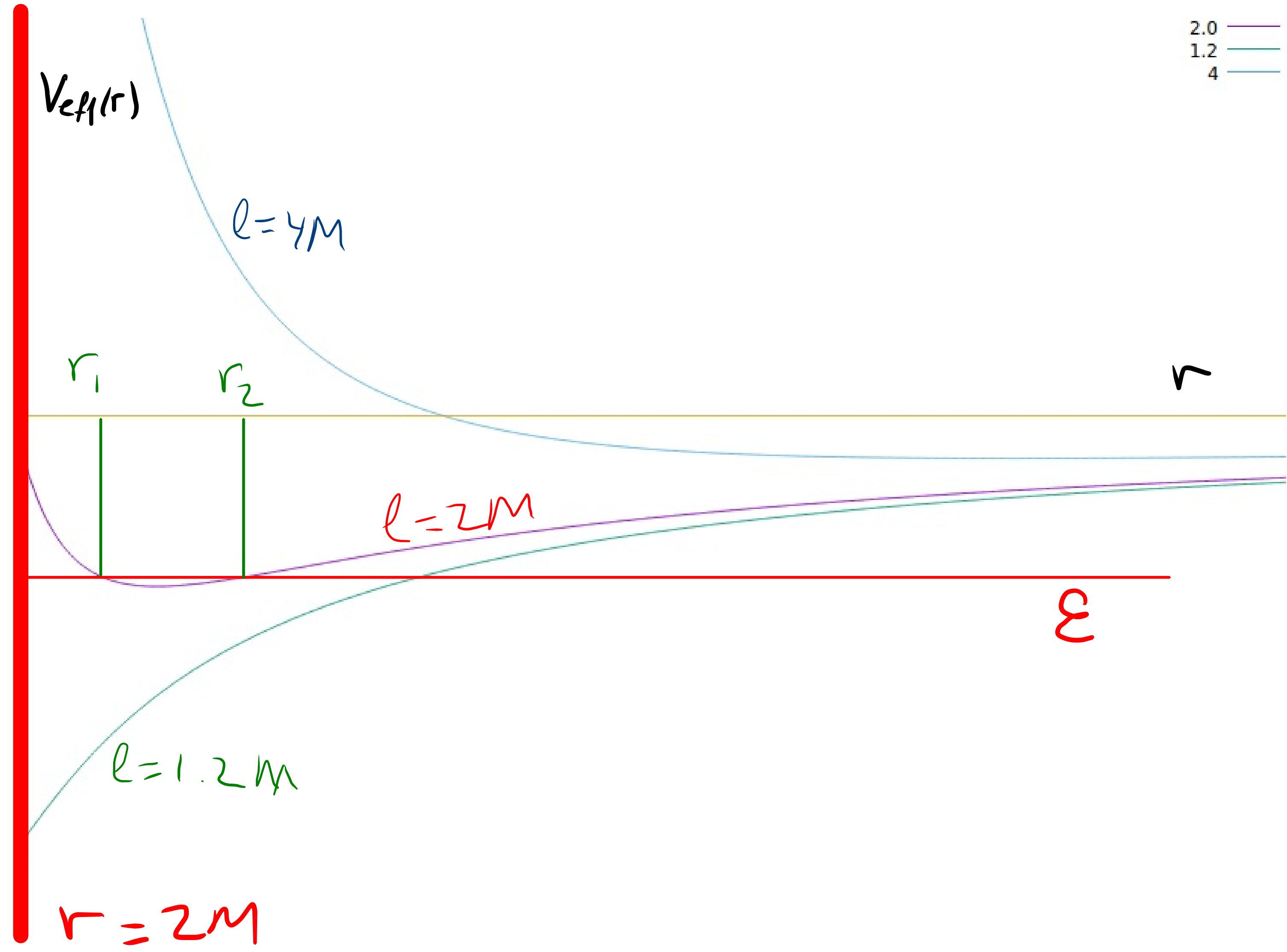
$$\Rightarrow \left(1-\frac{2M}{r}\right)\left[-e^2\left(1-\frac{2M}{r}\right)^{-2} + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-2}\frac{\ell^2}{r^2}\right] = -1$$

