The Schwarzschild Solution

Part II: Exterior Region $r > R_s$

The geometry outside a spherically symmetric star

- The metric in (t, r, θ, φ) coordinates:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

- The metric in (t, r, θ, φ) coordinates:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

- $g_{\mu\nu}$ independent of t and φ

$$\partial_t g_{\mu\nu} = 0$$

$$\partial_\varphi g_{\mu\nu} = 0$$

- The metric in (t, r, θ, φ) coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

- $g_{\mu\nu}$ independent of t and φ

$$\begin{aligned} \partial_t g_{\mu\nu} &= 0 \\ \partial_\varphi g_{\mu\nu} &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \xi &= \partial_t \\ \eta &= \partial_\varphi \end{aligned}$$

Killing Vector Fields

- The metric in (t, r, θ, φ) coordinates:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

- $g_{\mu\nu}$ independent of t and φ

$$\begin{matrix} \partial_t g_{\mu\nu} = 0 \\ \partial_\varphi g_{\mu\nu} = 0 \end{matrix} \Rightarrow \begin{matrix} \xi = \partial_t \\ \eta = \partial_\varphi \end{matrix} \Rightarrow \begin{matrix} e = -\xi^\mu u_\mu \\ l = \eta^\mu u_\mu \end{matrix}$$

conserved along geodesics

- The metric in (t, r, θ, φ) coordinates:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

- $g_{\mu\nu}$ independent of t and φ

$$\begin{matrix} \partial_t g_{\mu\nu} = 0 & & \xi = \partial_t & & e = -\xi^\mu u_\mu = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \\ & \Rightarrow & & \Rightarrow & \\ \partial_\varphi g_{\mu\nu} = 0 & & \eta = \partial_\varphi & & l = \eta^\mu u_\mu = r^2\sin^2\theta\,\frac{d\varphi}{d\lambda} \end{matrix}$$

λ: affine parameter of geodesic $\qquad \xi^\mu = (1,0,0,0) \qquad \eta^\mu = (0,0,0,1)$

- The metric in (t, r, θ, φ) coordinates:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\, d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

- $g_{\mu\nu}$ independent of t and φ

$$\begin{array}{lclcl} \partial_t g_{\mu\nu} = 0 & & \xi = \partial_t & & e = -\xi^\mu u_\mu = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \\ & \Rightarrow & & \Rightarrow & \\ \partial_\varphi g_{\mu\nu} = 0 & & \eta = \partial_\varphi & & l = \eta^\mu u_\mu = r^2\sin^2\theta\,\frac{d\varphi}{d\lambda} \end{array}$$

- timelike geodesic: $u^\mu u_\mu = -1 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$ $(\lambda \equiv \tau)$

  null geodesic: $u^\mu u_\mu = 0 = g_{\mu\nu}\frac{dx}{d\lambda}\frac{dx^\nu}{d\lambda}$

- $l = \text{const} \Rightarrow$ geodesics lie on a plane

choose coordinates, so that $\theta = \frac{\pi}{2}$, $\frac{d\theta}{d\lambda} = 0$, $\frac{d^2\theta}{d\lambda^2} = 0$

$\sin\theta = 1$

- $g_{\mu\nu}$ independent of t and φ

$$\left.\begin{array}{l} \partial_t g_{\mu\nu} = 0 \\ \partial_\varphi g_{\mu\nu} = 0 \end{array}\right. \Rightarrow \begin{array}{l} \xi = \partial_t \\ \eta = \partial_\varphi \end{array} \Rightarrow \begin{array}{l} e = -\xi^\mu u_\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \\ l = \eta^\mu u_\mu = r^2 \sin^2\theta \frac{d\varphi}{d\lambda} \end{array}$$

- timelike geodesic: $u^\mu u_\mu = -1 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$ $(\lambda \equiv \tau)$

  null geodesic: $u^\mu u_\mu = 0 = g_{\mu\nu}\frac{dx}{d\lambda}\frac{dx^\nu}{d\lambda}$

- Freely falling massive particles:

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \qquad l = r^2 \frac{d\varphi}{d\tau}$$

$$u^\mu u_\mu = -1$$

constants of their motion

- Freely falling massive particles:

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \qquad l = r^2 \frac{d\varphi}{d\tau}$$

$$u^\mu u_\mu = -1 \Rightarrow -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1$$

- Freely falling massive particles:

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \qquad \ell = r^2 \frac{d\varphi}{d\tau}$$

$$u^\mu u_\mu = -1 \Rightarrow -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1$$

$$\Rightarrow -\left(1-\frac{2M}{r}\right)^{-1} e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2} = -1$$

- Freely falling massive particles:

$$e = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \qquad l = r^2 \frac{d\varphi}{d\tau} \qquad V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

$$u^\mu u_\mu = -1 \Rightarrow \quad \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad \mathcal{E} = \frac{e^2 - 1}{2}$$

$$\Rightarrow -\left(1 - \frac{2M}{r}\right)^{-1} e^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2} \quad = -1$$

- Freely falling massive particles:

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \qquad \ell = r^2 \frac{d\varphi}{d\tau} \qquad V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

$$u^\mu u_\mu = -1 \Rightarrow \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad \mathcal{E} = \frac{e^2 - 1}{2}$$

- Freely falling massless particles:

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \qquad \ell^2 = r^2 \frac{d\varphi}{d\lambda}$$

→

- there is no proper time!
- λ affine parameter
- fix so that $p^\mu = u^\mu = \frac{dx^\mu}{d\lambda}$

$$u^\mu u_\mu = 0$$

↳ null tangent vector

- Freely falling massive particles:

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \qquad l = r^2\frac{d\varphi}{d\tau} \qquad V_{eff}(r) = -\frac{M}{r}+\frac{l^2}{2r^2}-\frac{Ml^2}{r^3}$$

$$u^\mu u_\mu = -1 \Rightarrow \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad \mathcal{E} = \frac{e^2-1}{2}$$

- Freely falling massless particles:

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \qquad l^2 = r^2\frac{d\varphi}{d\lambda}$$

$$u^\mu u_\mu = 0 \Rightarrow -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$

once null,
always null!

- Freely falling massive particles:

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \qquad \ell = r^2\frac{d\varphi}{d\tau} \qquad V_{eff}(r) = -\frac{M}{r}+\frac{\ell^2}{2r^2}-\frac{M\ell^2}{r^3}$$

$$u^\mu u_\mu = -1 \Rightarrow \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad \mathcal{E} = \frac{e^2-1}{2}$$

- Freely falling massless particles:

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \qquad \ell^2 = r^2\frac{d\varphi}{d\lambda}$$

$$u^\mu u_\mu = 0 \Rightarrow -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$

$$\Rightarrow -\left(1-\frac{2M}{r}\right)\left[\frac{e}{\left(1-\frac{2M}{r}\right)}\right]^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{\ell}{r^2}\right)^2 = 0$$

- Freely falling massive particles:

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \qquad l = r^2 \frac{d\varphi}{d\tau} \qquad V_{eff}(r) = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}$$

$$u^\mu u_\mu = -1 \Rightarrow \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad \mathcal{E} = \frac{e^2 - 1}{2}$$

- Freely falling massless particles:

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \qquad l^2 = r^2 \frac{d\varphi}{d\lambda}$$

$$u^\mu u_\mu = 0 \Rightarrow -\left(1 - \frac{2M}{r}\right)^{-1} e^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{l^2}{r^2} = 0$$

$$\Rightarrow -\left(1 - \frac{2M}{r}\right)\left[\frac{e}{\left(1 - \frac{2M}{r}\right)}\right]^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{l}{r^2}\right)^2 = 0$$

- Freely falling massive particles:

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \qquad \ell = r^2\frac{d\varphi}{d\tau} \qquad V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

$$u^\mu u_\mu = -1 \Rightarrow \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad \mathcal{E} = \frac{e^2-1}{2}$$

- Freely falling massless particles:

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \qquad \ell^2 = r^2\frac{d\varphi}{d\lambda}$$

$$u^\mu u_\mu = 0 \Rightarrow -\left(1-\frac{2M}{r}\right)^{-1} e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2} = 0$$

$$\Rightarrow -\frac{e^2}{\ell^2} + \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2}\left(1-\frac{2M}{r}\right) = 0$$

- Freely falling massive particles:

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \qquad \ell = r^2 \frac{d\varphi}{d\tau} \qquad V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

$$u^\mu u_\mu = -1 \Rightarrow \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad \mathcal{E} = \frac{e^2 - 1}{2}$$

- Freely falling massless particles:

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \qquad \ell^2 = r^2 \frac{d\varphi}{d\lambda} \qquad W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

$$u^\mu u_\mu = 0 \Rightarrow \frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r) \qquad b^2 = \frac{\ell^2}{e^2}$$

$$\Rightarrow -\frac{e^2}{\ell^2} + \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) = 0$$

- Freely falling massive particles:

$$e=\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \qquad l=r^2\frac{d\varphi}{d\tau} \qquad V_{eff}(r)=-\frac{M}{r}+\frac{l^2}{2r^2}-\frac{Ml^2}{r^3}$$

$$u^\mu u_\mu=-1 \Rightarrow \boxed{\mathcal{E}=\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+V_{eff}(r)} \qquad \mathcal{E}=\frac{e^2-1}{2}$$

- Freely falling massless particles:

$$e=\left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \qquad l^2=r^2\frac{d\varphi}{d\lambda} \qquad W_{eff}(r)=\frac{1}{r^2}\left(1-\frac{2M}{r}\right)$$

$$u^\mu u_\mu=0 \Rightarrow \boxed{\frac{1}{b^2}=\frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2+W_{eff}(r)} \qquad b^2=\frac{l^2}{e^2}$$

Orbits of massless particles

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \Rightarrow \quad \frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} e$$

$$\ell = r^2 \quad \frac{d\varphi}{d\lambda} \Rightarrow \quad \frac{d\varphi}{d\lambda} = \frac{\ell}{r^2}$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

$$W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

$$b^2 = \ell^2 / e^2$$

Reparameterizing $\ell\lambda \to \lambda$, ℓ-dependence vanishes

Only $b = \frac{\ell}{e}$ dependence

Orbits of massless particles

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \Rightarrow \frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} e$$

$$l = r^2 \frac{d\varphi}{d\lambda} \Rightarrow \frac{d\varphi}{d\lambda} = \frac{l}{r^2}$$

$$\frac{1}{b^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

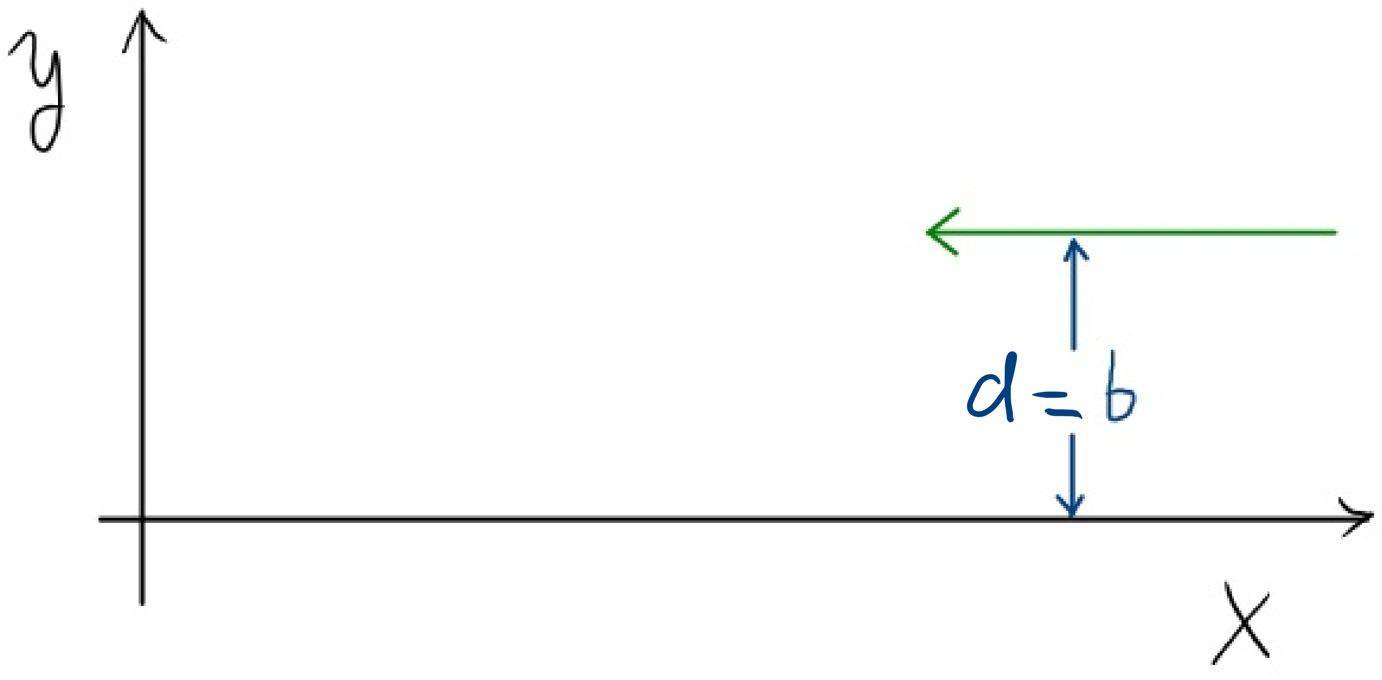
$$W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

$$b^2 = l^2/e^2$$

sign of l $\Rightarrow$ sign of $\frac{d\varphi}{d\lambda}$ (direction of φ - motion)

irrelevant if we simply want to study the shape of orbits

Orbits of massless particles



$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \Rightarrow \frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} e$$

$$\ell = r^2 \frac{d\varphi}{d\lambda} \Rightarrow \frac{d\varphi}{d\lambda} = \frac{\ell}{r^2}$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

$$W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

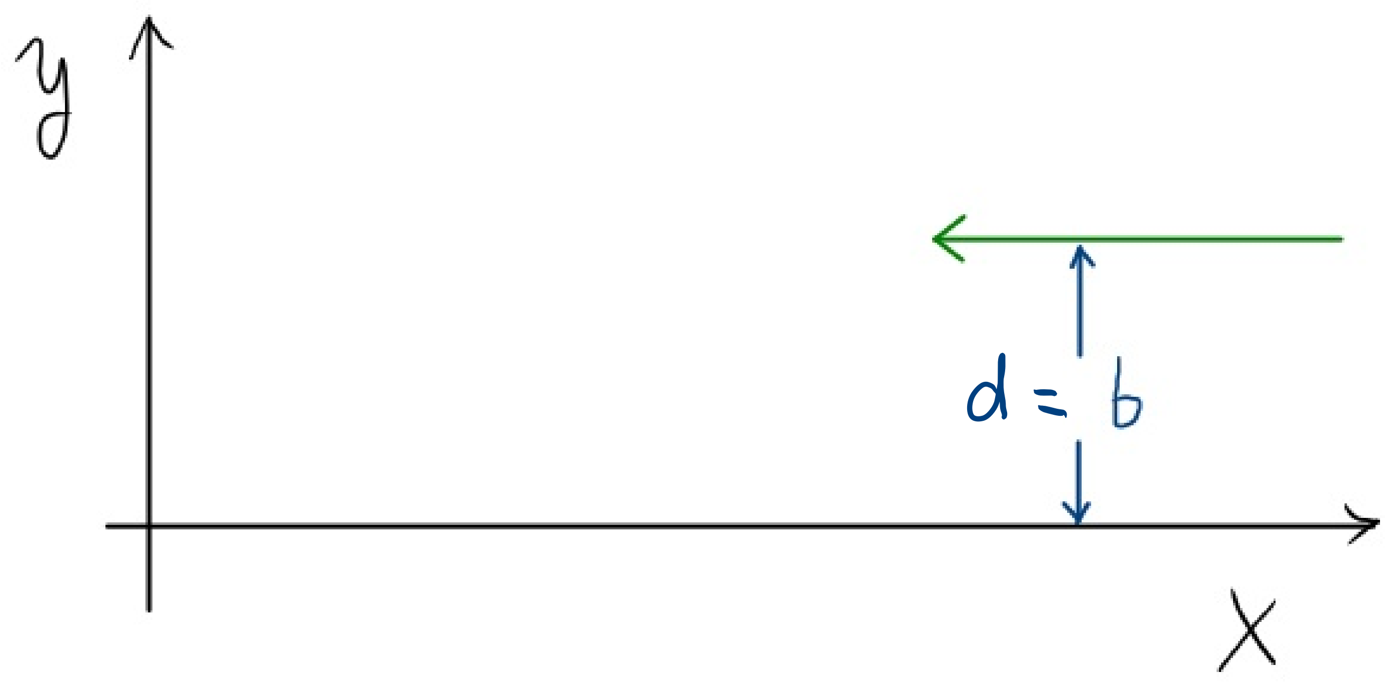
$$b^2 = \ell^2 / e^2$$

Light scattering: $r(0) \gg 2M$

$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

Orbits of massless particles



$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \Rightarrow \frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} e$$

$$\ell = r^2 \frac{d\varphi}{d\lambda} \Rightarrow \frac{d\varphi}{d\lambda} = \frac{\ell}{r^2}$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

$$W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

$$b^2 = \ell^2/e^2$$

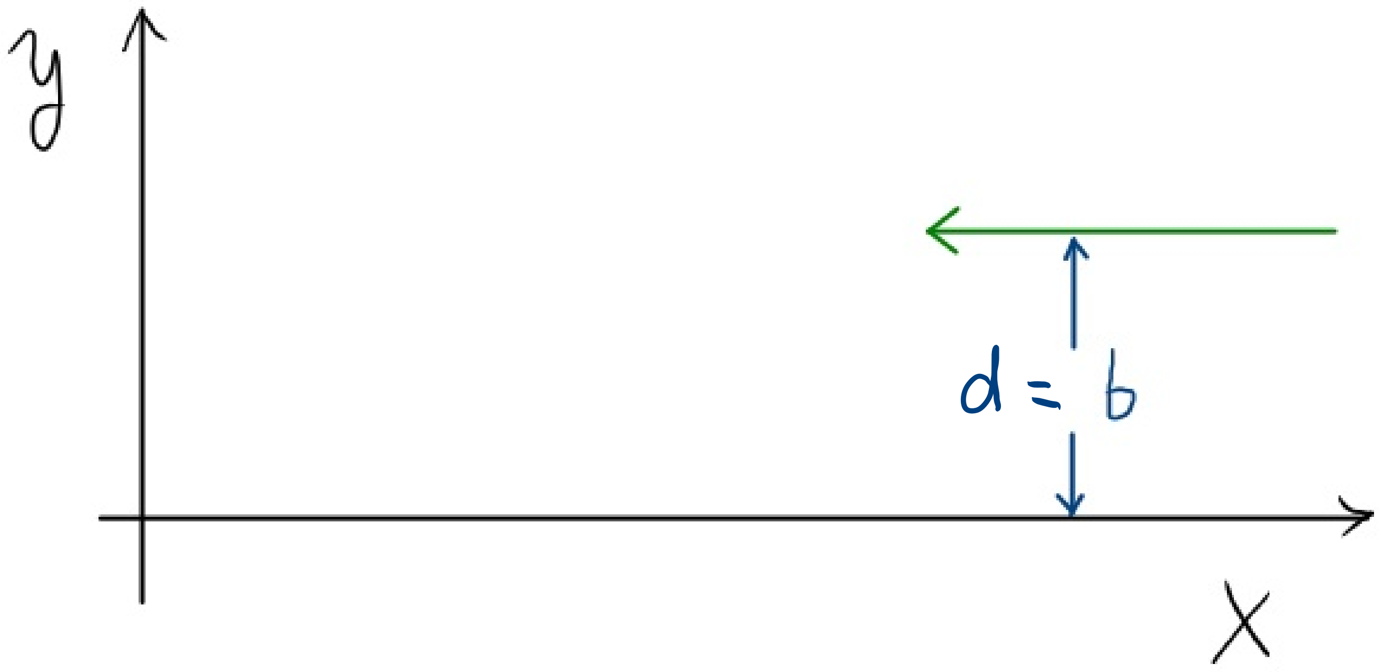
Light scattering: $r(0) >> 2M$

$$b = \left|\frac{\ell}{e}\right| = \frac{r^2\, d\varphi/d\lambda}{\left(1 - \frac{2M}{r}\right) dt/d\lambda} \approx r^2 \frac{d\varphi}{dt}$$

(here $d\varphi/dt > 0$)

$$\phi \approx \frac{d}{r} \Rightarrow \frac{d\varphi}{dt} = \frac{d\varphi}{dr}\frac{dr}{dt} = \frac{d}{dr}\left(\frac{d}{r}\right)\frac{dr}{dt}$$

Orbits of massless particles



$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \Rightarrow \frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} e$$

$$\ell = r^2 \frac{d\varphi}{d\lambda} \Rightarrow \frac{d\varphi}{d\lambda} = \frac{\ell}{r^2}$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

$$W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

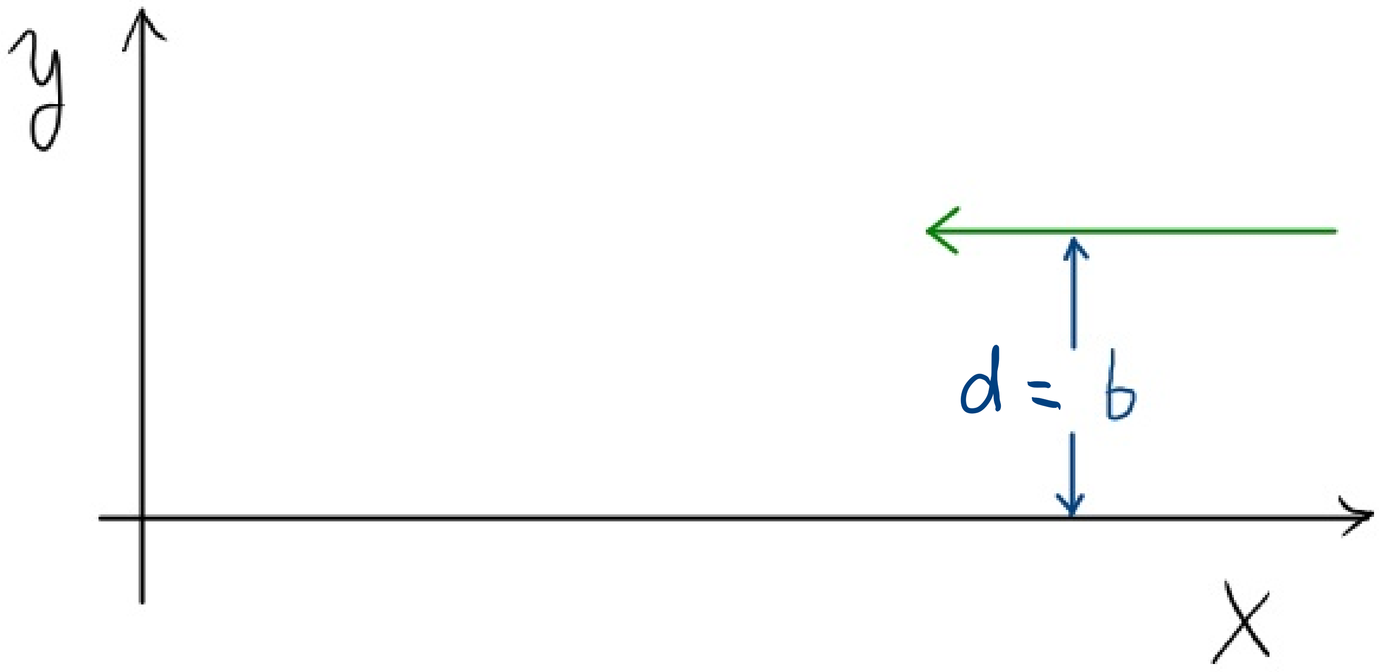
$$b^2 = \ell^2 / e^2$$

Light scattering: $r(0) \gg 2M$

$\frac{dr}{dt} \approx -1$ $\left(\text{photon} \approx \text{in radial direction}\right)$ $b = \left|\frac{\ell}{e}\right| = \frac{r^2 \, d\varphi/d\lambda}{\left(1 - \frac{2M}{r}\right) dt/d\lambda} \approx r^2 \frac{d\varphi}{dt}$ (here $d\varphi/dt > 0$)

$$\phi \approx \frac{d}{r} \Rightarrow \frac{d\varphi}{dt} = \frac{d\varphi}{dr}\frac{dr}{dt} = \frac{d}{dr}\left(\frac{d}{r}\right)\frac{dr}{dt} = \left(-\frac{d}{r^2}\right)(-1) = \frac{d}{r^2}$$

Orbits of massless particles



$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \Rightarrow \frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} e$$

$$\ell = r^2 \frac{d\varphi}{d\lambda} \Rightarrow \frac{d\varphi}{d\lambda} = \frac{\ell}{r^2}$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

$$W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

$$b^2 = \ell^2 / e^2$$

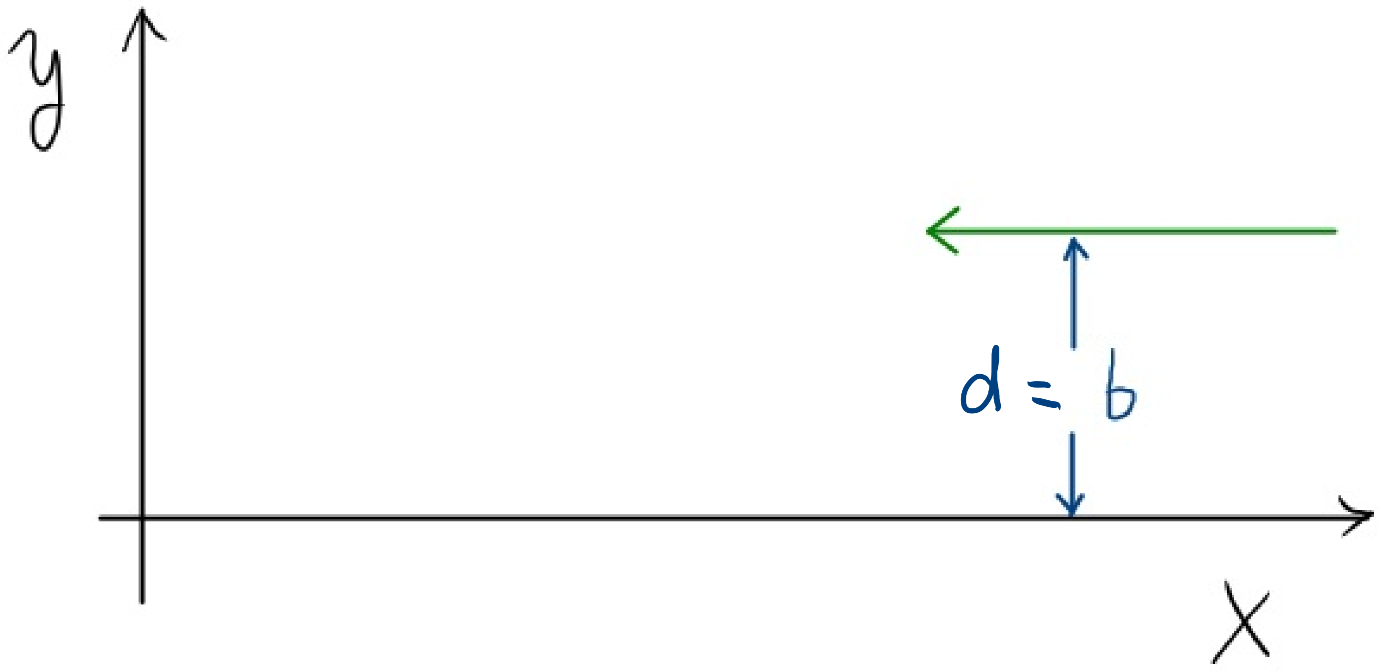
Light scattering: $r(0) >> 2M$

$\frac{dr}{dt} \approx -1$ $\left(\begin{matrix}\text{Photon}\\ \approx\\ \text{in radial direction}\end{matrix}\right)$ $\underline{\underline{b}} = \left|\frac{\ell}{e}\right| = \frac{r^2 d\varphi/d\lambda}{(1-\frac{2M}{r})\, dt/d\lambda} \approx r^2 \frac{d\varphi}{dt} \approx r^2 \cdot \frac{d}{r^2} = \underline{\underline{d}}$

$$\phi \approx \frac{d}{r} \Rightarrow \frac{d\varphi}{dt} = \frac{d\varphi}{dr}\frac{dr}{dt} = \frac{d}{dr}\left(\frac{d}{r}\right)\frac{dr}{dt} = \left(-\frac{d}{r^2}\right)(-1) = \frac{d}{r^2}$$

Orbits of massless particles

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \Rightarrow \frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} e$$



$$\ell = r^2 \quad \frac{d\varphi}{d\lambda} \Rightarrow \frac{d\varphi}{d\lambda} = \frac{\ell}{r^2}$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

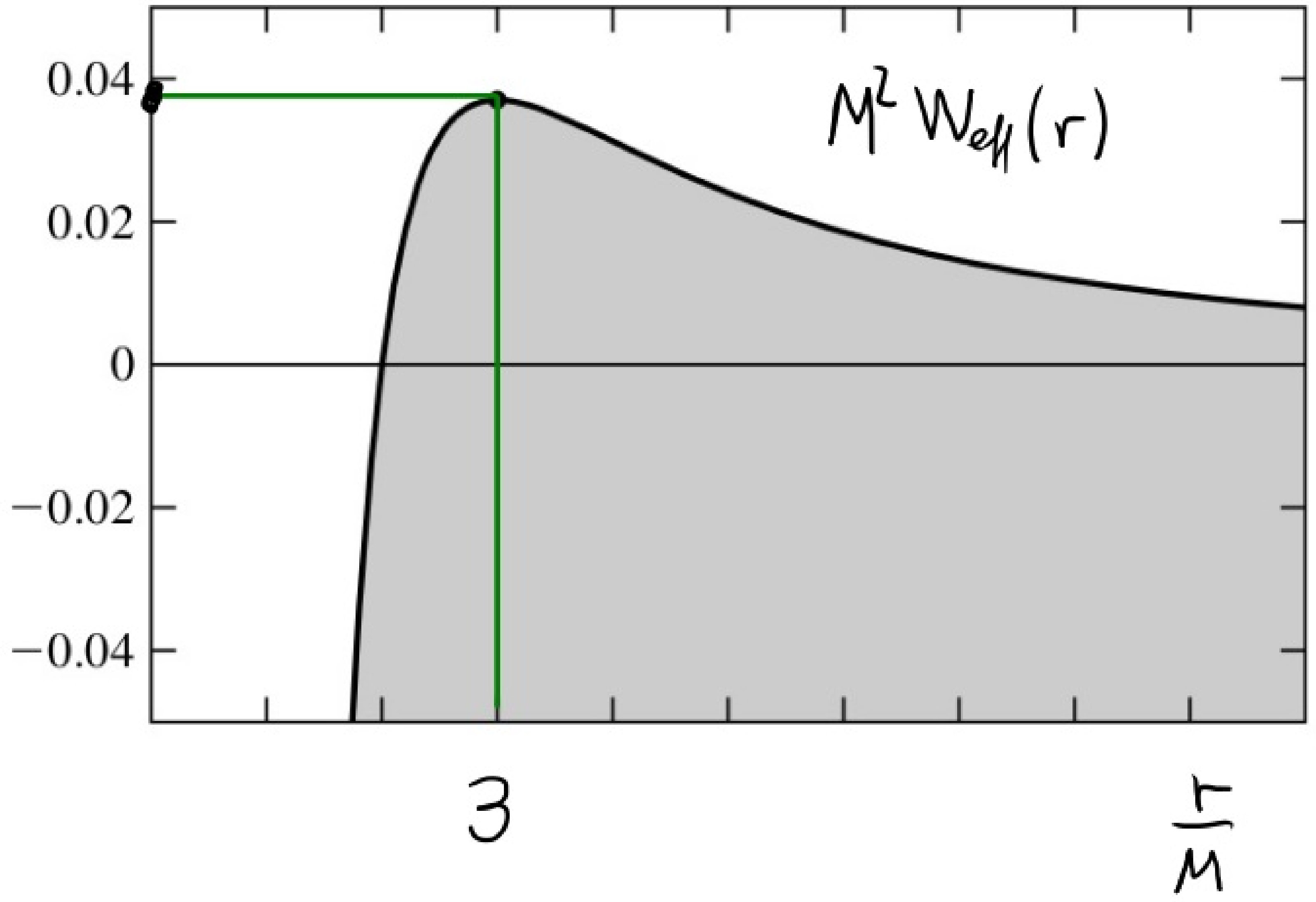
$$W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

$$b^2 = \ell^2 / e^2$$

Light scattering: b is the impact parameter

Orbits of massless particles

$\frac{1}{27}$



$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \Rightarrow \quad \frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$\ell = r^2 \quad \frac{d\varphi}{d\lambda} \Rightarrow \quad \frac{d\varphi}{d\lambda} = \frac{\ell}{r^2}$$

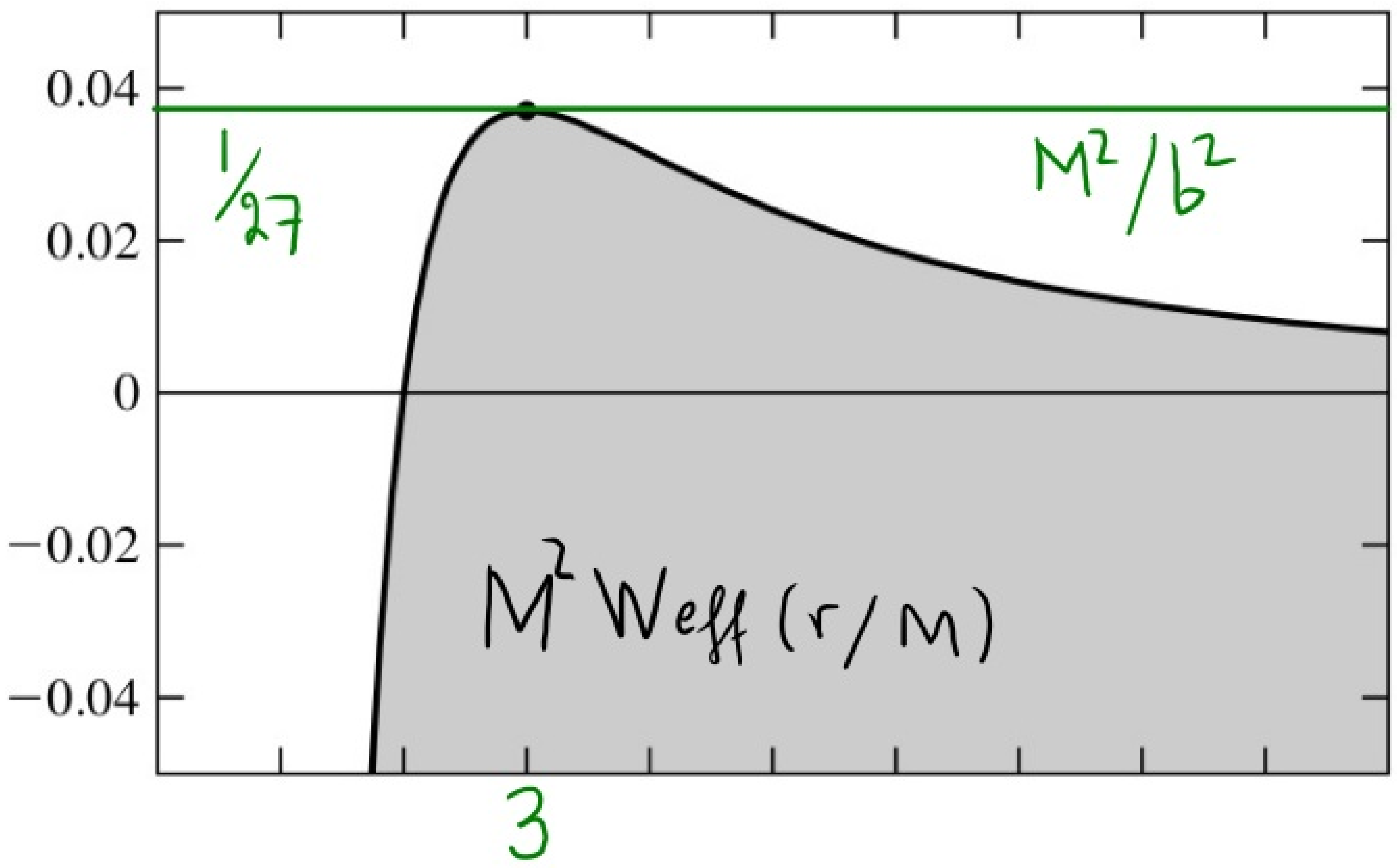
$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

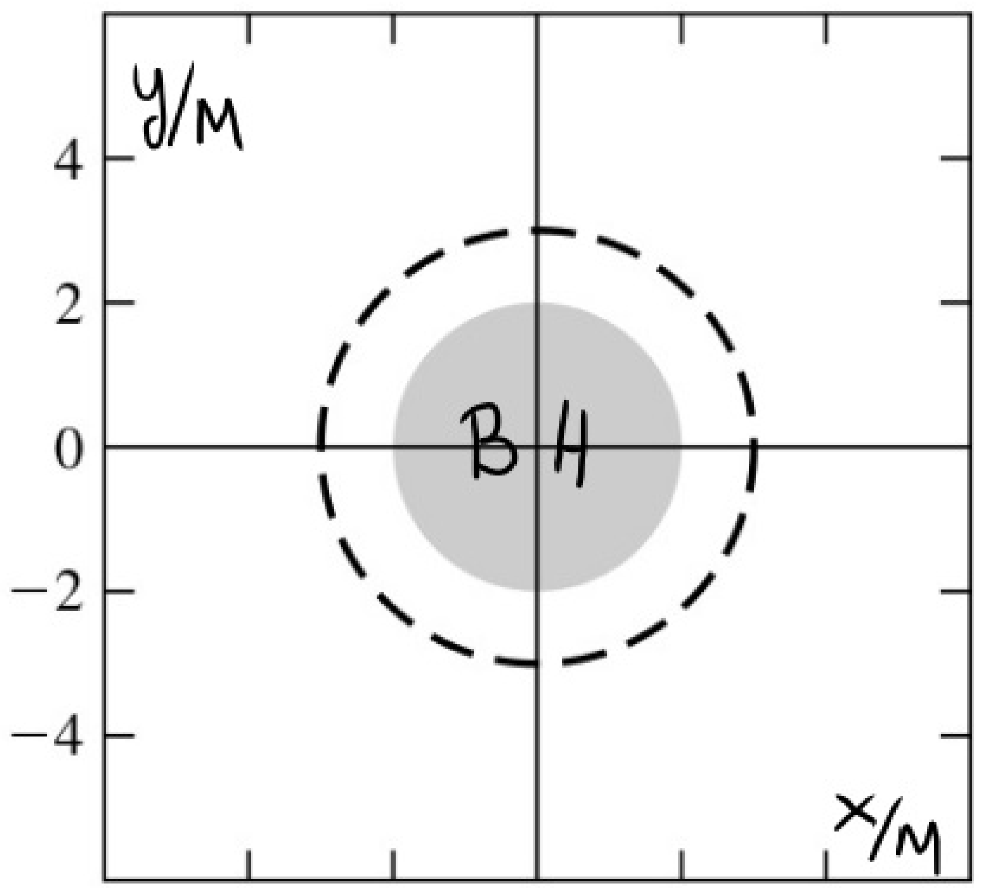
$$W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

$$b^2 = \ell^2 / e^2$$

$$\frac{d\,W_{eff}(r)}{dr} = \frac{6M}{r^4} - \frac{2}{r^2} = 0 \Rightarrow r_{max} = 3M$$

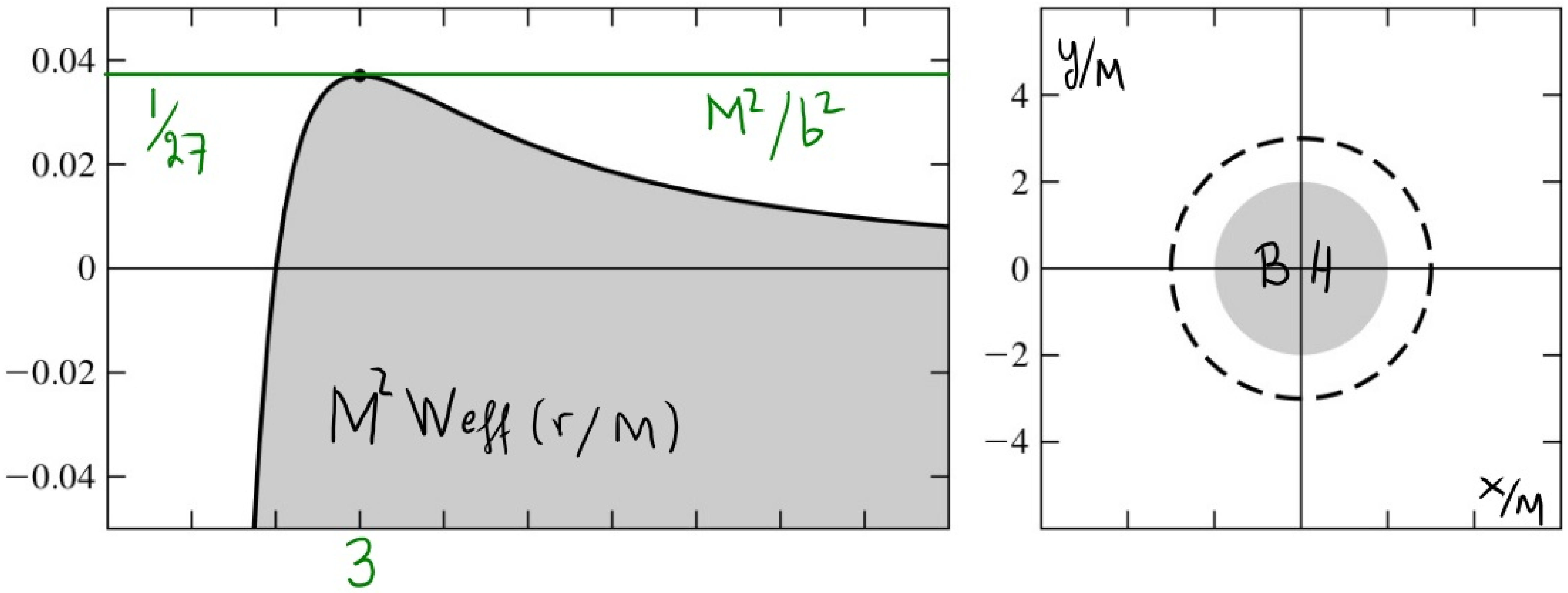
$$W_{eff}(r_{max}) = \frac{1}{27M^2}$$





Circular orbits: $b^2 = 27 M^2$ $r = 3M$

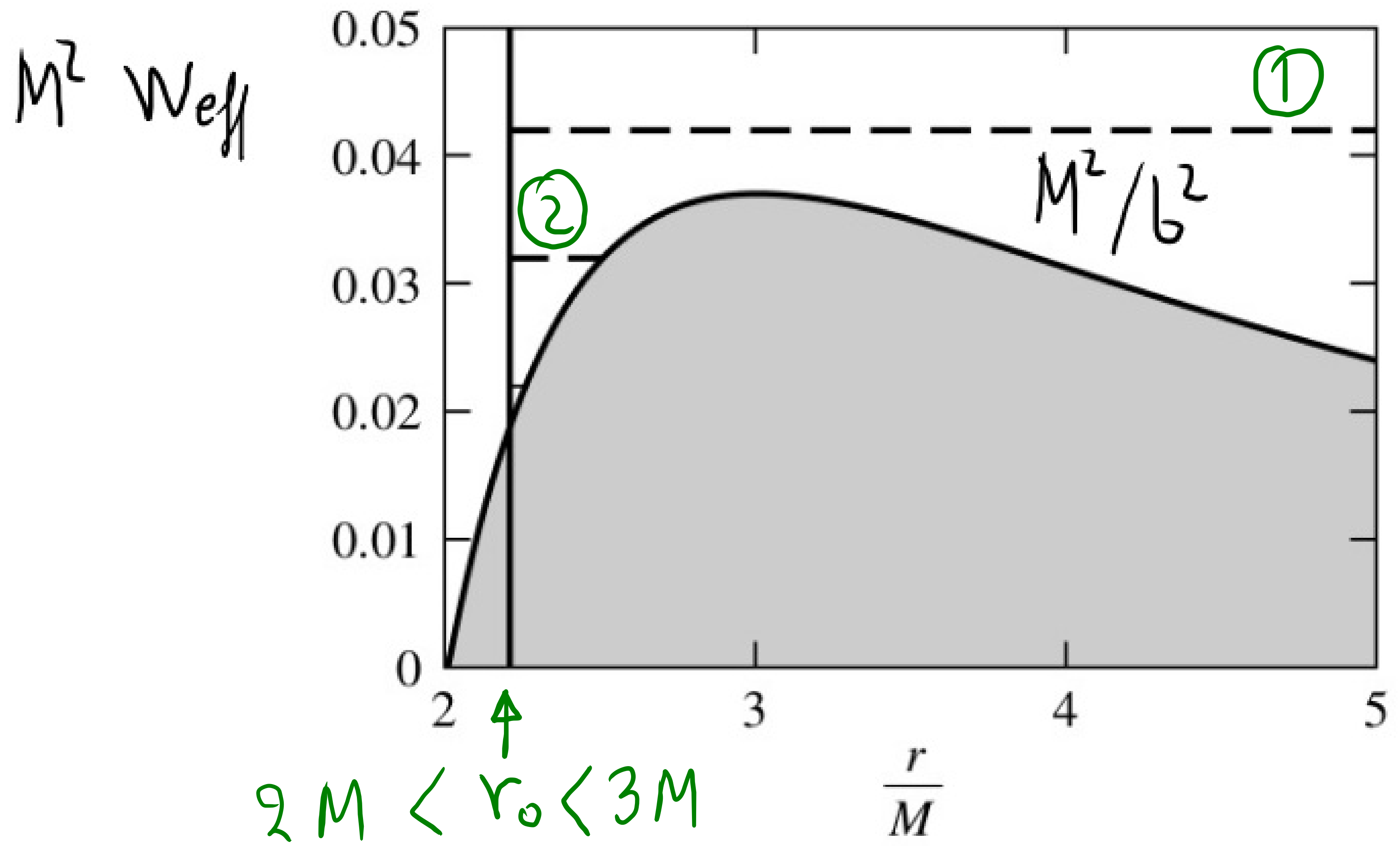
(unstable)