$$\Rightarrow -e^2\left(1-\frac{2M}{r}\right)^{-2} + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-2}\frac{\ell^2}{r^2} = -\left(1-\frac{2M}{r}\right)^{-1}$$

$$\Rightarrow -e^2 + \left(1-\frac{2M}{r}\right)^2\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2} = -\left(1-\frac{2M}{r}\right)$$

$$\Rightarrow e^2 - 1 = \left(1-\frac{2M}{r}\right)^2\left(\frac{dr}{d\tau}\right)^2 - \frac{2M}{r} + \frac{\ell^2}{r^2}$$

$$\Rightarrow \mathcal{E} = \frac{e^2-1}{2} = \frac{1}{2}\left(1-\frac{2M}{r}\right)^2\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} + \frac{\ell^2}{2r^2}$$



Using $V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2}$ we can analyze orbits as in the Newtonian case. We have bound orbits, scattering, and orbits falling on the singularity $r=2M$

We can see that $r=2M$ is a singularity, by calculating the Ricci scalar:

$$R = R^{\mu}{}_{\mu} = \frac{6M^2}{r(r-2M)^3}$$, so $r=2M$ is a singular point

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Massless particle: $u^{\mu} = \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, 0, \frac{d\phi}{d\lambda}\right)$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} = \text{const} \Rightarrow \frac{dt}{d\lambda} = \left(1-\frac{2M}{r}\right)^{-1} e$$

$$\ell = \left(1-\frac{2M}{r}\right) r^2 \frac{d\phi}{d\lambda} = \text{const} \Rightarrow \frac{d\phi}{d\lambda} = \left(1-\frac{2M}{r}\right)^{-1}\frac{\ell}{r^2}$$

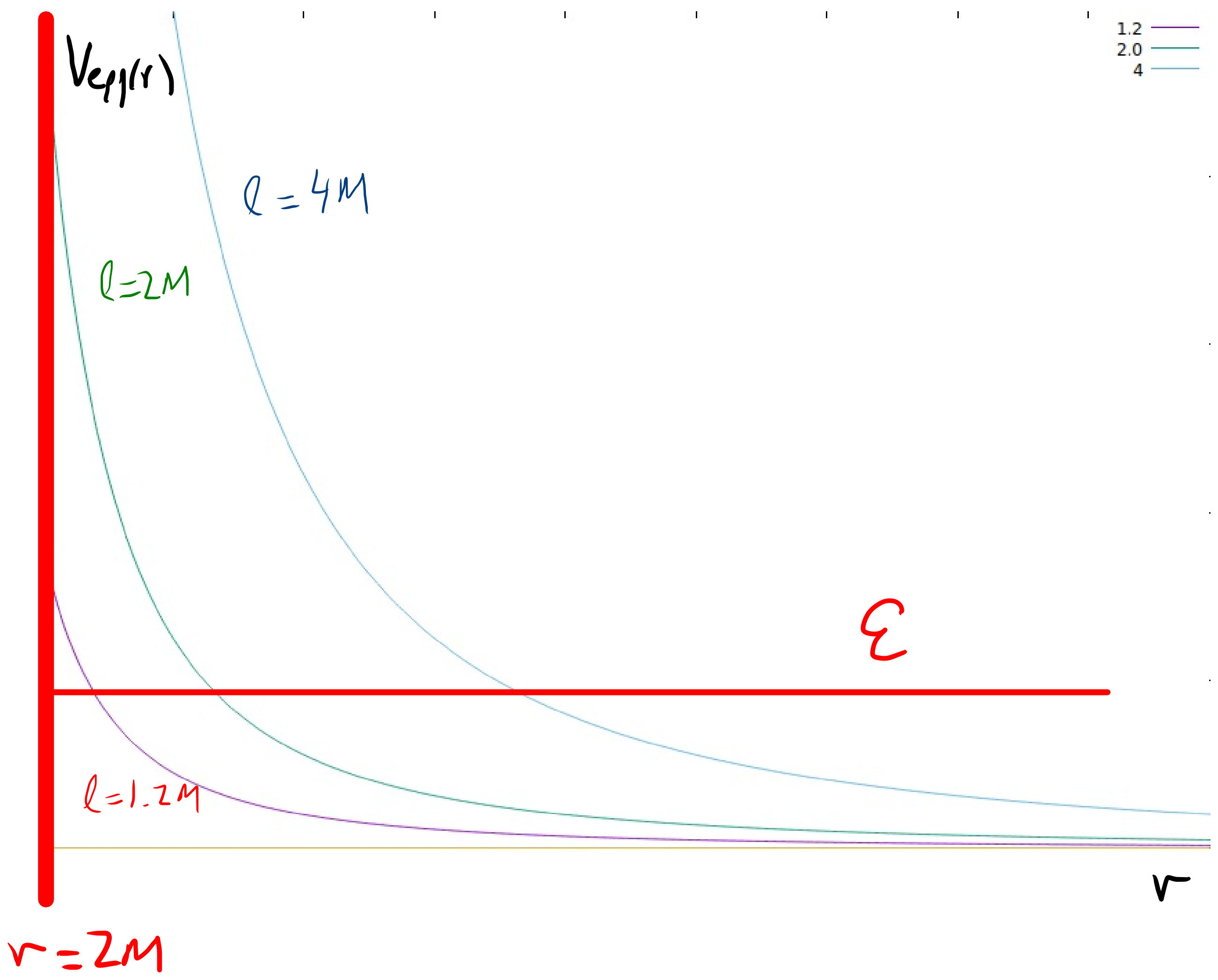
$$u_{\mu}u^{\mu} = 0 \Rightarrow -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)\left(\frac{dr}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right) r^2 \left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

$$\Rightarrow -\left(1-\frac{2M}{r}\right) e^2 \left(1-\frac{2M}{r}\right)^{-2} + \left(1-\frac{2M}{r}\right)(\dot{r})^2 + \left(1-\frac{2M}{r}\right) r^2 \left(1-\frac{2M}{r}\right)^{-2}\frac{\ell^2}{r^4} = 0$$

$$\Rightarrow -e^2\left(1-\frac{2M}{r}\right)^{-1} + \left(1-\frac{2M}{r}\right)(\dot{r})^2 + \left(1-\frac{2M}{r}\right)^{-1}\frac{\ell^2}{r^2} = 0$$

$$\Rightarrow -e^2 + \left(1-\frac{2M}{r}\right)^2(\dot{r})^2 + \frac{\ell^2}{r^2} = 0$$

$$\Rightarrow \mathcal{E} = \frac{e^2}{2} = \frac{1}{2}\left(1-\frac{2M}{r}\right)^2(\dot{r})^2 + \frac{\ell^2}{2r^2}$$



Anne starts at $t=0$ $r=R_1$, so when going up

$r(0) = R_1 = v \cdot 0 + c_1 \Rightarrow c_1 = R_1$

$r(t) = vt + R_1$

She arrives at $r(t_1) = R_2 \Rightarrow vt_1 + R_1 = R_2 \Rightarrow t_1 = \frac{R_2 - R_1}{v}$

Her 4 velocity is $u^\mu = (u^0, u^1, 0, 0) = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right)$

The proper time elapsed during her first trip upwards was:

$$\tau_1 = \int \sqrt{d\tau^2} = \int \left[|g_{00}(dt)^2 + g_{11}\,dr^2| \right]^{1/2} = \int_0^{t_1} \left[\left|g_{00}\left(\frac{dt}{dt}\right)^2 + g_{11}\left(\frac{dr}{dt}\right)^2\right| \right]^{1/2} dt$$

$$= \int_0^{t_1} \left[+\left(1-\frac{2M}{r}\right)\cdot 1 - \left(1-\frac{2M}{r}\right) v^2 \right]^{1/2} dt = \int_0^{t_1} (1-v^2)^{1/2} \sqrt{1-\frac{2M}{vt+R_1}}\; dt$$

$$= \frac{1}{v}(1-v^2)^{1/2}\left[(R_2(R_2-2M))^{1/2} - (R_1(R_1-2M))^{1/2} + M \ln \frac{\left(1+\sqrt{1-\frac{2M}{R_1}}\right)\left(-1+\sqrt{1-\frac{2M}{R_2}}\right)}{\left(-1+\sqrt{1-\frac{2M}{R_1}}\right)\left(1+\sqrt{1-\frac{2M}{R_2}}\right)} \right]$$