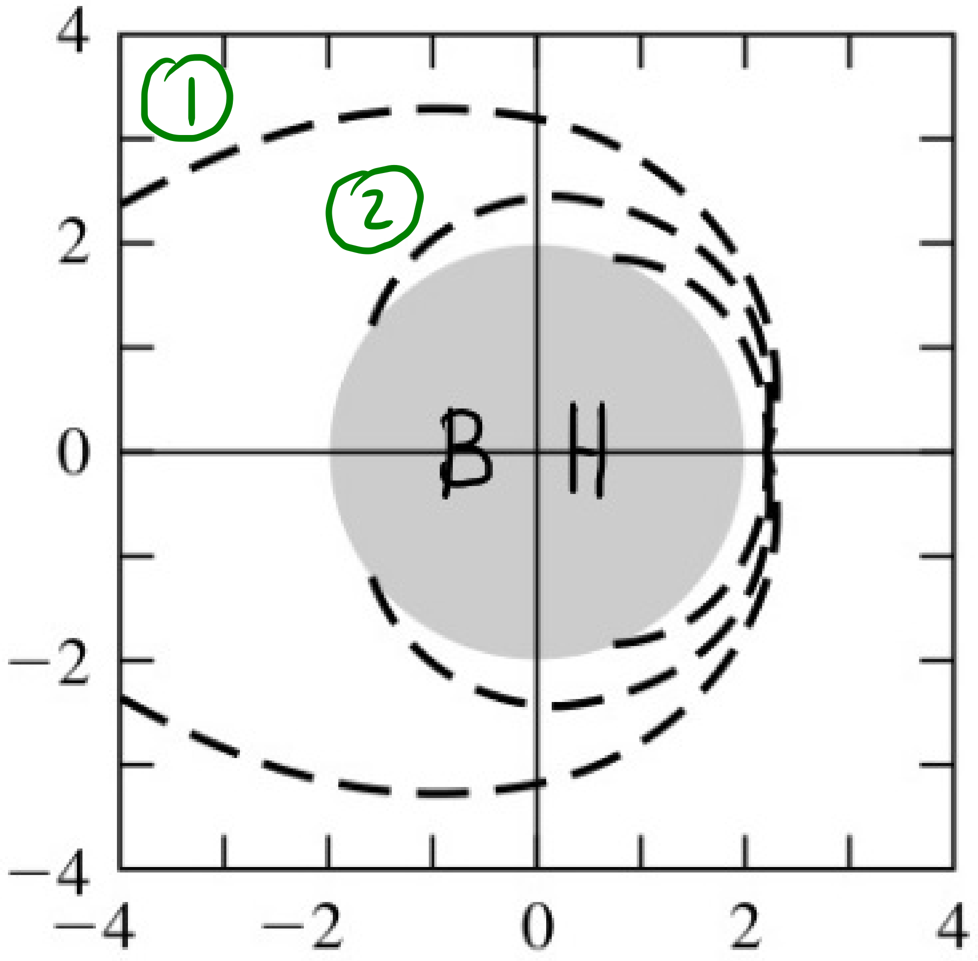


Circular orbits: $b^2 = 27 M^2$ $\quad r = 3M$

quite close... OK for black holes, not for stars

e.g $M_\odot \approx 1.5$ km $\Rightarrow$ $r \approx 4.5$ km

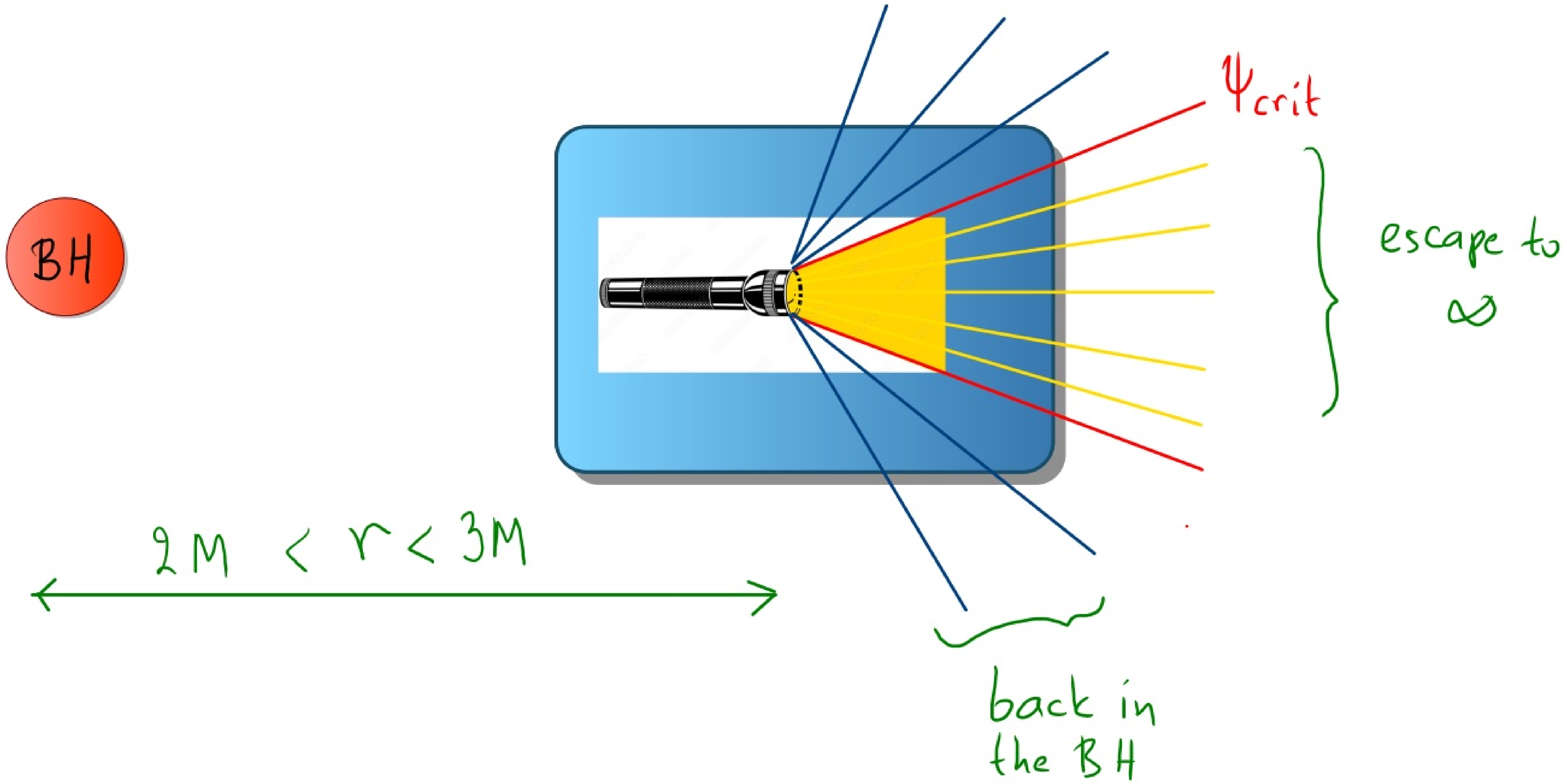




Light emitted in the $2M < r_0 < 3M$ region follows a curved orbit:

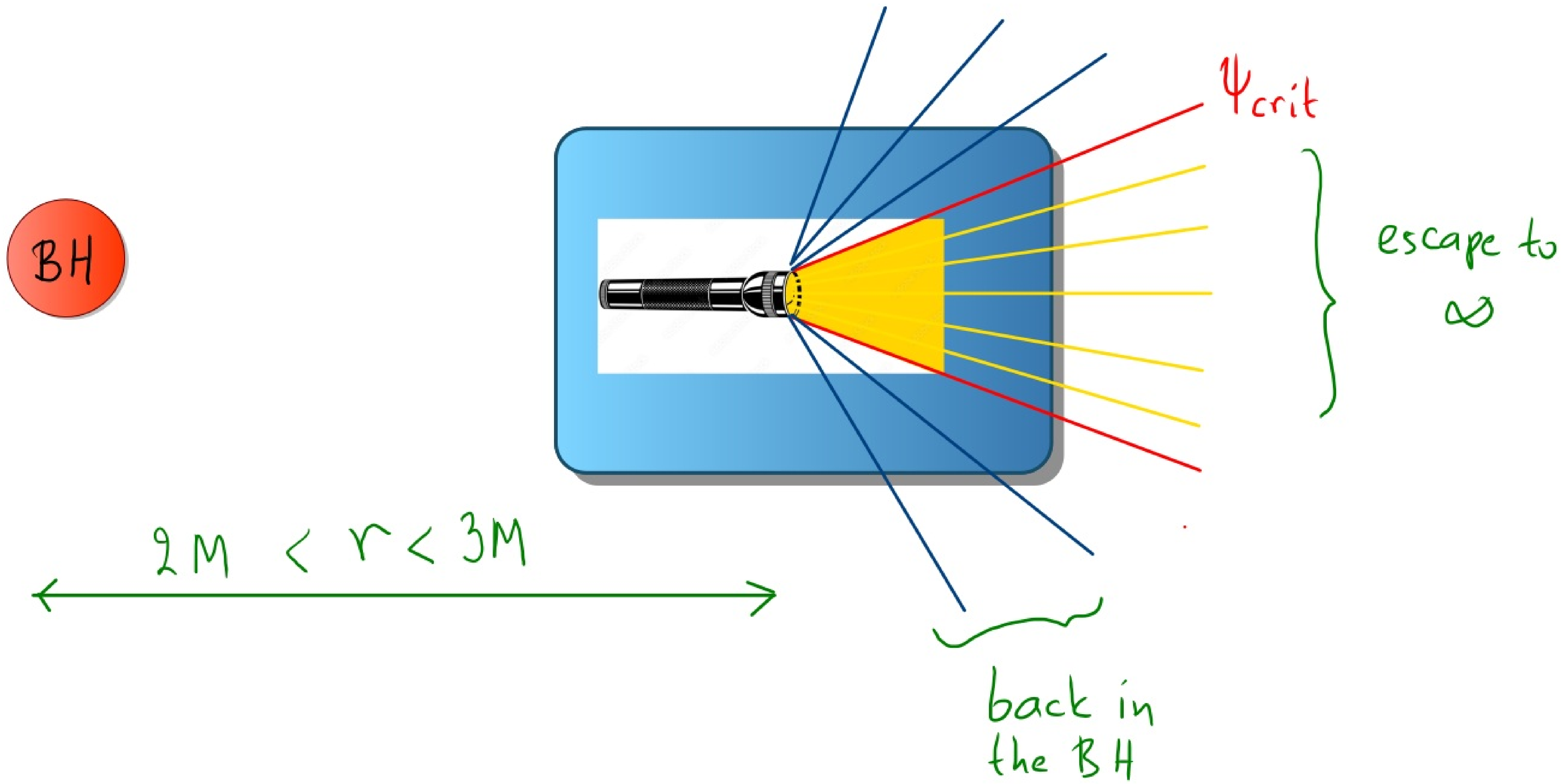
- may fall into the BH
- may escape to infinity

A stationary observer at $2M < R < 3M$



- How much light escapes to ∞?

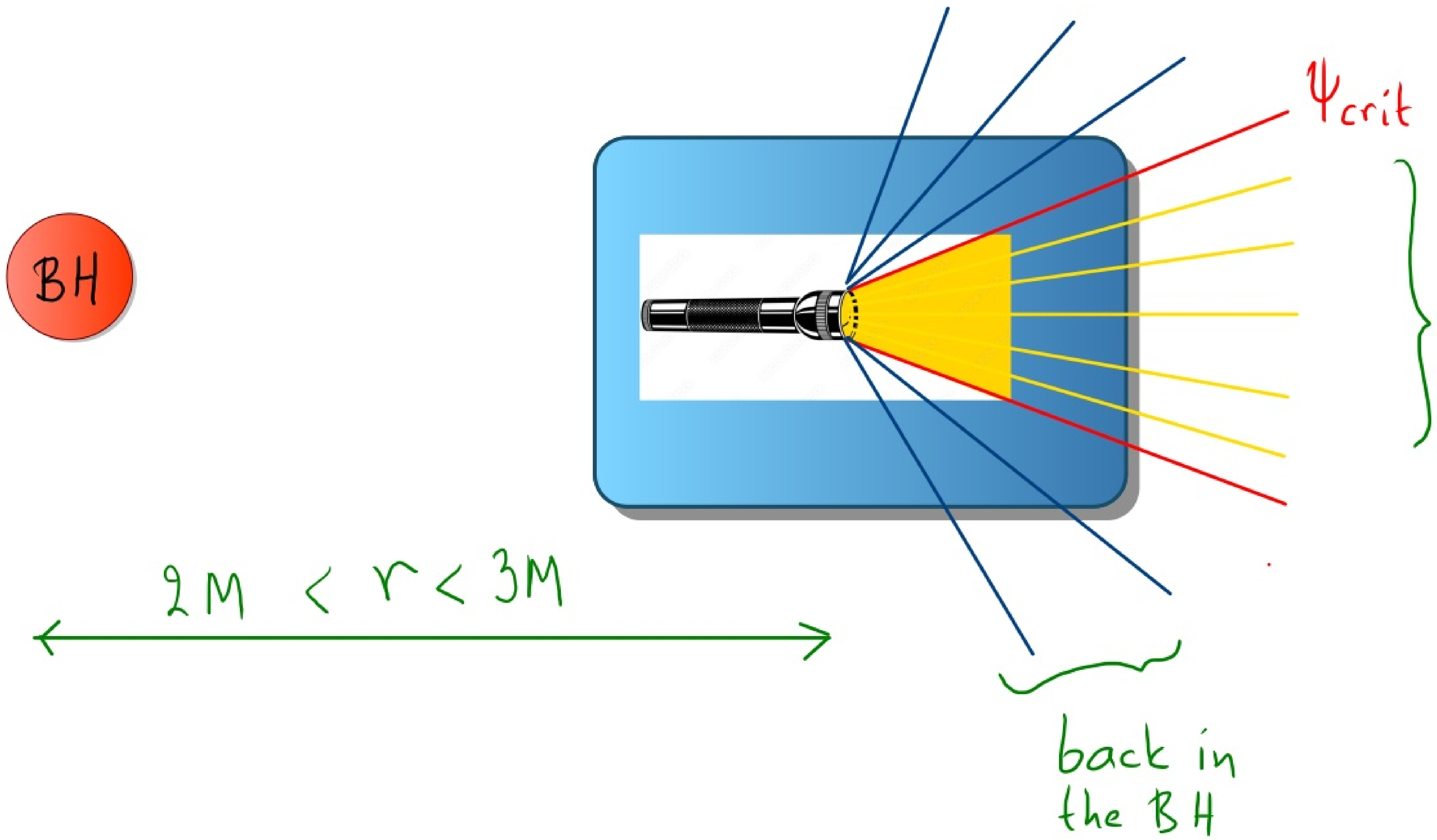
A stationary observer at $2M<R<3M$

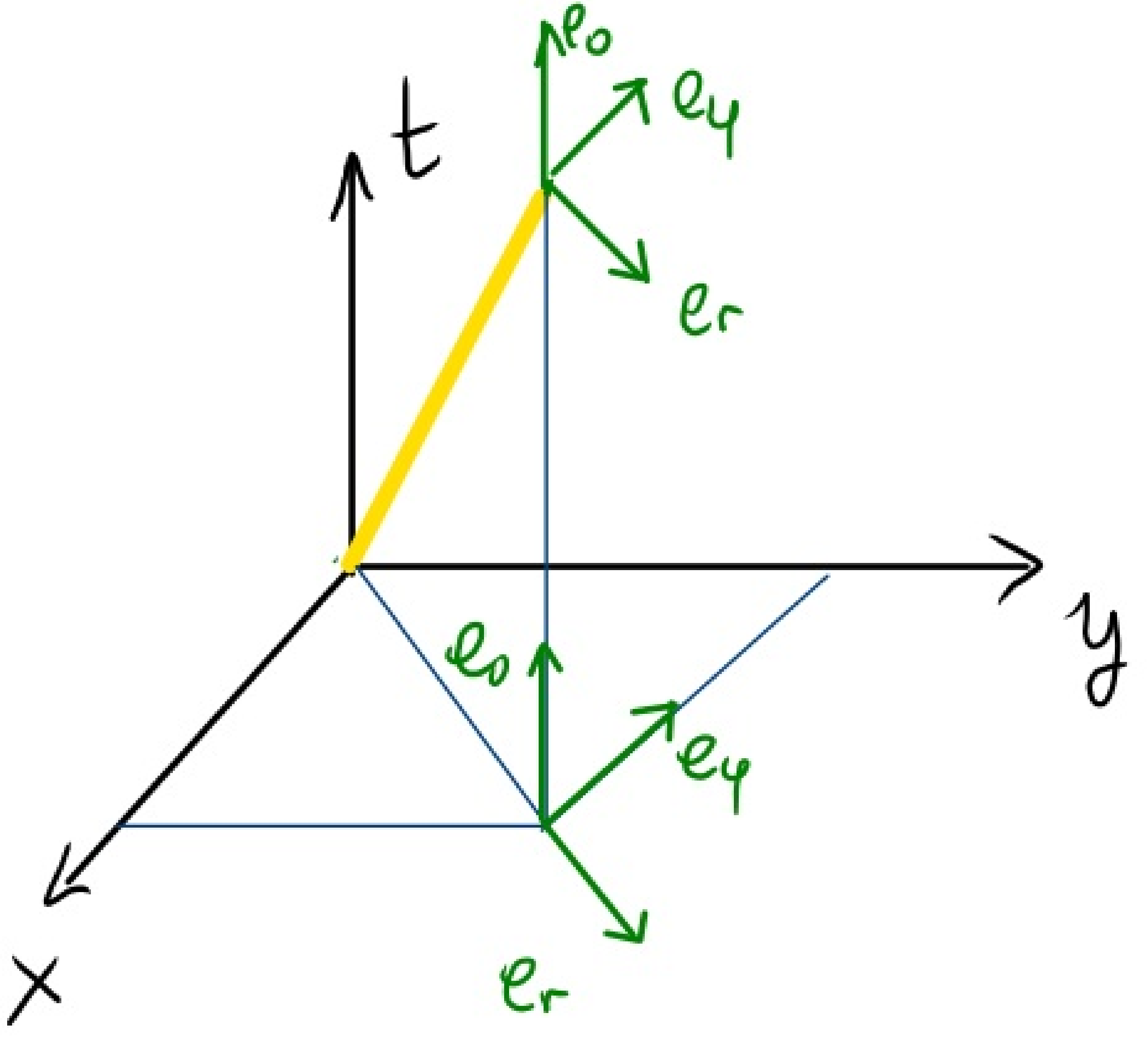


- How much light escapes to ∞? $\exists\ \psi_{crit}$ s.t $\begin{cases} |\phi|<\psi_{crit} \rightarrow \infty \\ |\phi|>\psi_{crit} \rightarrow BH \end{cases}$

e.g. radial light $\phi=0$: $b=\frac{\cancelto{0}{\ell}}{e}=0 \Rightarrow \frac{1}{b^2}=\infty$

A stationary observer at $2M<R<3M$



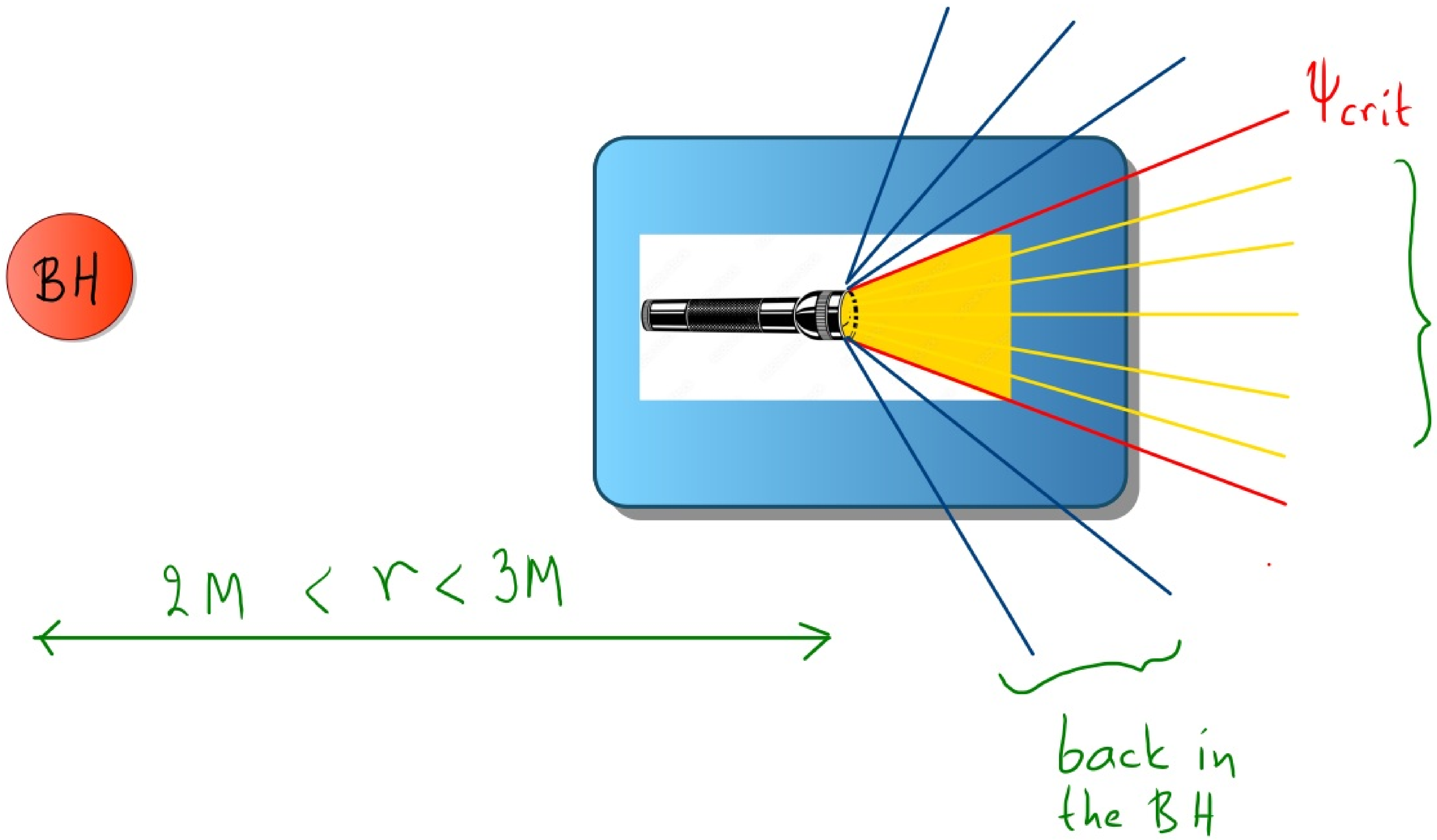


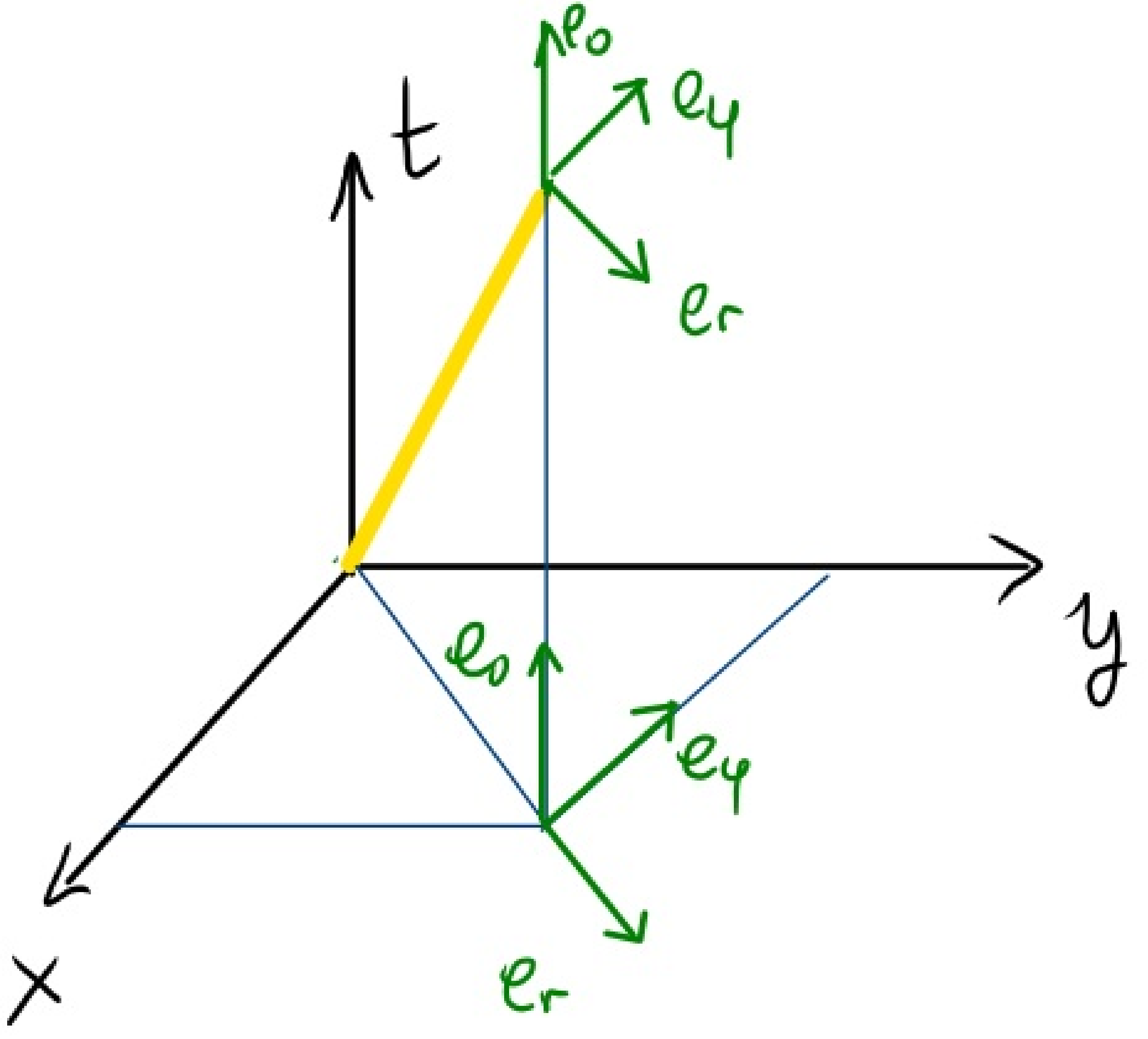
- How much light escapes to ∞?

stationary $\Rightarrow$ $u^\mu = \left(\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right) = \left(1-\frac{2M}{R}\right)^{-\frac{1}{2}}(\partial_t)^\mu$

$$u^\mu u_\mu = g_{\mu\nu}u^\mu u^\nu = g_{00}u^0u^0 = -\left(1-\frac{2M}{R}\right)\cdot\left(1-\frac{2M}{R}\right)^{-\frac{1}{2}}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} = -1$$

A stationary observer at $2M < R < 3M$





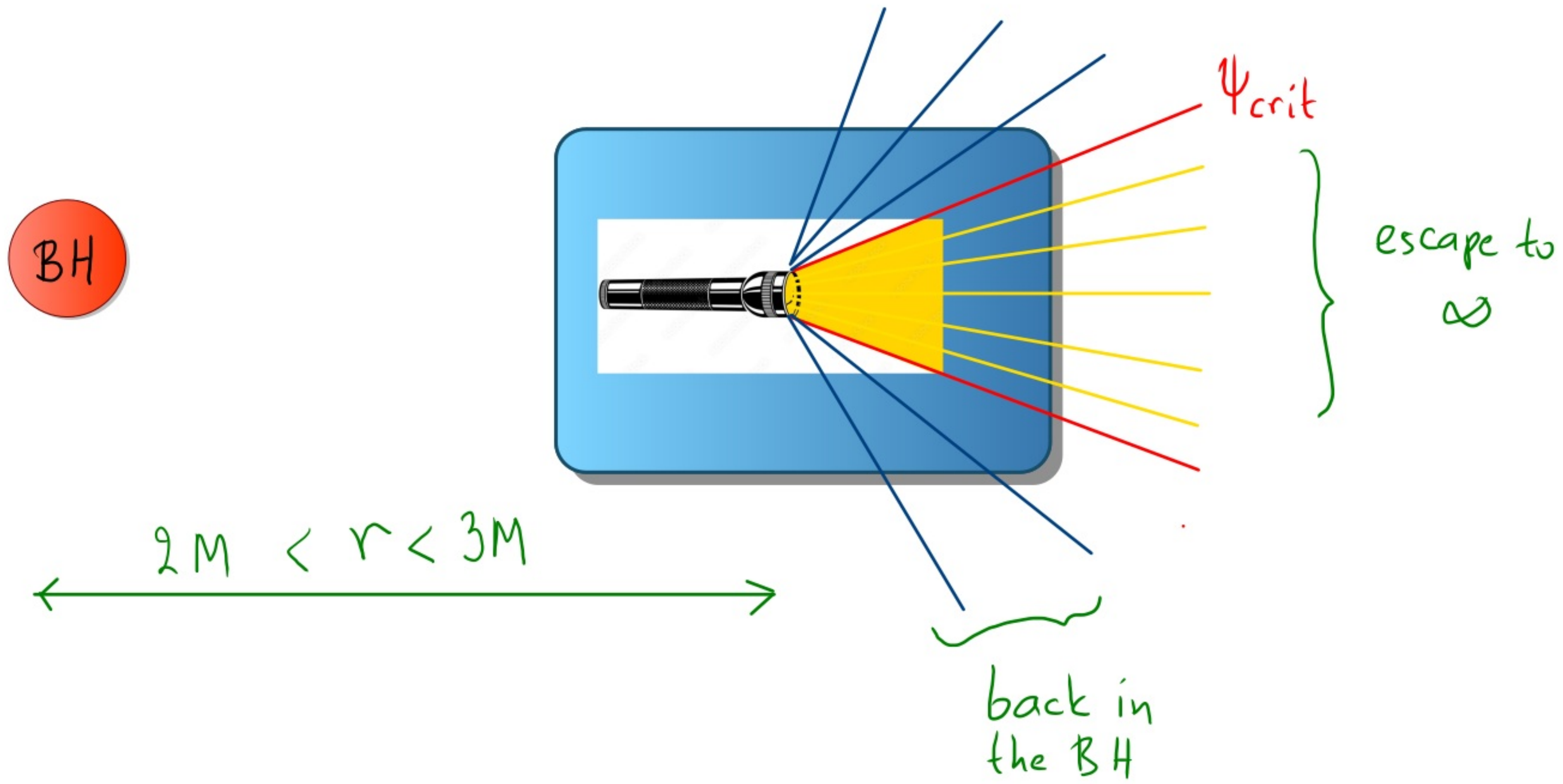
- How much light escapes to ∞?

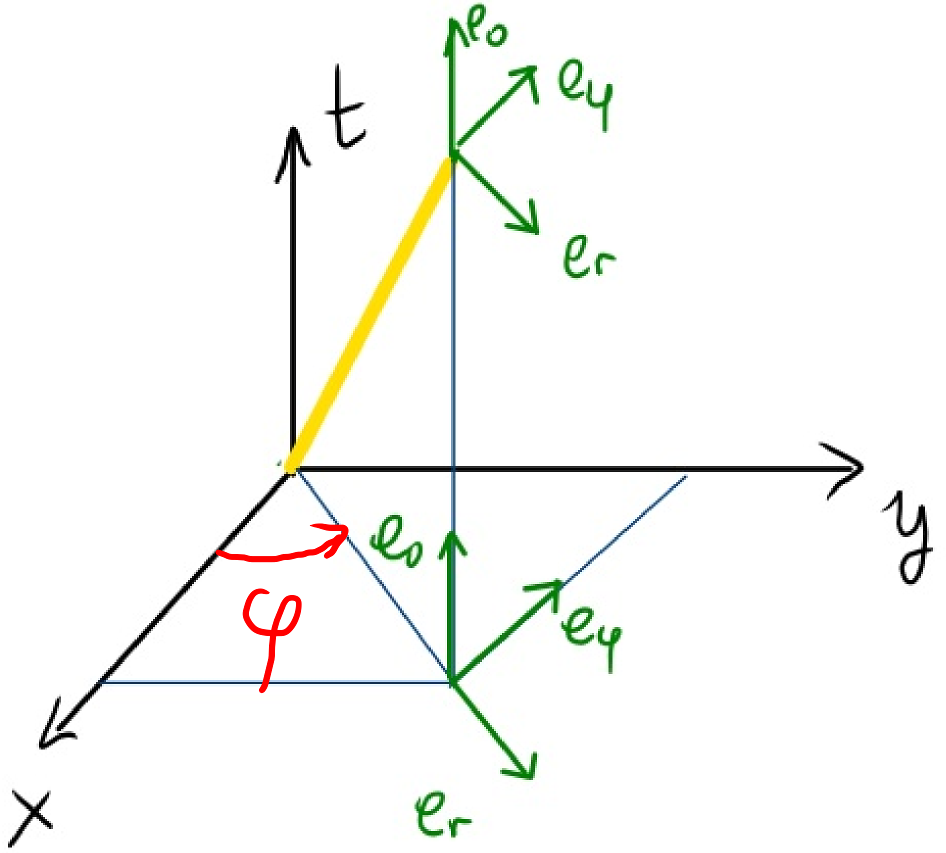
  stationary $\Rightarrow$ $u^\mu = \left(\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right) = \left(1-\frac{2M}{R}\right)^{-\frac{1}{2}} (\partial_t)^\mu$

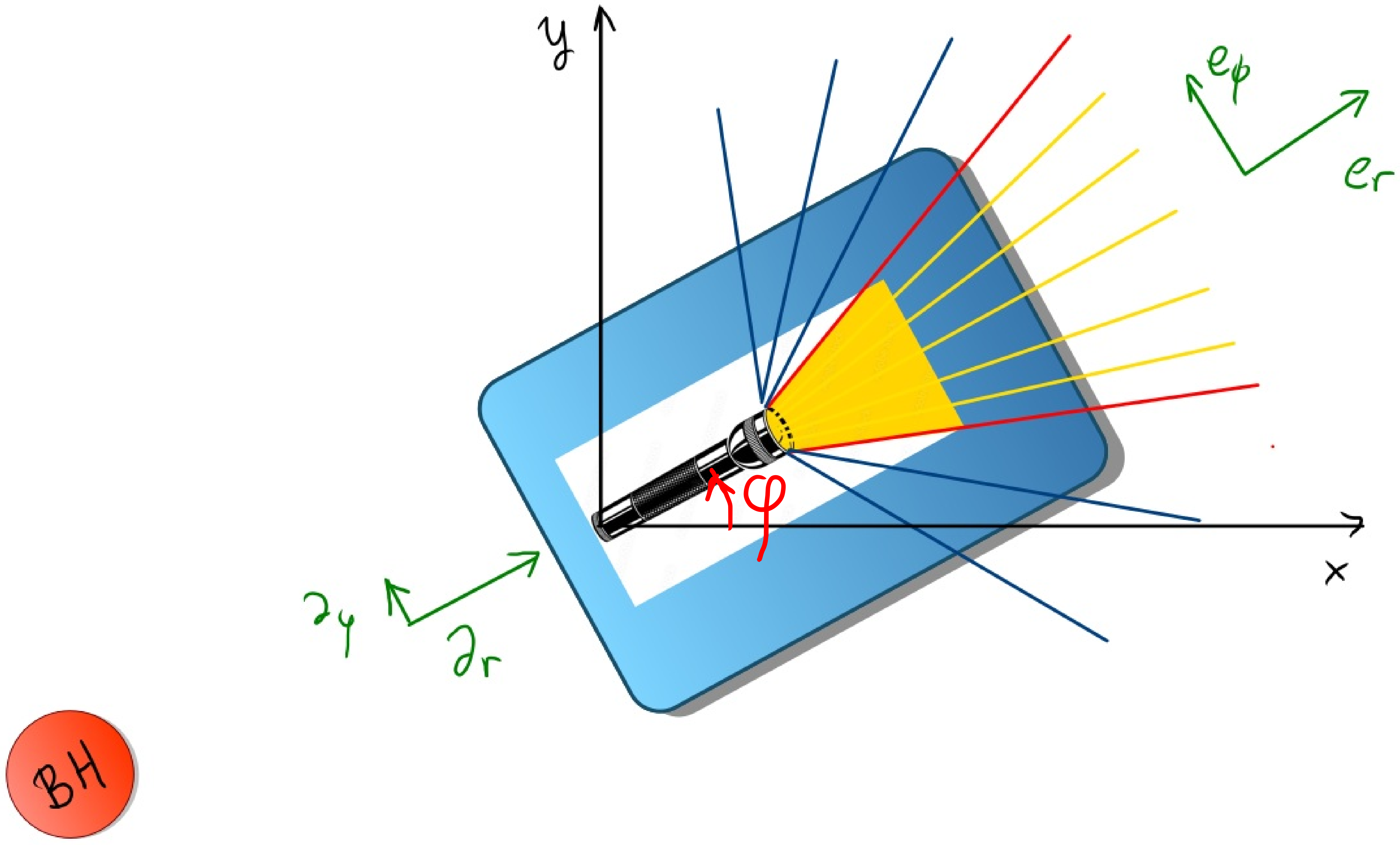
- local "lab": orthonormal basis $\{e_0, e_r, e_\theta, e_\phi\}$

  $e_0^\mu = u^\mu$

A stationary
observer
at
$2M<R<3M$





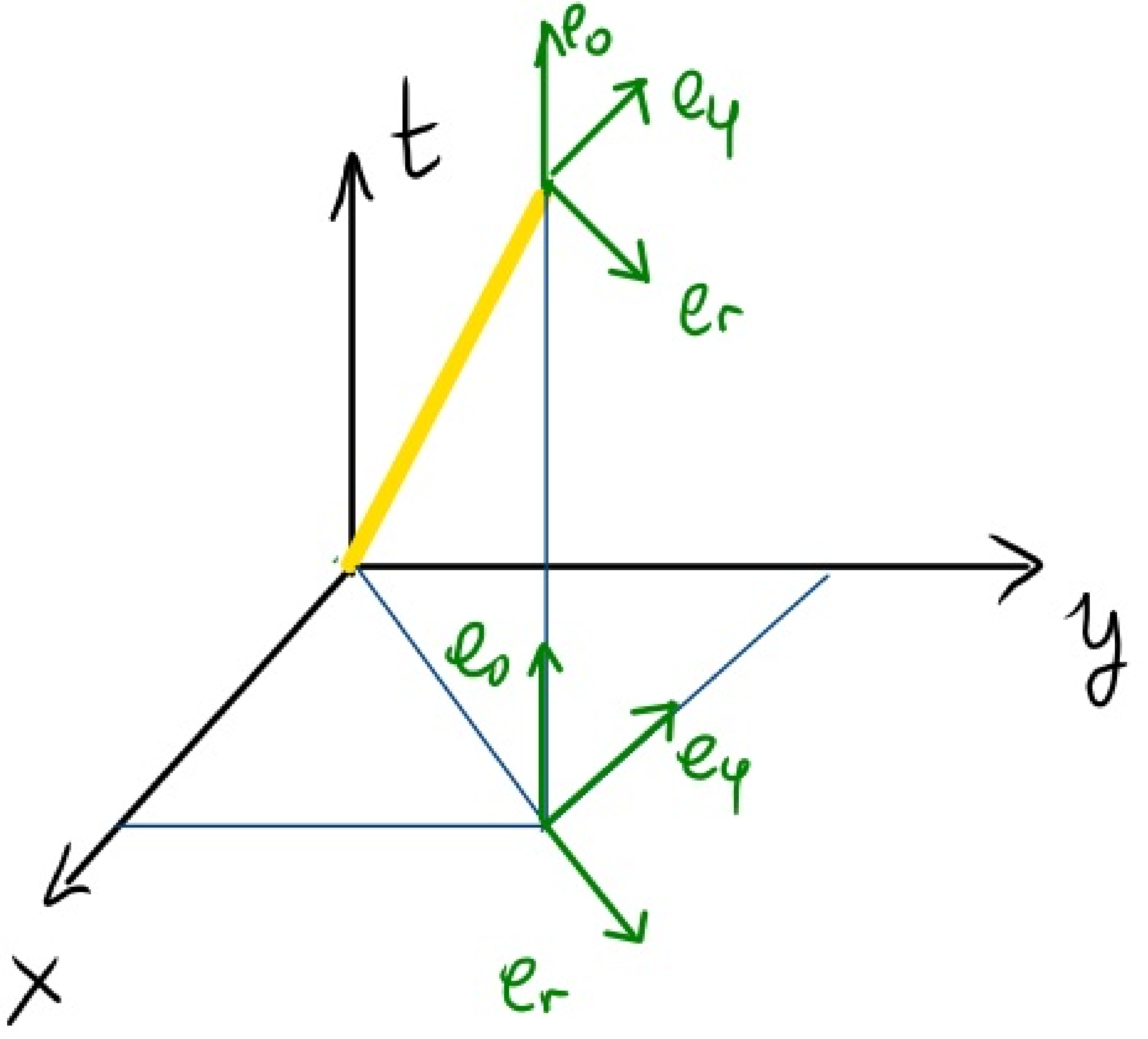


$$(e_r)^\mu = \left(0,\ \left(1-\frac{2M}{R}\right)^{\frac{1}{2}},\ 0,\ 0\right)$$

$$(e_r)^\mu (e_r)_\mu = g_{\mu\nu}\,(e_r)^\mu (e_r)^\nu$$

$$= g_{rr}\,(e_r)^r (e_r)^r$$

$$= \left(1-\frac{2M}{R}\right)^{-1}\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}\left(1-\frac{2M}{R}\right)^{\frac{1}{2}} = 1$$



- How much light escapes to ∞?