On her way back she follows the trajectory

$$r = -vt + c_2 \quad , \text{with } r(t_1) = R_2 \Rightarrow R_2 = -v\frac{R_2-R_1}{v} + c_2 \Rightarrow c_2 = 2R_2 - R_1$$

She arrives at $r = R_1$ when

$$r(t_2) = R_1 \Rightarrow -vt_2 + (2R_2 - R_1) = R_1 \Rightarrow t_2 = 2\frac{R_2-R_1}{v}$$

For George the elapsed time is

$$d\tau^2 = -ds^2 = -g_{00}dt^2 = +\left(1-\frac{2M}{R_1}\right)dt^2 \Rightarrow$$

$$\Delta\tau = \left(1-\frac{2M}{R_1}\right)^{1/2}\Delta t = \left(1-\frac{2M}{R_1}\right)^{1/2}(t_2-0) = \left(1-\frac{2M}{R_1}\right)^{1/2} 2\,\frac{R_2-R_1}{v}$$

For Anne, the trip coming back lasted for

$$\tau_2 = \int_{t_1}^{t_2} d\tau = \int_{t_1}^{t_2}\left[-g_{00}\left(\frac{dt}{dt}\right)^2 - g_{11}\left(\frac{dr}{dt}\right)^2\right]^{1/2} dt = \int_{t_1}^{t_2}(1-v^2)^{1/2}\sqrt{1-\frac{2M}{-vt+2R_2-R_1}}\;dt$$

$$= \tau_1 \qquad \text{(due to symmetry we should have anticipated it)}$$

Therefore, Anne aged by $2\tau_1$

The 4-velocity of Anne is $u^\mu = (u^0, u^1, 0, 0) = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right) = \frac{dt}{d\tau}\left(1, \frac{dr}{dt}, 0, 0\right)$

$$U^\mu = \frac{dt}{d\tau}\left(1, \pm v, 0, 0\right)$$

But we have calculated that on her way up

$$d\tau = (1-v^2)^{1/2}\left(1 - \frac{2M}{vt+R_1}\right)^{1/2} dt \Rightarrow \frac{dt}{d\tau} = (1-v^2)^{-1/2}\left(1 - \frac{2M}{vt+R_1}\right)^{-1/2}$$

$$\Rightarrow U^\mu = (1-v^2)^{-1/2}\left(1 - \frac{2M}{vt+R_1}\right)\left(1, v, 0, 0\right)$$

On her way down:

$$d\tau = (1-v^2)^{1/2}\left(1 - \frac{2M}{-vt+2R_2-R_1}\right)^{+1/2} dt \quad , \text{ so}$$

$$U^\mu = (1-v^2)^{-1/2}\left(1 - \frac{2M}{-vt+2R_2-R_1}\right)^{-1/2}\left(1, -v, 0, 0\right)$$

$$u_\mu u^\mu = g_{00}(u^0)^2 + g_{11}(u^1)^2$$

$$= (1-v^2)^{-1}\left(1-\frac{2M}{vt+R_1}\right)^{-1}\left(-\left(1-\frac{2M}{r}\right)+\left(1-\frac{2M}{r}\right)v^2\right)$$

$$= -\left(1-\frac{2M}{r}\right)\left(1-\frac{2M}{vt+R_1}\right)^{-1} = -\left(1-\frac{2M}{vt+R_1}\right)\left(1-\frac{2M}{vt+R_1}\right)^{-1} = -1 \text{ , as expected}$$

Without knowing $r(t)$, we would have

$$u_\mu u^\mu = g_{00}\left(\frac{dt}{d\tau}\right)^2 + g_{11}\left(\frac{dr}{d\tau}\right)^2 = -\left(1-\frac{2M}{r}\right)\left[\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2\right] = -1$$

$$\Rightarrow \left(\frac{dt}{d\tau}\right)^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)$$

which relates $\left(\frac{dt}{d\tau}\right)^2$ to $\left(\frac{dr}{d\tau}\right)^2$ $\forall\, t$.