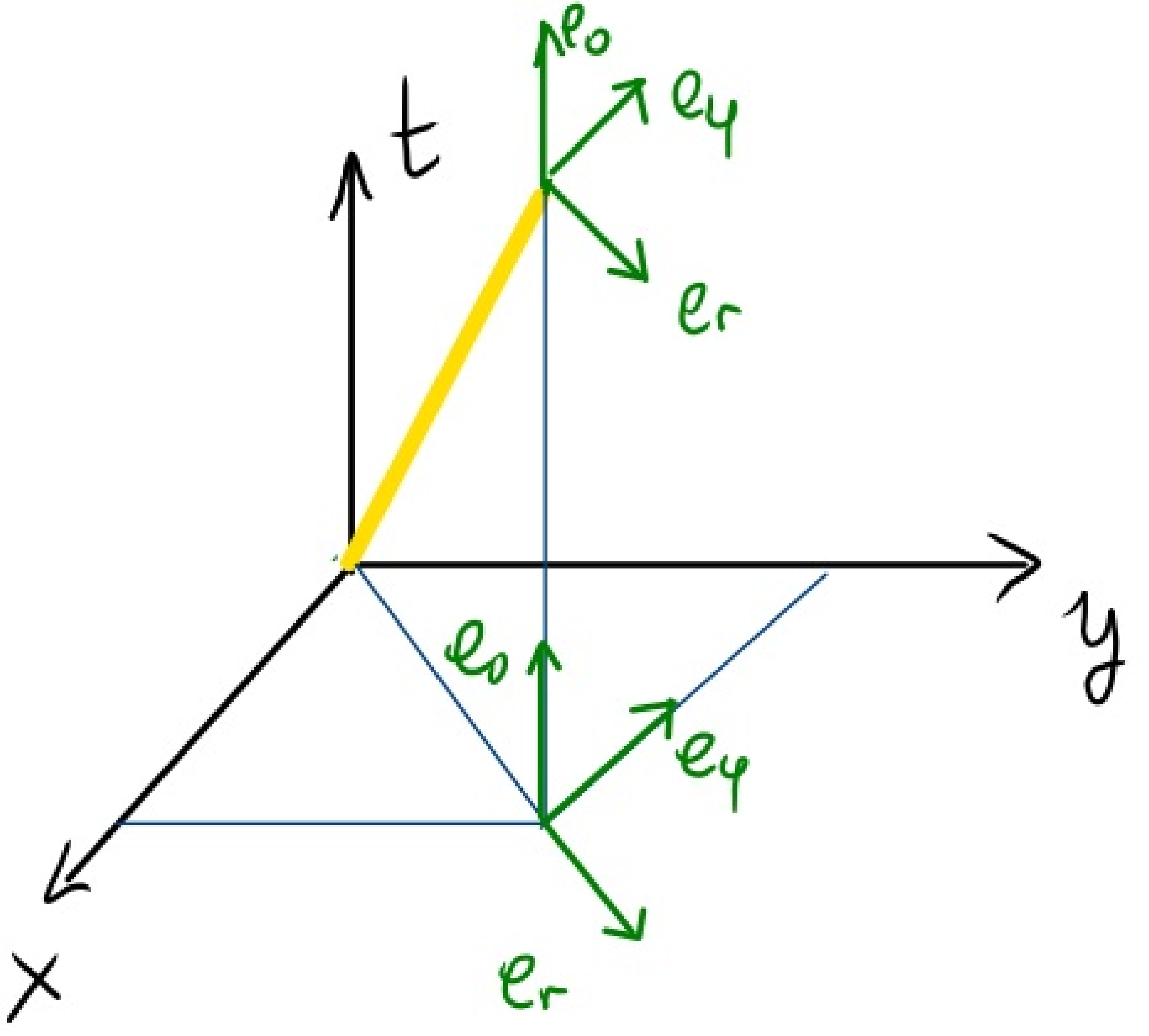
  stationary $\Rightarrow$ $u^\mu = \left(\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right) = \left(1-\frac{2M}{R}\right)^{-\frac{1}{2}}(\partial_t)^\mu$

- local "lab": orthonormal basis $\{e_0, e_r, e_\theta, e_\phi\}$

  $e_0^\mu = u^\mu$

$$(e_r)^\mu = \left(0, \left(1-\frac{2M}{R}\right)^{\frac{1}{2}}, 0, 0\right)$$

$$(e_\phi)^\mu = \left(0, 0, 0, \frac{1}{R}\right)$$



$$e_\phi \cdot e_\phi = g_{\mu\nu}(e_\phi)^\mu (e_\phi)^\nu =$$

$$= g_{\phi\phi}(e_\phi)^\phi (e_\phi)^\phi = R^2 \frac{1}{R}\cdot\frac{1}{R} = 1$$

- How much light escapes to ∞?

  stationary $\Rightarrow$ $u^\mu = \left(\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right) = \left(1-\frac{2M}{R}\right)^{-\frac{1}{2}}(\partial_t)^\mu$

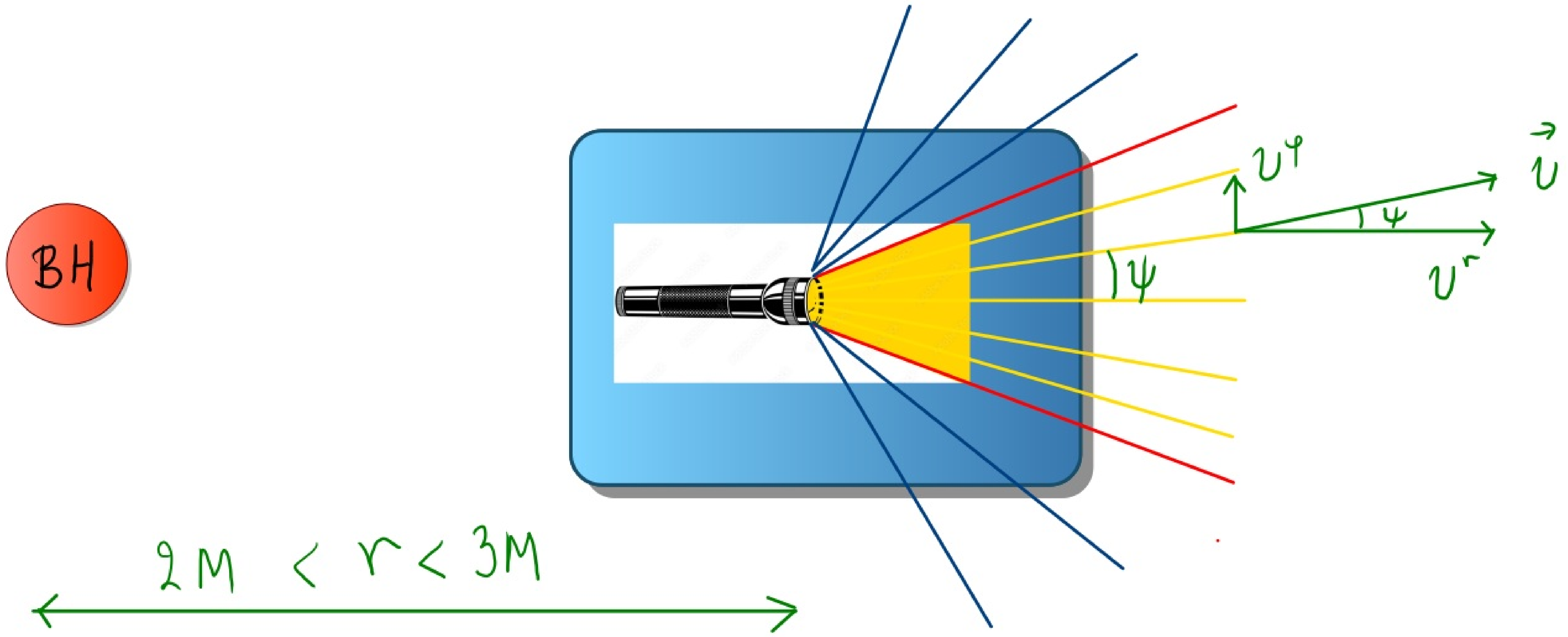
- local "lab": orthonormal basis $\{e_0, e_r, e_\theta, e_\phi\}$

  $e_0^\mu = u^\mu$

$$(e_r)^\mu = \left(0,\ \left(1-\frac{2M}{R}\right)^{\frac{1}{2}},\ 0,\ 0\right)$$

$$(e_\varphi)^\mu = \left(0,\ 0,\ 0,\ \frac{1}{R}\right)$$

$$\tan\psi = \frac{v^\varphi}{v^r} = \frac{v\cdot e_\varphi}{v\cdot e_r}$$



- How much light escapes to ∞?

  stationary $\Rightarrow$ $u^\mu = \left(\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right) = \left(1-\frac{2M}{R}\right)^{-\frac{1}{2}} (\partial_t)^\mu$

- local "lab": orthonormal basis $\{e_0, e_r, e_\theta, e_\phi\}$

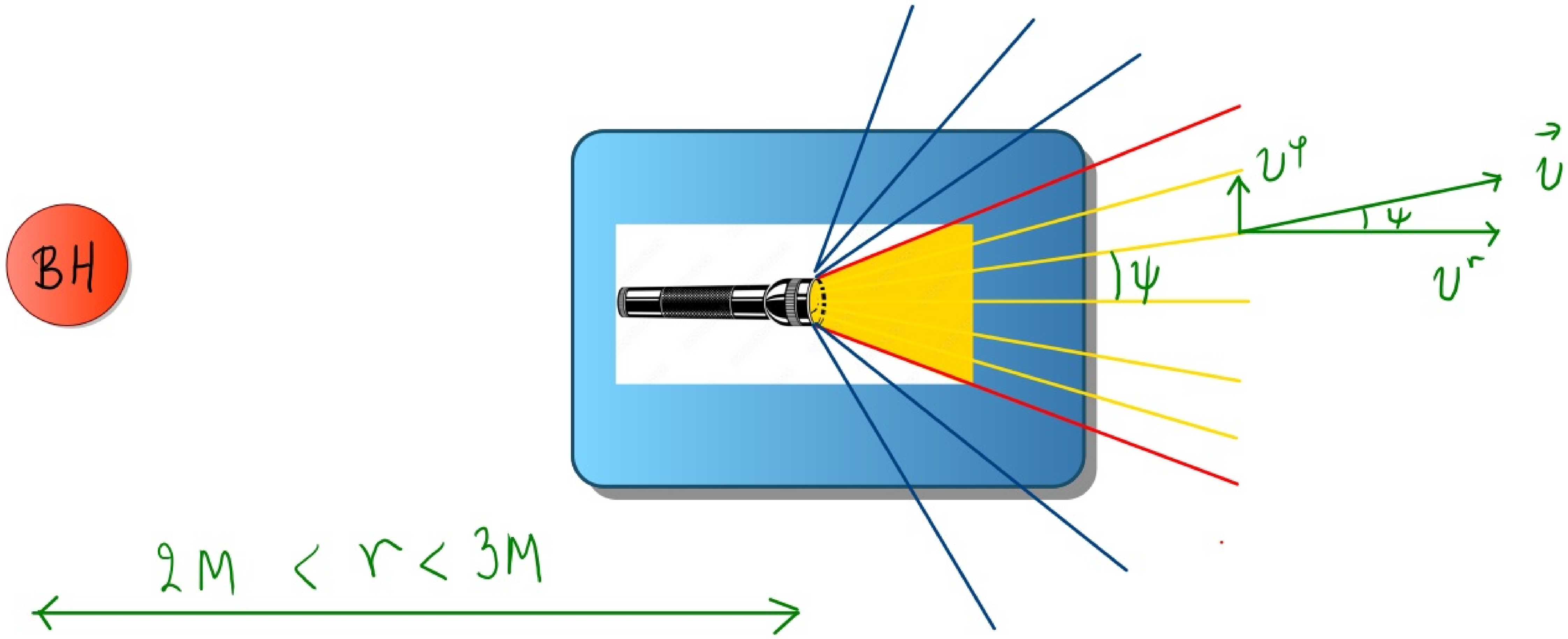
  $e_0^\mu = u^\mu$

$$(e_r)^\mu = \left(0, \left(1-\frac{2M}{R}\right)^{\frac{1}{2}}, 0, 0\right)$$

$$(e_\phi)^\mu = \left(0, 0, 0, \frac{1}{R}\right)$$

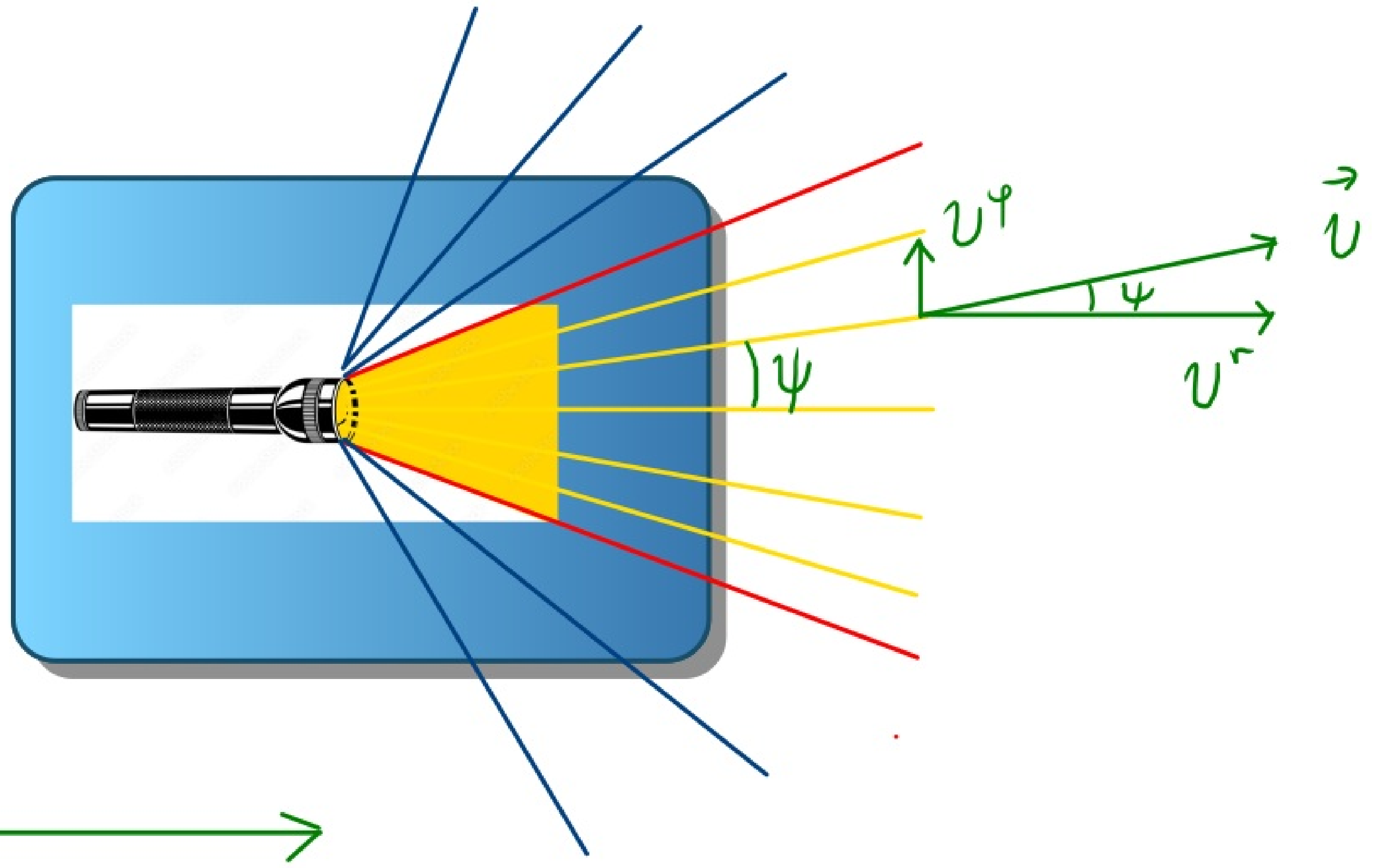
$$\tan\psi = \frac{v^\phi}{v^r} = \frac{v\cdot e_\phi}{v\cdot e_r}$$

$$= \frac{g_{\phi\phi}\, v^\phi (e_\phi)^\phi}{g_{rr}\, v^r (e_r)^r}$$



$$(e_r)^\mu = \left(0,\ \left(1-\frac{2M}{R}\right)^{\frac{1}{2}},\ 0,\ 0\right)$$

$$(e_\phi)^\mu = \left(0,\ 0,\ 0,\ \frac{1}{R}\right)$$



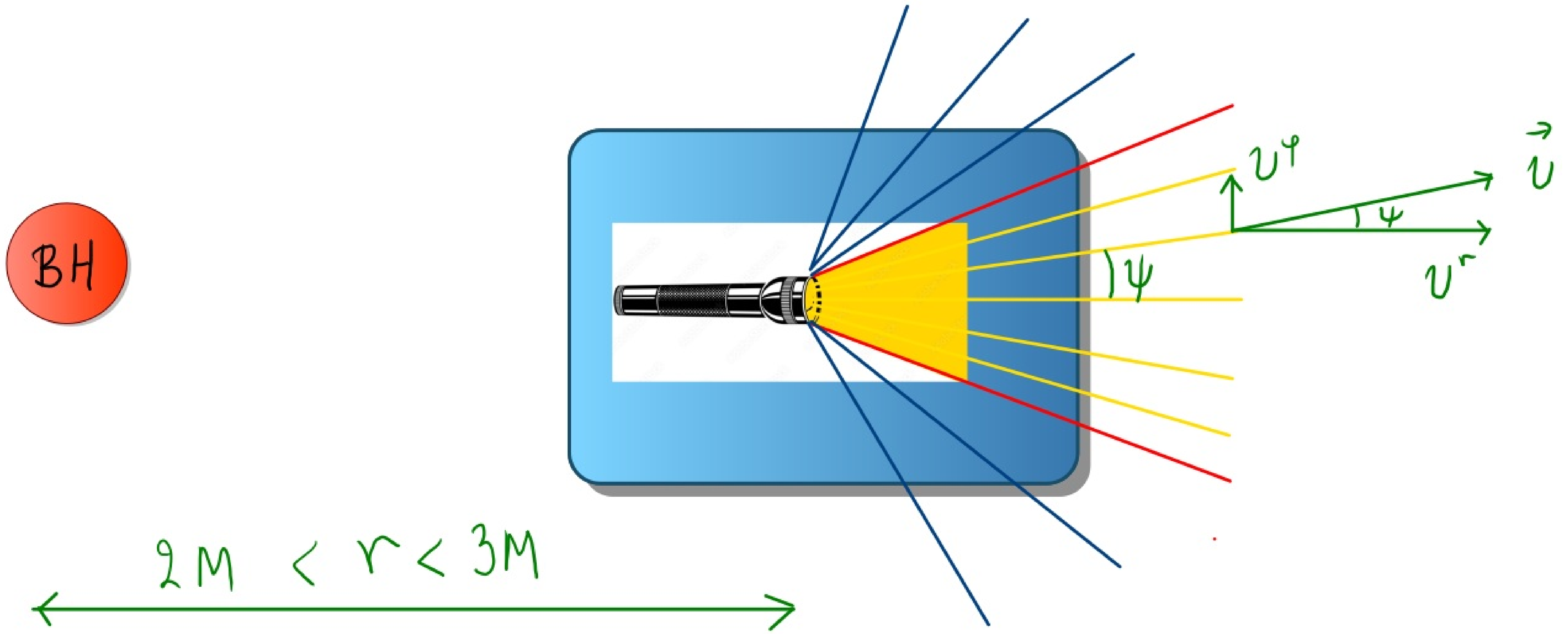
$2M < r < 3M$

$$\tan\psi = \frac{v^\psi}{v^r} = \frac{v\cdot e_\phi}{v\cdot e_r}$$

$$= \frac{g_{\phi\phi}\, v^\phi (e_\phi)^\phi}{g_{rr}\, v^r (e_r)^r} = \frac{R^2 \cdot \frac{d\phi}{d\lambda} \cdot \frac{1}{R}}{\left(1-\frac{2M}{R}\right)^{-1} \cdot \frac{dr}{d\lambda} \cdot \left(1-\frac{2M}{R}\right)^{\frac{1}{2}}}$$

$$(e_r)^\mu = \left(0,\ \left(1-\frac{2M}{R}\right)^{\frac{1}{2}},\ 0,\ 0\right)$$

$$(e_\phi)^\mu = \left(0,\ 0,\ 0,\ \frac{1}{R}\right)$$



$$\tan\psi = R\left(1-\frac{2M}{R}\right)^{\frac{1}{2}} \frac{d\phi/d\lambda}{dr/d\lambda}$$

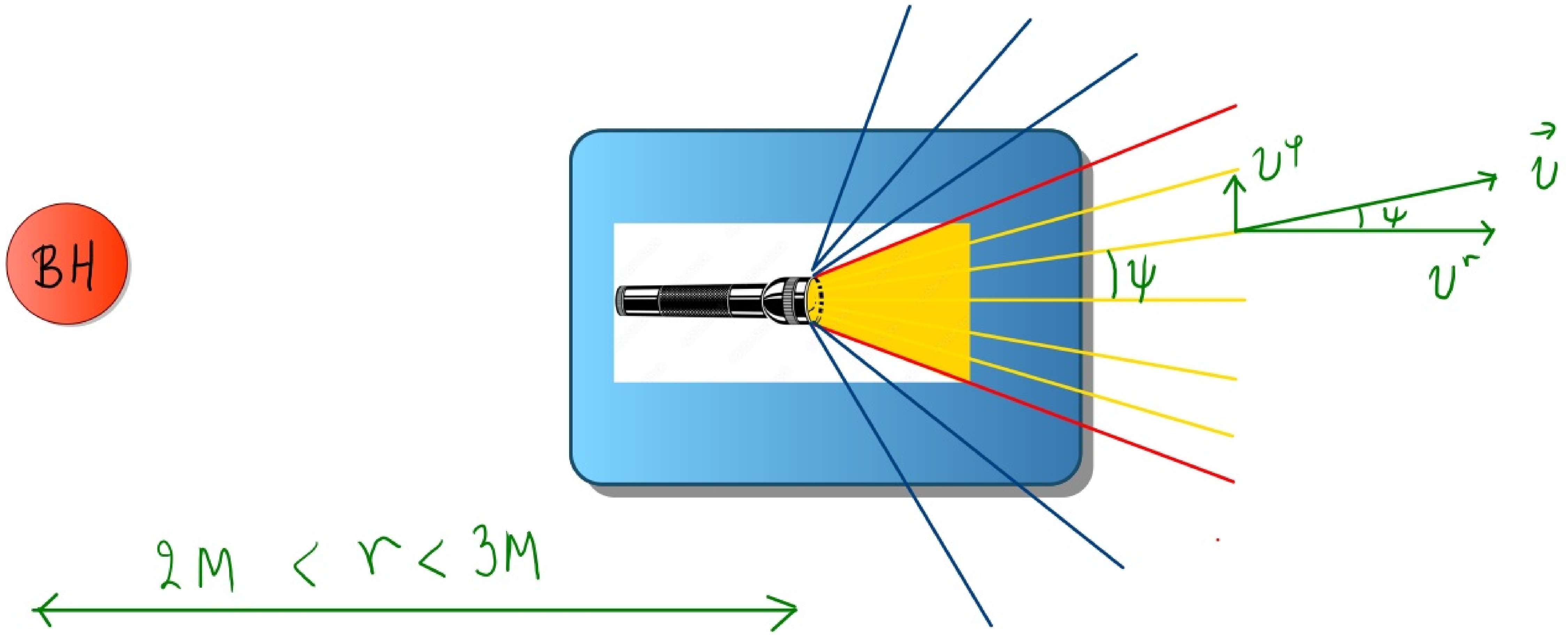
$$= \frac{g_{\phi\phi}\, v^\phi\, (e_\phi)^\phi}{g_{rr}\, v^r\, (e_r)^r} = \frac{R^2 \cdot \frac{d\phi}{d\lambda} \cdot \frac{1}{R}}{\left(1-\frac{2M}{R}\right)^{-1} \cdot \frac{dr}{d\lambda} \cdot \left(1-\frac{2M}{R}\right)^{\frac{1}{2}}}$$

$$(e_r)^\mu = \left(0,\ \left(1-\frac{2M}{R}\right)^{\frac{1}{2}},\ 0,\ 0\right)$$

$$(e_\phi)^\mu = \left(0,\ 0,\ 0,\ \frac{1}{R}\right)$$

$$\tan\psi = R\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}\frac{d\phi/d\lambda}{dr/d\lambda}$$

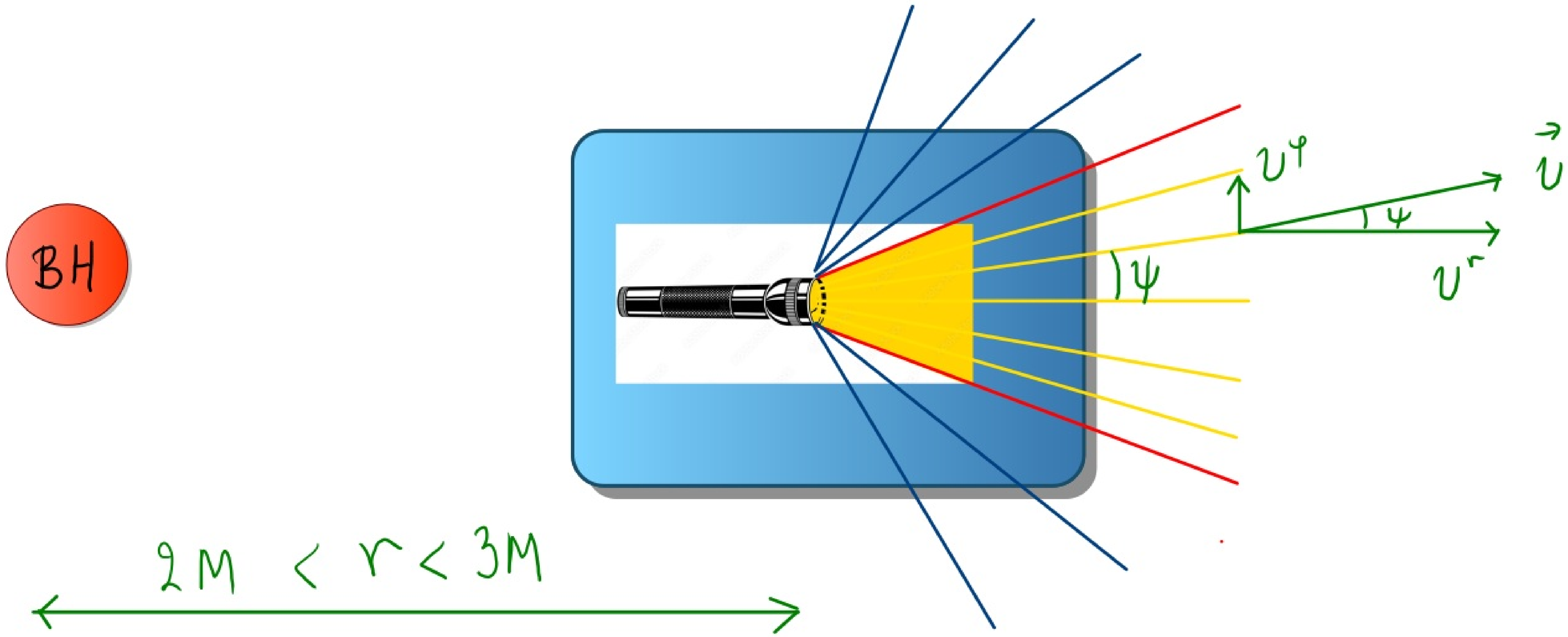
$$\ell = R^2\frac{d\phi}{d\lambda} \Rightarrow \frac{d\phi}{d\lambda} = \frac{\ell}{R^2}$$



$$(e_r)^\mu = \left(0,\ \left(1-\frac{2M}{R}\right)^{\frac{1}{2}},\ 0,\ 0\right)$$

$$(e_\phi)^\mu = \left(0,\ 0,\ 0,\ \frac{1}{R}\right)$$

$$\tan\psi = R\left(1-\frac{2M}{R}\right)^{\frac{1}{2}} \frac{d\phi/d\lambda}{dr/d\lambda}$$

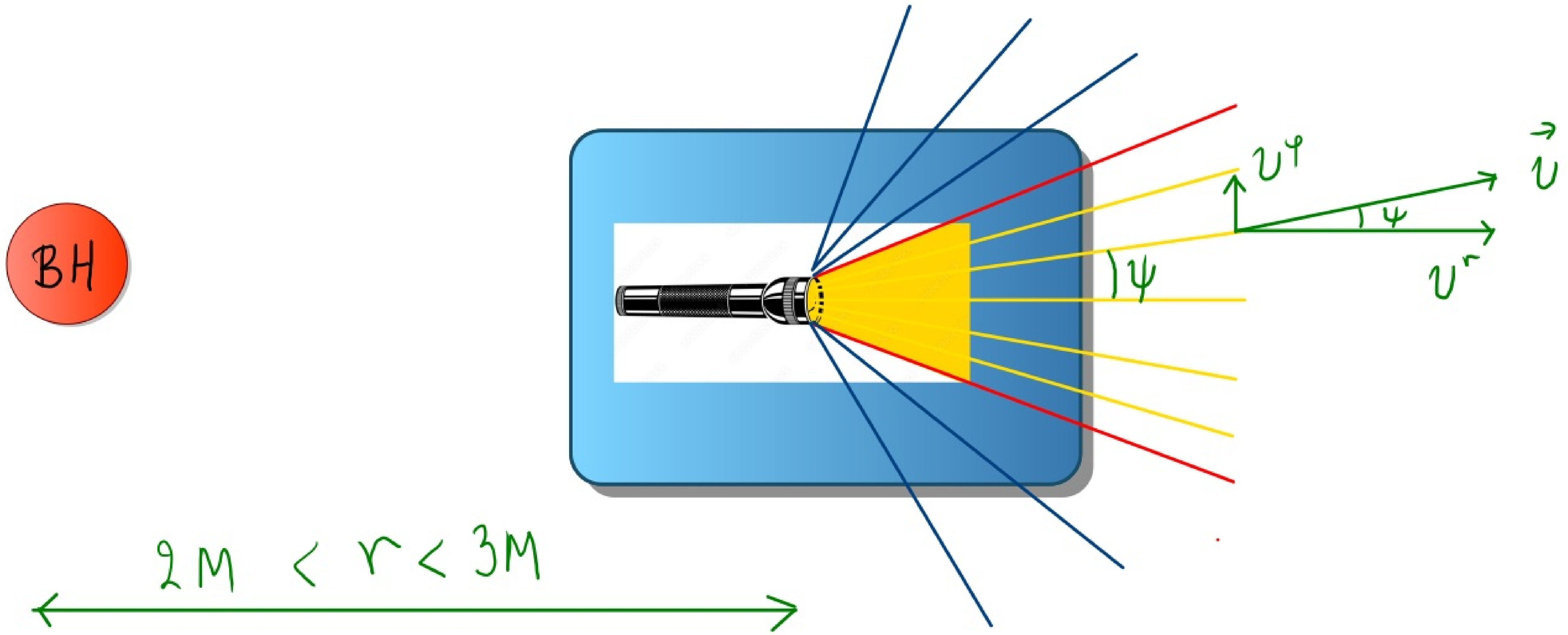


$$\ell = R^2 \frac{d\phi}{d\lambda} \Rightarrow \frac{d\phi}{d\lambda} = \frac{\ell}{R^2}$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(R) \Rightarrow \frac{dr}{d\lambda} = \ell\left[\frac{1}{b^2} - \frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{\frac{1}{2}}$$

$$(e_r)^\mu = \left(0,\ \left(1-\frac{2M}{R}\right)^{\frac{1}{2}},\ 0,\ 0\right)$$

$$(e_\varphi)^\mu = \left(0,\ 0,\ 0,\ \frac{1}{R}\right)$$



$$\tan\psi = R\left(1-\frac{2M}{R}\right)^{\frac{1}{2}} \frac{d\phi/d\lambda}{dr/d\lambda}$$

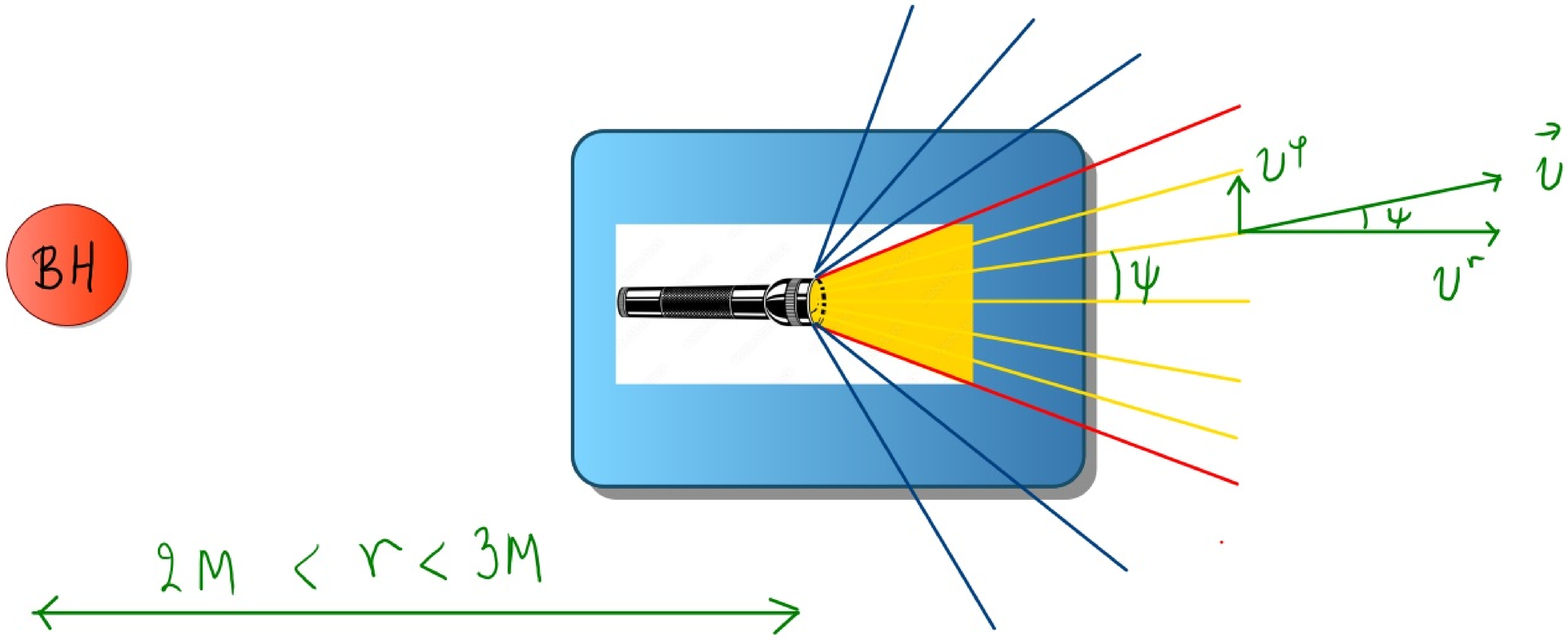
$$\ell = R^2 \frac{d\phi}{d\lambda} \Rightarrow \frac{d\phi}{d\lambda} = \frac{\ell}{R^2}$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(R) \Rightarrow \frac{dr}{d\lambda} = \ell\left[\frac{1}{b^2} - \frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{\frac{1}{2}}$$

$$\frac{d\phi/d\lambda}{dr/d\lambda} = \frac{\ell/R^2}{\ell\left[\frac{1}{b^2} - \frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{\frac{1}{2}}} = \frac{1}{R^2}\left[\frac{1}{b^2} - \frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$

$$(e_r)^\mu = \left(0,\ \left(1-\frac{2M}{R}\right)^{\frac{1}{2}},\ 0,\ 0\right)$$

$$(e_\phi)^\mu = \left(0,\ 0,\ 0,\ \frac{1}{R}\right)$$



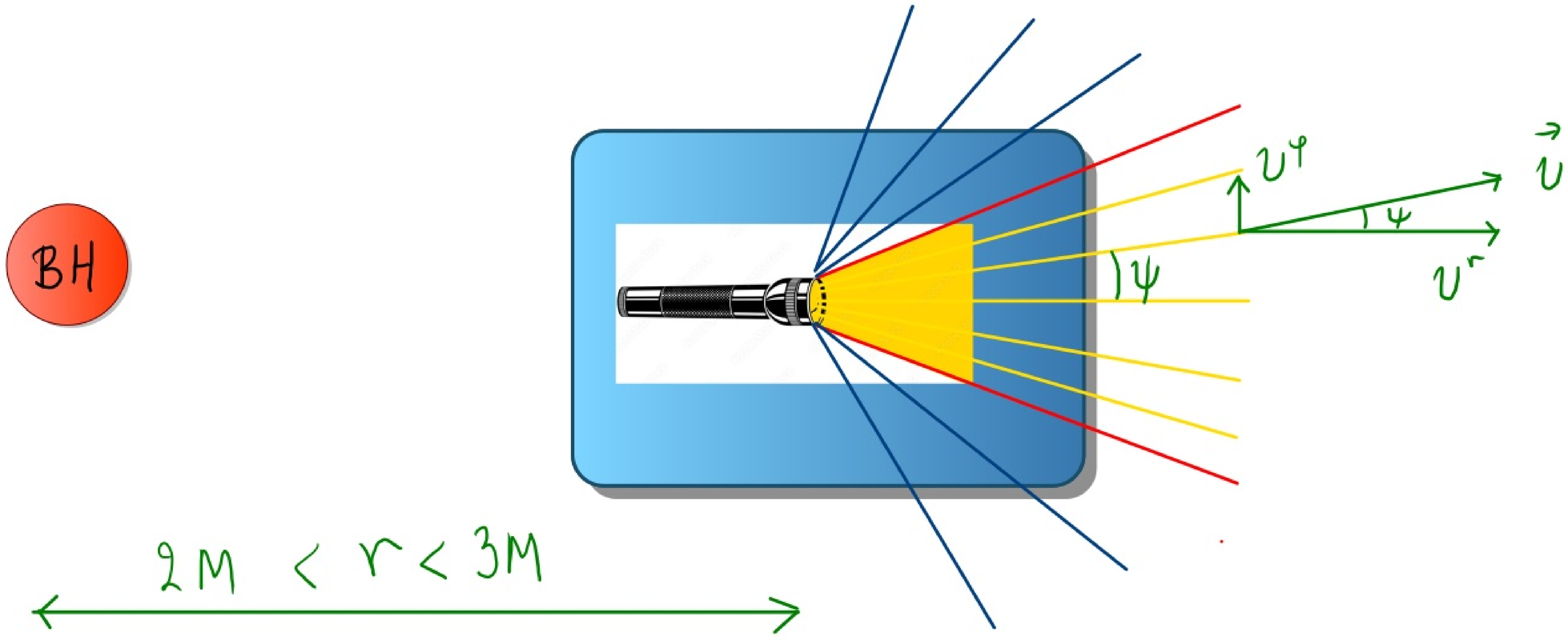
$$\tan\psi = R\left(1-\frac{2M}{R}\right)^{\frac{1}{2}} \frac{d\phi/d\lambda}{dr/d\lambda}$$

$$= R\left(1-\frac{2M}{R}\right)^{\frac{1}{2}} \frac{1}{R^2}\left[\frac{1}{b^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$

$$\frac{d\phi/d\lambda}{dr/d\lambda} = \frac{\ell/R^2}{\ell\left[\frac{1}{b^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{\frac{1}{2}}} = \frac{1}{R^2}\left[\frac{1}{b^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$

$$(e_r)^\mu = \left(0,\ \left(1-\frac{2M}{R}\right)^{\frac{1}{2}},\ 0,\ 0\right)$$

$$(e_\phi)^\mu = \left(0,\ 0,\ 0,\ \frac{1}{R}\right)$$



$$\tan\psi = \frac{1}{R}\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}\left[\frac{1}{b^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$

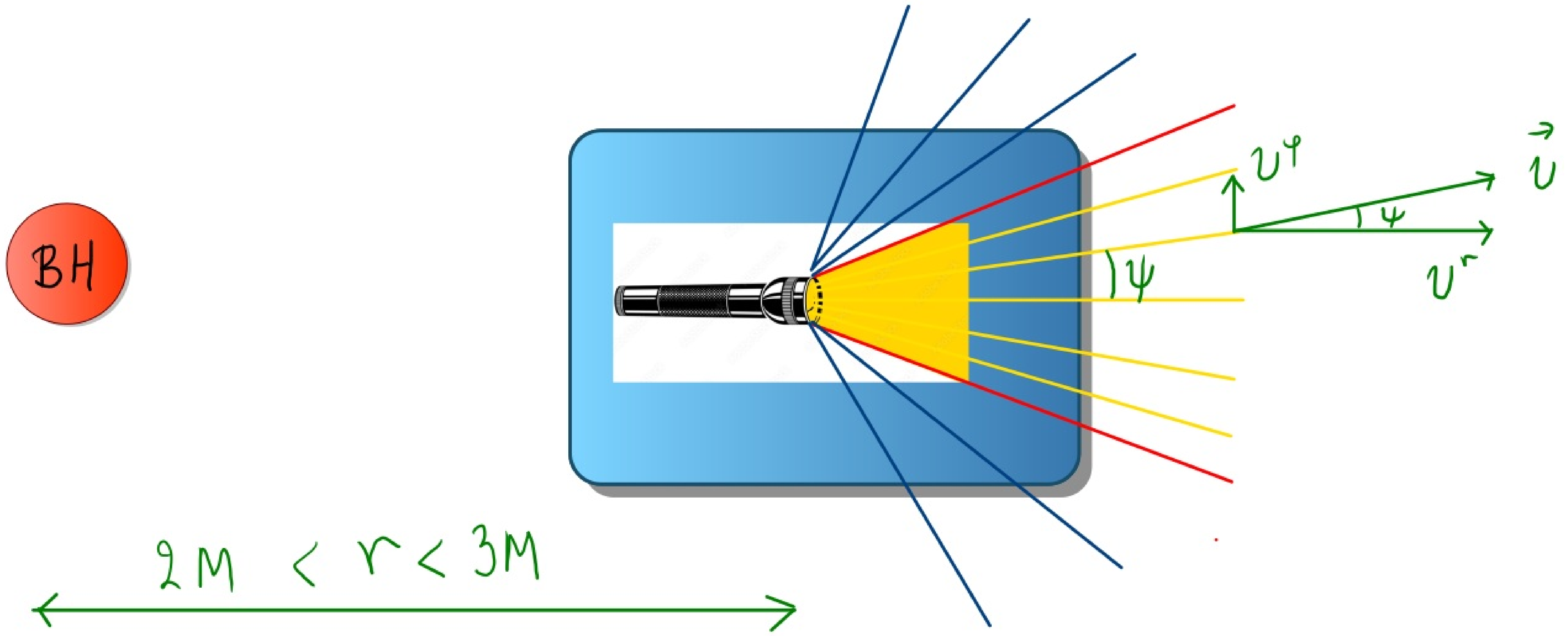
$$= R\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}\frac{1}{R^2}\left[\frac{1}{b^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$

$$\frac{d\phi/d\lambda}{dr/d\lambda} = \frac{\ell/R^2}{\ell\left[\frac{1}{b^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{\frac{1}{2}}} = \frac{1}{R^2}\left[\frac{1}{b^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$

$$(e_r)^\mu = \left(0, \left(1-\frac{2M}{R}\right)^{\frac{1}{2}}, 0, 0\right)$$

$$(e_\varphi)^\mu = \left(0, 0, 0, \frac{1}{R}\right)$$

$$\tan\psi = \frac{1}{R}\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}\left[\frac{1}{b^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$



↳ careful w/ signs

we have assumed $\frac{d\phi}{d\lambda} > 0$ and $\frac{dr}{d\lambda} > 0$

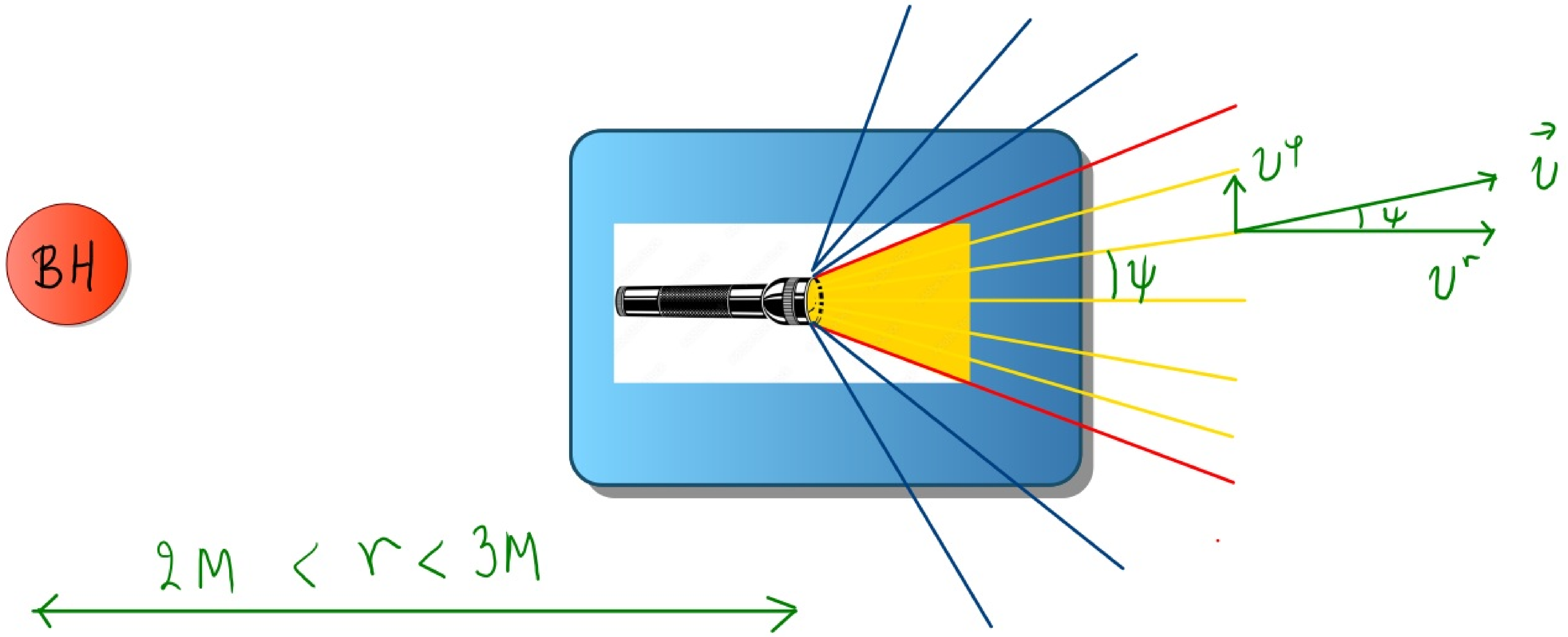
valid for $0 \leq \psi \leq \frac{\pi}{2}$

for $\psi > \frac{\pi}{2}$ $\frac{dr}{d\lambda} < 0$, need $(-)$ sign

$$(e_r)^\mu = \left(0,\ \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}},\ 0,\ 0\right)$$

$$(e_\varphi)^\mu = \left(0,\ 0,\ 0,\ \frac{1}{R}\right)$$

$$\tan\psi = \frac{1}{R}\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}\left[\frac{1}{b^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$



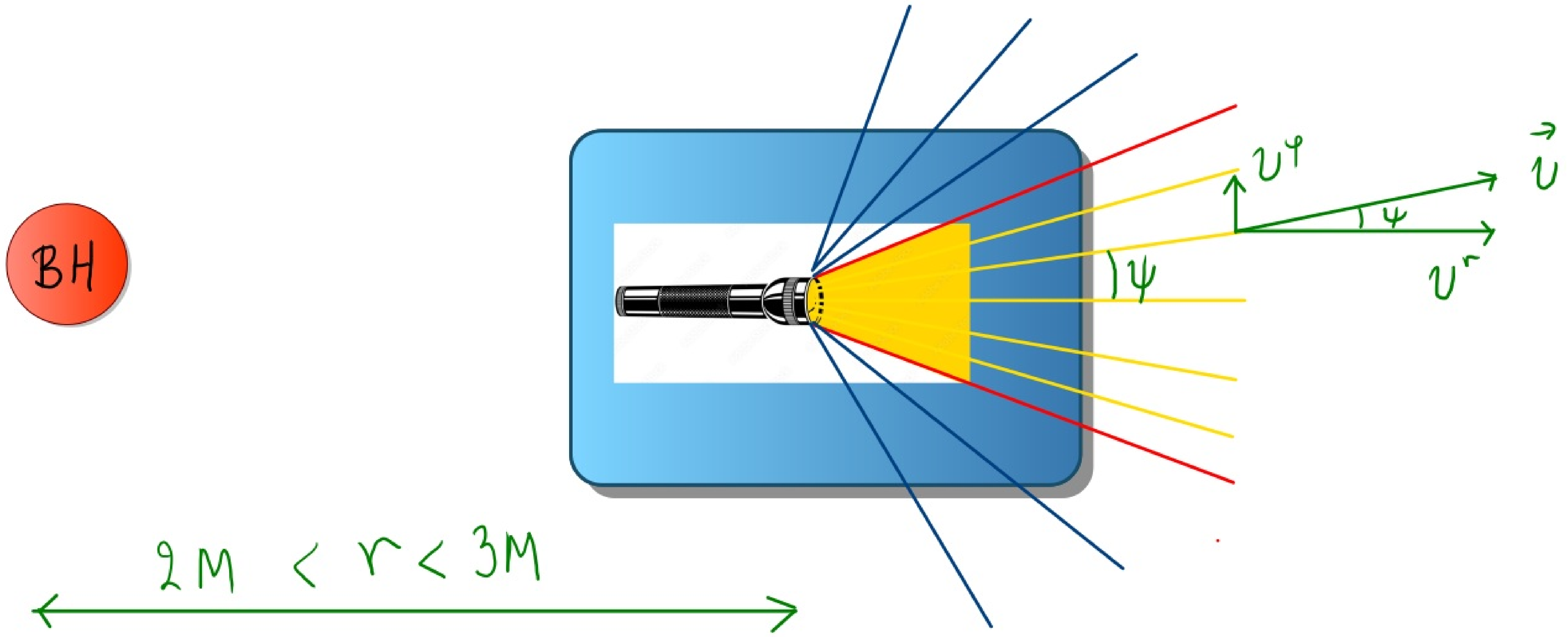
Light rays barely escape when $\frac{1}{b^2} = \text{Well} \Rightarrow b^2 = 27M^2$

b is varied with ψ, so ψ_{crit} when $b^2 = 27\,M^2 \Rightarrow$

$$\tan\psi_{crit} = \frac{1}{R}\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}\left[\frac{1}{27M^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$

$$(e_r)^\mu = \left(0,\ \left(1-\frac{2M}{R}\right)^{\frac{1}{2}},\ 0,\ 0\right)$$

$$(e_\varphi)^\mu = \left(0,\ 0,\ 0,\ \frac{1}{R}\right)$$



$$\tan\psi = \frac{1}{R}\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}\left[\frac{1}{b^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$

Light rays barely escape when $\frac{1}{b^2} = W_{eff} \Rightarrow b^2 = 27M^2$

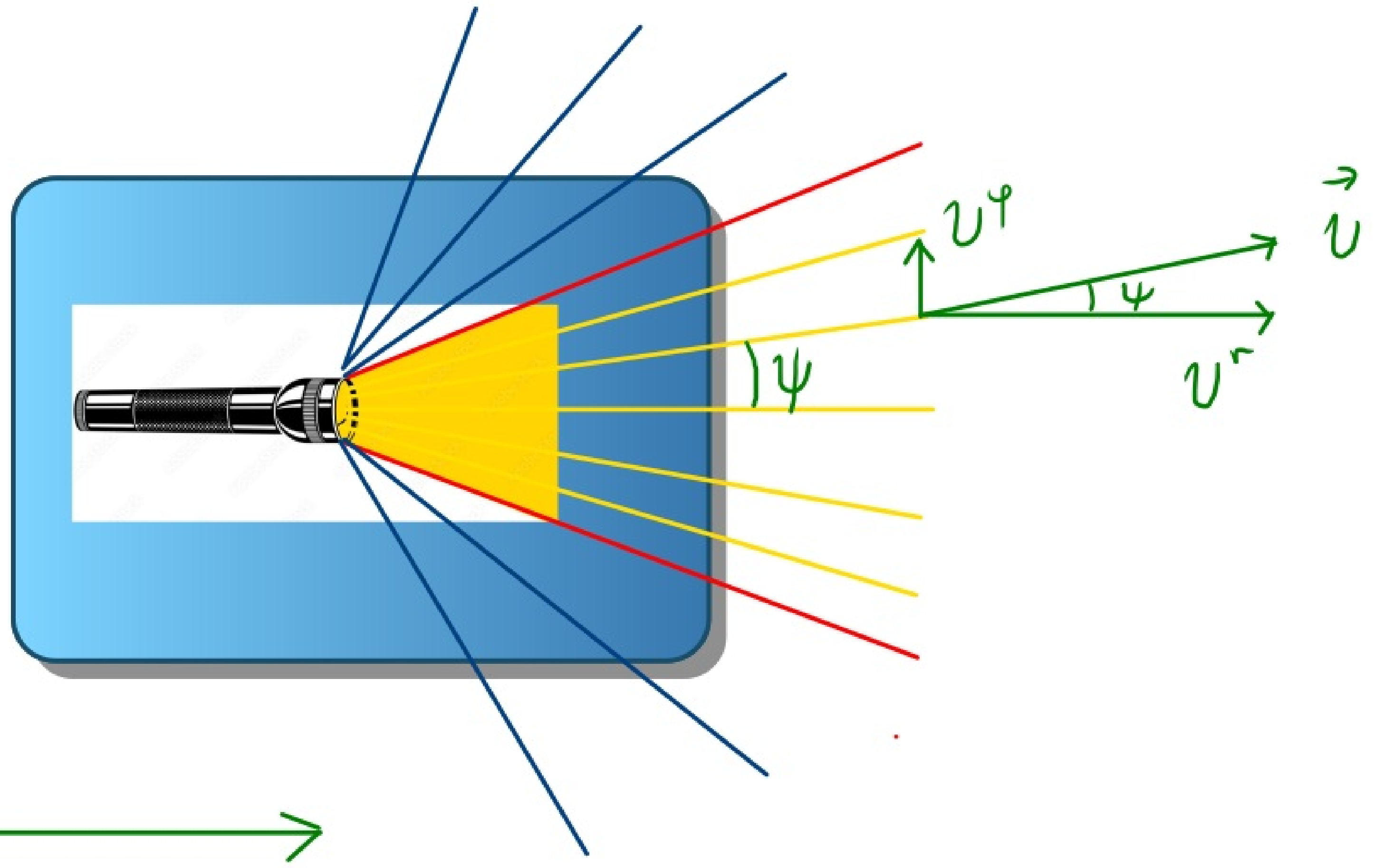
b is varied with ψ, so ψ_{crit} when $b^2 = 27M^2 \Rightarrow$

$$\tan\psi_{crit} = \frac{1}{R}\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}\left[\frac{1}{27M^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}} \simeq \frac{1}{0^{1/2}}$$

When $R=3M$, $\tan\psi_{crit} = +\infty \Rightarrow \psi_{crit} = \frac{\pi}{2}$

$$(e_r)^\mu = \left(0, \left(1-\frac{2M}{R}\right)^{\frac{1}{2}}, 0, 0\right)$$

$$(e_\varphi)^\mu = \left(0, \quad 0 \quad, 0, \frac{1}{R}\right)$$



$$\tan\psi = \frac{1}{R}\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}\left[\frac{1}{b^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$

$$\xleftrightarrow{2M < r < 3M}$$

Light rays barely escape when $\frac{1}{b^2} = W_{eff} \Rightarrow b^2 = 27M^2$

b is varied with ψ, so ψ_{crit} when $b^2 = 27M^2 \Rightarrow$

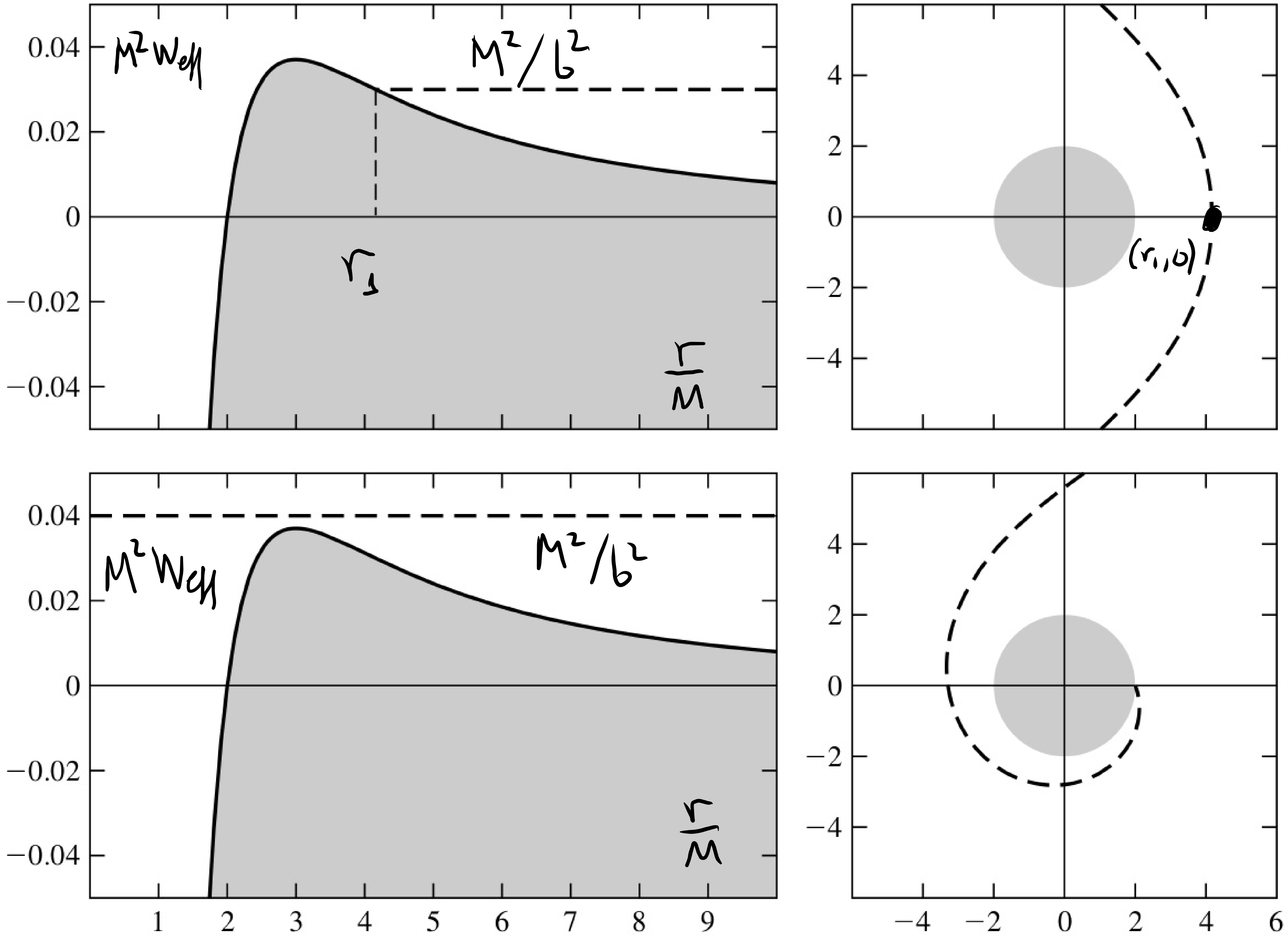
$$\tan\psi_{crit} = \frac{1}{R}\underbrace{\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}}_{=0}\left[\frac{1}{27M^2}-\frac{1}{R^2}\left(1-\frac{2M}{R}\right)\right]^{-\frac{1}{2}}$$

When $R = 2M$, $\tan\psi_{crit} = 0 \Rightarrow \psi_{crit} = 0$

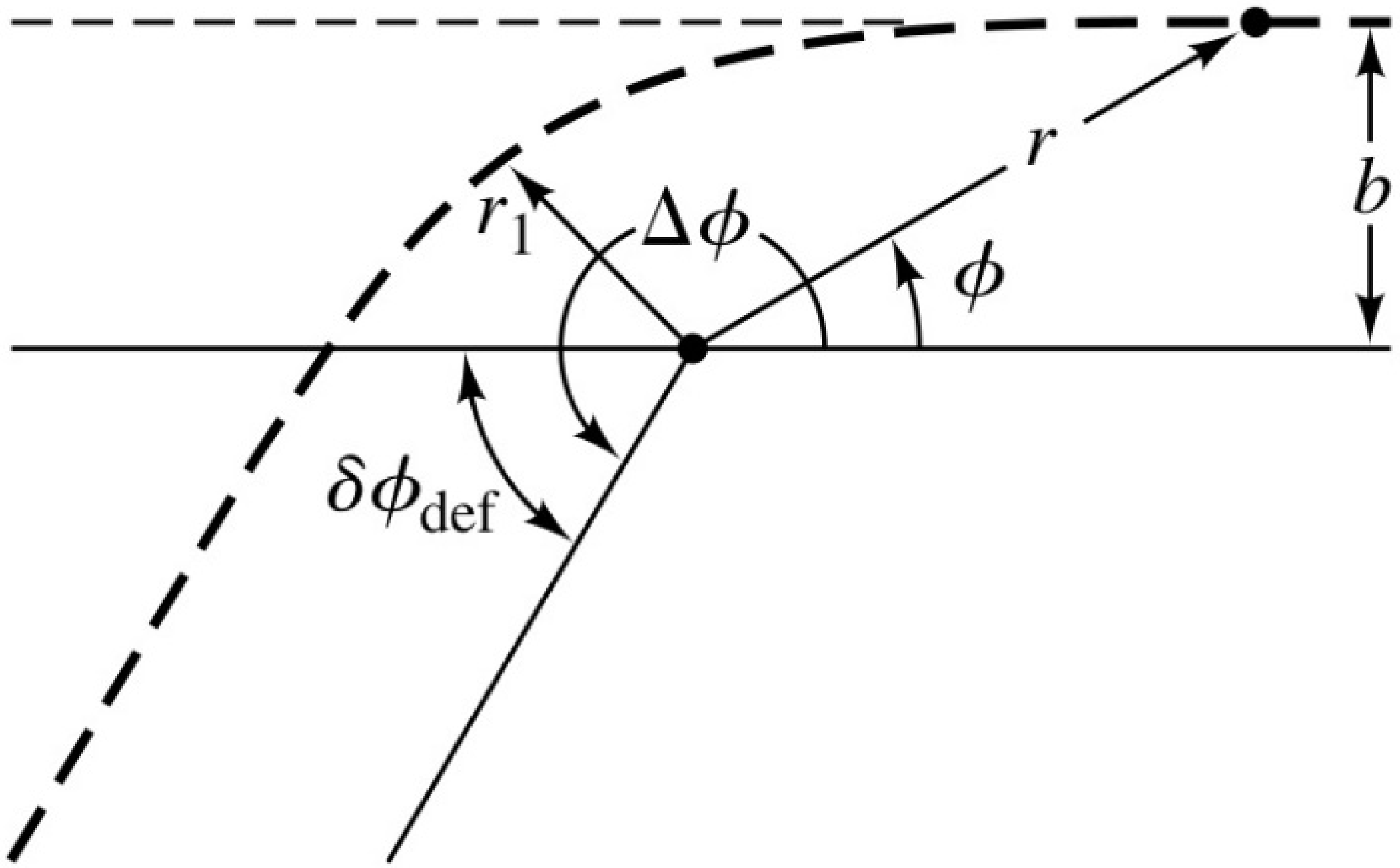
No light escapes to ∞,
no communication

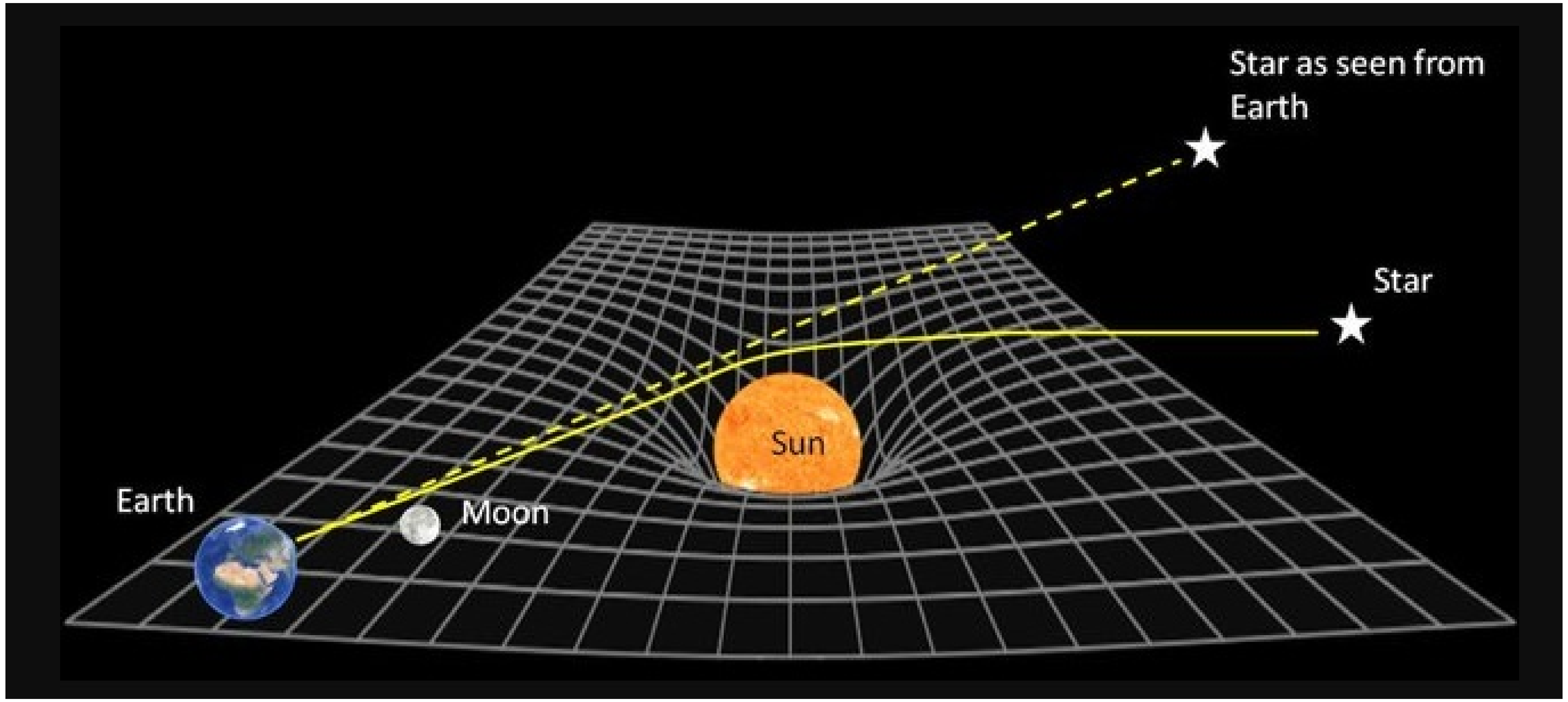
Light scattering:
deflection of light

Absorbed by the BH

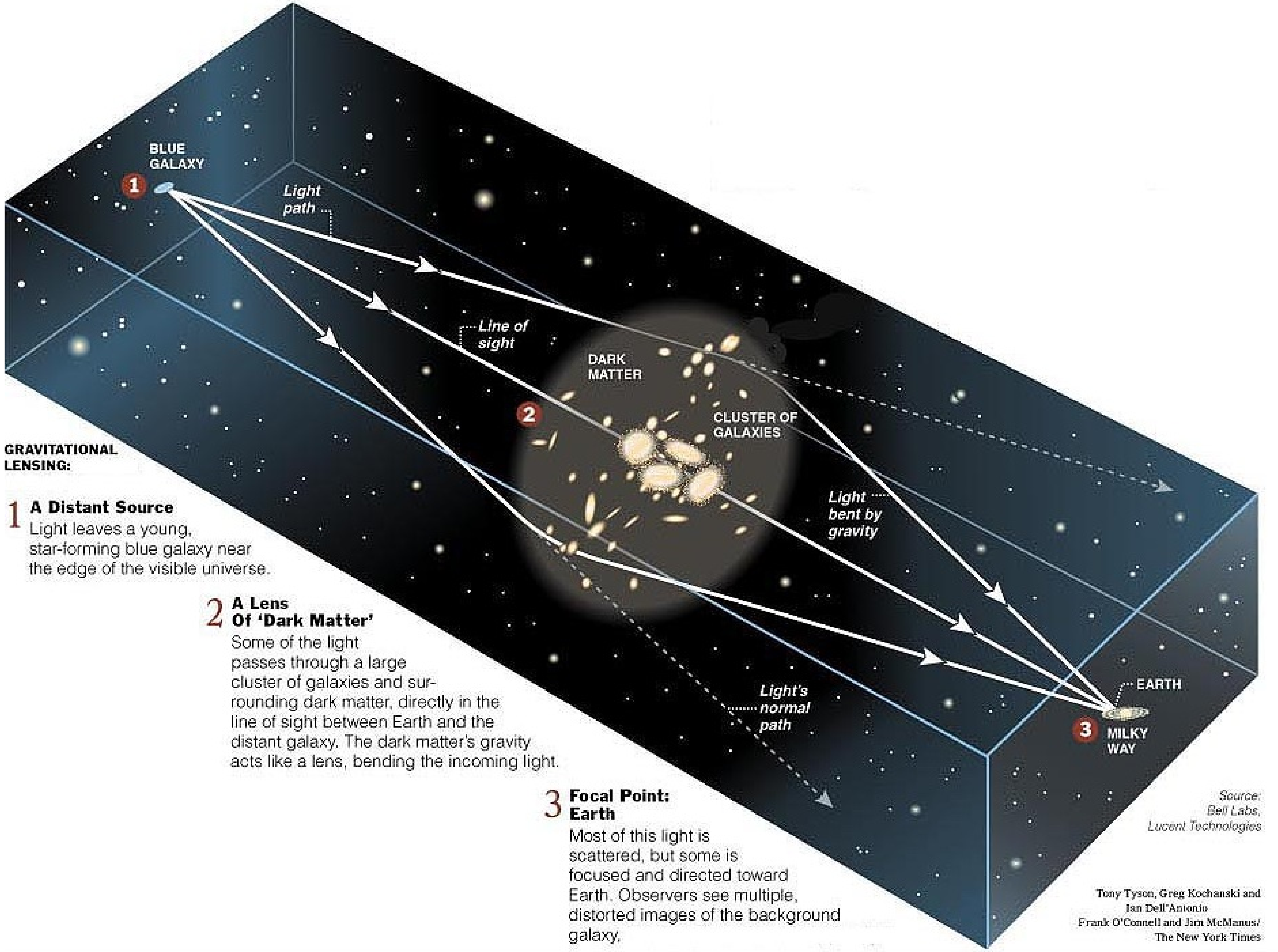


Deflection of light





Deflection of light



GRAVITATIONAL LENSING:

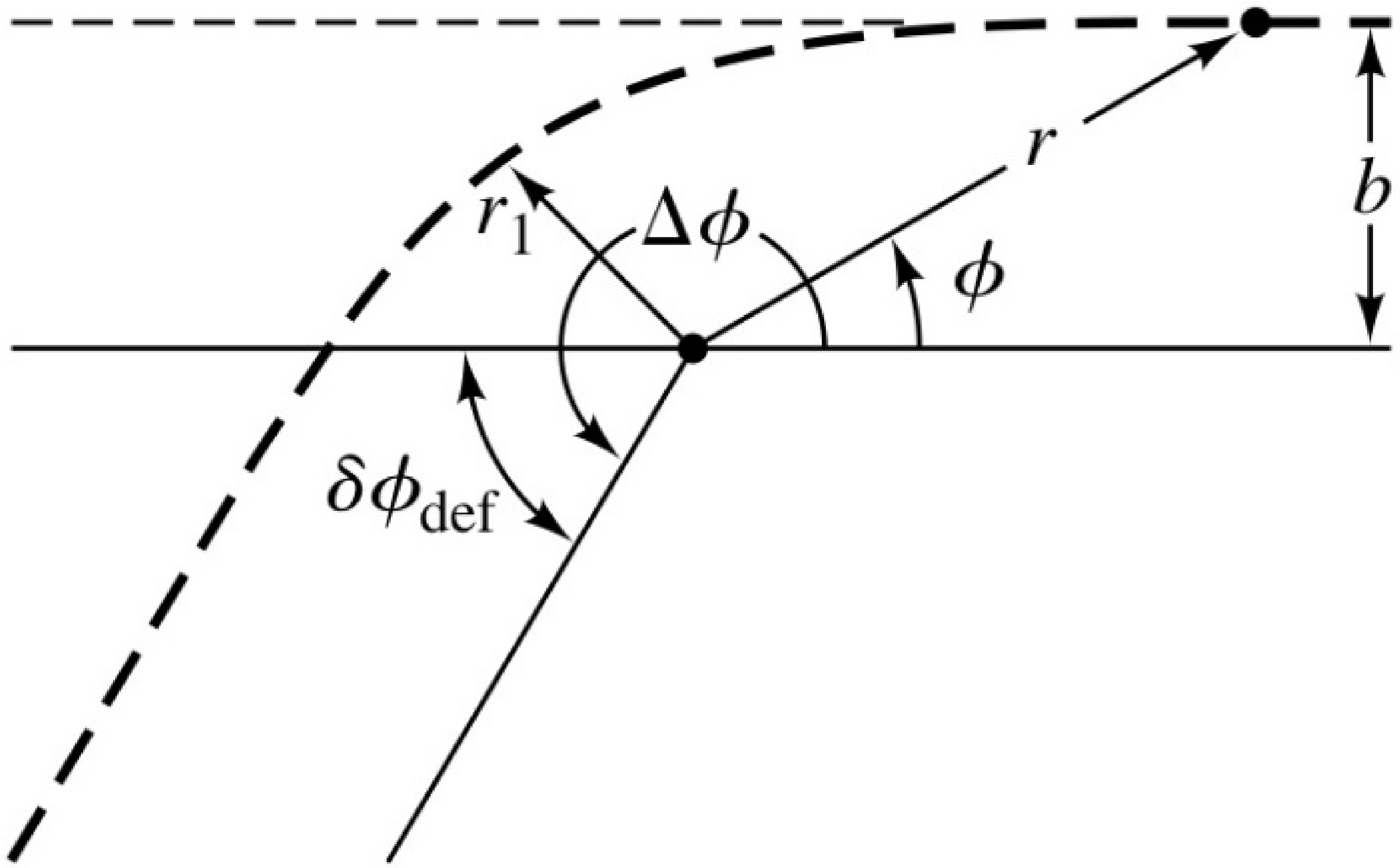
1 **A Distant Source**
Light leaves a young, star-forming blue galaxy near the edge of the visible universe.

2 **A Lens Of 'Dark Matter'**
Some of the light passes through a large cluster of galaxies and surrounding dark matter, directly in the line of sight between Earth and the distant galaxy. The dark matter's gravity acts like a lens, bending the incoming light.

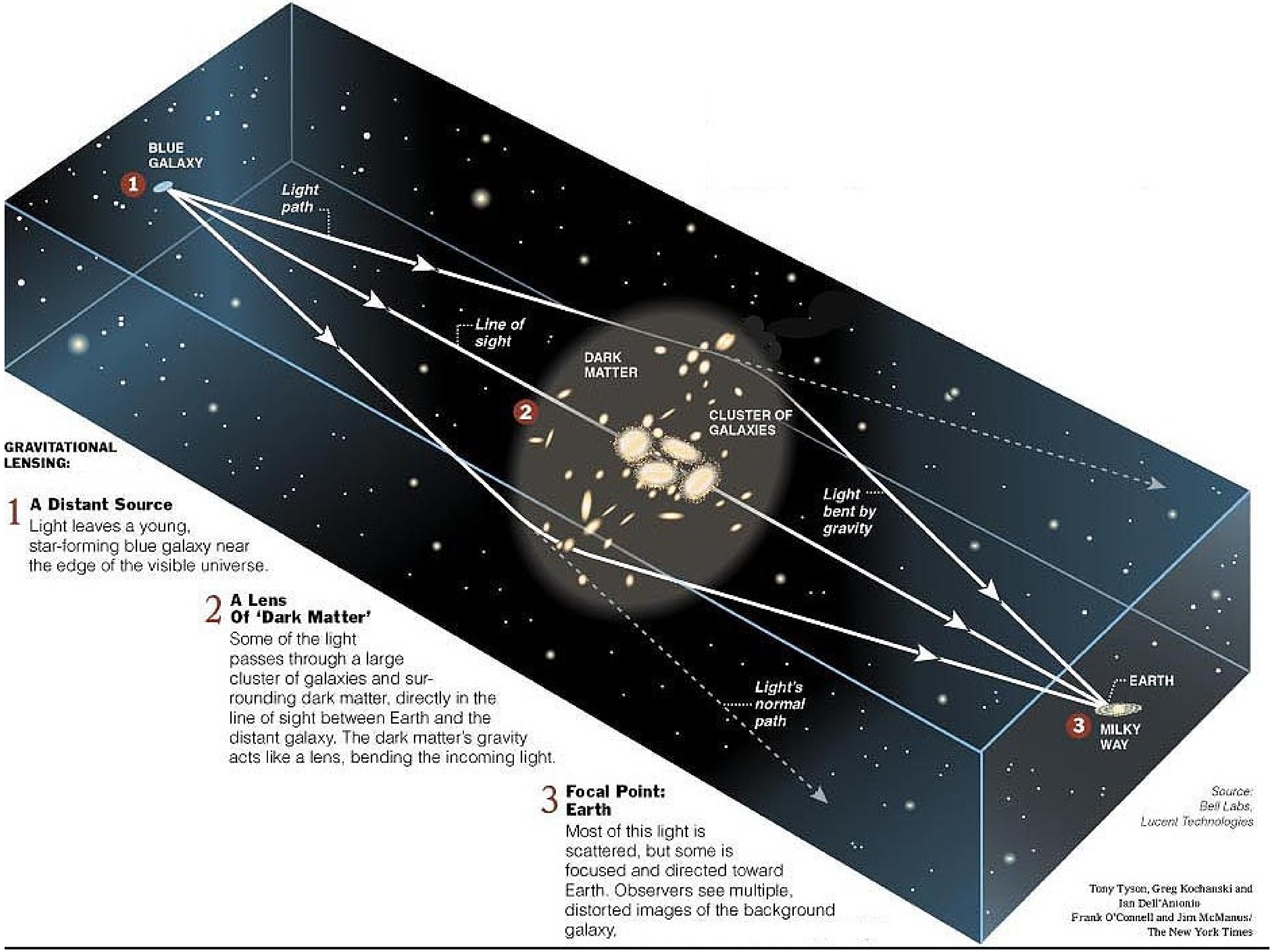
3 **Focal Point: Earth**
Most of this light is scattered, but some is focused and directed toward Earth. Observers see multiple, distorted images of the background galaxy.

Source: Bell Labs, Lucent Technologies

Tony Tyson, Greg Kochanski and Ian Dell'Antonio
Frank O'Connell and Jim McManus/The New York Times



Deflection of light



GRAVITATIONAL LENSING:

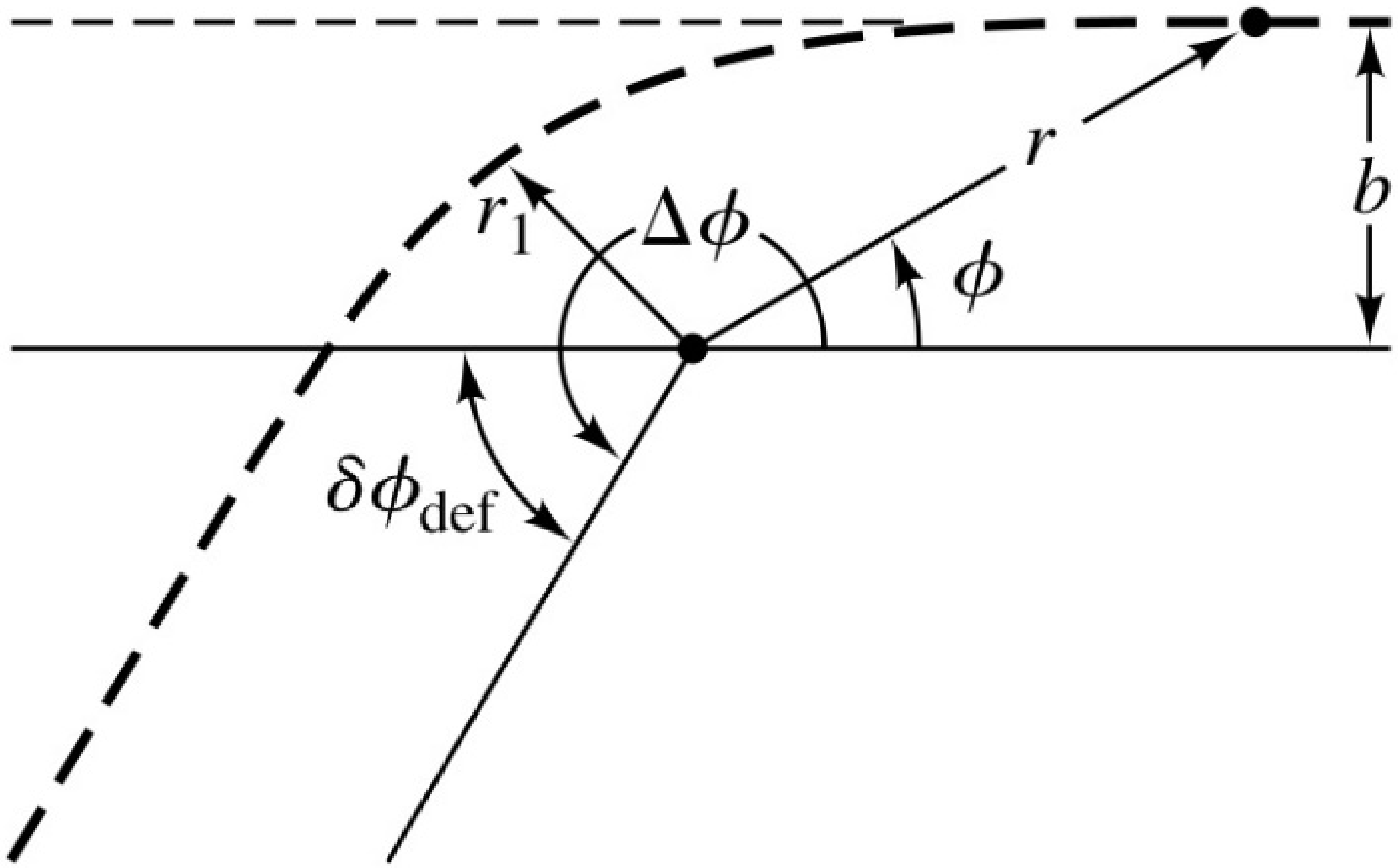
1 **A Distant Source**
Light leaves a young, star-forming blue galaxy near the edge of the visible universe.

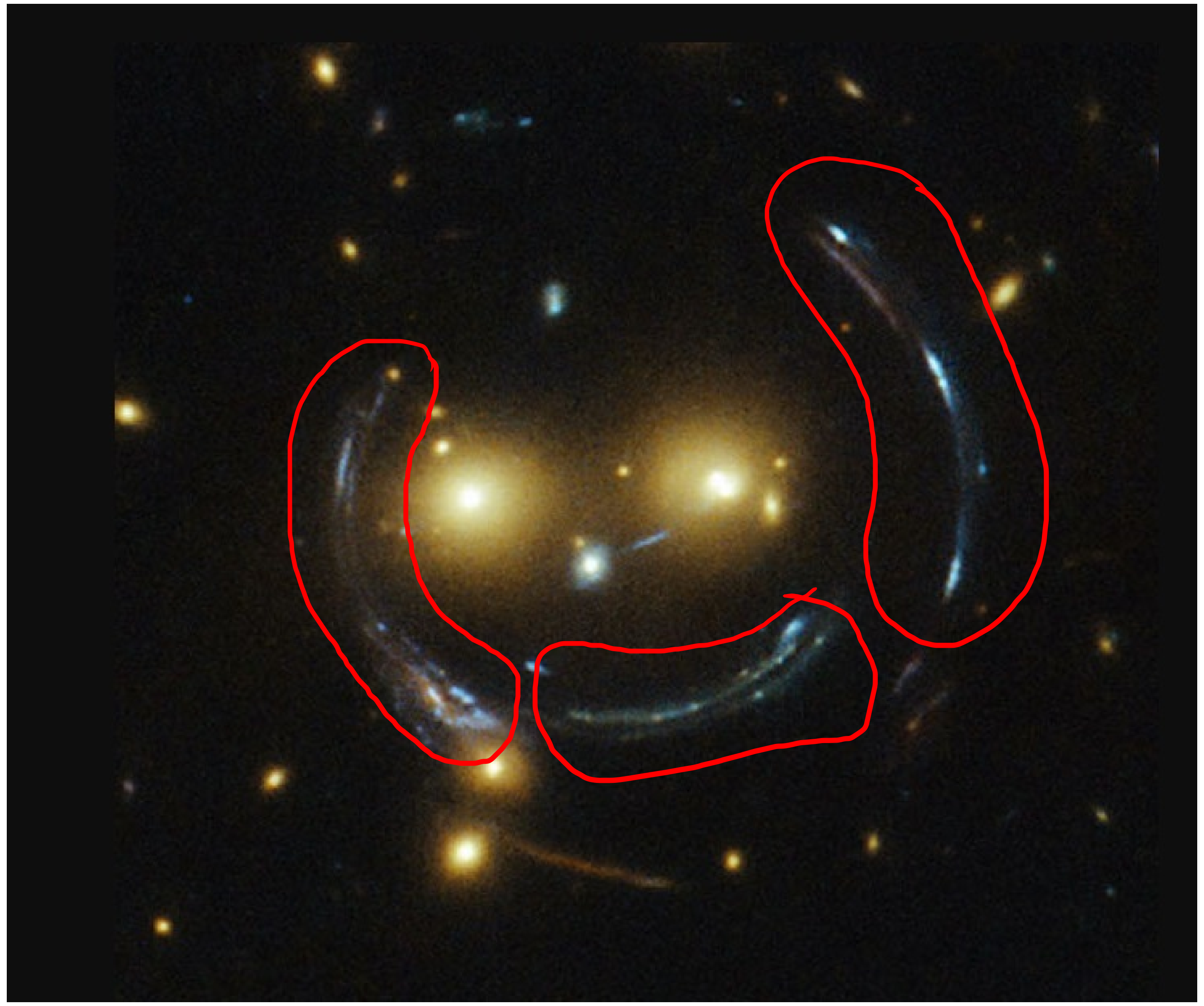
2 **A Lens Of 'Dark Matter'**
Some of the light passes through a large cluster of galaxies and surrounding dark matter, directly in the line of sight between Earth and the distant galaxy. The dark matter's gravity acts like a lens, bending the incoming light.

3 **Focal Point: Earth**
Most of this light is scattered, but some is focused and directed toward Earth. Observers see multiple, distorted images of the background galaxy.