7. Two particles fall radially in from infinity in the Schwarzschild geometry. One starts with $e = 1$, the other with $e = 2$. A stationary observer at $r = 6M$ measures the speed of each when they pass by. How much faster is the second particle moving at that point?

Hartle 9.7

The 4-velocity of the observer is $u^\mu = \left[\left(1-\frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right] = \left(1-\frac{2M}{r}\right)^{-1/2} \xi^\mu$

$\xi = \partial_t$ the KVF

The energy of the particle, as measured by the observer is:

$E = -P_\mu u^\mu = -m\, U_\mu u^\mu = -m\left(1-\frac{2M}{r}\right)^{-1/2} U_\mu \xi^\mu \qquad \xi^\mu = [1,0,0,0]$

$\Rightarrow e = \frac{E}{m} = -\left(1-\frac{2M}{r}\right)^{-1/2} U_\mu \xi^\mu$ is conserved, and

$U_\mu \xi^\mu = g_{00} U^0 \xi^0 = -\left(1-\frac{2M}{r}\right) U^0$, so

$e = \left(1-\frac{2M}{r}\right)^{+1/2} U^0 \Rightarrow U^0 = \left(1-\frac{2M}{r}\right)^{-1/2} e$

But $U^0 = \gamma = (1-v^2)^{1/2} \Rightarrow \gamma = \left(1-\frac{2M}{r}\right)^{-1/2} e \Rightarrow \gamma^2 = \left(1-\frac{2M}{r}\right)^{-1} e^2 \Rightarrow \frac{1}{\gamma^2} = \left(1-\frac{2M}{r}\right) e^{-2}$

$$\Rightarrow v^2 = 1 - \frac{1}{\gamma^2} = 1 - \left(1 - \frac{2M}{r}\right) e^{-2} = 1 - \left(1 - \frac{2M}{6M}\right) e^{-2} = 1 - \frac{2}{3e^2}$$

$$v = \left(1 - \frac{2}{3e^2}\right)^{1/2}$$

$$v_1 = \left(1 - \frac{2}{3 \cdot 1}\right)^{1/2} = \left(\frac{1}{3}\right)^{1/2} = \left(\frac{2}{6}\right)^{1/2}$$

$$v_2 = \left(1 - \frac{2}{3 \cdot 2^2}\right)^{1/2} = \left(1 - \frac{1}{6}\right)^{1/2} = \left(\frac{5}{6}\right)^{1/2}$$

$$\frac{v_2}{v_1} = \frac{\sqrt{5}}{\sqrt{2}}$$

8. A spaceship is moving without power in a circular orbit about a black hole of mass M. (The exterior geometry is the Schwarzschild geometry.) The Schwarzschild radius of the orbit is $7M$.

(a) What is the period of the orbit as measured by an observer at infinity?

(b) What is the period of the orbit as measured by a clock in the spaceship?

Hartle 9.8

From (9.46) of Hartle's book $\Omega = \frac{d\phi}{dt} = \left(\frac{M}{r^3}\right)^{\frac{1}{2}}$

This is the angular velocity as measured by the observer at infinity

The corresponding period is

$$T_\infty = \frac{2\pi}{\Omega} = \frac{2\pi}{\left(\frac{M}{r^3}\right)^{\frac{1}{2}}} = \frac{2\pi}{\left[\frac{M}{(7M)^3}\right]^{1/2}} = 2\pi\, 7^{\frac{3}{2}} M$$

The period as measured by the spaceship needs the angular velocity w.r.t. to its proper time:

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = \Omega \frac{dt}{d\tau}$$

We have a circular orbit at $\theta = \pi/2$

$$u_\mu u^\mu = -1 \Rightarrow g_{\mu\nu} u^\mu u^\nu = g_{00} \left(\frac{dt}{d\tau}\right)^2 + g_{33} \left(\frac{d\phi}{d\tau}\right)^2 = -\left(1-\frac{2M}{R}\right)\left(\frac{dt}{d\tau}\right)^2 + R^2 \frac{d\phi}{d\tau^2} = -1$$

$$\Rightarrow \left(1-\frac{2M}{R}\right)\left(\frac{dt}{d\tau}\right)^2 - R^2\left(\frac{d\phi}{dt}\,\frac{dt}{d\tau}\right)^2 = 1$$

$$\Rightarrow \left(\frac{dt}{d\tau}\right)^2 \left[\left(1-\frac{2M}{R}\right) - R^2 \Omega^2\right] = 1 \quad \Rightarrow$$

$$\Rightarrow \left(\frac{dt}{d\tau}\right)^2 \left[1 - \frac{2M}{R} - R^2 \frac{M}{R^3}\right] = 1$$

$$\Rightarrow \left(\frac{dt}{d\tau}\right)^2 \left(1 - \frac{3M}{R}\right) = 1 \Rightarrow \left(\frac{dt}{d\tau}\right)^2 = \left(1-\frac{3M}{R}\right)^{-1} = \left(1-\frac{3M}{7M}\right)^{-1} = \left(\frac{4}{7}\right)^{-1} = \frac{7}{4}$$

$$\Rightarrow \frac{dt}{d\tau} = \frac{\sqrt{7}}{2}$$

So $\dfrac{d\phi}{d\tau} = \Omega \dfrac{dt}{d\tau} = \left(\dfrac{M}{r^3}\right)^{\frac{1}{2}} \dfrac{dt}{d\tau} = \left(\dfrac{M}{7^3 M^3}\right)^{\frac{1}{2}} \cdot \dfrac{\sqrt{7}}{2} = 7^{-\frac{3}{2}}\, 7^{1/2} \cdot 2^{-1} M^{-1} = \dfrac{1}{14M}$, $P = \dfrac{2\pi}{d\phi/d\tau}$

10. Find the linear velocity of a particle in a circular orbit of radius R in the Schwarzschild geometry that would be measured by a stationary observer stationed at one point on the orbit. What is its value at the ISCO?

Hartle 9.10

The stationary observer has 4-velocity

$$u^\mu = \left[\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$$

The 4-velocity of the particle is U^μ and

$$E = -m\, U_\mu u^\mu \quad , \quad U^\mu = \left[\left(1-\frac{3M}{R}\right)^{-1/2}, 0, 0, \left(1-\frac{3M}{R}\right)^{-1/2}\left(\frac{M}{R^3}\right)^{1/2}\right] \quad \begin{pmatrix}\text{Hartle} \\ (9.47)\end{pmatrix}$$

$$\frac{E}{m} = \gamma = g_{00} U^0 u^0 = -\left(1-\frac{2M}{R}\right)\left(1-\frac{3M}{R}\right)^{-1/2}\left(1-\frac{2M}{R}\right)^{-1/2} = \left(1-\frac{2M}{R}\right)^{1/2}\left(1-\frac{3M}{R}\right)^{-1/2}$$

Then $\gamma = (1-v^2)^{-1/2} \Rightarrow v = \left(1-\frac{1}{\gamma^2}\right)^{1/2} = \left(\frac{M}{R}\right)^{1/2}\left(1-\frac{2M}{R}\right)^{-1/2}$

② Consider the 2-dimensional metric

$$ds^2 = \left(1-\frac{2M}{r}\right)(-dt^2+dr^2)$$

- Compute $\Gamma^\mu{}_{\nu\rho}$
- Compute $R^\mu{}_{\nu\rho\lambda}$, $R_{\mu\nu}$, R, $G_{\mu\nu}$
- Write down the explicit form of the geodesic equations

$$\ddot{x}^\mu = -\Gamma^\mu{}_{\nu\rho}\,\dot{x}^\nu \dot{x}^\rho$$

- Use the conserved quantity e^2, and make appropriate rescalings, to write down the equations in the form:

$$\ddot{r} = -\frac{1}{r^2}\left[\left(1-\frac{2}{r}\right)^{-1}\dot{r}^2 + \left(1-\frac{2}{r}\right)^{-3} e^2\right] \qquad \dot{t} = \left(1-\frac{2}{r}\right)^{-1} e$$