Source: Bell Labs, Lucent Technologies

Tony Tyson, Greg Kochanski and Ian Dell'Antonio
Frank O'Connell and Jim McManus/ The New York Times



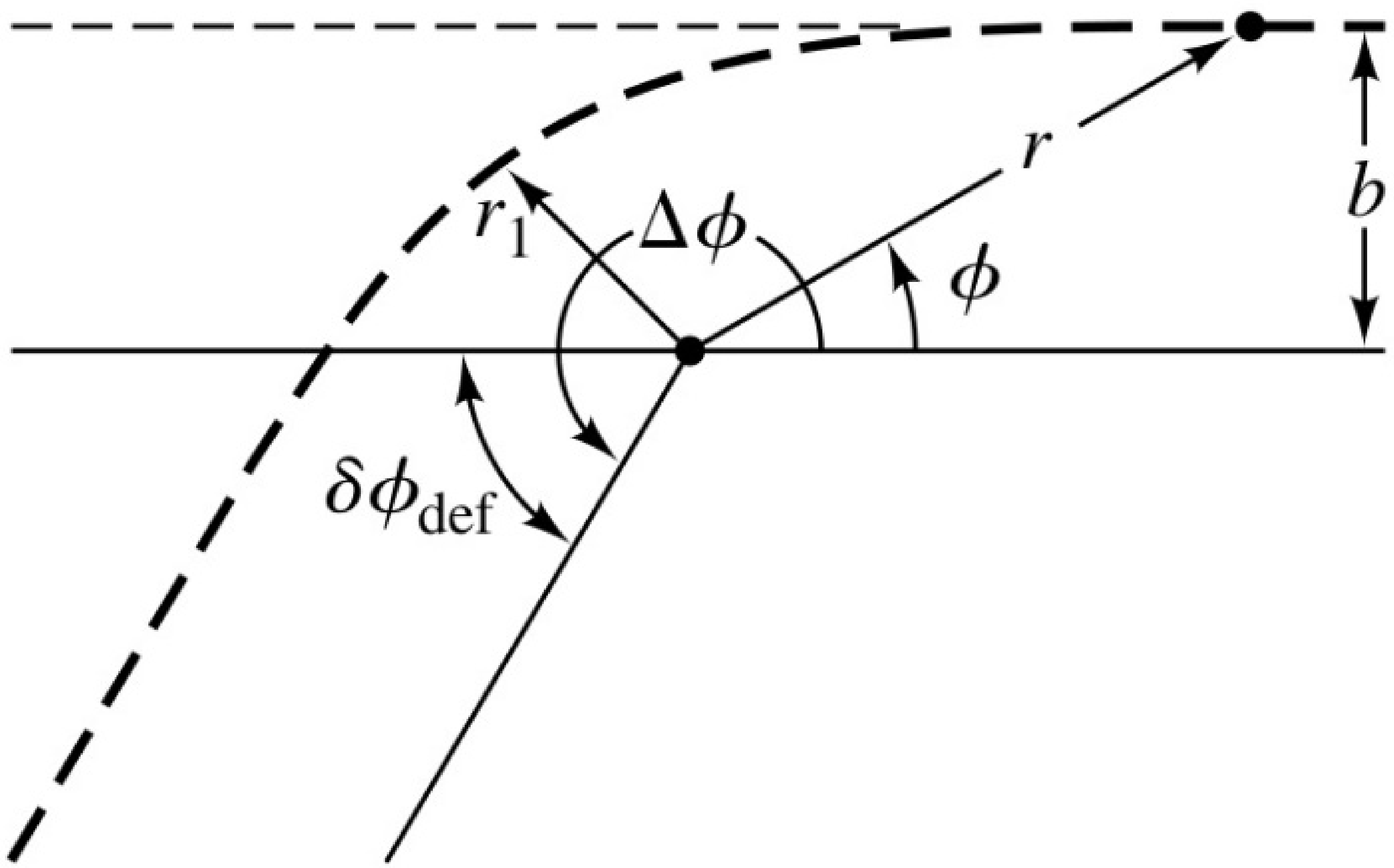


Deflection of light

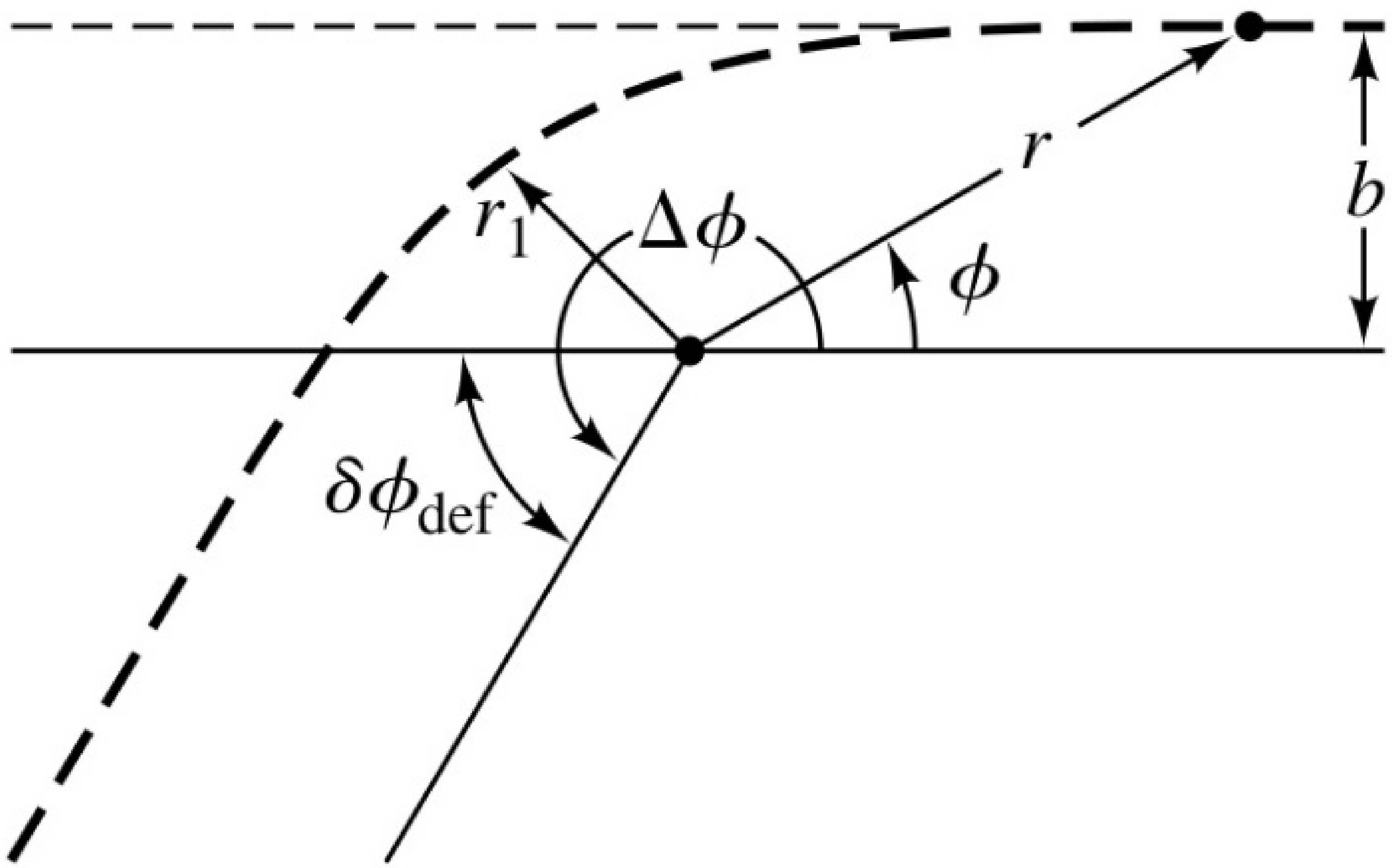
As before:

$$\ell = r^2 \frac{d\phi}{d\lambda} \Rightarrow \frac{d\phi}{d\lambda} = \frac{\ell}{r^2}$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right) + W_{eff}(r) \Rightarrow \frac{dr}{d\lambda} = \pm \ell\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{\frac{1}{2}}$$



Deflection of light



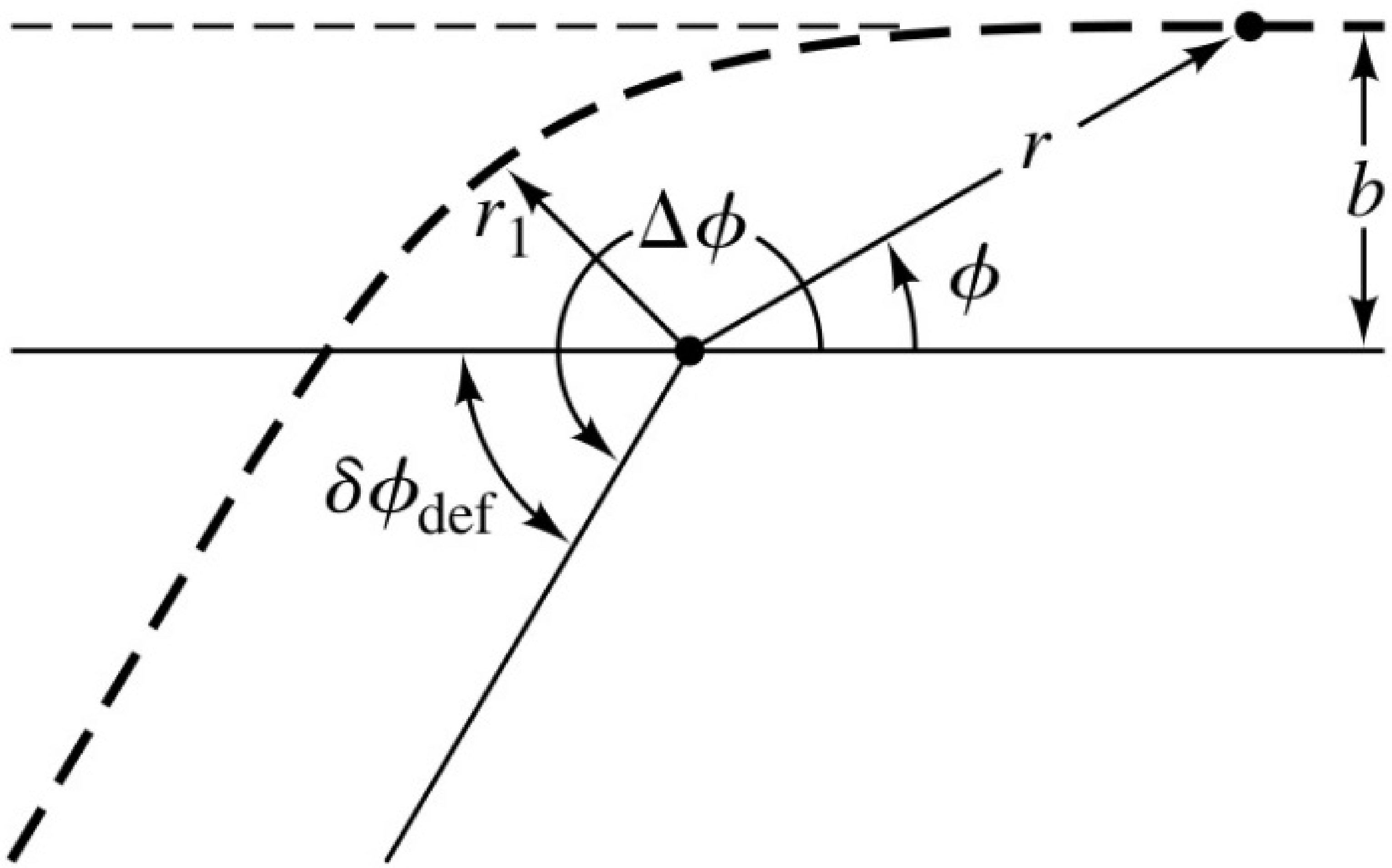
As before:

$$\ell = r^2 \frac{d\phi}{d\lambda} \Rightarrow \frac{d\phi}{d\lambda} = \frac{\ell}{r^2}$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right) + W_{eff}(r) \Rightarrow \frac{dr}{d\lambda} = \pm \ell \left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{\frac{1}{2}}$$

$$\Rightarrow \frac{d\phi}{dr} = \frac{d\phi/d\lambda}{dr/d\lambda} = \pm \frac{1}{r}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-\frac{1}{2}}$$

Deflection of light



As before:

$$l = r^2 \frac{d\phi}{d\lambda} \Rightarrow \frac{d\phi}{d\lambda} = \frac{l}{r^2}$$

$$\frac{1}{b^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right) + W_{\text{eff}}(r) \Rightarrow \frac{dr}{d\lambda} = \pm l\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{\frac{1}{2}}$$

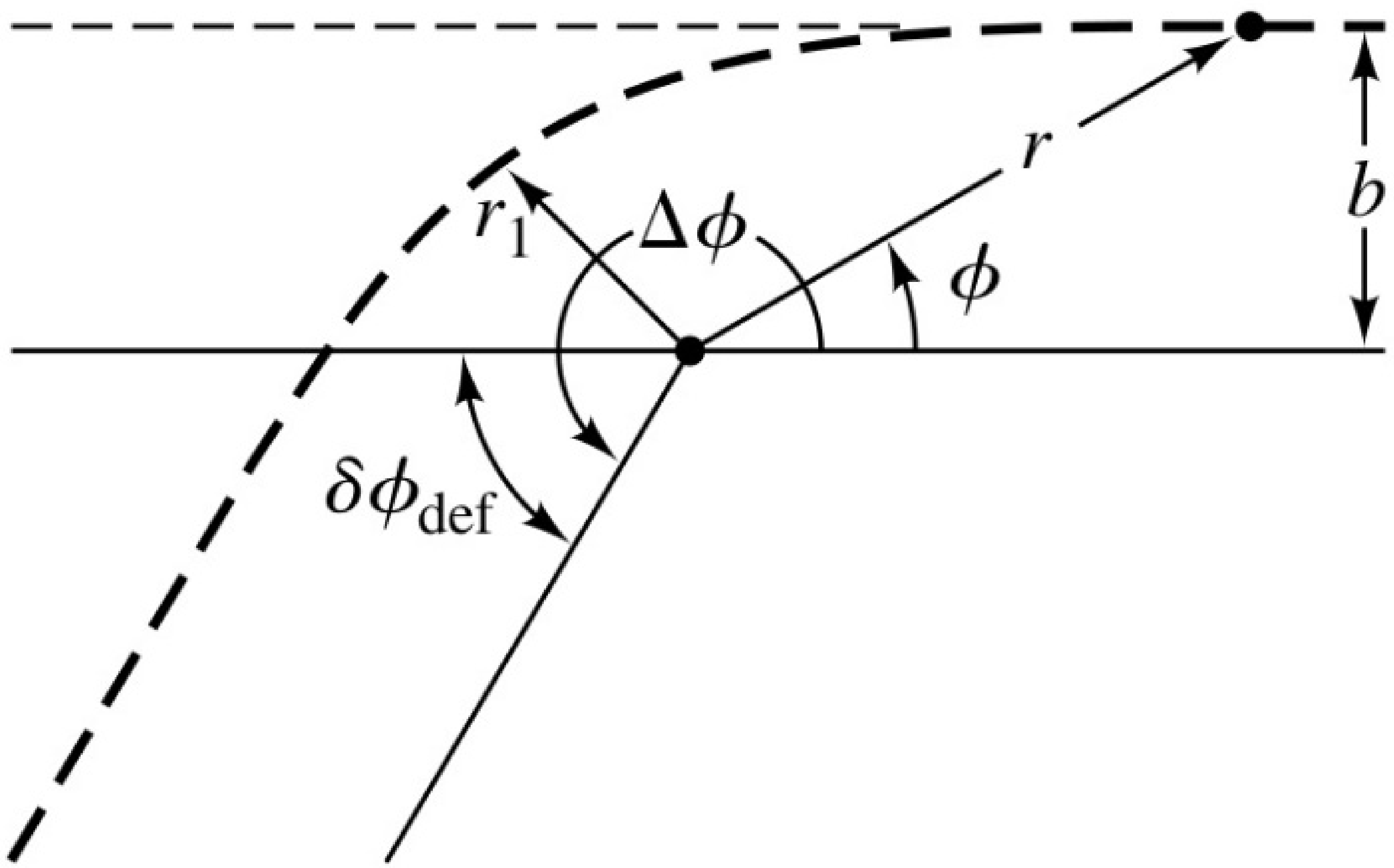
$$\Rightarrow \frac{d\phi}{dr} = \frac{d\phi/d\lambda}{dr/d\lambda} = \pm \frac{1}{r}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-\frac{1}{2}} \Rightarrow$$

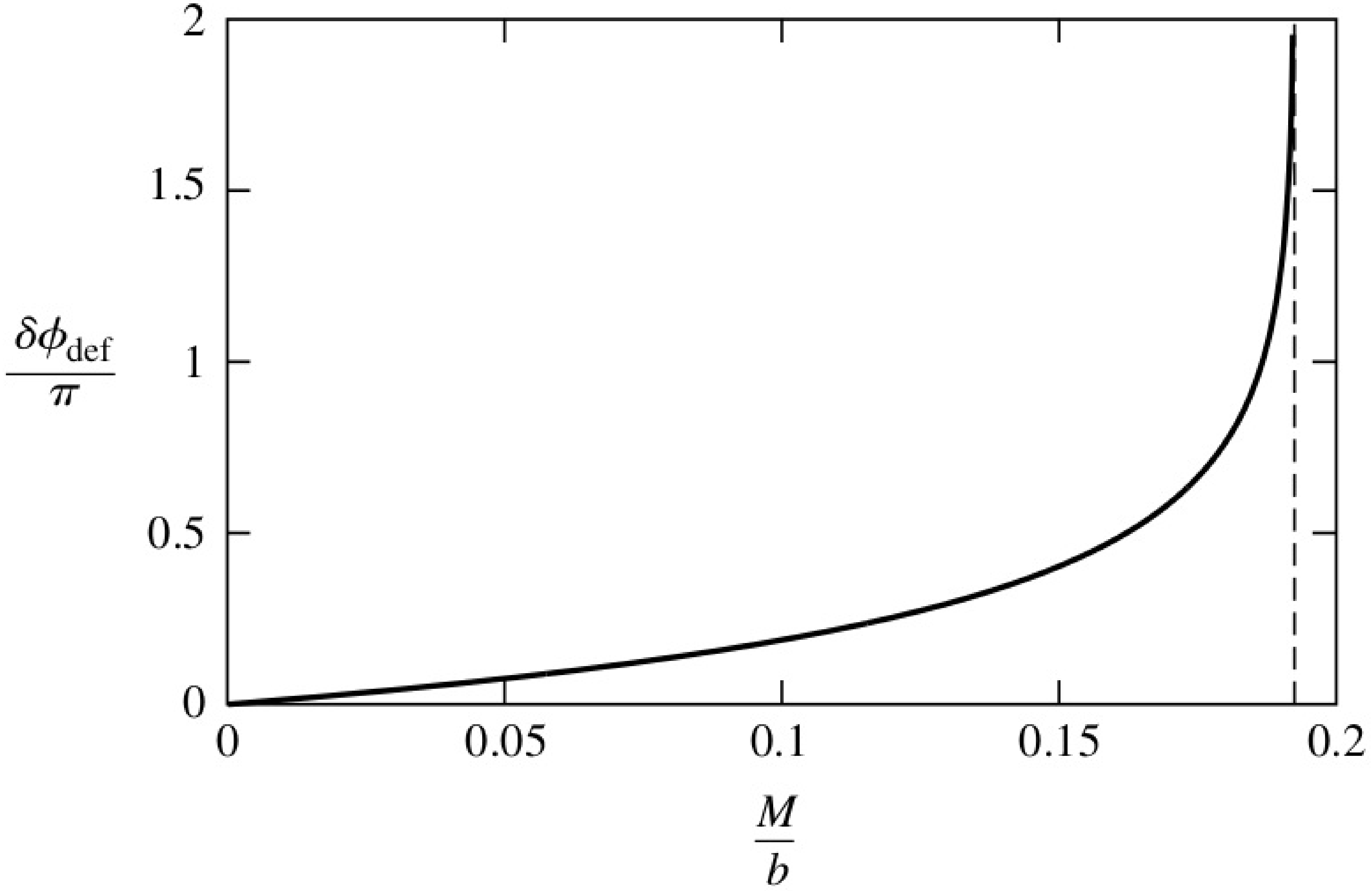
$$\Delta\phi = 2\int_{r_1}^{\infty} \frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-1/2} \quad , \quad \frac{1}{b^2} = W_{\text{eff}}(\overset{\text{turning point}}{r_1})$$

Deflection of light

$$\Delta\phi = 2\int_{r_1}^{\infty} \frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-\frac{1}{2}}$$

$$\delta\phi_{def} = \Delta\phi - \pi$$





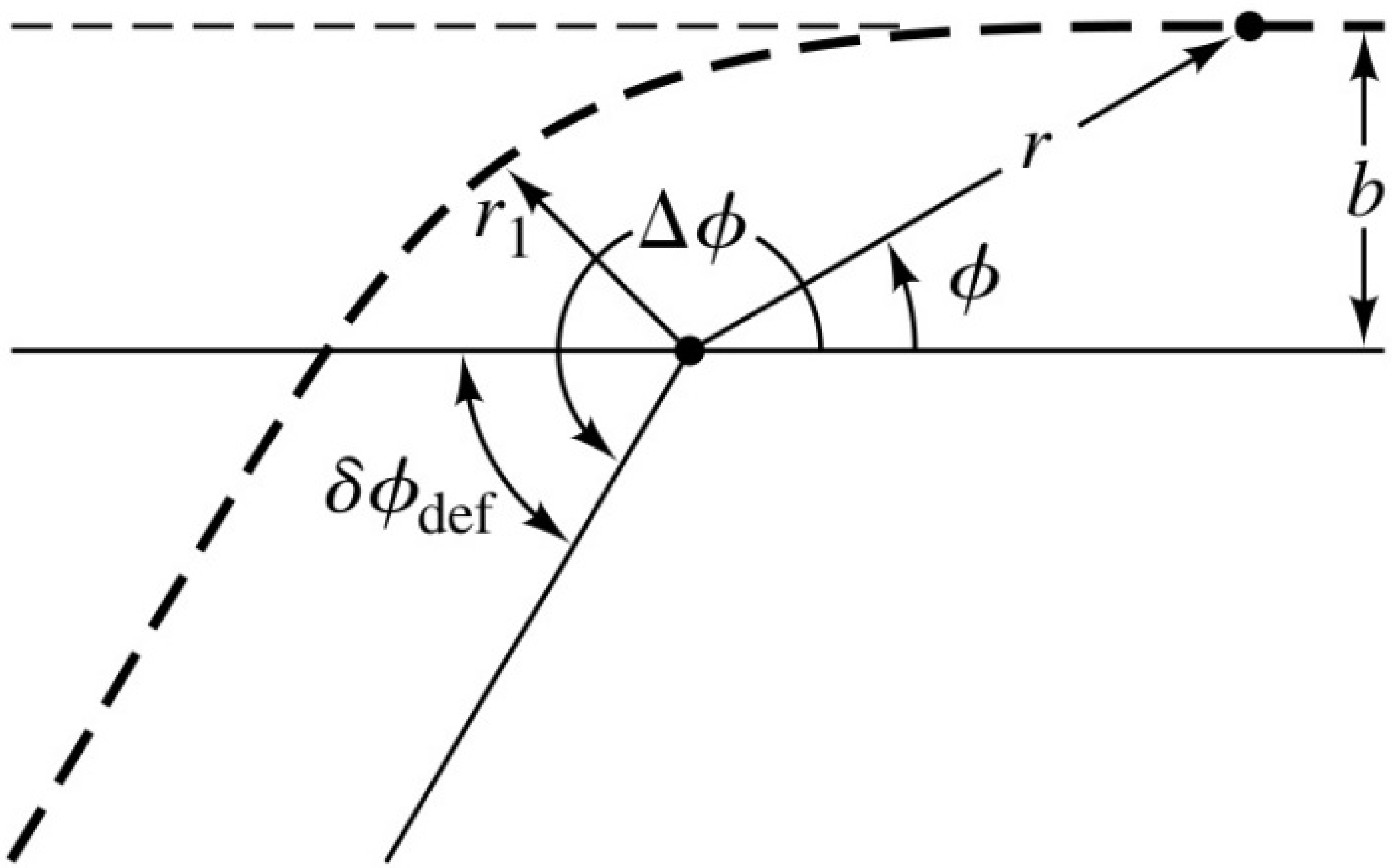
A function of $\frac{M}{b}$

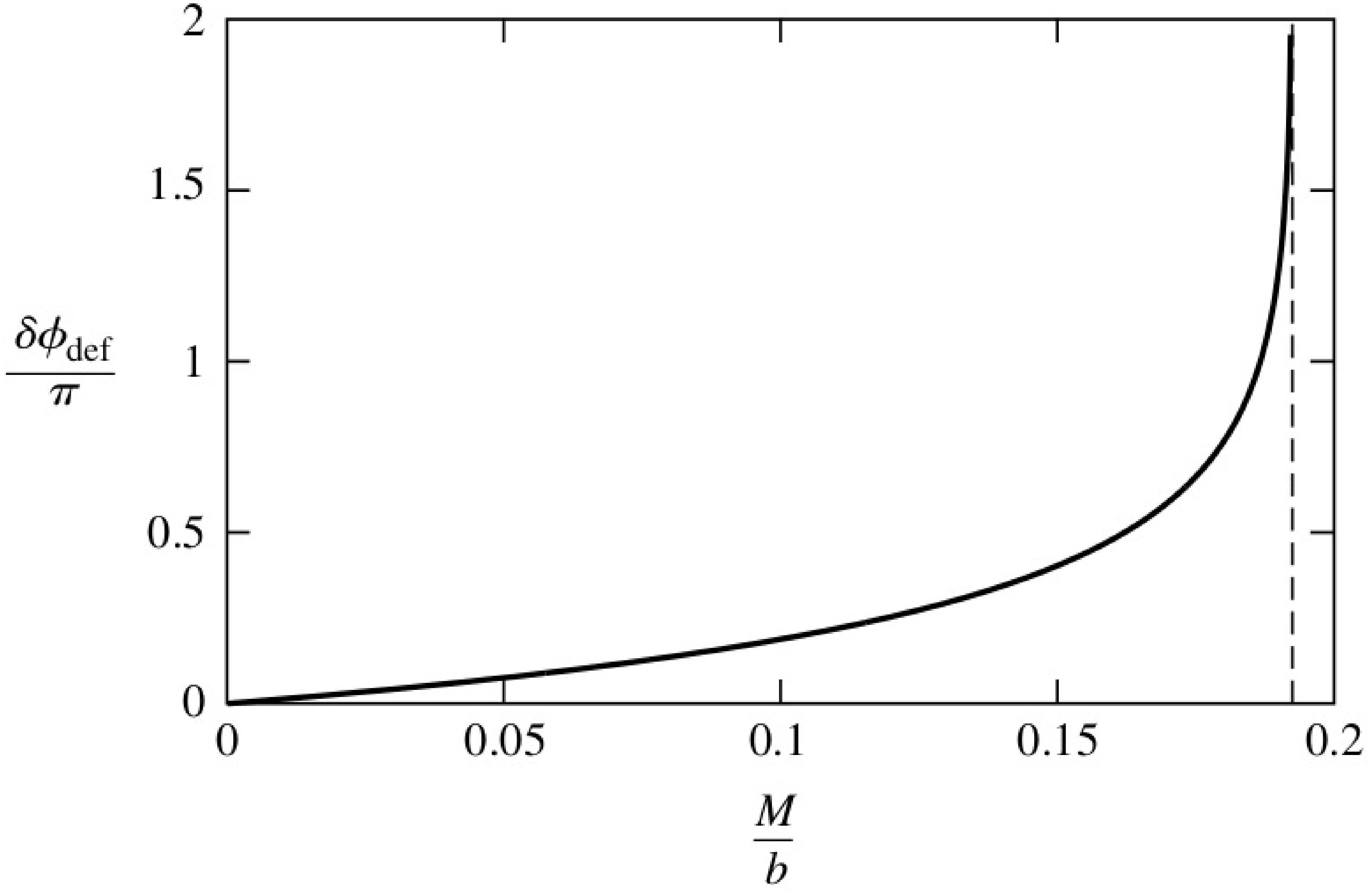
$$b \gg M \Rightarrow \frac{\delta\phi}{\pi} \ll 1$$

Deflection of light

$$\Delta\phi = 2\int_{r_1}^{\infty} \frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-\frac{1}{2}}$$

$$\delta\phi_{def} = \Delta\phi - \pi$$





A function of $\frac{M}{b}$

$$b \gg M \Rightarrow \frac{\delta\phi}{\pi} \ll 1$$

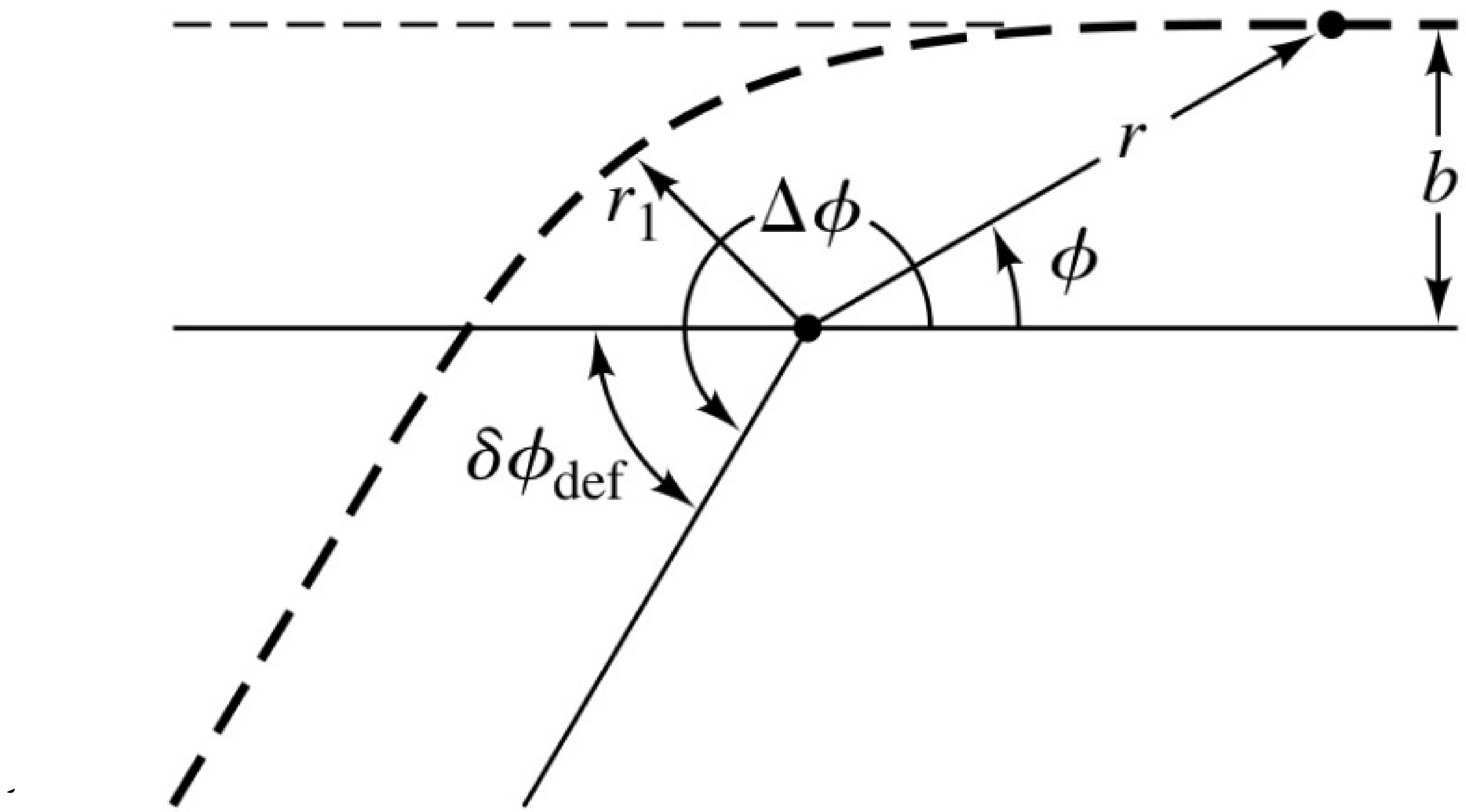
For the sun: $M_\odot = 1.5$ km

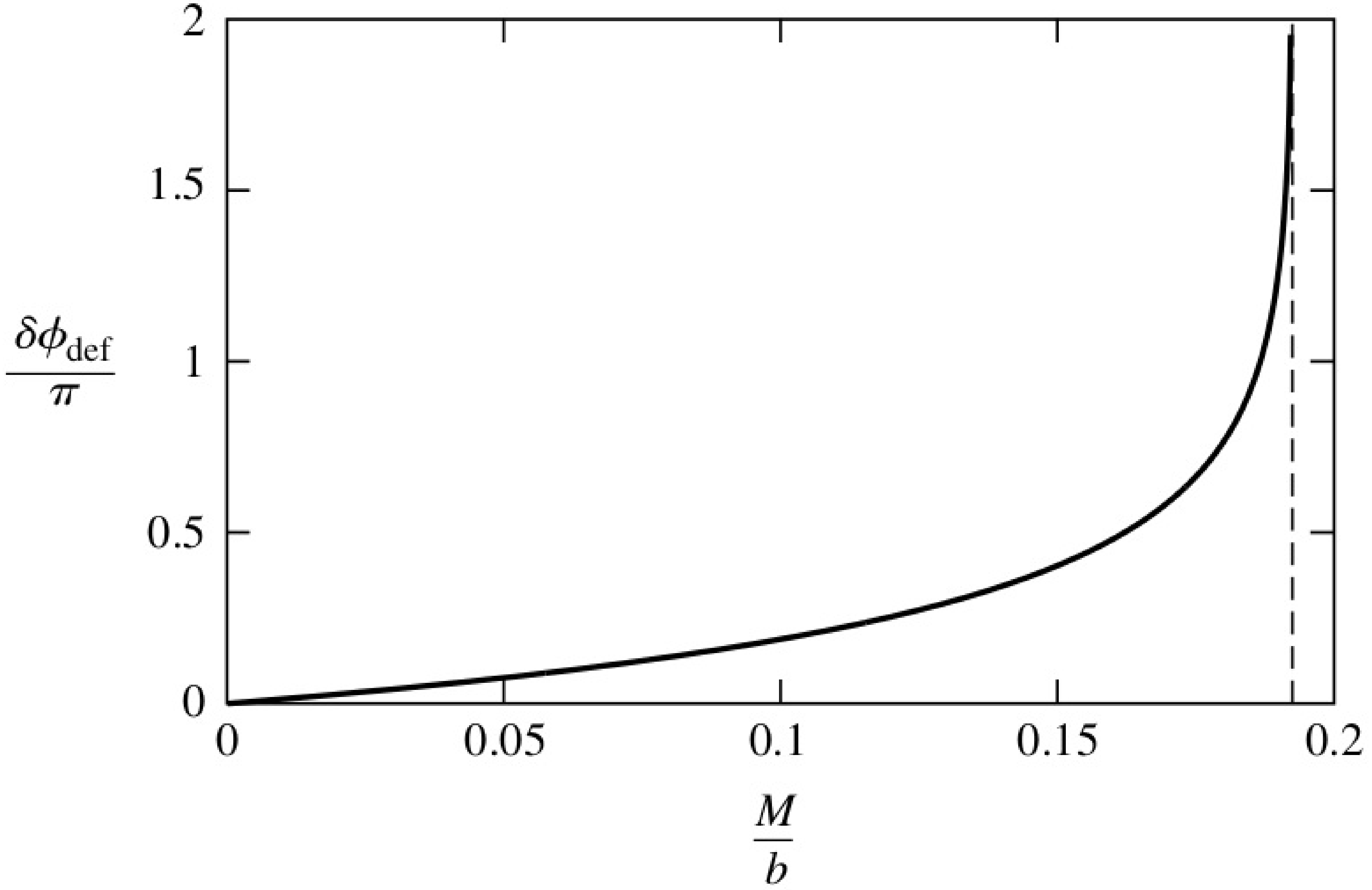
$$b_{min} = R_\odot = 7\times10^5 \text{ km}$$

Deflection of light

$$\Delta\phi = 2\int_{r_1}^{\infty} \frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-\frac{1}{2}}$$

$$\delta\phi_{def} = \Delta\phi - \pi$$





A function of $\frac{M}{b} \sim \frac{1.5}{7\times10^5} \sim 10^{-6}$

$b \gg M \Rightarrow \frac{\delta\phi}{\pi} \ll 1$

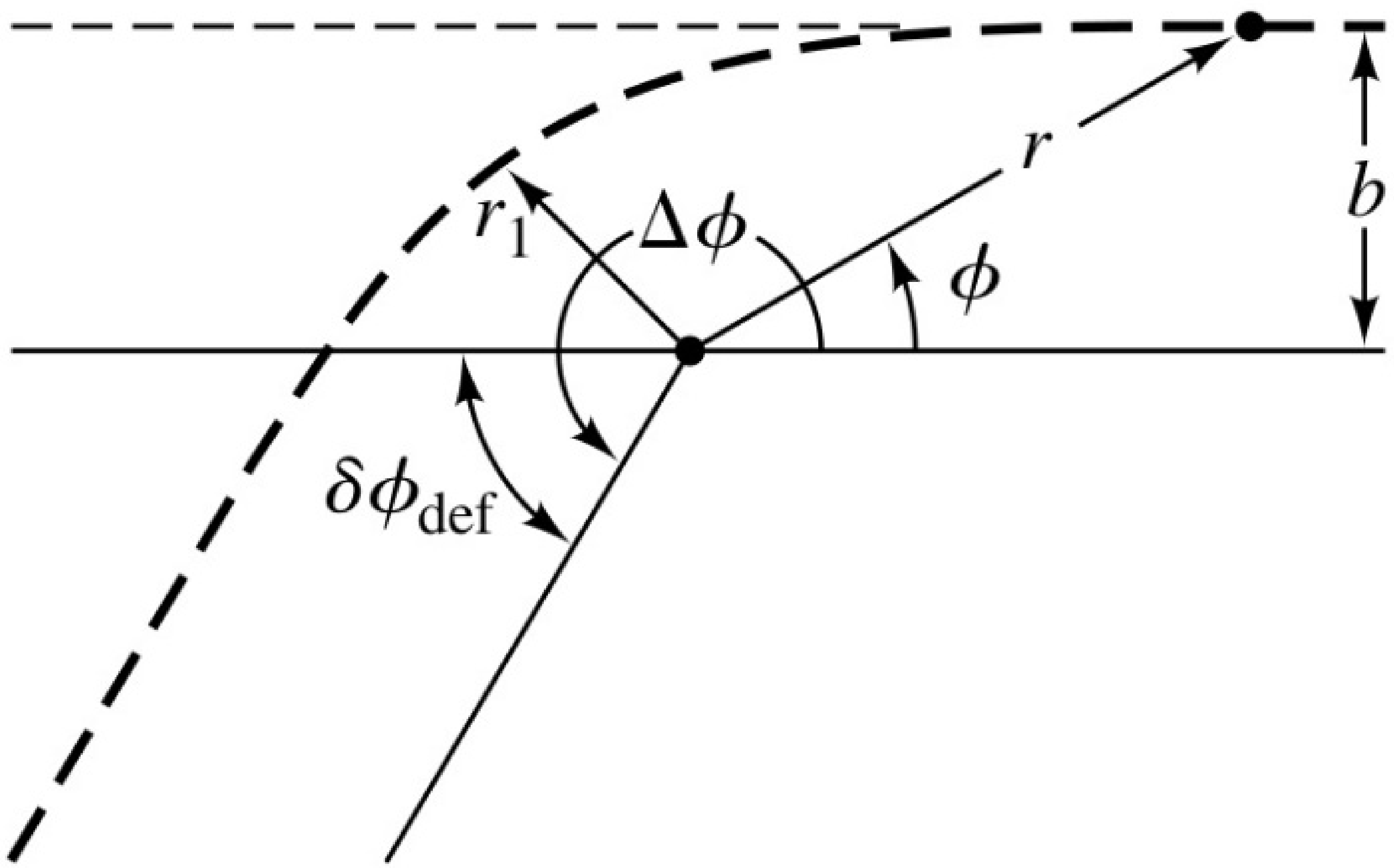
For the sun: $M_\odot = 1.5$ km

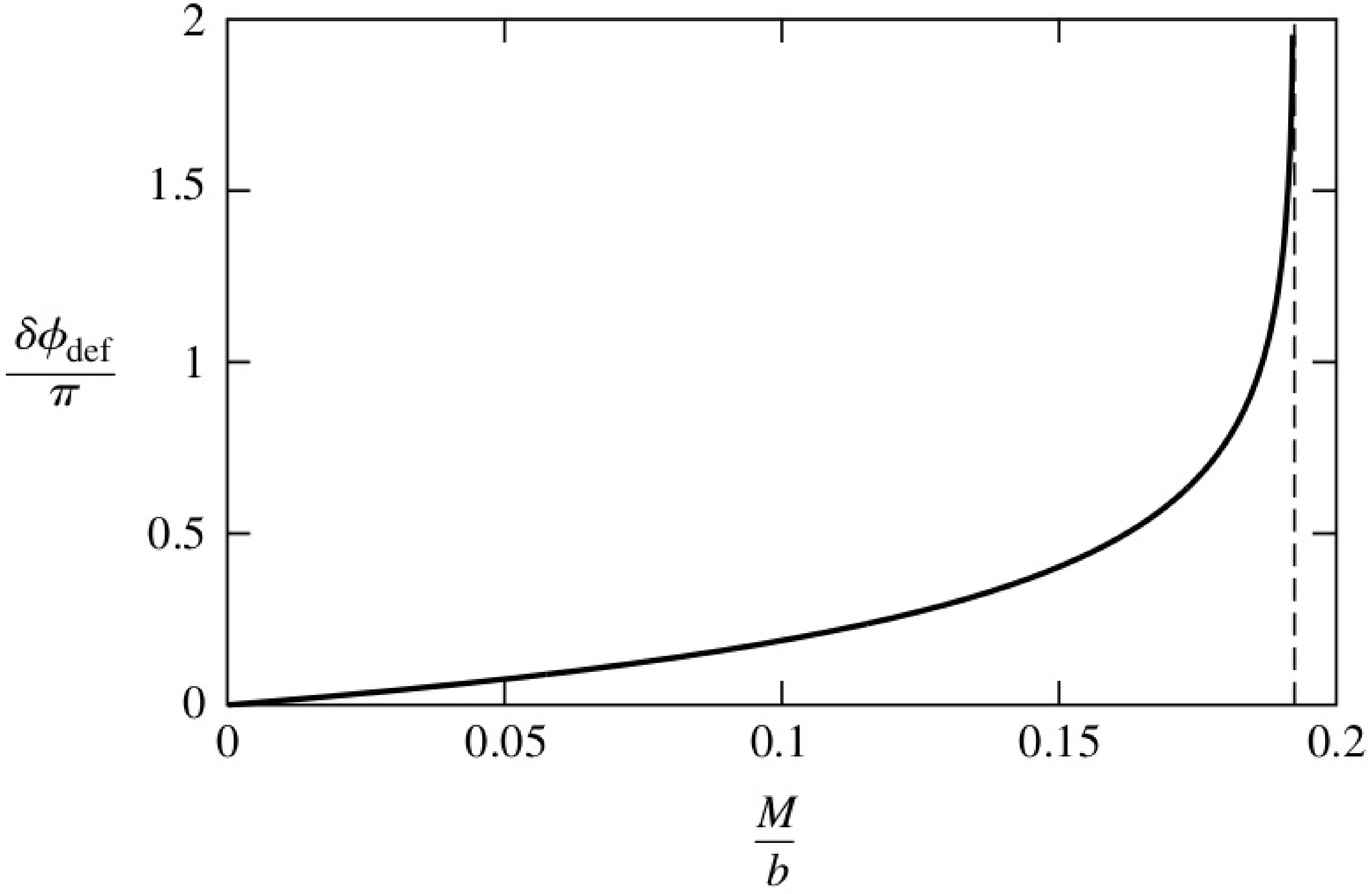
$b_{min} = R_\odot = 7\times10^5$ km

Deflection of light

$$\Delta\phi = 2\int_{r_1}^{\infty} \frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-\frac{1}{2}}$$

$$\delta\phi_{def} = \Delta\phi - \pi$$





A function of $\frac{M}{b} \sim \frac{1.5}{7\times10^5} \sim 10^{-6}$

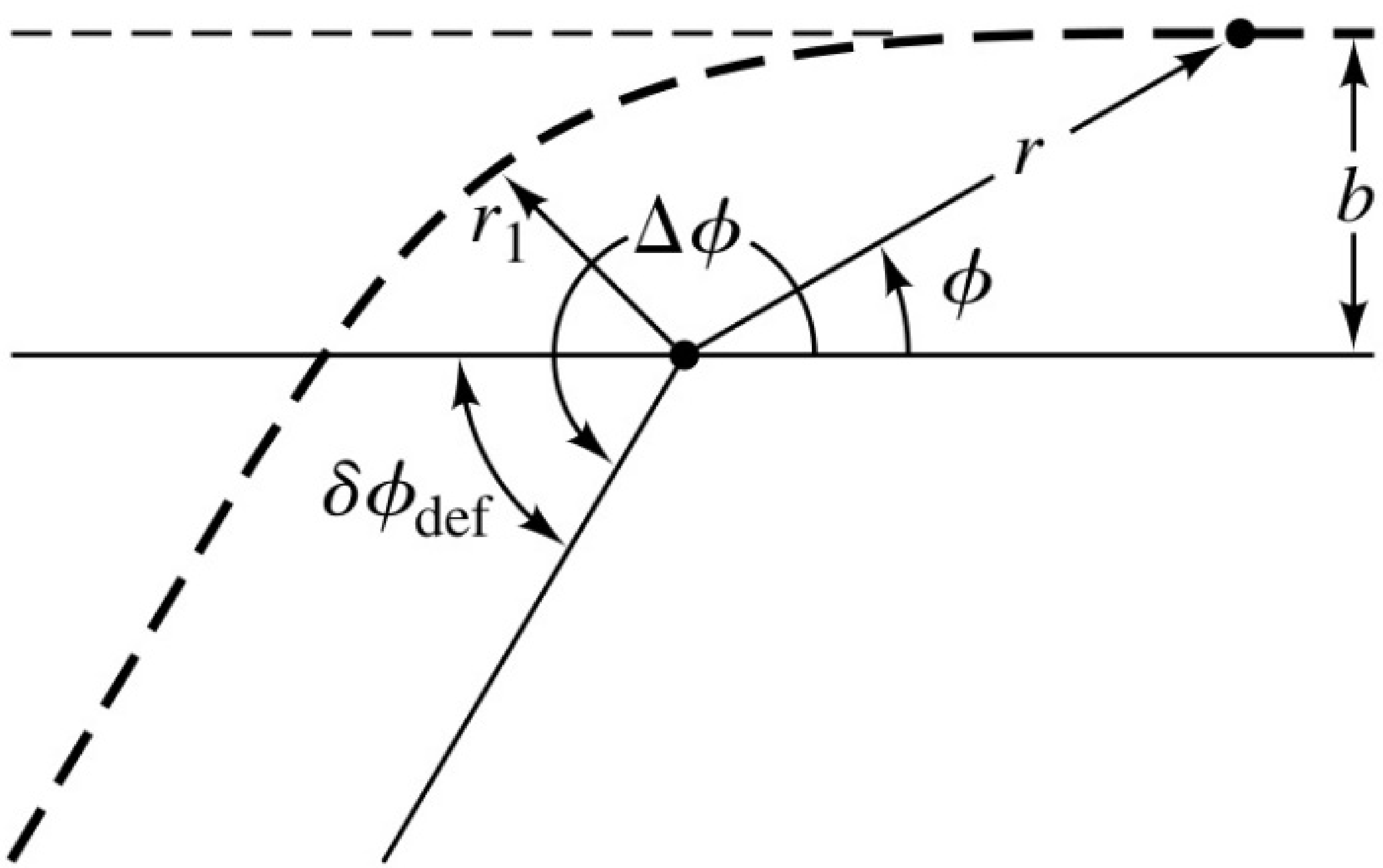
$b \gg M \Rightarrow \frac{\delta\phi}{\pi} \ll 1$

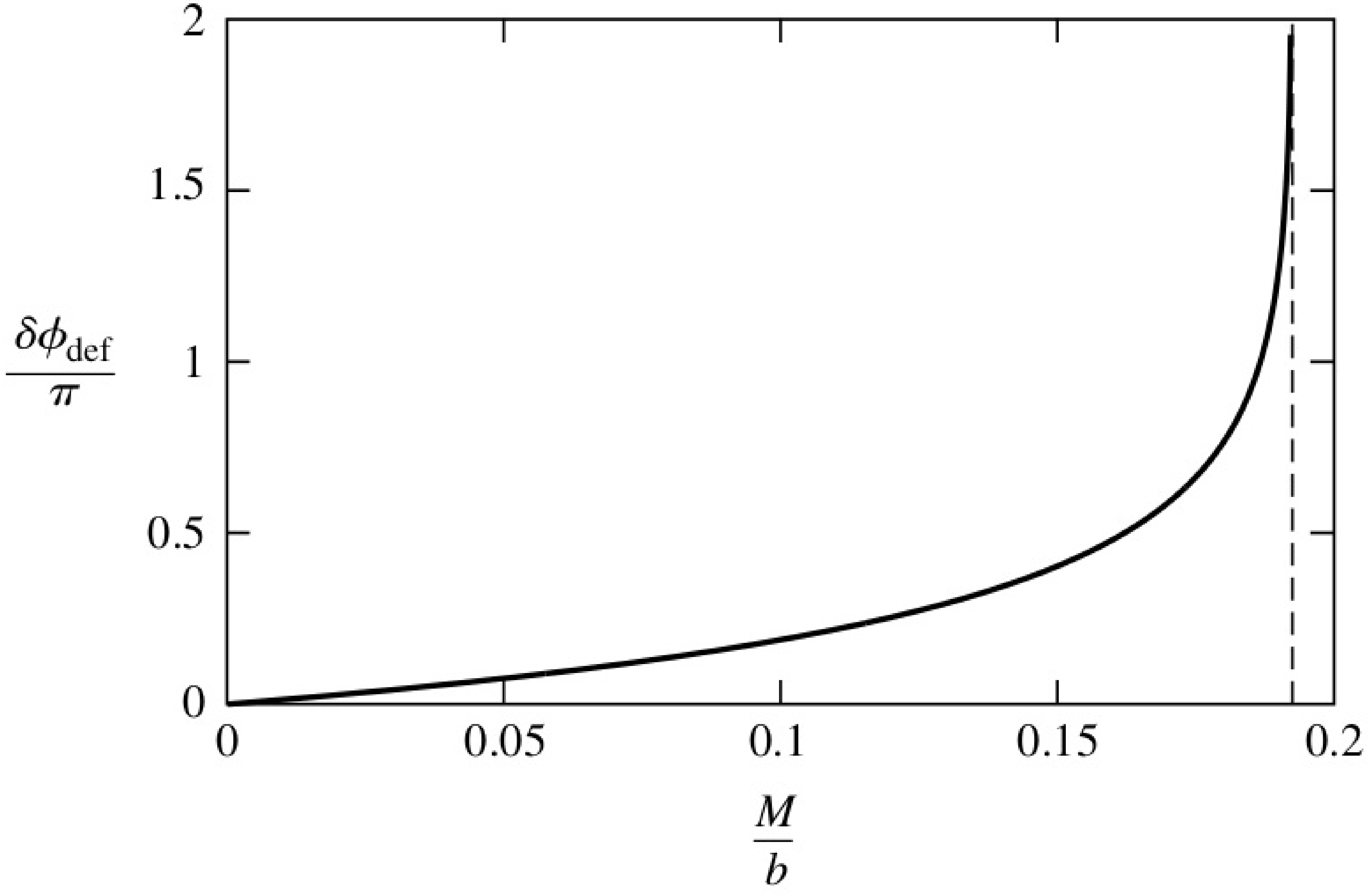
$\Rightarrow \delta\phi \approx \frac{4M}{b}$ $(\approx 1.7''$ for the sun$)$

Deflection of light

$$\Delta\phi = 2\int_{r_1}^{\infty} \frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-\frac{1}{2}}$$

$$\delta\phi_{def} = \Delta\phi - \pi$$





A function of $\frac{M}{b}$

As b decreases, $\delta\phi$ increases,

and

$b \to \sqrt{27}M$, $\delta\phi \to \infty$

(photon injected to circular orbit)

Calculate the orbits in the Schwarzschild Geometry $(r > r_s)$

- We have analyzed the geometric properties of the orbits of freely falling particles

- To calculate $x^\mu(\tau)$, we have to compute the geodesics:
  - solve the geodesic equations

Calculate the orbits in the Schwarzschild Geometry ($r>r_s$)

- We have analyzed the geometric properties of the orbits of freely falling particles
- To calculate $x^\mu(\tau)$, we have to compute the geodesics:
  - solve the geodesic equations
- A difficult task for a general metric, simplified here due to isometries ($\Rightarrow$ conserved quantities)

- We need to compute Christoffel symbols:
  - use $\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu})$, or
  - variational principle (problem for lecture 5)

- We need to compute Christoffel symbols:
  - use $\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu} \right)$, or
  - variational principle (problem for lecture 5)

$$\Gamma^{0}{}_{10} = -\frac{M}{2Mr-r^2} \qquad \Gamma^{1}{}_{00} = \frac{M(r-2M)}{r^3} \qquad \Gamma^{1}{}_{11} = \frac{M}{2Mr-r^2}$$

$$\Gamma^{1}{}_{22} = 2M-r \qquad \Gamma^{1}{}_{33} = (2M-r)\sin^2\theta \qquad \Gamma^{2}{}_{21} = \frac{1}{r}$$

$$\Gamma^{2}{}_{33} = -\sin\theta\cos\theta \qquad \Gamma^{3}{}_{31} = \frac{1}{r} \qquad \Gamma^{3}{}_{32} = \cot\theta$$

(the other are zero, or related by $\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu}$)

- We need to compute Christoffel symbols:
  - use $\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu} \right)$, or
  - variational principle (problem for lecture 5)

$$\Gamma^{0}{}_{10} = - \frac{M}{2Mr - r^{2}} \qquad \Gamma^{1}{}_{00} = \frac{M(r-2M)}{r^{3}} \qquad \Gamma^{1}{}_{11} = \frac{M}{2Mr - r^{2}}$$

$$\Gamma^{1}{}_{22} = 2M - r \qquad \Gamma^{1}{}_{33} = (2M-r)\sin^{2}\theta \qquad \Gamma^{2}{}_{21} = \frac{1}{r}$$

$$\Gamma^{2}{}_{33} = - \sin\theta \cos\theta \qquad \Gamma^{3}{}_{31} = \frac{1}{r} \qquad \Gamma^{3}{}_{32} = \cot\theta$$

- Geodesic equations: $\frac{d^{2}x^{\mu}}{d\lambda^{2}} + \Gamma^{\mu}{}_{\nu\rho} \frac{dx^{\nu}}{d\lambda} \frac{dx^{\rho}}{d\lambda} = 0$

- We can also compute the Riemann tensor (problem in Lecture 6) from $R^{\sigma}{}_{\lambda\mu\nu} = \partial_{\mu}\Gamma_{\nu\lambda}^{\sigma} - \partial_{\nu}\Gamma_{\mu\lambda}^{\sigma} + \Gamma_{\mu\rho}^{\sigma}\Gamma_{\nu\lambda}^{\rho} - \Gamma_{\nu\rho}^{\sigma}\Gamma_{\mu\lambda}^{\rho}$

- We can also compute the Riemann tensor (problem in Lecture 6) from $R^{\sigma}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\nu\lambda} - \partial_\nu \Gamma^{\sigma}_{\mu\lambda} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda} - \Gamma^{\sigma}_{\nu\rho}\Gamma^{\rho}_{\mu\lambda}$

$$R^{0}{}_{110} = \frac{2M}{r^2(2M-r)} \qquad R^{0}{}_{220} = \frac{M}{r} \qquad R^{0}{}_{330} = \frac{M}{r}\sin^2\theta$$

$$R^{1}{}_{010} = \frac{2M(2M-r)}{r^4} \qquad R^{1}{}_{221} = \frac{M}{r} \qquad R^{1}{}_{331} = \frac{M}{r}\sin^2\theta$$

$$R^{2}{}_{020} = \frac{M(r-2M)}{r^4} \qquad R^{2}{}_{121} = \frac{M}{r^2(2M-r)} \qquad R^{2}{}_{332} = -\frac{2M}{R}\sin^2\theta$$

$$R^{3}{}_{030} = \frac{M(r-2M)}{r^4} \qquad R^{3}{}_{131} = \frac{M}{r^2(2M-r)} \qquad R^{3}{}_{230} = \frac{2M}{r}$$

(the other are zero, or related by symmetries)

- We can also compute the Riemann tensor (problem in Lecture 6) from $R^{\sigma}{}_{\lambda\mu\nu} = \partial_\mu \Gamma_{\nu\lambda}^{\sigma} - \partial_\nu \Gamma_{\mu\lambda}^{\sigma} + \Gamma_{\mu\rho}^{\sigma}\Gamma_{\nu\lambda}^{\rho} - \Gamma_{\nu\rho}^{\sigma}\Gamma_{\mu\lambda}^{\rho}$

$$R_{\sigma\lambda\mu\nu} = g_{\sigma\rho} R^{\rho}{}_{\lambda\mu\nu}$$

$$R_{1010} = -\frac{2M}{r^3} \qquad R_{2020} = \frac{M}{r^2}(r-2M) \qquad R_{2121} = \frac{M}{2M-r}$$

$$R_{3030} = \frac{M}{r^2}(r-2M)\sin^2\theta \qquad R_{3131} = \frac{M}{2M-r}\sin^2\theta \qquad R_{3232} = 2Mr\sin^2\theta$$

More symmetries here: $R_{\mu\nu\rho\lambda} = R_{[\mu\nu][\rho\lambda]}$, $R_{\mu\nu\rho\lambda} = R_{\rho\lambda\mu\nu}$
(only non-zero components shown)

- We can also compute the Riemann tensor (problem in Lecture 6) from

$$R^{\sigma}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\nu\lambda} - \partial_\nu \Gamma^{\sigma}_{\mu\lambda} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda} - \Gamma^{\sigma}_{\nu\rho}\Gamma^{\rho}_{\mu\lambda}$$

$$R_{\sigma\lambda\mu\nu} = g_{\sigma\rho} R^{\rho}{}_{\lambda\mu\nu}$$

$$R_{\mu\nu} = 0$$

(yeah! solution to the vacuum Einstein equations)

- We can also compute the Riemann tensor (problem in Lecture 6) from $R^{\sigma}{}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\sigma}_{\nu\lambda} - \partial_{\nu}\Gamma^{\sigma}_{\mu\lambda} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda} - \Gamma^{\sigma}_{\nu\rho}\Gamma^{\rho}_{\mu\lambda}$

$$R_{\sigma\lambda\mu\nu} = g_{\sigma\rho} R^{\rho}{}_{\lambda\mu\nu}$$

$$R_{\mu\nu} = 0 \qquad \Rightarrow \qquad R = g^{\mu\nu} R_{\mu\nu} = 0$$

- We can also compute the Riemann tensor (problem in Lecture 6) from $R^{\sigma}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\nu\lambda} - \partial_\nu \Gamma^{\sigma}_{\mu\lambda} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda} - \Gamma^{\sigma}_{\nu\rho}\Gamma^{\rho}_{\mu\lambda}$

$$R_{\sigma\lambda\mu\nu} = g_{\sigma\rho} R^{\rho}{}_{\lambda\mu\nu}$$

$$R_{\mu\nu} = 0 \quad \Rightarrow \quad R = g^{\mu\nu} R_{\mu\nu} = 0$$

But: $K = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} = \dfrac{48 M^2}{r^6}$ (Kretschmann scalar)

$\hookrightarrow$
- blows up as $r \to 0$
- regular at $r = 2M$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$$

$$\ddot{t} = -\dot{t}\,\dot{r}\,\frac{2M}{r^2}\;\frac{1}{1-\frac{2M}{r}}$$

$$\ddot{r} = -\dot{t}^2\,\frac{M}{r^2}\left(1-\frac{2M}{r}\right) + \dot{r}^2\,\frac{M}{r^2}\,\frac{1}{1-\frac{2M}{r}} + \dot{\theta}^2\,r\left(1-\frac{2M}{r}\right) + \dot{\phi}^2\,r\sin^2\theta\left(1-\frac{2M}{r}\right)$$

$$\ddot{\theta} = -\dot{r}\,\dot{\theta}\,\frac{2}{r} + \dot{\phi}^2\cos\theta\,\sin\theta$$

$$\ddot{\phi} = -\dot{r}\,\dot{\phi}\,\frac{2}{r} - \dot{\theta}\,\dot{\phi}\cot\theta$$

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu{}_{\nu\rho}\,\frac{dx^\mu}{d\lambda}\,\frac{dx^\rho}{d\lambda}$$

Geodesic Equations: $\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$

$$\ddot{t} = -\dot{t}\,\dot{r}\,\frac{2M}{r^2}\,\frac{1}{1-\frac{2M}{r}}$$

$$\ddot{r} = -\dot{t}^2\,\frac{M}{r^2}\left(1-\frac{2M}{r}\right) + \dot{r}^2\,\frac{M}{r^2}\,\frac{1}{1-\frac{2M}{r}} + \dot{\theta}^2\,r\left(1-\frac{2M}{r}\right) + \dot{\phi}^2\,r\sin^2\theta\left(1-\frac{2M}{r}\right)$$

$$\ddot{\theta} = -\dot{r}\,\dot{\theta}\,\frac{2}{r} + \dot{\phi}^2\cos\theta\,\sin\theta$$

$$\ddot{\phi} = -\dot{r}\,\dot{\phi}\,\frac{2}{r} - \dot{\theta}\,\dot{\phi}\cot\theta$$

1st integrals:

$$e = \left(1-\frac{2M}{r}\right)\dot{t}$$

$$l = r^2\sin^2\theta\;\dot{\phi}$$

$$u^\mu u_\mu = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \end{cases}$$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$$

$$\ddot{t} = -\dot{t}\,\dot{r}\,\frac{2M}{r^2}\,\frac{1}{1-\frac{2M}{r}}$$

$$\ddot{r} = -\dot{t}^2\,\frac{M}{r^2}\left(1-\frac{2M}{r}\right) + \dot{r}^2\,\frac{M}{r^2}\,\frac{1}{1-\frac{2M}{r}} + \dot{\theta}^2\,r\left(1-\frac{2M}{r}\right) + \dot{\phi}^2\,r\sin^2\theta\left(1-\frac{2M}{r}\right)$$

$$\ddot{\theta} = -\dot{r}\,\dot{\theta}\,\frac{2}{r} + \dot{\phi}^2\cos\theta\,\sin\theta$$

$$\ddot{\phi} = -\dot{r}\,\dot{\phi}\,\frac{2}{r} - \dot{\theta}\,\dot{\phi}\cot\theta$$

1st integrals:

$$e = \left(1-\frac{2M}{r}\right)\dot{t}$$

$$u^\mu u_\mu = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \end{cases}$$

$$\ell = r^2\sin^2\theta\,\dot{\phi} \Rightarrow \text{planar motion} \quad \theta = \frac{\pi}{2} \Rightarrow \dot{\theta} = \ddot{\theta} = 0$$

$$\sin\theta = 1$$

Geodesic Equations: $\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$

$$\ddot{t} = -\dot{t}\,\dot{r}\,\frac{2M}{r^2}\,\frac{1}{1-\frac{2M}{r}}$$

$$\ddot{r} = -\dot{t}^2\,\frac{M}{r^2}\left(1-\frac{2M}{r}\right) + \dot{r}^2\,\frac{M}{r^2}\,\frac{1}{1-\frac{2M}{r}} + \cancel{\dot{\theta}^2}\,r\left(1-\frac{2M}{r}\right) + \dot{\phi}^2\,r\,\cancel{\sin^2}\theta\left(1-\frac{2M}{r}\right)$$

$$\cancel{\ddot{\theta} = -\dot{r}\dot{\theta}\,\frac{2}{r} + \dot{\phi}^2\cos\theta\,\sin\theta}$$

$$\ddot{\phi} = -\dot{r}\dot{\phi}\,\frac{2}{r} - \cancel{\dot{\theta}}\,\dot{\phi}\cot\theta$$

1st integrals:

$$e = \left(1-\frac{2M}{r}\right)\dot{t}$$

$$u^\mu u_\mu = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \end{cases}$$

$$\ell = r^2\,\cancel{\sin^2}\theta\;\dot{\phi}$$

$\Rightarrow$ planar motion $\theta = \frac{\pi}{2} \Rightarrow \dot{\theta} = \ddot{\theta} = 0$

$\sin\theta = 1$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right) \qquad \theta = \frac{\pi}{2}$$

$$\ddot{t} = -\dot{t}\,\dot{r}\,\frac{2M}{r^2}\;\frac{1}{1-\frac{2M}{r}}$$

$$\ddot{r} = -\dot{t}^2\,\frac{M}{r^2}\left(1-\frac{2M}{r}\right) + \dot{r}^2\,\frac{M}{r^2}\,\frac{1}{1-\frac{2M}{r}} + \dot{\phi}^2 r\left(1-\frac{2M}{r}\right)$$

$$\ddot{\phi} = -\dot{r}\,\dot{\phi}\,\frac{2}{r}$$

1st integrals:

$$e = \left(1-\frac{2M}{r}\right)\dot{t}$$

$$l = r^2\dot{\phi}$$

$$u^\mu u_\mu = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \end{cases}$$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right) \qquad \theta = \frac{\pi}{2}$$

$$\ddot{t} = -\dot{t}\,\dot{r}\,\frac{2M}{r^2}\,\frac{1}{1-\frac{2M}{r}}$$

$$\ddot{r} = -\dot{t}^2\,\frac{M}{r^2}\left(1-\frac{2M}{r}\right) + \dot{r}^2\,\frac{M}{r^2}\,\frac{1}{1-\frac{2M}{r}} + \dot{\phi}^2 r\left(1-\frac{2M}{r}\right)$$

$$\ddot{\phi} = -\dot{r}\,\dot{\phi}\,\frac{2}{r}$$

1st integrals:

$$e = \left(1-\frac{2M}{r}\right)\dot{t} \Rightarrow \dot{t} = e\left(1-\frac{2M}{r}\right)^{-1}$$

$$\ell = r^2\dot{\phi} \Rightarrow \dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \end{cases}$$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right) \qquad \theta = \frac{\pi}{2}$$

$$\ddot{t} = -\boxed{\dot{t}}\,\dot{r}\,\frac{2M}{r^2}\;\frac{1}{1-\frac{2M}{r}}$$

$$\ddot{r} = -\boxed{\dot{t}^2}\,\frac{M}{r^2}\left(1-\frac{2M}{r}\right) + \dot{r}^2\,\frac{M}{r^2}\,\frac{1}{1-\frac{2M}{r}} + \boxed{\dot{\phi}^2}\,r\left(1-\frac{2M}{r}\right)$$

$$\ddot{\phi} = -\dot{r}\,\boxed{\dot{\phi}}\,\frac{2}{r}$$

1st integrals:

$$e = \left(1-\frac{2M}{r}\right)\dot{t} \Rightarrow \boxed{\dot{t}} = e\left(1-\frac{2M}{r}\right)^{-1}$$

$$u^\mu u_\mu = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \end{cases}$$

$$\ell = r^2\dot{\phi} \Rightarrow \boxed{\dot{\phi}} = \frac{\ell}{r^2}$$

Geodesic Equations:

$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$ $\theta = \frac{\pi}{2}$

$$\ddot{t} = - \dot{r}\, e \frac{2M}{r^2} \left(1 - \frac{2M}{r}\right)^{-2}$$

$$\ddot{r} = - e^2 \frac{M}{r^2} \left(1 - \frac{2M}{r}\right)^{-1} + \dot{r}^2 \frac{M}{r^2} \left(1 - \frac{2M}{r}\right)^{-1} + \frac{\ell^2}{r^3} \left(1 - \frac{2M}{r}\right)$$

$$\ddot{\phi} = - \dot{r} \frac{2\ell}{r^3}$$

1st integrals:

$$e = \left(1 - \frac{2M}{r}\right) \dot{t} \Rightarrow \dot{t} = e \left(1 - \frac{2M}{r}\right)^{-1}$$

$$\ell = r^2 \dot{\phi} \Rightarrow \dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \end{cases}$$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right) \qquad \theta = \frac{\pi}{2}$$

$$\ddot{t} = - \dot{r}\, e\, \frac{2M}{r^2} \left(1 - \frac{2M}{r}\right)^{-2}$$

redundant: derivative of (1)

$$\ddot{r} = - e^2 \frac{M}{r^2} \left(1 - \frac{2M}{r}\right)^{-1} + \dot{r}^2 \frac{M}{r^2} \left(1 - \frac{2M}{r}\right)^{-1} + \frac{\ell^2}{r^3} \left(1 - \frac{2M}{r}\right)$$

$$\ddot{\phi} = - \dot{r} \frac{2\ell}{r^3}$$

redundant: derivative of (2)

1st integrals:

$$e = \left(1 - \frac{2M}{r}\right) \dot{t} \Rightarrow \dot{t} = e \left(1 - \frac{2M}{r}\right)^{-1} \quad (1)$$

$$u^\mu u_\mu = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \end{cases}$$

$$\ell = r^2 \dot{\phi} \Rightarrow \dot{\phi} = \frac{\ell}{r^2} \quad (2)$$

Geodesic Equations: $\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$ $\quad \theta = \frac{\pi}{2}$

Solve:

$$\ddot{r} = - \frac{e^2}{1-\frac{2M}{r}} \frac{M}{r^2} + \dot{r}^2 \frac{M}{r^2} \frac{1}{1-\frac{2M}{r}} + \frac{\ell^2}{r^3}\left(1-\frac{2M}{r}\right)$$

$$\dot{t} = \frac{e}{1-\frac{2M}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \begin{cases} -1 \\ 0 \end{cases}$$

Geodesic Equations: $\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$ $\theta = \frac{\pi}{2}$

Solve:

$$\dot{V} = -\frac{e^2}{1-\frac{2M}{r}} \frac{M}{r^2} + \dot{r}^2 \frac{M}{r^2} \frac{1}{1-\frac{2M}{r}} + \frac{\ell^2}{r^3}\left(1 - \frac{2M}{r}\right)$$

$$\dot{r} = V$$

$$\dot{t} = \frac{e}{1-\frac{2M}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \begin{cases} -1 \\ 0 \end{cases}$$

Geodesic Equations:

$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$ $\qquad \theta = \frac{\pi}{2}$

Solve:

$$\dot{v} = -\frac{e^2}{1-\frac{2M}{r}}\frac{M}{r^2} + \dot{r}^2\frac{M}{r^2}\frac{1}{1-\frac{2M}{r}} + \frac{\ell^2}{r^3}\left(1-\frac{2M}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2M}{r}}$$

+ initial conditions → unique solution

$(t(0), r(0), \dot{r}(0), \phi(0))$

$$\dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \begin{cases} -1 \\ 0 \end{cases}$$

used only in initial conditions, then it is conserved

(conservation of inner product of parallel transported vectors)

Geodesic Equations: $\left(\dot{x}^\mu \equiv \dfrac{dx^\mu}{d\lambda} \right)$ $\quad \theta = \dfrac{\pi}{2}$

Solve:

$$\dot{V} = - \frac{e^2}{1-\frac{2M}{r}} \frac{M}{r^2} + \dot{r}^2 \frac{M}{r^2} \frac{1}{1-\frac{2M}{r}} + \frac{\ell^2}{r^3}\left(1 - \frac{2M}{r}\right)$$

$$\dot{r} = V$$

$$\dot{t} = \frac{e}{1-\frac{2M}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \begin{cases} -1 \\ 0 \end{cases}$$

Rescale to obtain dimensionless quantities

t, r, τ, M : units of length, so

$t = M\tilde{t}$, $r = M\tilde{r}$, $\tau = M\tilde{\tau}$: $\tilde{t}, \tilde{r}, \tilde{\tau}$ dimensionless

so, take

$t \to Mt \quad r \to Mr \quad \tau \to M\tau$ (the new ones dimensionless)

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right) \qquad \theta = \frac{\pi}{2}$$

Solve:

$$\dot{v} = - \frac{e^2}{1-\frac{2M}{r}} \frac{M}{r^2} + \dot{r}^2 \frac{M}{r^2} \frac{1}{1-\frac{2M}{r}} + \frac{\ell^2}{r^3}\left(1 - \frac{2M}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2M}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \begin{cases} -1 \\ 0 \end{cases}$$

Rescale to obtain dimensionless quantities

$$\frac{d}{d\tau} \rightarrow \frac{1}{M}\frac{d}{d\tau} \Rightarrow v = \dot{r}\,,\ \dot{t} \text{ dimensionless}$$

$$\dot{v} = \frac{dv}{d\tau} \rightarrow \frac{1}{M}\frac{dv}{d\tau} = M^{-1}\dot{v}$$

$$\dot{\phi} = \frac{d\phi}{d\tau} \rightarrow \frac{1}{M}\frac{d\phi}{d\tau} = M^{-1}\dot{\phi}$$

$t \rightarrow Mt \qquad r \rightarrow Mr \qquad \tau \rightarrow M\tau$ (the new ones dimensionless)

Geodesic Equations:

$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$ $\theta = \frac{\pi}{2}$

Solve:

$$\dot{v} = - \frac{e^2}{1-\frac{2M}{r}} \frac{M}{r^2} + \dot{r}^2 \frac{M}{r^2} \frac{1}{1-\frac{2M}{r}} + \frac{\ell^2}{r^3}\left(1 - \frac{2M}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2M}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \left\{ \begin{array}{l} -1 \\ 0 \end{array} \right.$$

Rescale to obtain dimensionless quantities

$\frac{d}{d\tau} \to \frac{1}{M}\frac{d}{d\tau} \Rightarrow v = \dot{r}$, $\dot{t}$ dimensionless

$$\dot{v} = \frac{dv}{d\tau} \to \frac{1}{M}\frac{dv}{d\tau} = M^{-1}\dot{v}$$

$$\dot{\phi} = \frac{d\phi}{d\tau} \to \frac{1}{M}\frac{d\phi}{d\tau} = M^{-1}\dot{\phi}$$

$t \to Mt$ $r \to Mr$ $\tau \to M\tau$ (the new ones dimensionless)

$\dot{v} \to M^{-1}\dot{v}$ $\dot{\phi} \to M^{-1}\dot{\phi}$

Geodesic Equations:

$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$ $\theta = \frac{\pi}{2}$

Solve:

$$\dot{v} = -\frac{e^2}{1-\frac{2M}{r}}\frac{M}{r^2} + \dot{r}^2\frac{M}{r^2}\frac{1}{1-\frac{2M}{r}} + \frac{\ell^2}{r^3}\left(1-\frac{2M}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2M}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \begin{cases} -1 \\ 0 \end{cases}$$

Rescale to obtain dimensionless quantities

$$\ell = r^2\dot{\phi} \rightarrow M^2 r^2 M^{-1}\dot{\phi} = M\ell$$

$$e = \left(1-\frac{2M}{r}\right)\dot{t} \rightarrow \left(1-\frac{2M}{Mr}\right)\dot{t} = \left(1-\frac{2}{r}\right)\dot{t} = e$$

$t \rightarrow Mt$ $r \rightarrow Mr$ $\tau \rightarrow M\tau$ (the new ones dimensionless)

$\dot{v} \rightarrow M^{-1}\dot{v}$ $\dot{\phi} \rightarrow M^{-1}\dot{\phi}$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right) \qquad \theta = \frac{\pi}{2}$$

Solve:

$$\dot{v} = - \frac{e^2}{1-\frac{2M}{r}} \frac{M}{r^2} + \dot{r}^2 \frac{M}{r^2} \frac{1}{1-\frac{2M}{r}} + \frac{\ell^2}{r^3}\left(1 - \frac{2M}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2M}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \begin{cases} -1 \\ 0 \end{cases}$$

Rescale to obtain dimensionless quantities

$e \to e \quad \phi \to \phi \quad \dot{r} \to \dot{r} \quad \dot{t} \to \dot{t} \quad v \to v$

$t \to Mt \quad r \to Mr \quad \tau \to M\tau$ (the new ones dimensionless)

$\dot{v} \to M^{-1}\dot{v} \quad \dot{\phi} \to M^{-1}\dot{\phi} \quad \ell \to M\ell$

Geodesic Equations:

$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right) \qquad \theta = \frac{\pi}{2}$

Solve:

$$\frac{\dot{v}}{M} = -\frac{e^2}{1-\frac{2M}{Mr}}\frac{M}{M^2 r^2} + \dot{r}^2 \frac{M}{M^2 r^2}\frac{1}{1-\frac{2M}{Mr}} + \frac{M^2 \ell^2}{M^3 r^3}\left(1-\frac{2M}{Mr}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2M}{Mr}}$$

$$\frac{\dot{\phi}}{M} = \frac{M\ell}{M^2 r^2}$$

$$u^\mu u_\mu = \begin{cases} -1 \\ 0 \end{cases}$$

$e \to e \quad \phi \to \phi \quad \dot{r} \to \dot{r} \quad \dot{t} \to \dot{t} \quad v \to v$

$t \to Mt \quad r \to Mr \quad \tau \to M\tau$

$\dot{v} \to M^{-1}\dot{v} \quad \dot{\phi} \to M^{-1}\dot{\phi} \quad \ell \to M\ell$

Geodesic Equations: $\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$ $\quad \theta = \frac{\pi}{2}$

Solve:

$$\dot{v} = -\frac{e^2}{1-\frac{2}{r}}\frac{1}{r^2} + \dot{r}^2\frac{1}{r^2}\frac{1}{1-\frac{2}{r}} + \frac{\ell^2}{r^3}\left(1-\frac{2}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \begin{cases} -1 \\ 0 \end{cases}$$

Restore to geometrized units after solving for dimensionless:

$e \to e \quad \phi \to \phi \quad \dot{r} \to \dot{r} \quad \dot{t} \to \dot{t} \quad v \to v$

$t \to M^{-1}t \quad r \to M^{-1}r \quad \tau \to M^{-1}\tau$

$\dot{v} \to M\,\dot{v} \quad \dot{\phi} \to M\,\dot{\phi} \quad \ell \to M^{-1}\ell$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right) \qquad \theta = \frac{\pi}{2}$$

Solve:

$$\dot{v} = \frac{\dot{r}^2 - e^2}{r^2\left(1-\frac{2}{r}\right)} + \frac{\ell^2}{r^3}\left(1-\frac{2}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

$$u^\mu u_\mu = \left\{ \begin{matrix} -1 \\ 0 \end{matrix} \right.$$

Restore M after solving those equations by

$e \to e \qquad \phi \to \phi \qquad \dot{r} \to \dot{r} \qquad \dot{t} \to \dot{t} \qquad v \to v$

$t \to t/M$	$r \to r/M$	$\tau \to \tau/M$
$\dot{v} \to M\,\dot{v}$	$\dot{\phi} \to M\,\dot{\phi}$	$\ell \to \ell/M$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau} \right) \qquad \theta = \frac{\pi}{2}$$

Solve:

$$\dot{V} = \frac{\dot{r}^2 - e^2}{r^2\left(1-\frac{2}{r}\right)} + \frac{\ell^2}{r^3}\left(1-\frac{2}{r}\right)$$

$$\dot{r} = V$$

$$\dot{t} = \frac{e}{1-\frac{2}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

Massive particle: $\lambda \to \tau$

$$u^\mu u_\mu = -1 \Rightarrow$$

$$\mathcal{E} = \frac{1}{2}(\dot{r})^2 + V_{eff}(r) \Rightarrow$$

$$\dot{r} = \pm\left[2\left(\mathcal{E} - V_{eff}(r)\right)\right]^{1/2}$$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}\right) \qquad \theta = \frac{\pi}{2}$$

Solve:

$$\dot{V} = \frac{\dot{r}^2 - e^2}{r^2\left(1-\frac{2}{r}\right)} + \frac{\ell^2}{r^3}\left(1-\frac{2}{r}\right)$$

$$\dot{r} = V$$

$$\dot{t} = \frac{e}{1-\frac{2}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

Massive particle: $\lambda \to \tau$

$$u^\mu u_\mu = -1 \Rightarrow$$

$$\mathcal{E} = \frac{1}{2}(\dot{r})^2 + V_{eff}(r) \Rightarrow$$

$$\dot{r} = \pm\left[2\left(\mathcal{E} - V_{eff}(r)\right)\right]^{1/2}$$

- choose ℓ, $\mathcal{E} = \frac{e^2-1}{2}$

Geodesic Equations: $\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau} \right)$ $\theta = \frac{\pi}{2}$

Solve:

$$\dot{V} = \frac{\dot{r}^2 - e^2}{r^2\left(1 - \frac{2}{r}\right)} + \frac{\ell^2}{r^3}\left(1 - \frac{2}{r}\right)$$

$$\dot{r} = V$$

$$\dot{t} = \frac{e}{1 - \frac{2}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

Massive particle: $\lambda \to \tau$

$u^\mu u_\mu = -1 \Rightarrow$

$\mathcal{E} = \frac{1}{2}(\dot{r})^2 + V_{eff}(r) \Rightarrow$

$\dot{r} = \pm \left[2 \left(\mathcal{E} - V_{eff}(r) \right) \right]^{1/2}$

- choose ℓ, $\mathcal{E} = \frac{e^2 - 1}{2}$
- choose $\phi(0)$, $r(0)$

Geodesic Equations:

$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau} \right)$ $\theta = \frac{\pi}{2}$

Solve:

$$\dot{v} = \frac{\dot{r}^2 - e^2}{r^2\left(1-\frac{2}{r}\right)} + \frac{\ell^2}{r^3}\left(1 - \frac{2}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

$$\dot{r}(0) = \pm \left[2\left(\mathcal{E} - V_{eff}(r(0)) \right) \right]^{1/2}$$

← choose also the sign +/-

- choose ℓ, $\mathcal{E} = \frac{e^2-1}{2}$
- choose $\phi(0)$, $r(0)$
- $\dot{\phi}(0)$, $\dot{r}(0) \equiv v(0)$ determined by $\ell, \mathcal{E}, r(0)$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right) \qquad \theta = \frac{\pi}{2}$$

Solve:

$$\dot{V} = \frac{\dot{r}^2 - e^2}{r^2\left(1 - \frac{2}{r}\right)} + \frac{\ell^2}{r^3}\left(1 - \frac{2}{r}\right)$$

$$\dot{r} = V$$

$$\dot{t} = \frac{e}{1 - \frac{2}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

Massless Particles:

$$u^\mu u_\mu = 0 \Rightarrow$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

$$b^2 = \frac{\ell^2}{e^2}$$

Geodesic Equations: $\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$ $\theta = \frac{\pi}{2}$

Solve:

$$\dot{v} = \frac{\dot{r}^2 - e^2}{r^2\left(1-\frac{2}{r}\right)} + \frac{\ell^2}{r^3}\left(1-\frac{2}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

Massless Particles:

$u^\mu u_\mu = 0 \Rightarrow$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

$$b^2 = \frac{\ell^2}{e^2}$$

Rescale λ: $\ell\lambda \to \lambda$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right) \qquad \theta = \frac{\pi}{2}$$

Solve:

$$\dot{v} = \frac{\dot{r}^2 - e^2}{r^2\left(1-\frac{2}{r}\right)} + \frac{\ell^2}{r^3}\left(1-\frac{2}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

Massless Particles:

$$u^\mu u_\mu = 0 \Rightarrow$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

$$b^2 = \frac{\ell^2}{e^2}$$

Rescale λ: $\ell \lambda \to \lambda$

$$\frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 \to \left(\frac{dr}{d\lambda}\right)^2$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \to \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda/\ell} = e \cdot \ell$$

Geodesic Equations: $\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$ $\quad \theta = \frac{\pi}{2}$

Solve:

$$\dot{v} = \frac{\dot{r}^2 - e^2}{r^2\left(1-\frac{2}{r}\right)} + \frac{\ell^2}{r^3}\left(1-\frac{2}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1-\frac{2}{r}}$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

Massless Particles:

$$u^\mu u_\mu = 0 \Rightarrow$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

$$b^2 = \frac{\ell^2}{e^2} \rightarrow \frac{\ell^2}{\ell^2 e^2} = \frac{1}{e^2}$$

Rescale λ: $\ell \lambda \rightarrow \lambda$

$$\frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 \rightarrow \left(\frac{dr}{d\lambda}\right)^2$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \rightarrow \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda/\ell} = e \cdot \ell$$

Geodesic Equations:

$$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}\right) \qquad \theta = \frac{\pi}{2}$$

Solve:

$$\ell^2 \dot{v} = \frac{\ell^2 \dot{r}^2 - e^2 \ell^2}{r^2\left(1-\frac{2}{r}\right)} + \frac{\ell^2}{r^3}\left(1-\frac{2}{r}\right)$$

$$\ell \dot{r} = \ell v$$

$$\ell \dot{t} = \frac{\ell e}{1-\frac{2}{r}}$$

$$\ell \dot{\phi} = \frac{\ell}{r^2}$$

Massless Particles:

$$u^\mu u_\mu = 0 \Rightarrow$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

$$b^2 = \frac{\ell^2}{e^2} \rightarrow \frac{\ell^2}{\ell^2 e^2} = \frac{1}{e^2}$$

$$\frac{d}{d\lambda} \rightarrow \ell \frac{d}{d\lambda}$$

Rescale λ: $\ell \lambda \rightarrow \lambda$

$$\dot{v} = \frac{d^2 r}{d\lambda^2} \rightarrow \ell^2 \dot{v}$$

$$\frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 \rightarrow \left(\frac{dr}{d\lambda}\right)^2$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \rightarrow \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda/\ell} = e \cdot \ell$$

Geodesic Equations:

$\left(\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \right)$ $\theta = \frac{\pi}{2}$

Solve:

$$\dot{v} = \frac{\dot{r}^2 - e^2}{r^2\left(1 - \frac{2}{r}\right)} + \frac{1}{r^3}\left(1 - \frac{2}{r}\right)$$

$$\dot{r} = v$$

$$\dot{t} = \frac{e}{1 - \frac{2}{r}}$$

$$\dot{\phi} = \frac{1}{r^2}$$

- choose e, $r(0)$, $\phi(0)$, $t(0)$

$\Rightarrow v(0) = \dot{r}(0) = \pm \left[e^2 - W_{eff}(r(0)) \right]^{1/2}$

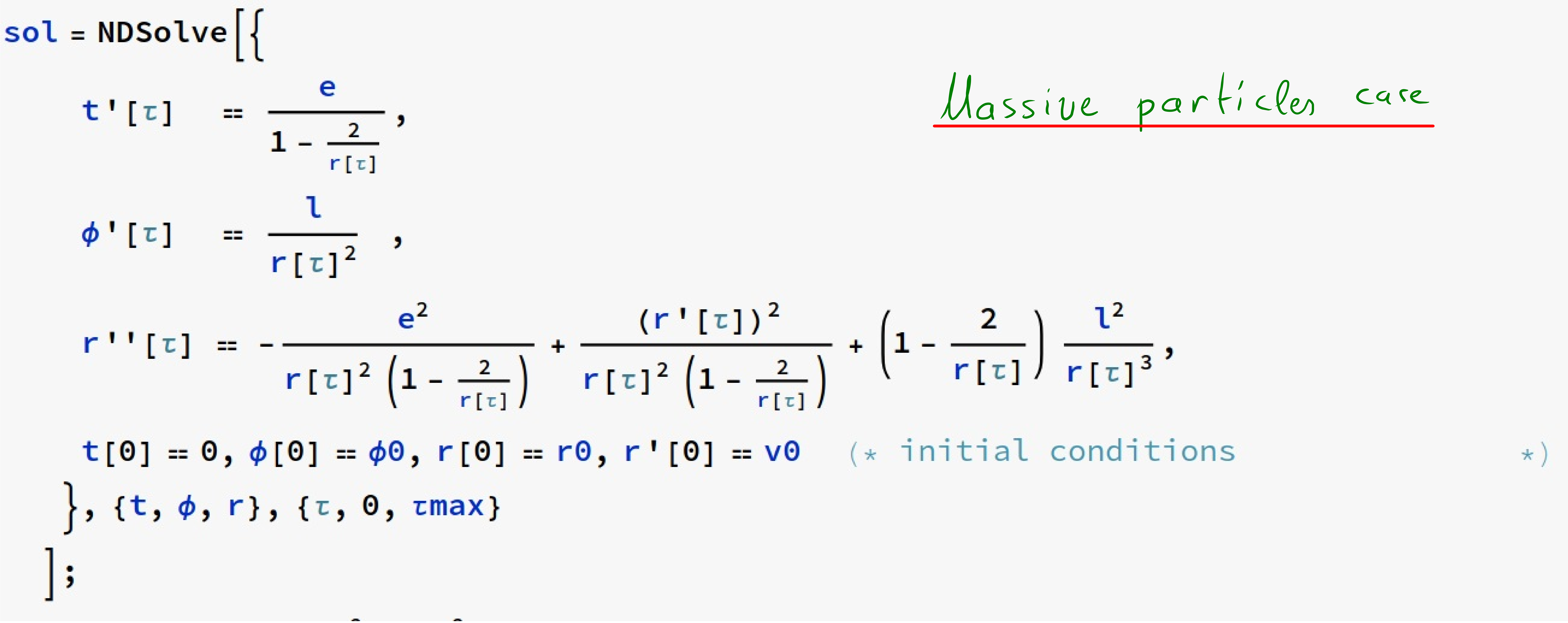
↳ choose sign

Numerical solutions

- Runge-Kutta method

Numerical solutions

- Runge-Kutta method
- Mathematica: (download notebook from website → Lecture 9)

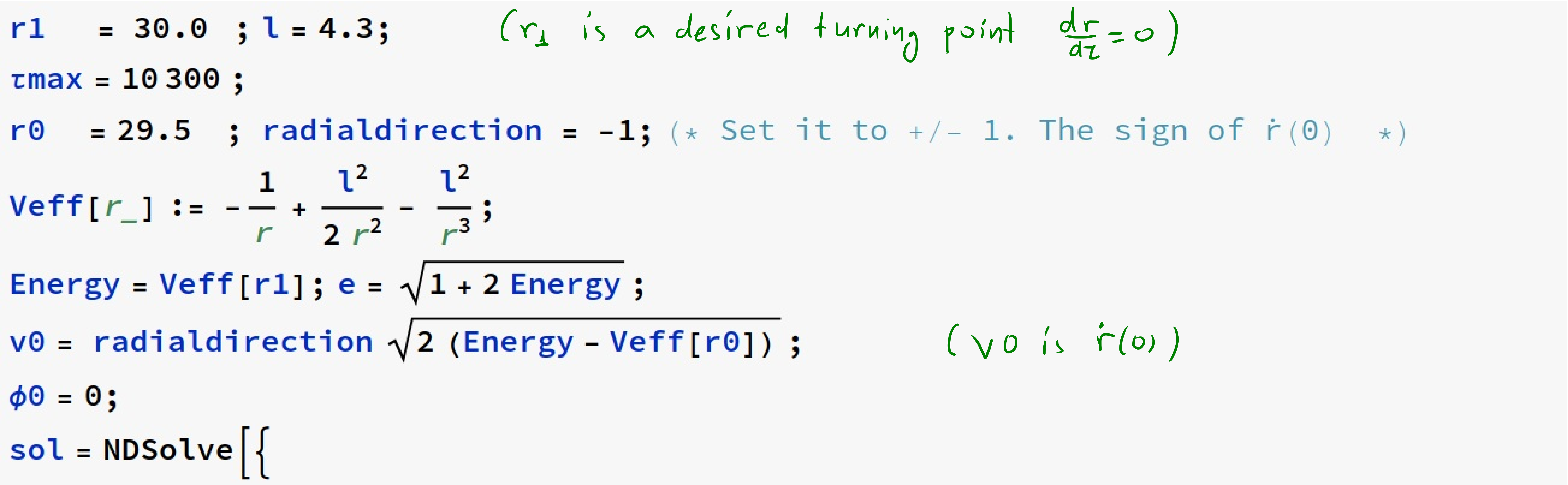


Massive particles case

```
sol = NDSolve[{
    t'[τ]  == e/(1 - 2/r[τ]),
    ϕ'[τ]  == l/r[τ]^2 ,
    r''[τ] == -e^2/(r[τ]^2 (1 - 2/r[τ])) + (r'[τ])^2/(r[τ]^2 (1 - 2/r[τ])) + (1 - 2/r[τ]) l^2/r[τ]^3,
    t[0] == 0, ϕ[0] == ϕ0, r[0] == r0, r'[0] == v0   (* initial conditions          *)
   }, {t, ϕ, r}, {τ, 0, τmax}
  ];
```