Christoffel Symbols:

$\Gamma^0{}_{1,0} \quad \frac{M}{r(-2M+r)}$

$\Gamma^1{}_{0,0} \quad \frac{M}{r(-2M+r)}$

$\Gamma^1{}_{1,1} \quad \frac{M}{r(-2M+r)}$

Riemann Tensor:

$R^0{}_{1,1,0} \quad \frac{2M(M-r)}{r^2(-2M+r)^2}$

$R^1{}_{0,1,0} \quad \frac{2M(M-r)}{r^2(-2M+r)^2}$

Ricci Tensor:

$R_{0,0} \quad \frac{2M(M-r)}{r^2(-2M+r)^2}$

$R_{1,1} \quad \frac{2M(-M+r)}{r^2(-2M+r)^2}$

Curvature Scalar:

$R = -\frac{4M(M-r)}{r(-2M+r)^3}$

$G_{\mu\nu} = 0$

Geodesic Equations:

$t_{\tau\tau} \quad + \quad -\frac{2Mr_\tau t_\tau}{2Mr-r^2} \quad = 0$

$r_{\tau\tau} \quad + \quad -\frac{M(r_\tau^2+t_\tau^2)}{(2M-r)r} \quad = 0$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)}$$

Christoffel Symbols:

$\Gamma^0{}_{1,0} \quad \frac{M}{r(-2M+r)}$

$\Gamma^1{}_{0,0} \quad \frac{M}{r(-2M+r)}$

$\Gamma^1{}_{1,1} \quad \frac{M}{r(-2M+r)}$

Riemann Tensor:

$R^0{}_{1,1,0} \quad \frac{2M(M-r)}{r^2(-2M+r)^2}$

$R^1{}_{0,1,0} \quad \frac{2M(M-r)}{r^2(-2M+r)^2}$

Ricci Tensor:

$R_{0,0} \quad \frac{2M(M-r)}{r^2(-2M+r)^2}$

$R_{1,1} \quad \frac{2M(-M+r)}{r^2(-2M+r)^2}$

Curvature Scalar:

$R = -\frac{4M(M-r)}{r(-2M+r)^3}$

$G_{\mu\nu} = 0$

Geodesic Equations:

$t_{\tau\tau} \quad + \quad -\frac{2Mr_\tau t_\tau}{2Mr-r^2} \quad = 0$

$r_{\tau\tau} \quad + \quad -\frac{M(r_\tau^2+t_\tau^2)}{(2M-r)r} \quad = 0$

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(-dt^2 + dr^2 + r^2\,d\varphi^2\right)$$

Christoffel Symbols:

$\Gamma^{0}{}_{1,0}$ $\quad \dfrac{M}{r\,(-2M+r)}$

$\Gamma^{1}{}_{0,0}$ $\quad \dfrac{M}{r\,(-2M+r)}$

$\Gamma^{1}{}_{1,1}$ $\quad \dfrac{M}{r\,(-2M+r)}$

$\Gamma^{1}{}_{2,2}$ $\quad \dfrac{r\,(-M+r)}{2M-r}$

$\Gamma^{2}{}_{2,1}$ $\quad \dfrac{-M+r}{r\,(-2M+r)}$

Riemann Tensor:

$R^{0}{}_{1,1,0}$ $\quad \dfrac{2M\,(M-r)}{r^2\,(-2M+r)^2}$

$R^{0}{}_{2,2,0}$ $\quad \dfrac{M\,(-M+r)}{(-2M+r)^2}$

$R^{1}{}_{0,1,0}$ $\quad \dfrac{2M\,(M-r)}{r^2\,(-2M+r)^2}$

$R^{1}{}_{2,2,1}$ $\quad -\dfrac{M\,r}{(-2M+r)^2}$

$R^{2}{}_{0,2,0}$ $\quad \dfrac{M\,(-M+r)}{r^2\,(-2M+r)^2}$

$R^{2}{}_{1,2,1}$ $\quad \dfrac{M}{r\,(-2M+r)^2}$

Ricci Tensor:

$R_{0,0}$ $\quad \dfrac{M\,(M-r)}{r^2\,(-2M+r)^2}$

$R_{1,1}$ $\quad \dfrac{M\,(-2M+3r)}{r^2\,(-2M+r)^2}$

$R_{2,2}$ $\quad \dfrac{M^2}{(-2M+r)^2}$

Curvature Scalar:

$$R = -\frac{2M\,(M-2r)}{r\,(-2M+r)^3}$$

Einstein Tensor:

$G_{0,0}$ $\quad \dfrac{M}{r\,(-2M+r)^2}$

$G_{1,1}$ $\quad \dfrac{M\,(-M+r)}{r^2\,(-2M+r)^2}$

$G_{2,2}$ $\quad \dfrac{2M\,(M-r)}{(-2M+r)^2}$

Geodesic Equations:

$$t_{\tau\tau} + -\frac{2M\,r_\tau\,t_\tau}{2Mr-r^2} = 0$$

$$r_{\tau\tau} + -\frac{-r^3\,\theta_\tau^2 + M\left(r_\tau^2 + t_\tau^2 + r^2\,\theta_\tau^2\right)}{(2M-r)\,r} = 0$$

$$\theta_{\tau\tau} + -\frac{2\,(M-r)\,r_\tau\,\theta_\tau}{r\,(-2M+r)} = 0$$

* * *

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2\,d\varphi^2$$

Christoffel Symbols:

$\Gamma^{0}{}_{1,0}$ $\quad \dfrac{M}{r\,(-2M+r)}$

$\Gamma^{1}{}_{0,0}$ $\quad \dfrac{M\,(-2M+r)}{r^3}$

$\Gamma^{1}{}_{1,1}$ $\quad \dfrac{M}{2Mr-r^2}$

$\Gamma^{1}{}_{2,2}$ $\quad 2M - r$

$\Gamma^{2}{}_{2,1}$ $\quad \dfrac{1}{r}$

Riemann Tensor:

$R^{0}{}_{1,1,0}$ $\quad -\dfrac{2M}{r^2\,(-2M+r)}$

$R^{0}{}_{2,2,0}$ $\quad \dfrac{M}{r}$

$R^{1}{}_{0,1,0}$ $\quad \dfrac{2M\,(2M-r)}{r^4}$

$R^{1}{}_{2,2,1}$ $\quad \dfrac{M}{r}$

$R^{2}{}_{0,2,0}$ $\quad \dfrac{M\,(-2M+r)}{r^4}$

$R^{2}{}_{1,2,1}$ $\quad \dfrac{M}{(2M-r)\,r^2}$

Ricci Tensor:

$R_{0,0}$ $\quad \dfrac{M\,(2M-r)}{r^4}$

$R_{1,1}$ $\quad \dfrac{M}{r^2\,(-2M+r)}$

$R_{2,2}$ $\quad -\dfrac{2M}{r}$

Curvature Scalar:

$R = 0$

Geodesic Equations:

$$t_{\tau\tau} + -\frac{2M\,r_\tau\,t_\tau}{2Mr-r^2} = 0$$

$$r_{\tau\tau} + \frac{M\,r_\tau^2}{2Mr-r^2} - \frac{M\,(2M-r)\,t_\tau^2}{r^3} + (2M-r)\,\theta_\tau^2 = 0$$

$$\theta_{\tau\tau} + \frac{2\,r_\tau\,\theta_\tau}{r} = 0$$

Einstein Tensor:

$G_{0,0}$ $\quad \dfrac{M\,(2M-r)}{r^4}$

$G_{1,1}$ $\quad \dfrac{M}{r^2\,(-2M+r)}$

$G_{2,2}$ $\quad -\dfrac{2M}{r}$