Numerical solutions

- Runge-Kutta method
- Mathematica: Initial conditions, massive case

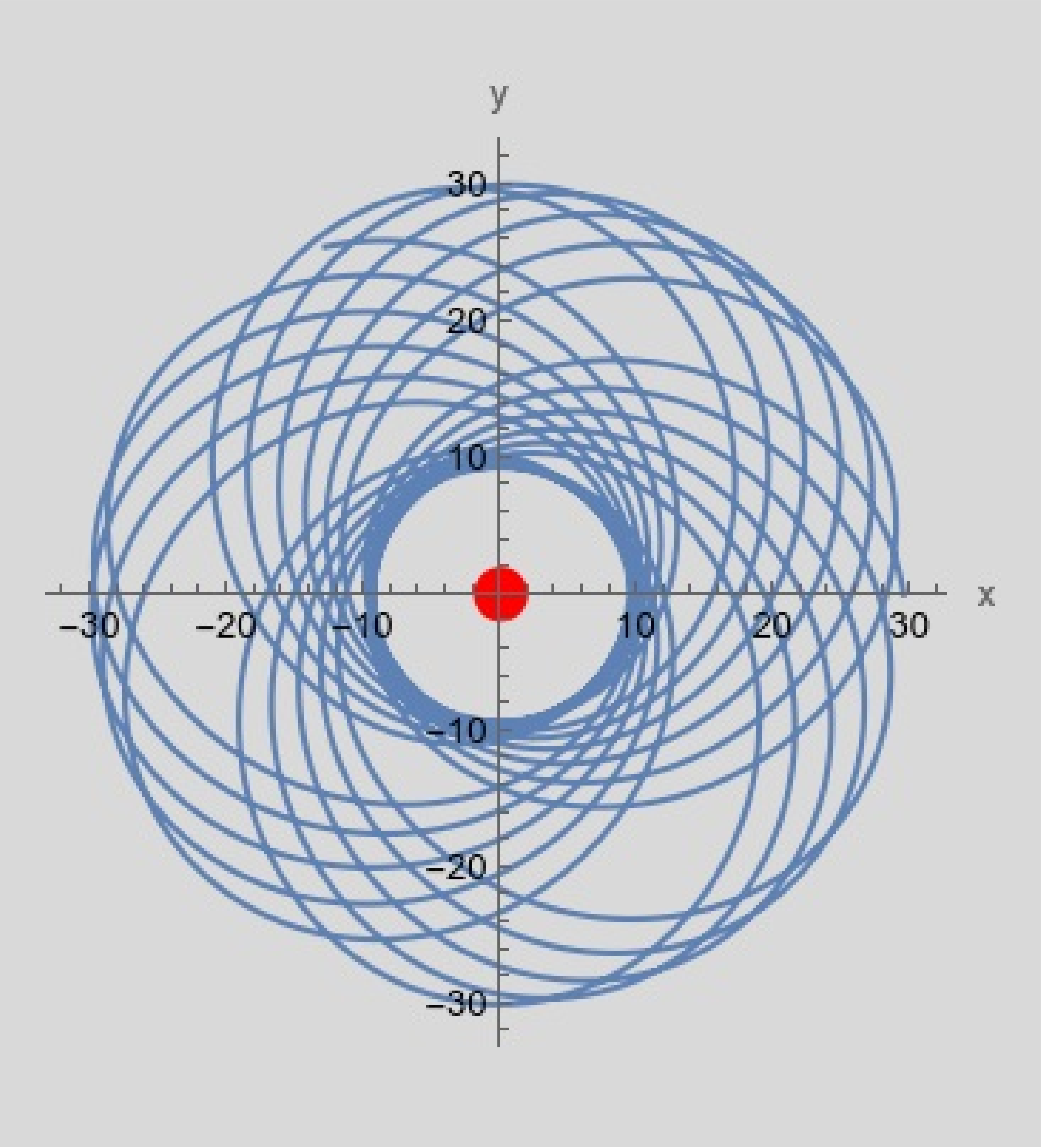


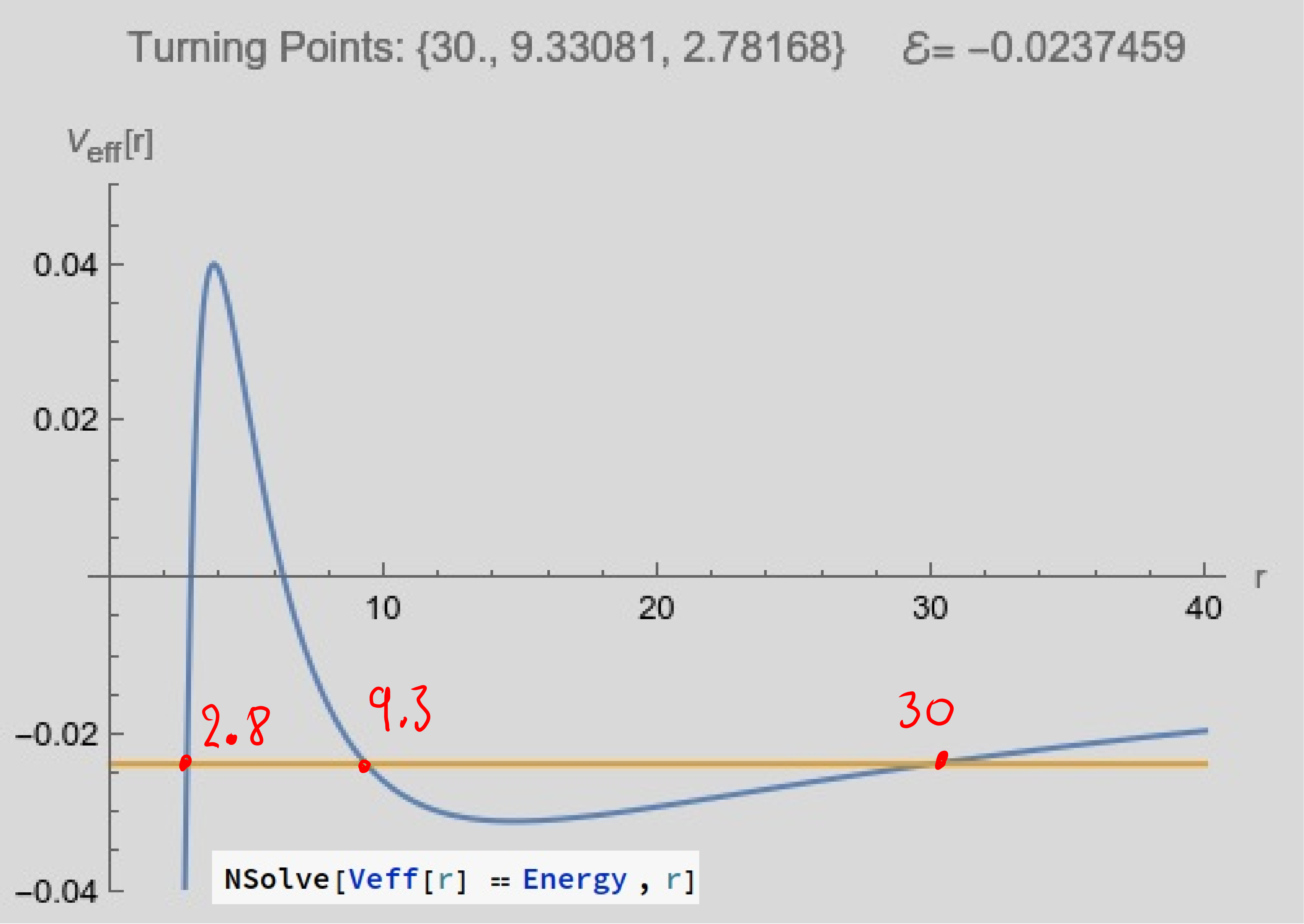
```
r1   = 30.0  ; l = 4.3;
τmax = 10 300 ;
r0   = 29.5   ; radialdirection = -1; (* Set it to +/- 1. The sign of ṙ(0)   *)
Veff[r_] := -1/r + l^2/(2 r^2) - l^2/r^3;
Energy = Veff[r1]; e = Sqrt[1 + 2 Energy] ;
v0 = radialdirection Sqrt[2 (Energy - Veff[r0])] ;
ϕ0 = 0;
sol = NDSolve[{
```

(r_1 is a desired turning point $\frac{dr}{d\tau}=0$)

($v0$ is $\dot{r}(0)$)

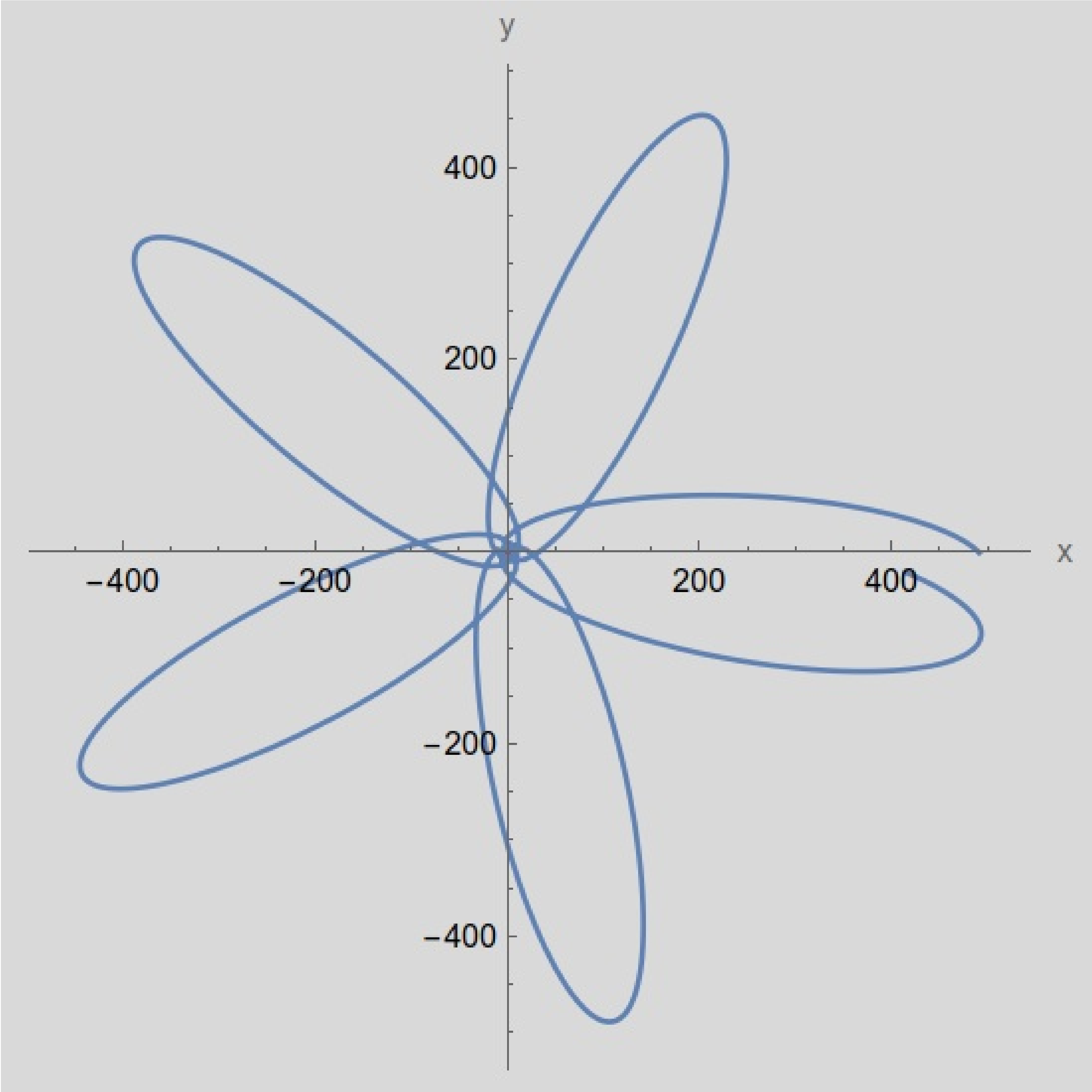
$\ell = 4.3$

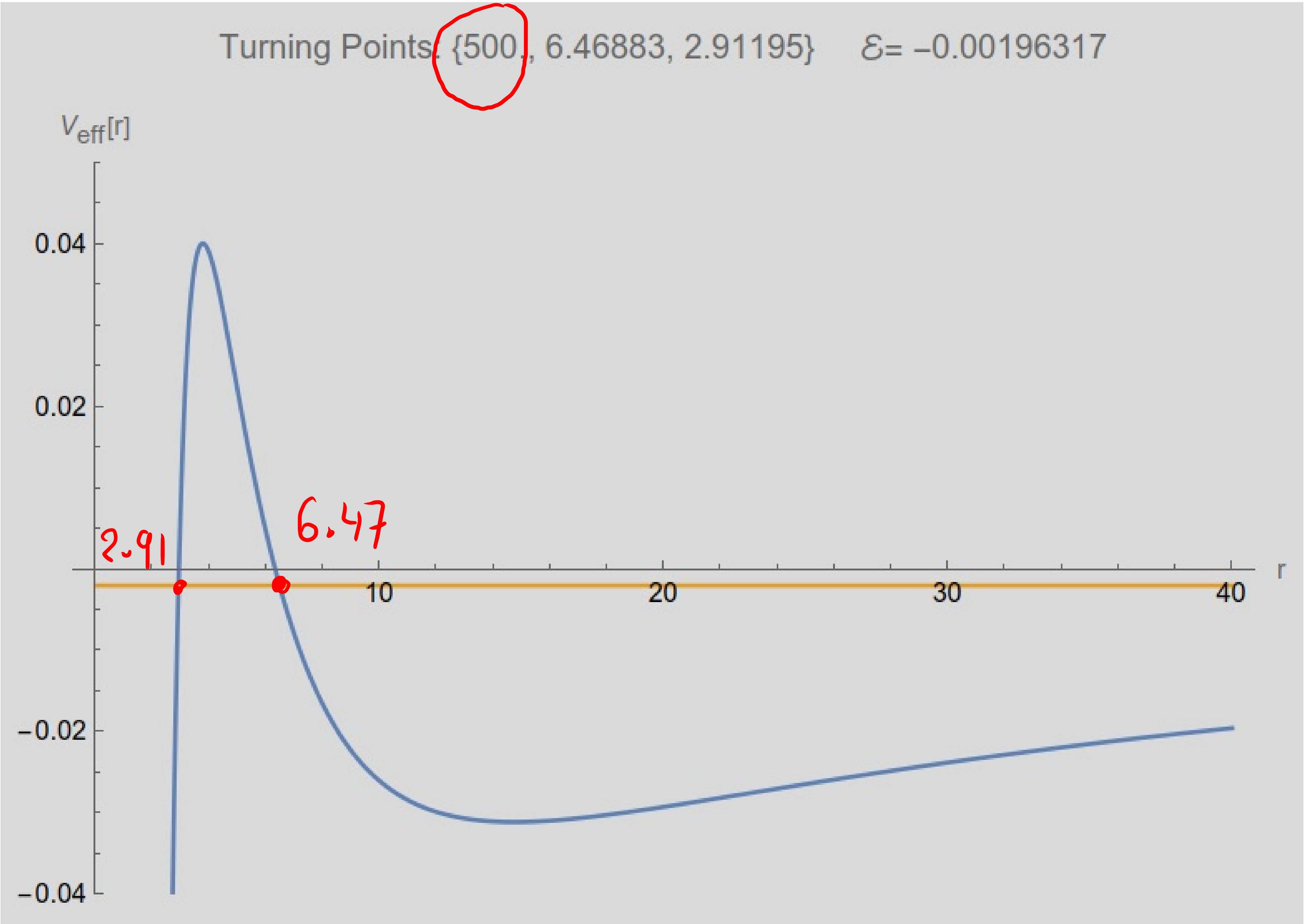



Turning Points: {30., 9.33081, 2.78168} $\mathcal{E}$= −0.0237459

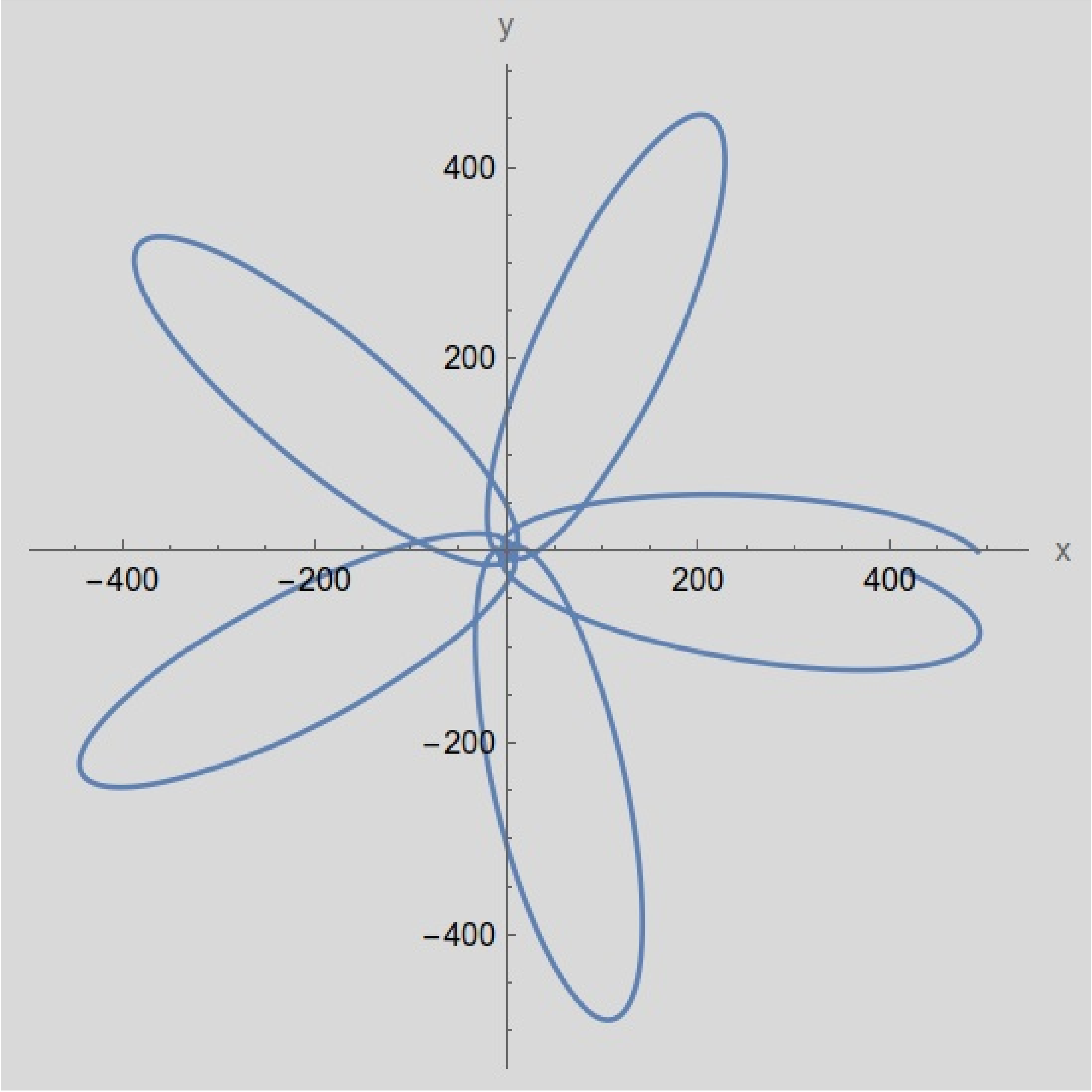
```
NSolve[Veff[r] == Energy , r]
```

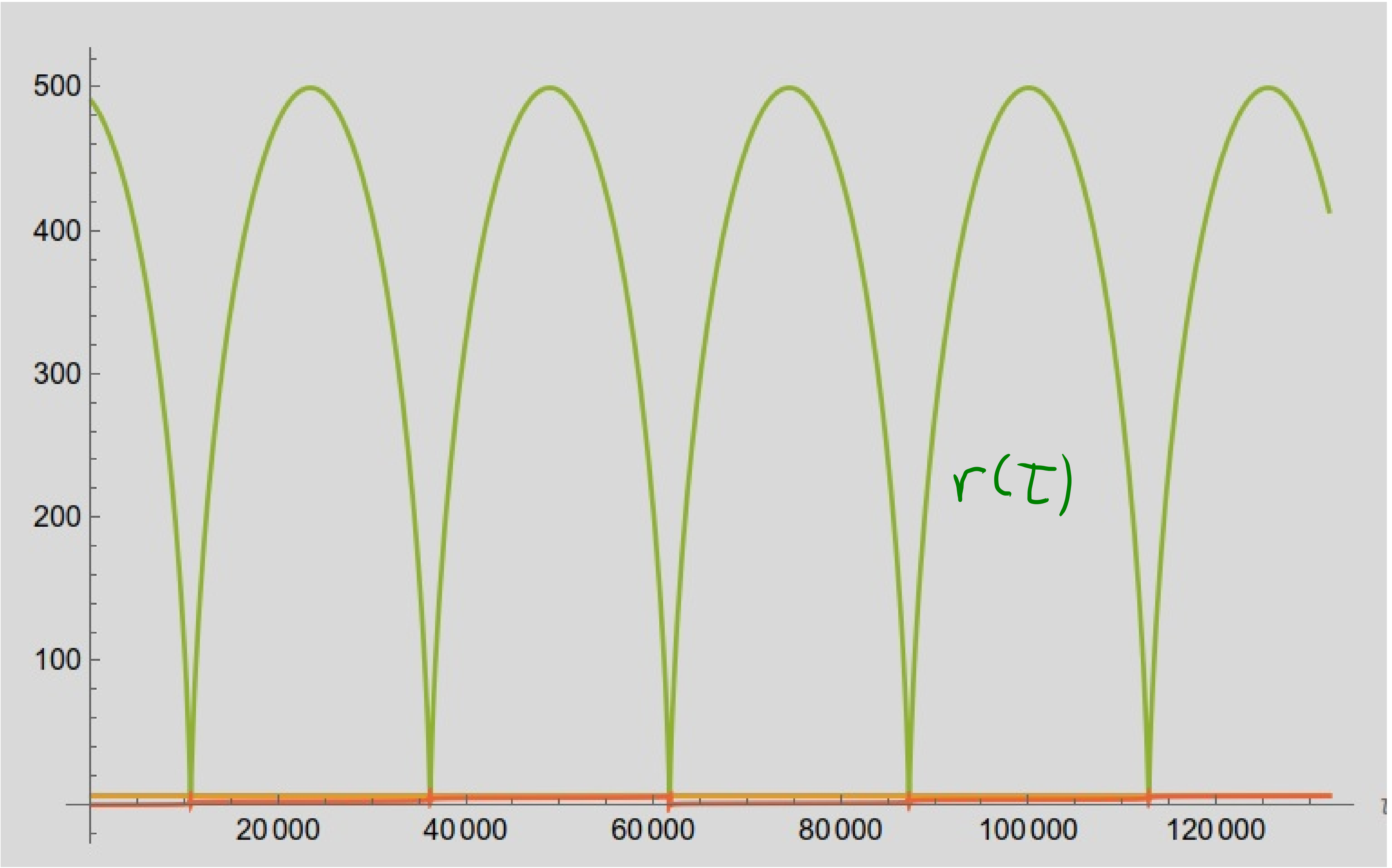
$\ell = 4.3$



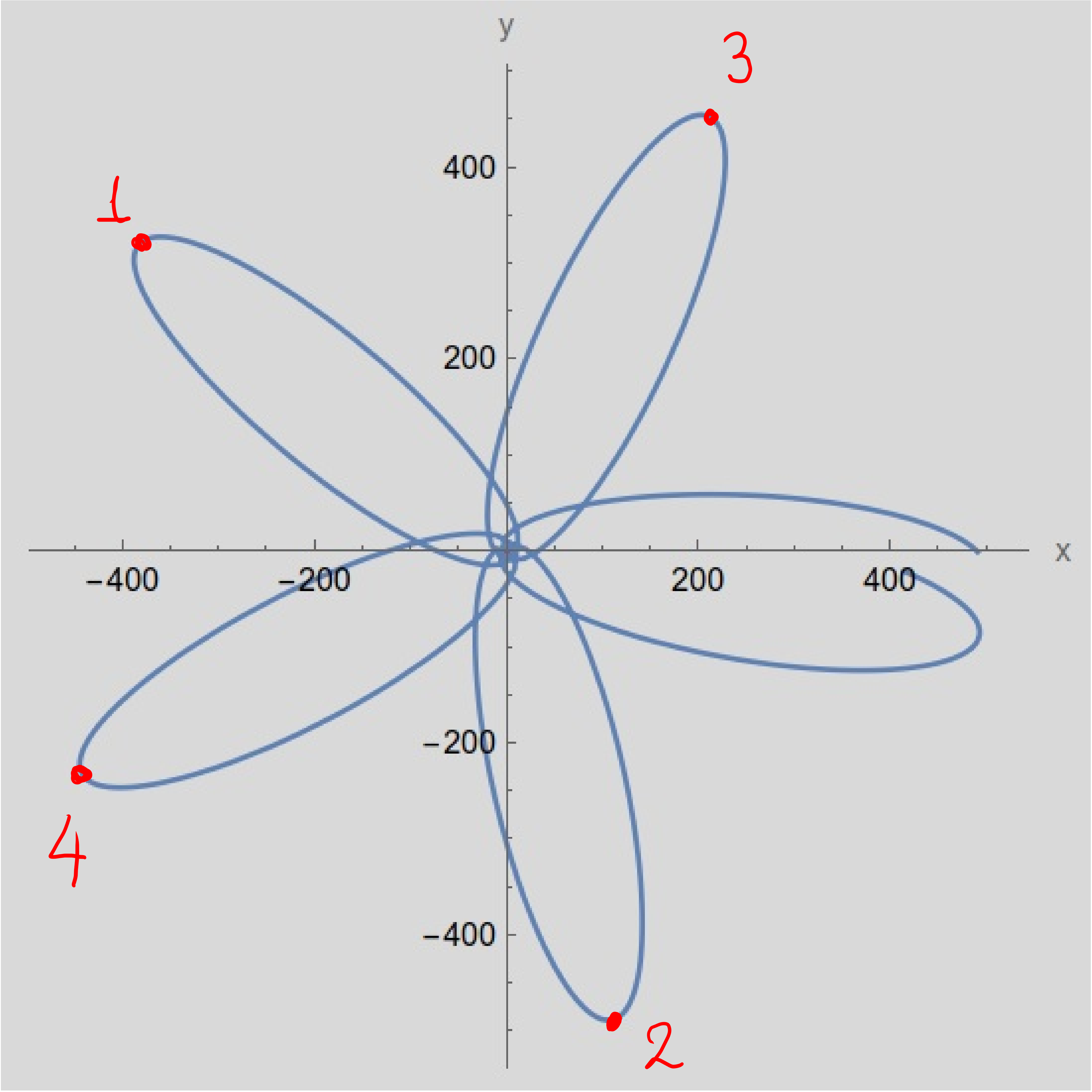

Turning Points {500., 6.46883, 2.91195} ℰ= −0.00196317

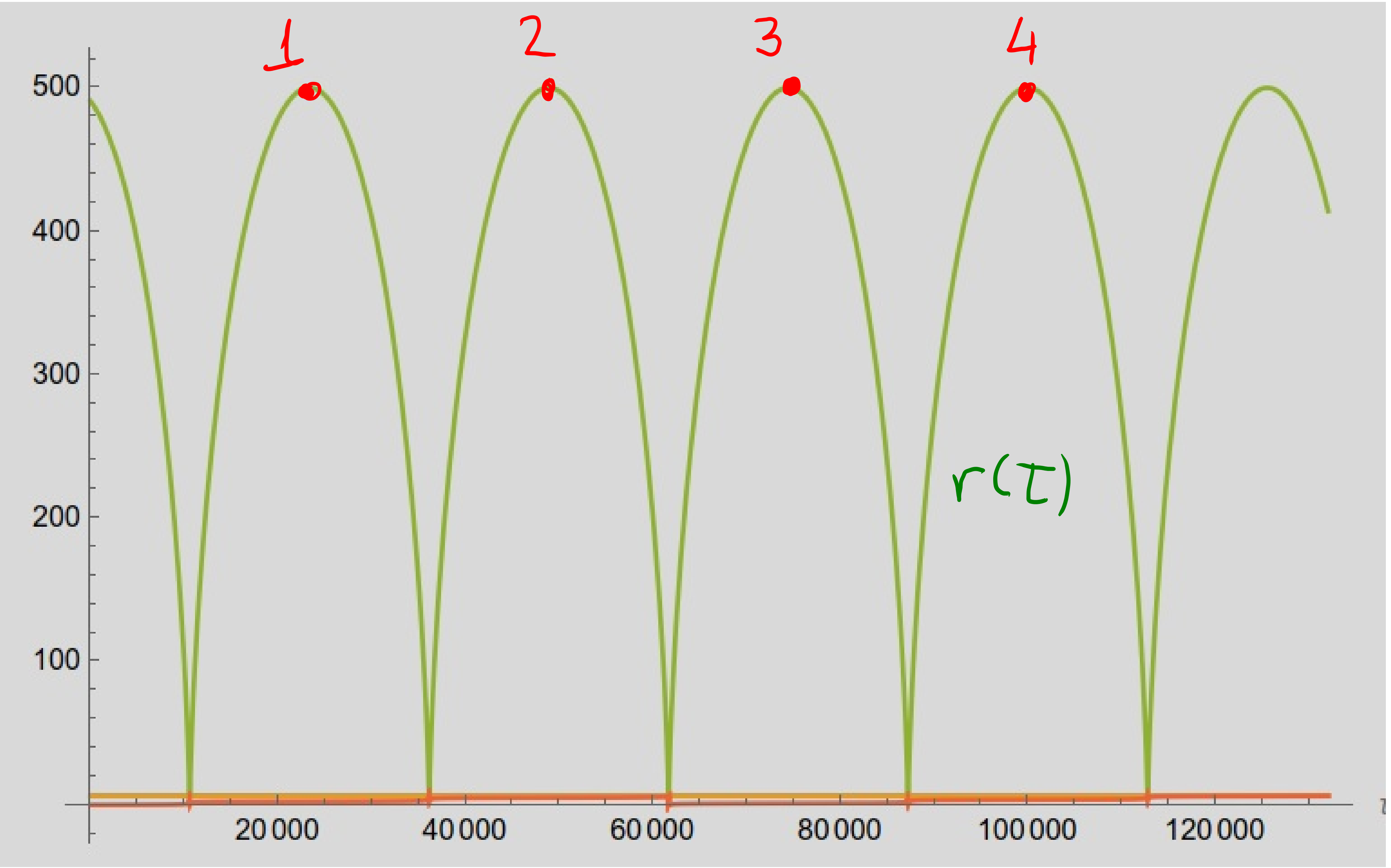
$\ell = 4.3$

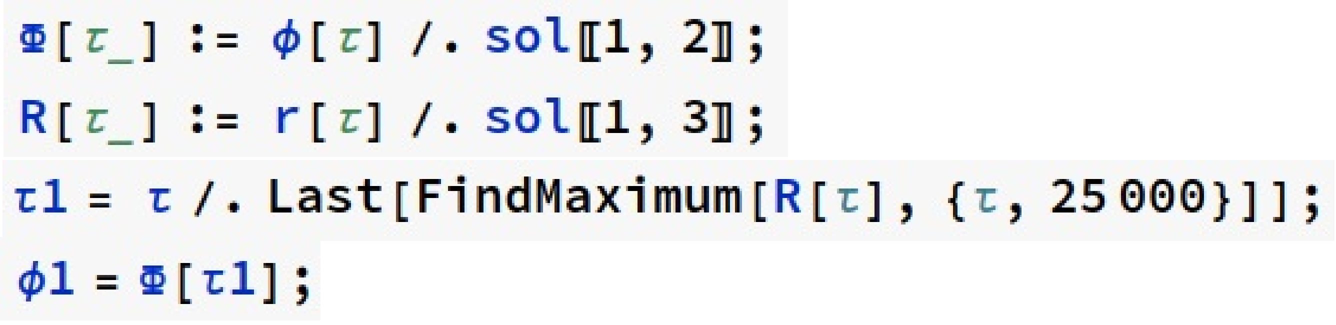

y
400
200
-200
-400
x
-400
-200
200
400


500
400
300
200
100
20000
40000
60000
80000
100000
120000
r(τ)

$\ell = 4.3$

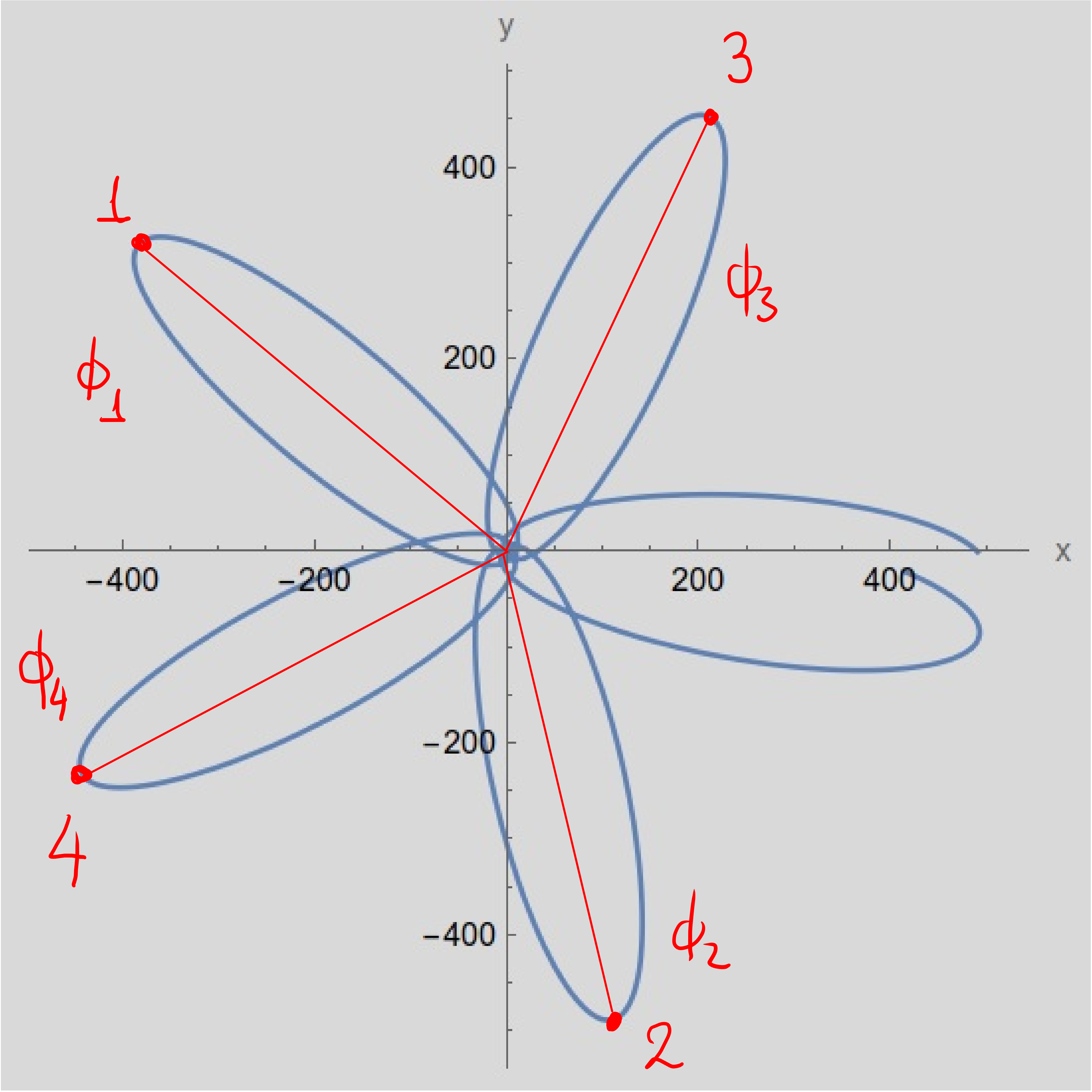


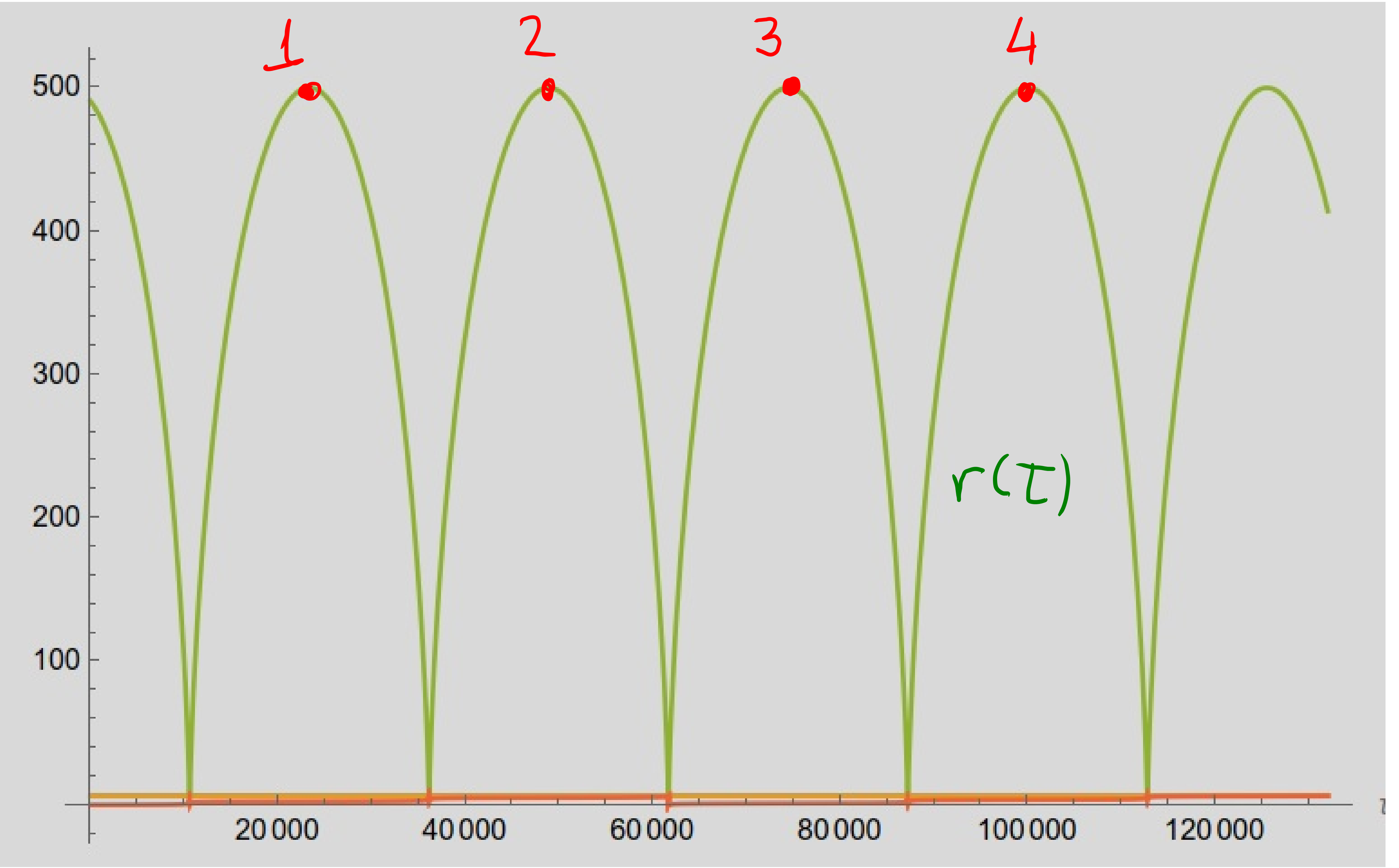




```
Φ[τ_] := ϕ[τ] /. sol⟦1, 2⟧;
R[τ_] := r[τ] /. sol⟦1, 3⟧;
τ1 = τ /. Last[FindMaximum[R[τ], {τ, 25 000}]];
ϕ1 = Φ[τ1];
```

$\ell = 4.3$





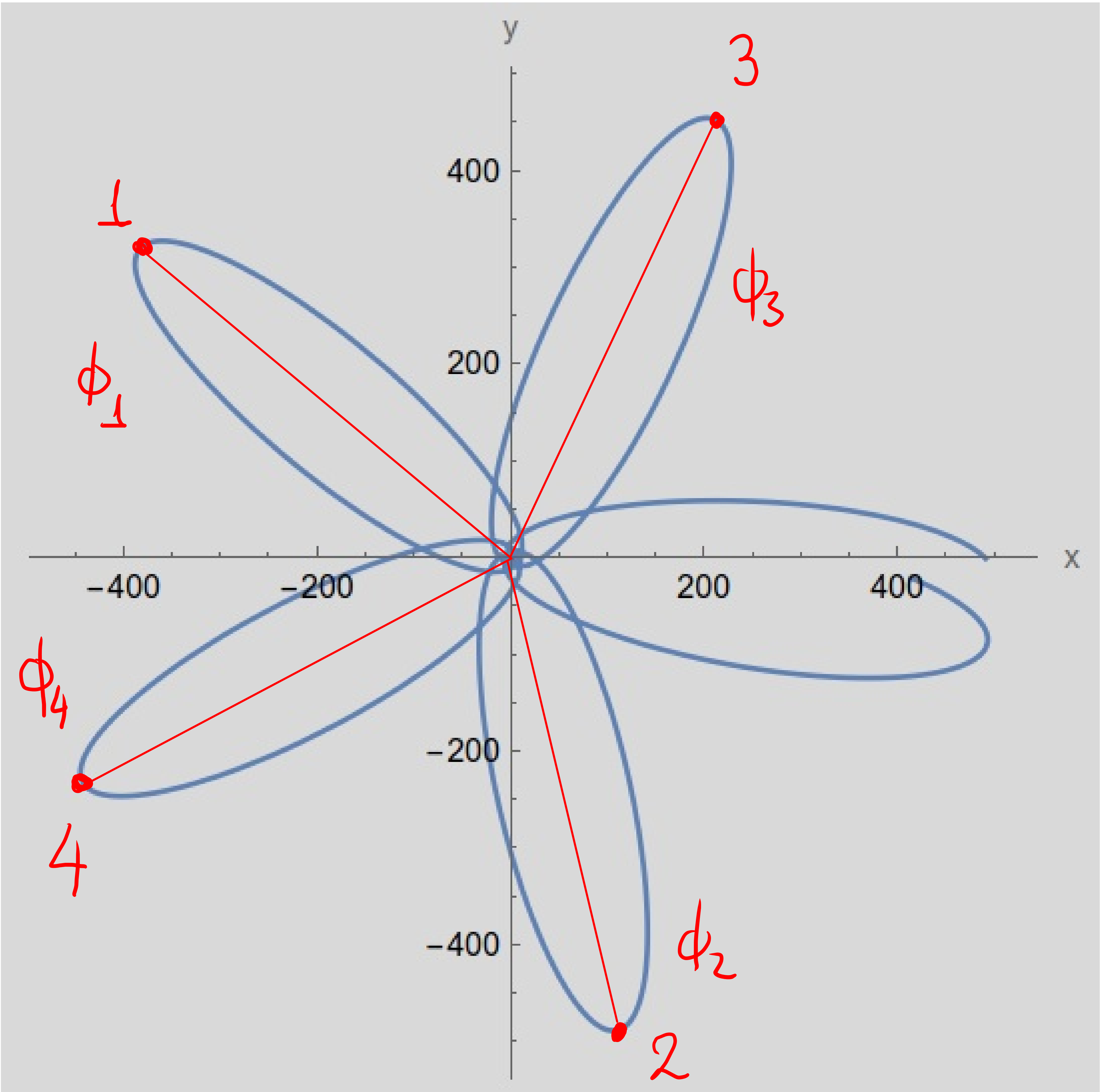
$\phi_1 = 140.223^\circ$

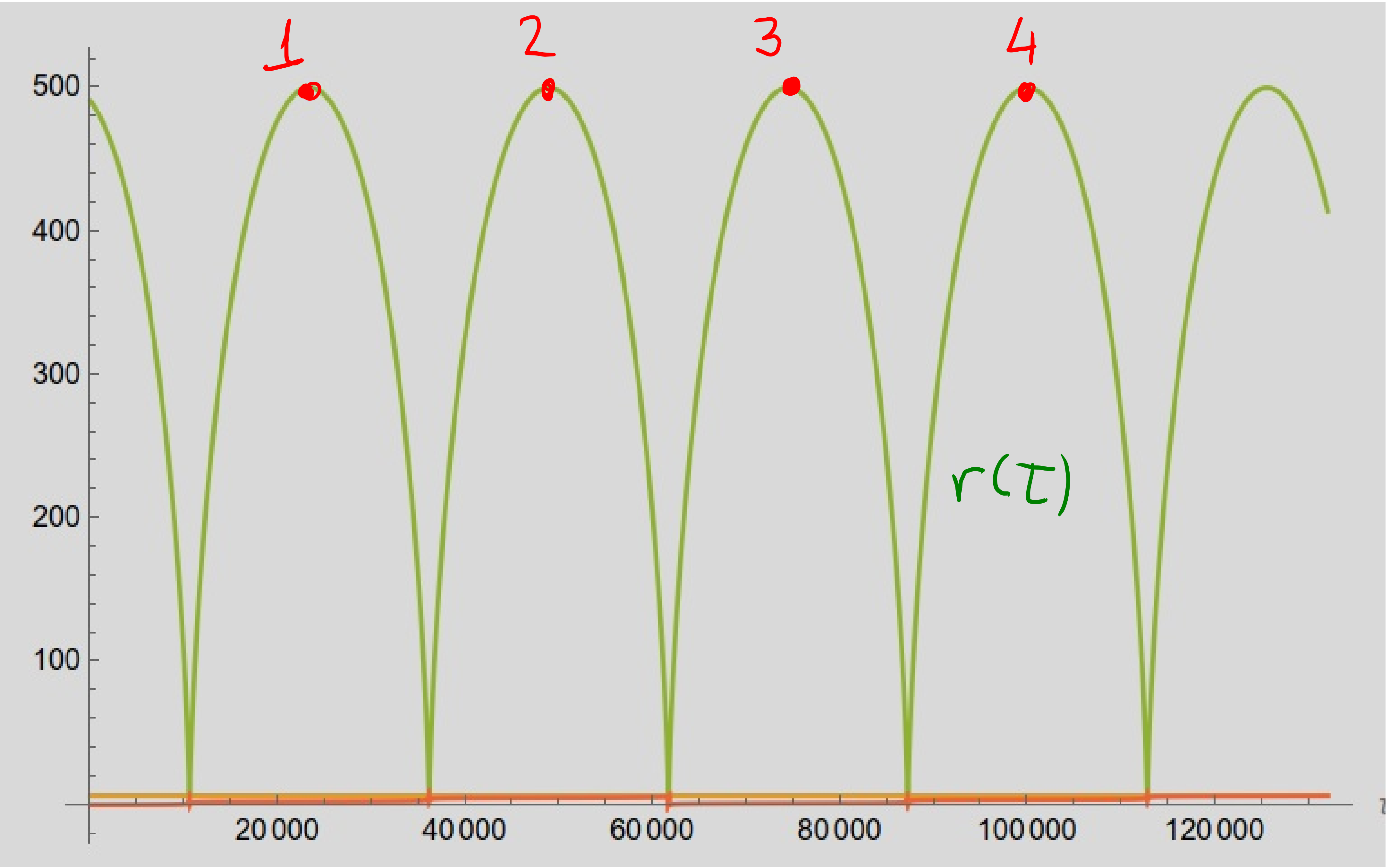
$\phi_2 = 282.656^\circ$

$\phi_3 = 65.089^\circ$

$\phi_4 = 207.522^\circ$

$\ell = 4.3$





Precession angle:

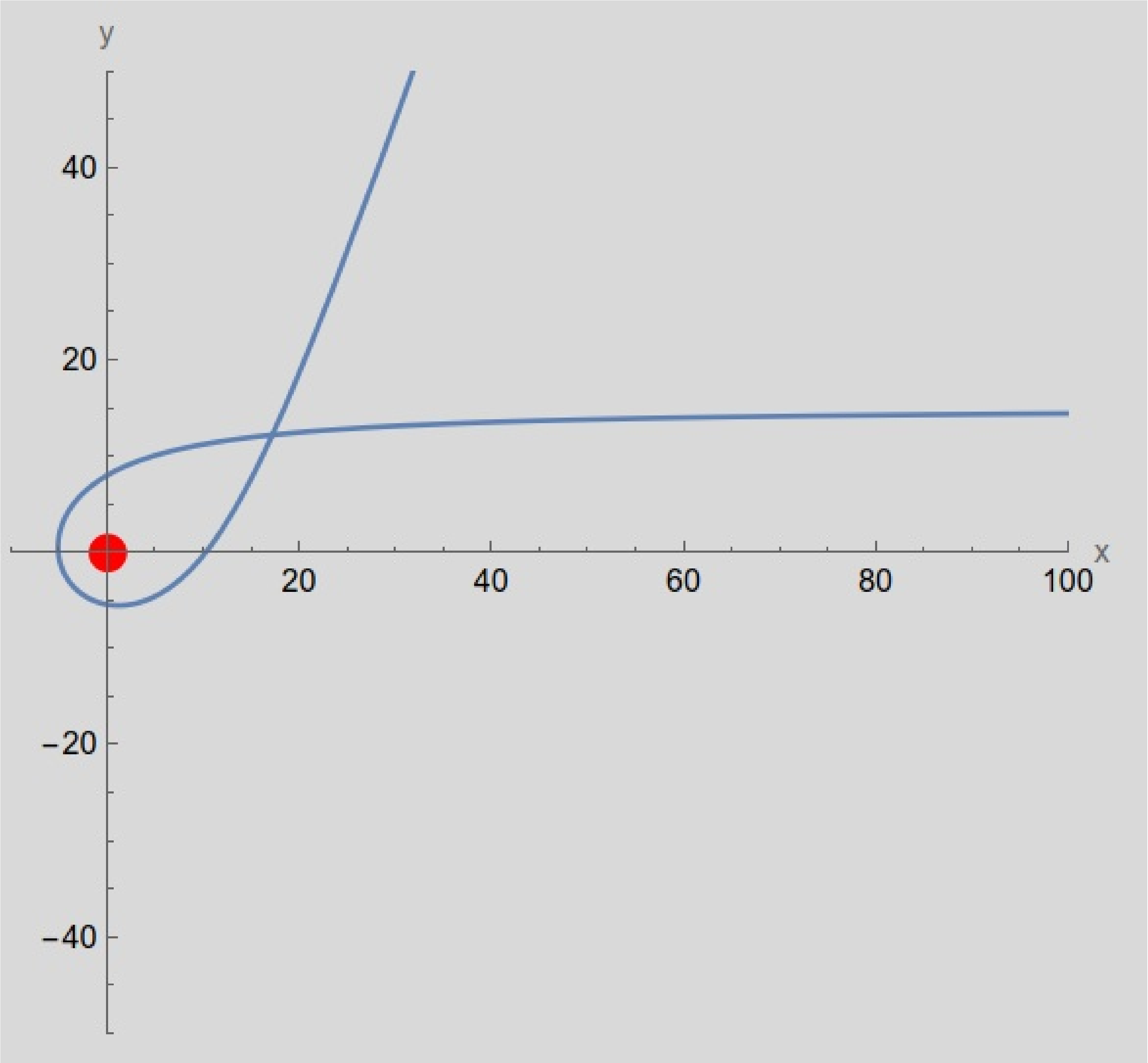
$$\delta\phi = \phi_{i+1} - \phi_i - 2\pi = 142.433^\circ$$

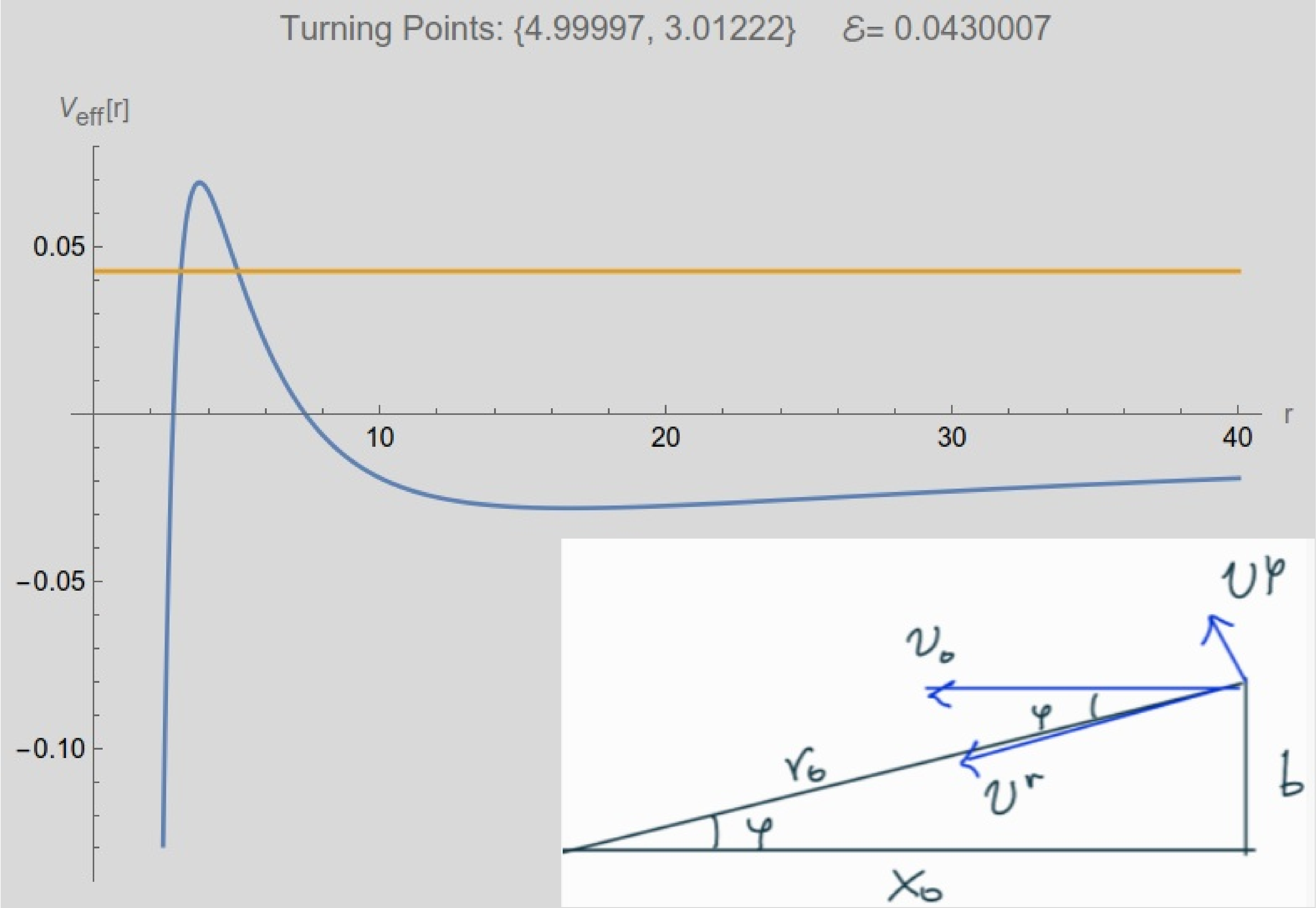
$\phi_1 = 140.223^\circ$

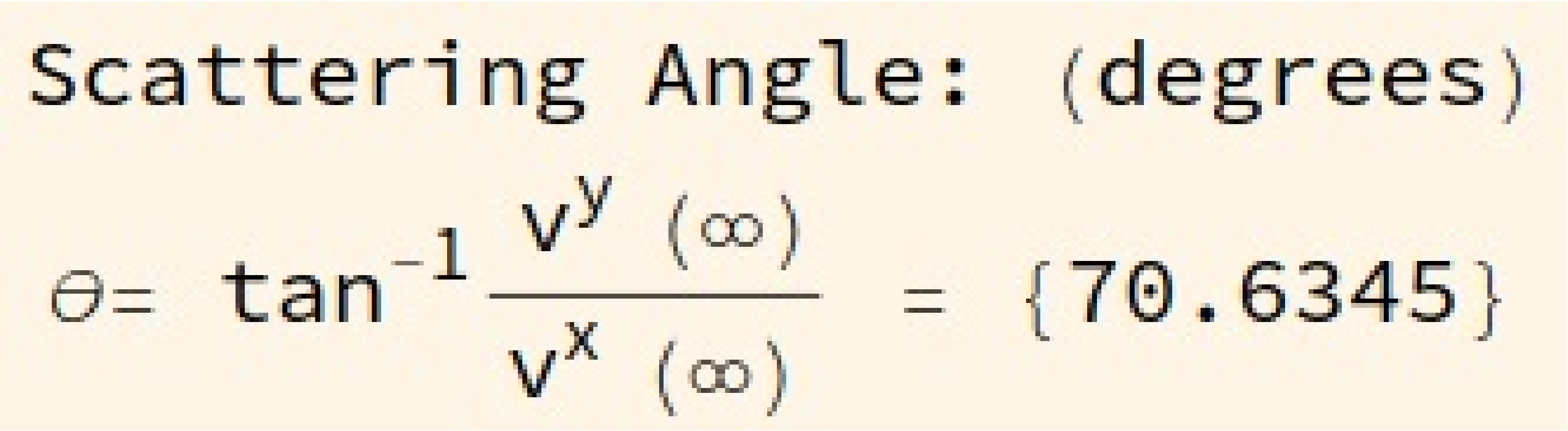
$\phi_2 = 282.656^\circ$

$\phi_3 = 65.089^\circ$

$\phi_4 = 207.522^\circ$





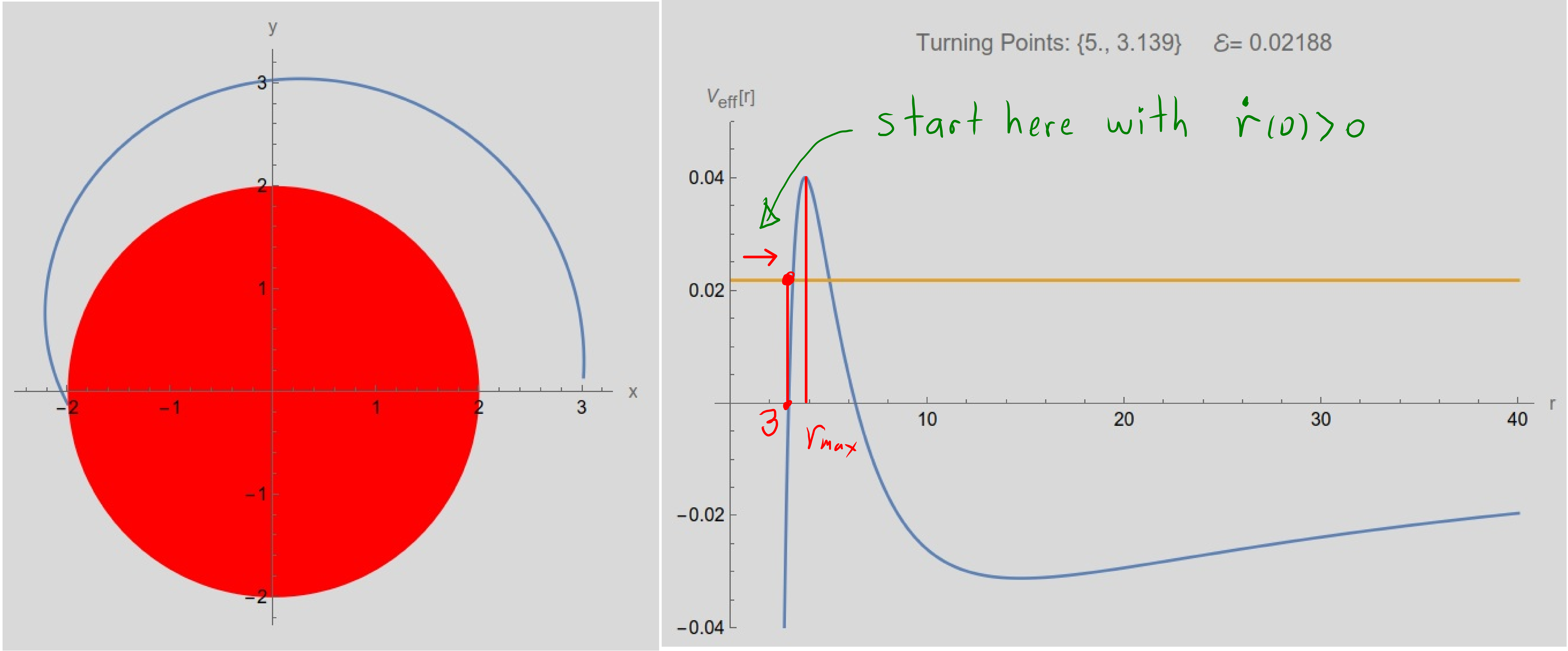

Scattering Angle: (degrees)

$$\theta = \tan^{-1}\frac{v^y\ (\infty)}{v^x\ (\infty)} = \{70.6345\}$$

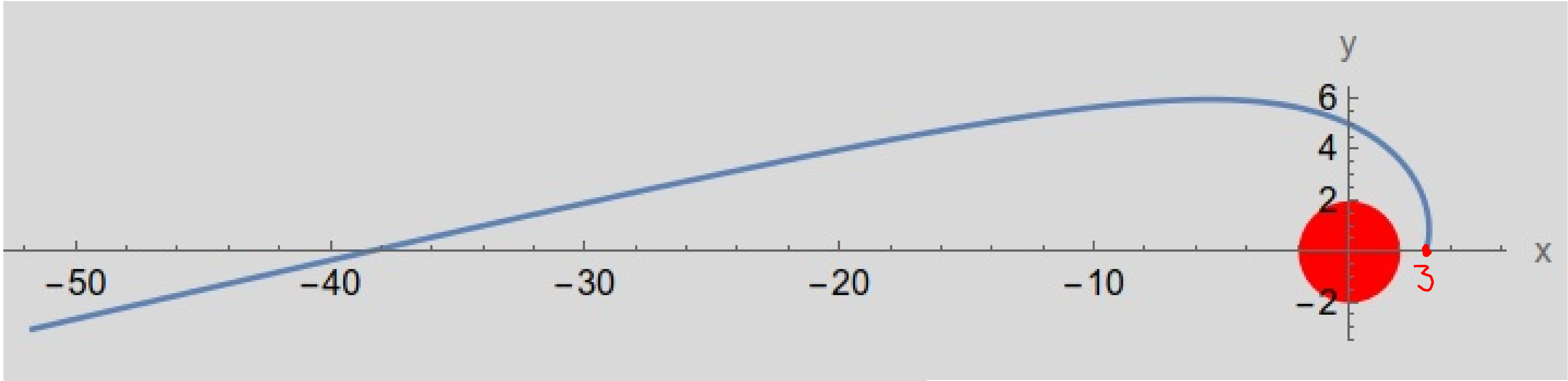
Scattering: set $b = 15$

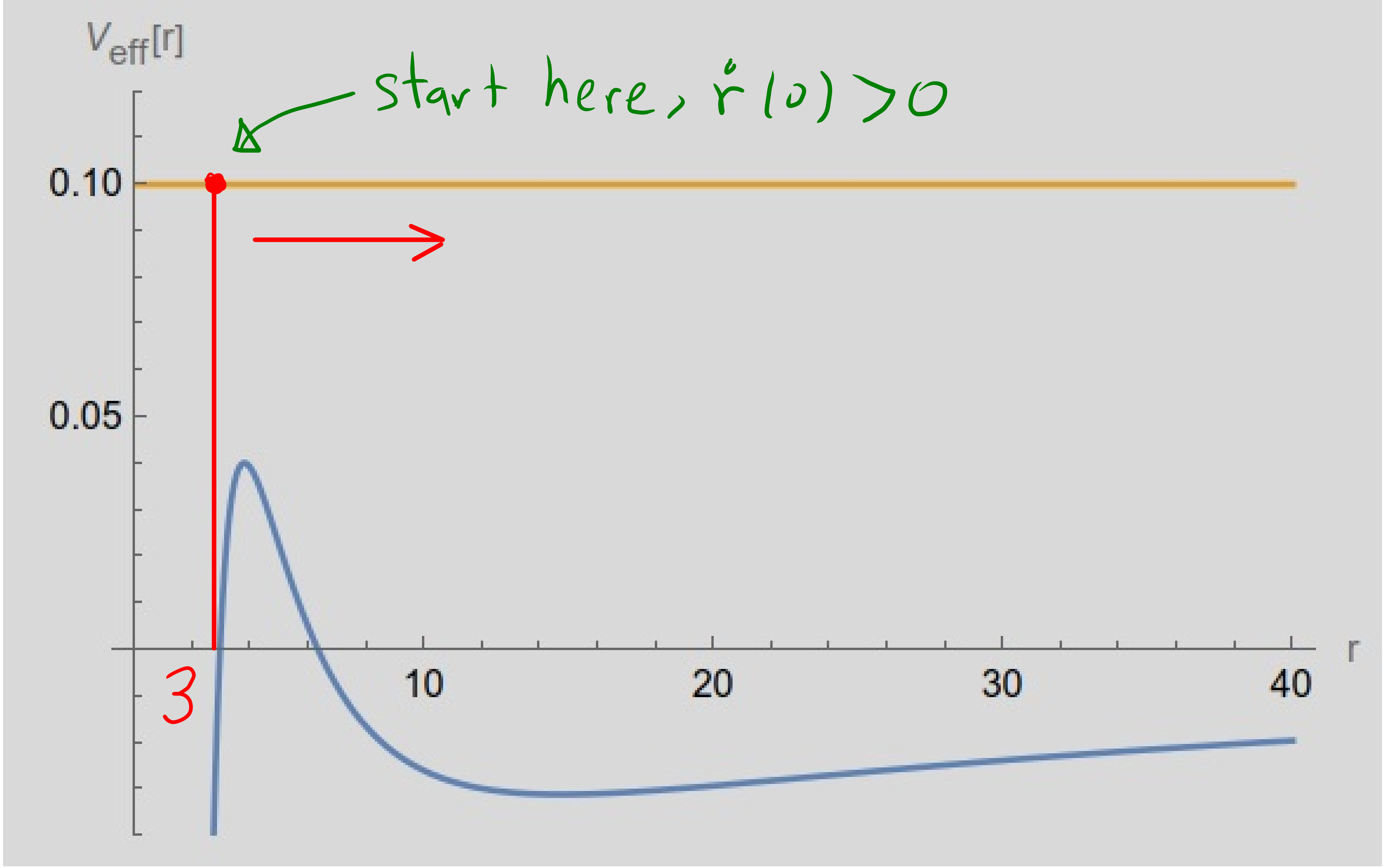
$x_0 = 500$

$v_0 = 0.3$



Cannot escape from the $2M < r < r_{max}$ region when $\mathcal{E} < V_{eff}(r_{max})$

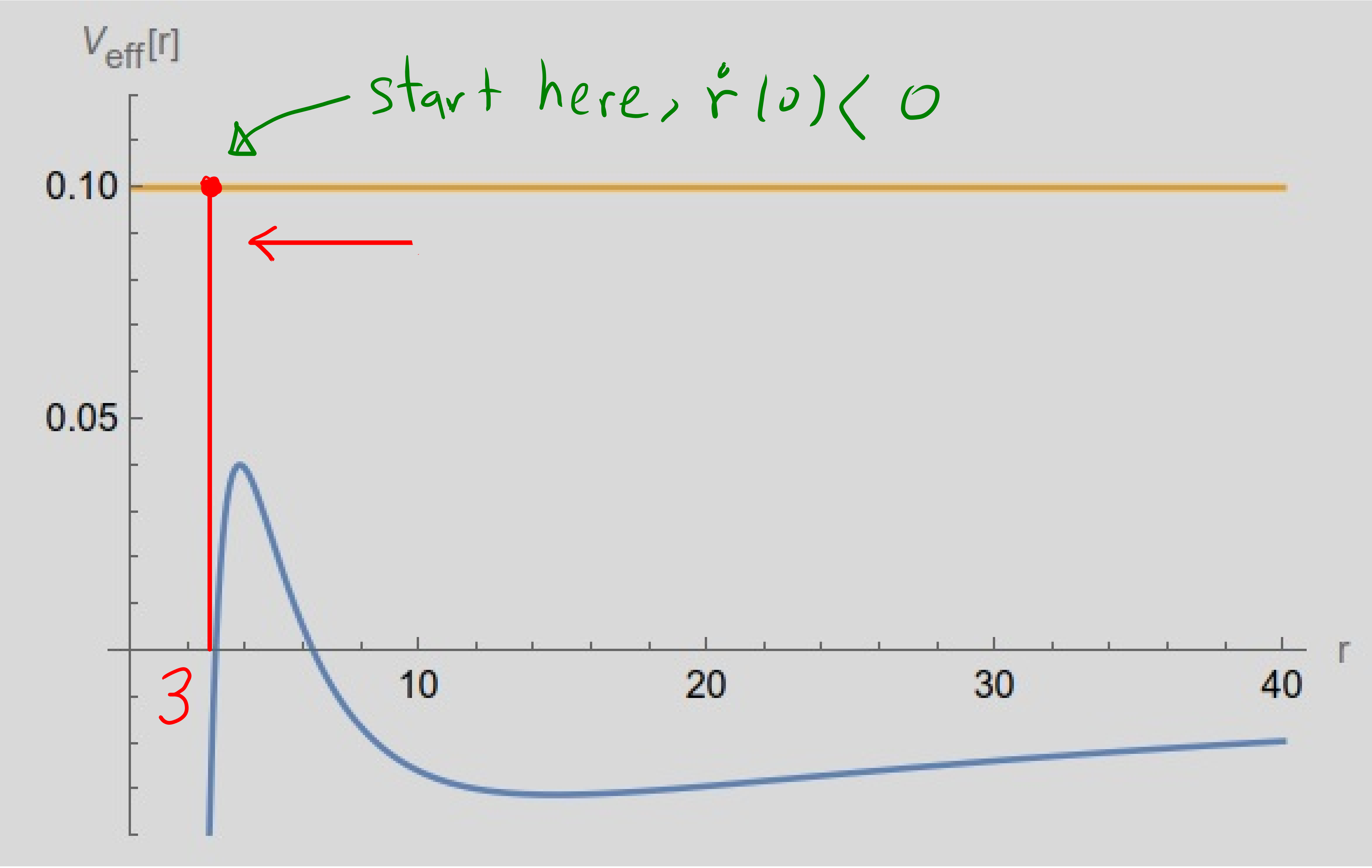


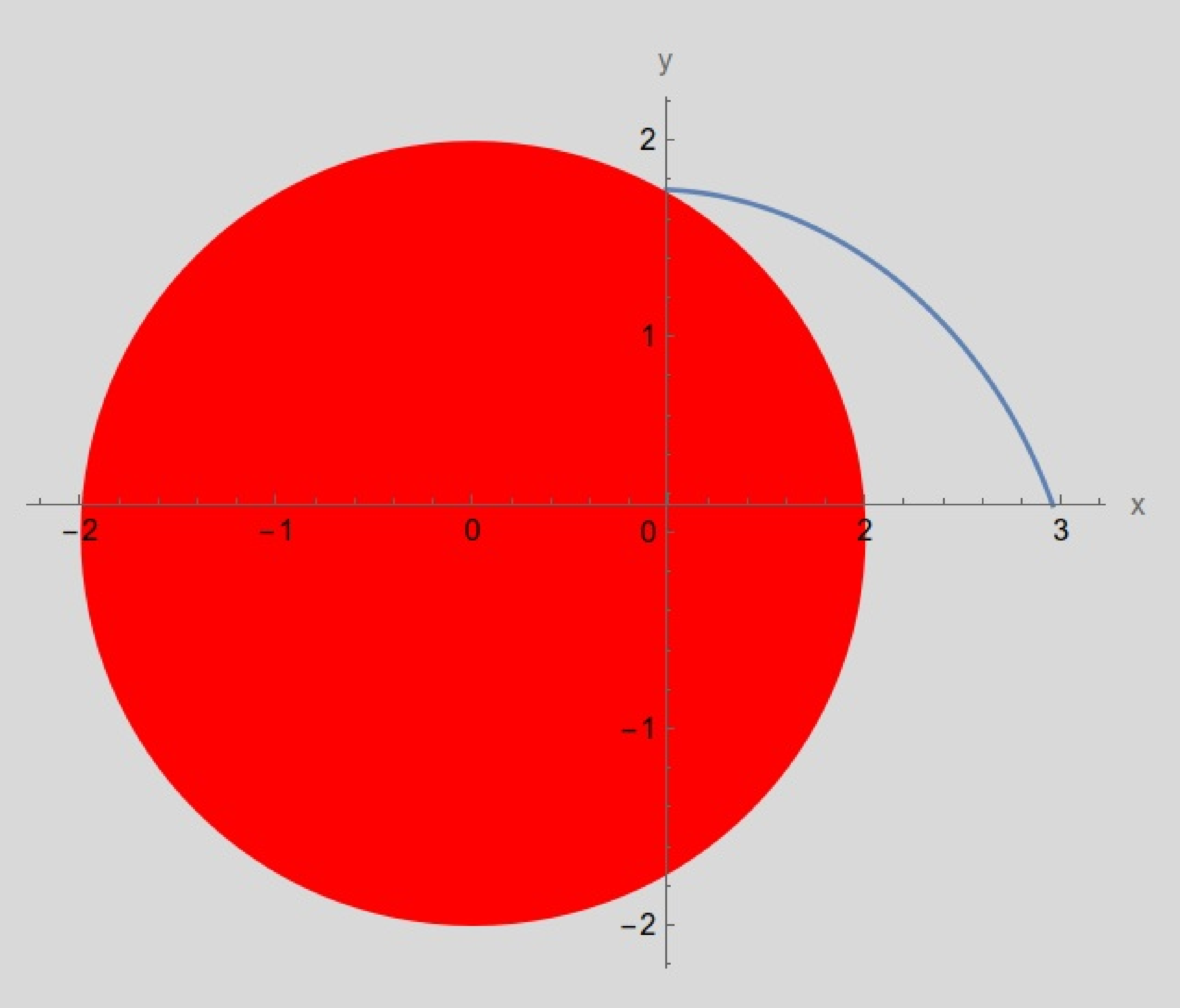


Escape from the $2M < r < r_{max}$ region when

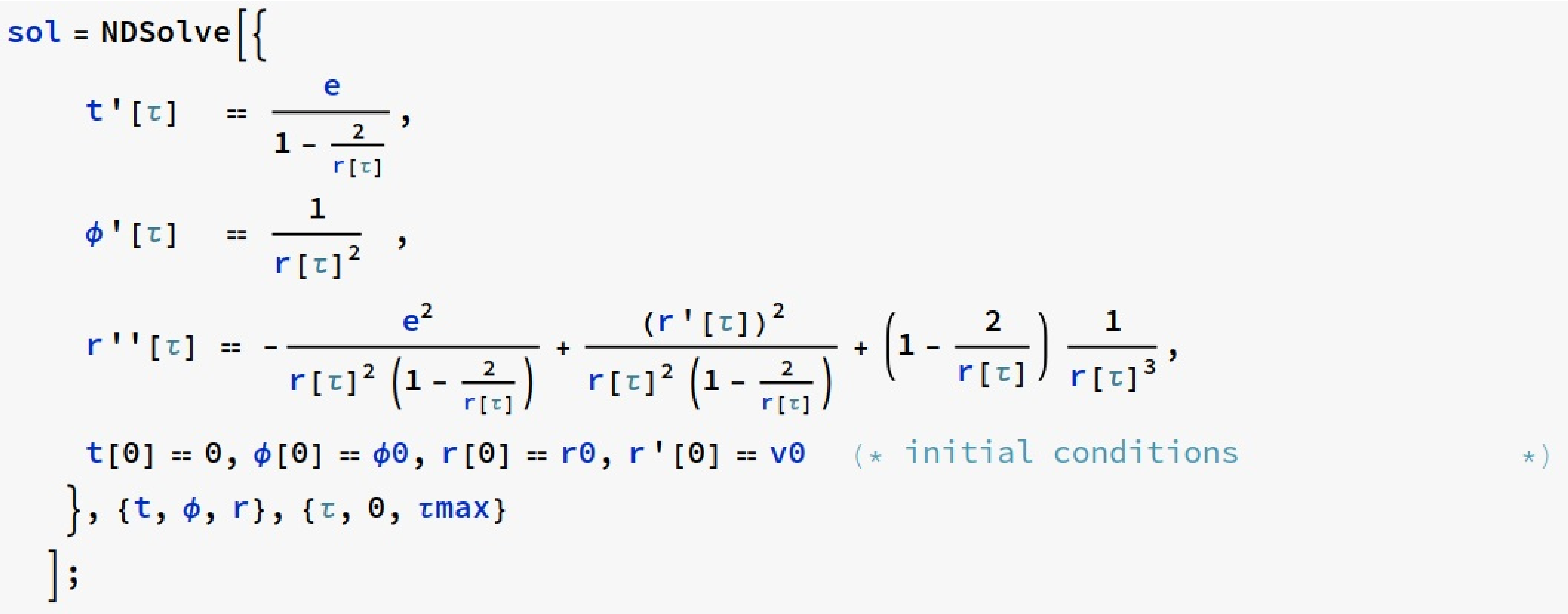
$\mathcal{E} > V(r_{max})$, $\dot{r}(0)>0$

If we go in the wrong direction ... black fate


start here, $\dot{r}(0) < 0$



Massless Particles



sol = NDSolve[{

$$t'[\tau] == \frac{e}{1-\frac{2}{r[\tau]}},$$

$$\phi'[\tau] == \frac{1}{r[\tau]^2},$$

$$r''[\tau] == -\frac{e^2}{r[\tau]^2\left(1-\frac{2}{r[\tau]}\right)} + \frac{(r'[\tau])^2}{r[\tau]^2\left(1-\frac{2}{r[\tau]}\right)} + \left(1-\frac{2}{r[\tau]}\right)\frac{1}{r[\tau]^3},$$

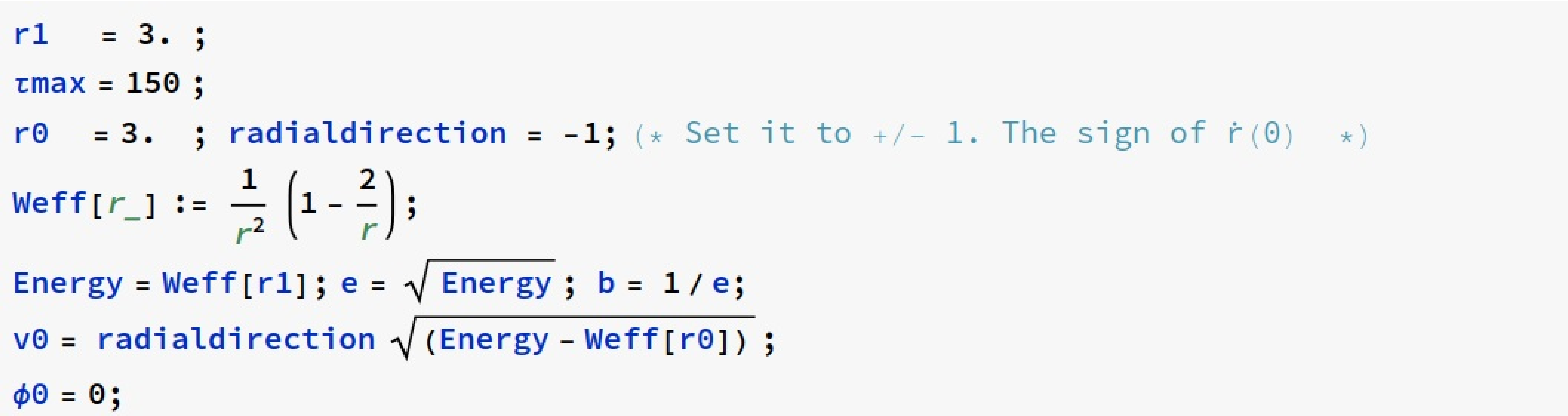
t[0] == 0, ϕ[0] == ϕ0, r[0] == r0, r'[0] == v0 (* initial conditions *)

}, {t, ϕ, r}, {τ, 0, τmax}

];

Same as for massive, simply $\ell \rightarrow 1$ (ℓ has been scaled away)

Massless Particles



```
r1    = 3. ;
τmax = 150 ;
r0   = 3.   ; radialdirection = -1; (* Set it to +/- 1. The sign of ṙ(0)   *)
Weff[r_] := 1/r^2 (1 - 2/r);
Energy = Weff[r1]; e = √Energy ; b = 1/e;
v0 = radialdirection √(Energy - Weff[r0]) ;
ϕ0 = 0;
```

Initial conditions: Set

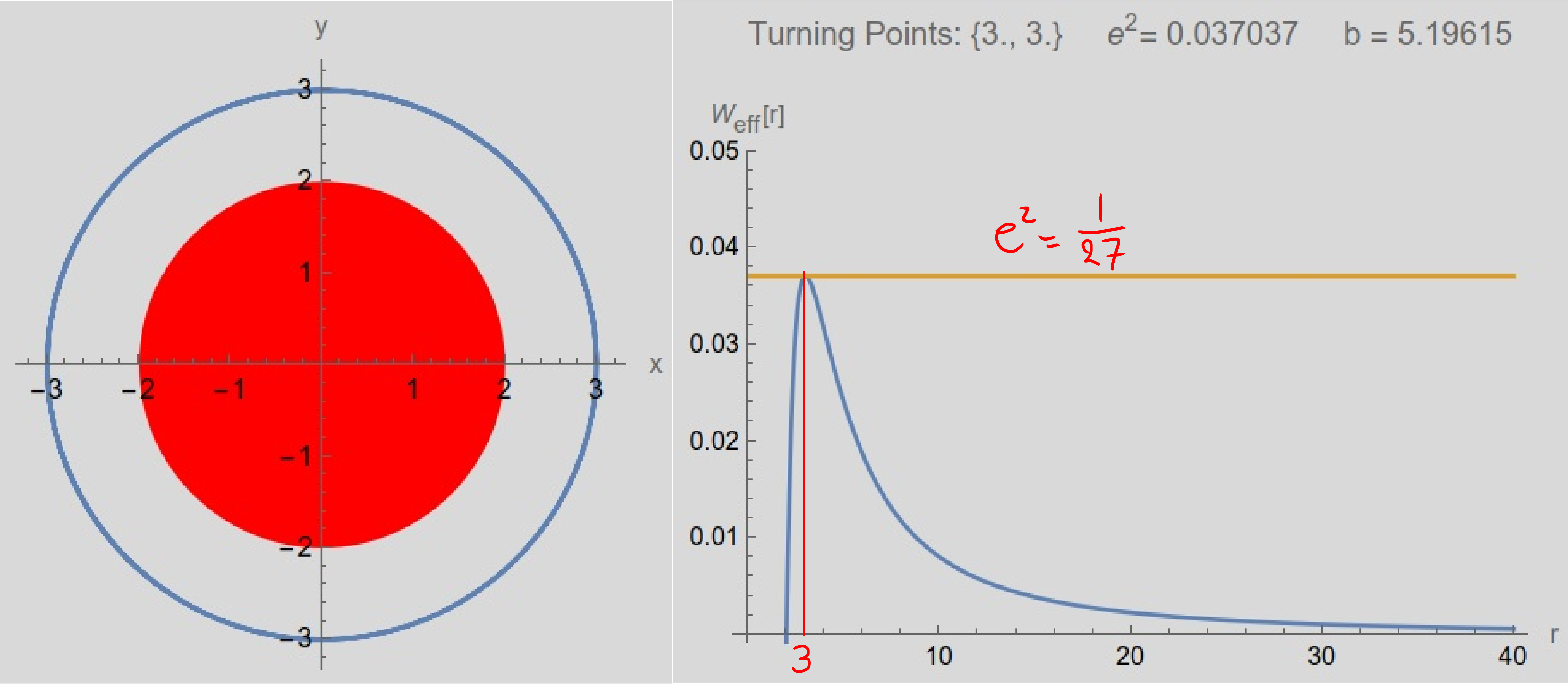
r_1 : a turning point → $e = \{W_{eff}(r_1)\}^{1/2}$

r_0 : $r(0)$

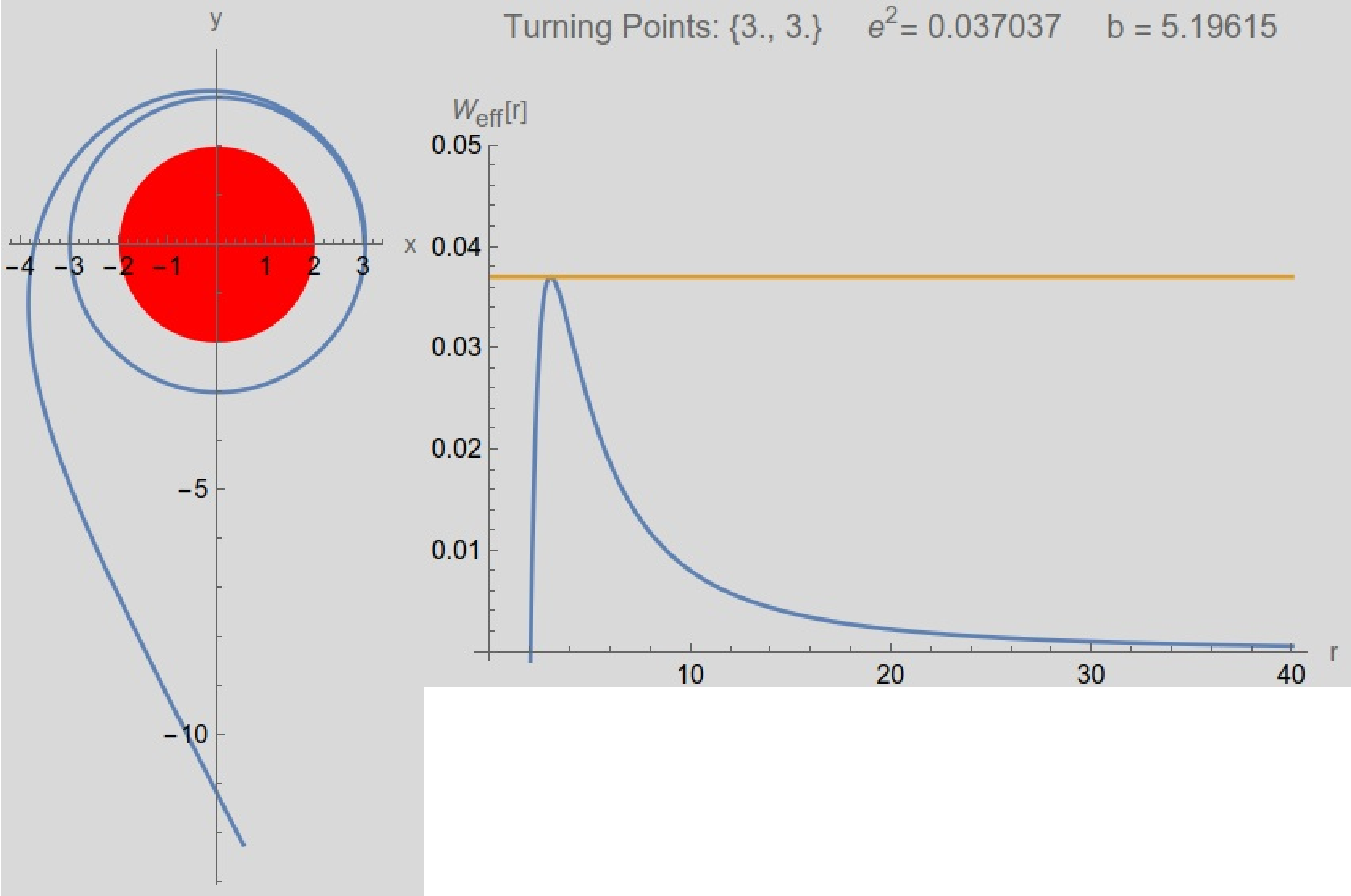
V_0 : $\dot{r}(0) = \pm \{E - W_{eff}(r_0)\}^{1/2}$

ϕ_0 : $\phi(0) = 0$

Circular orbits: $r1 = 3$, $r(0) = 3$

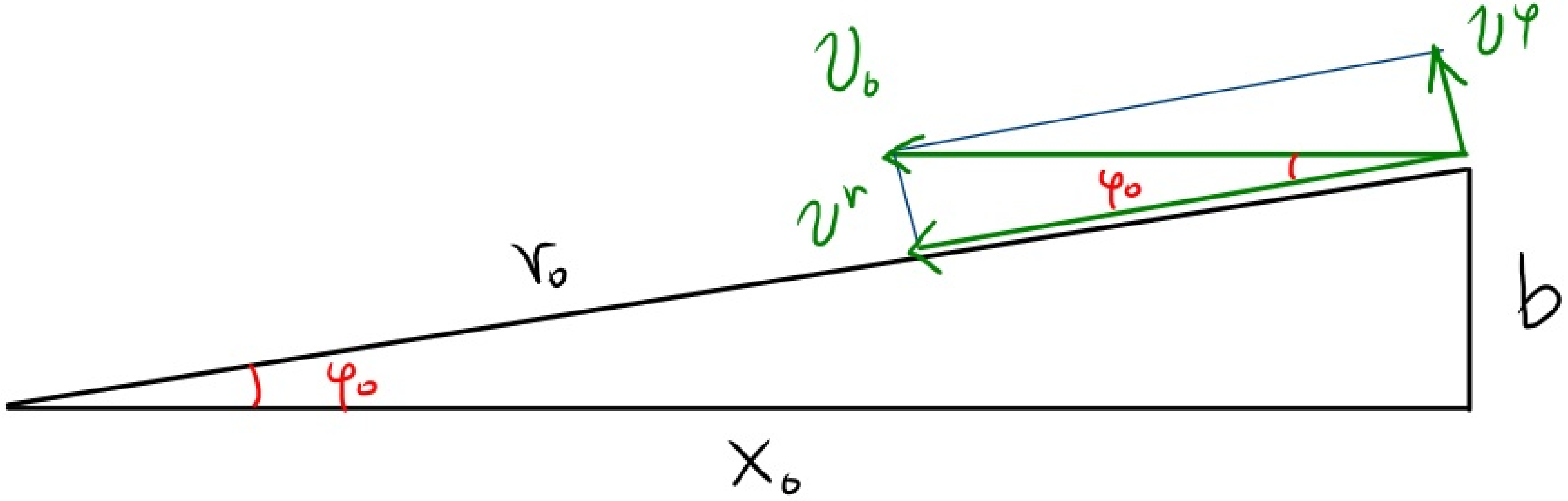


Circular orbits are unstable: $r_1 \to 3.0001$ $\qquad r(0) = 3.0001$



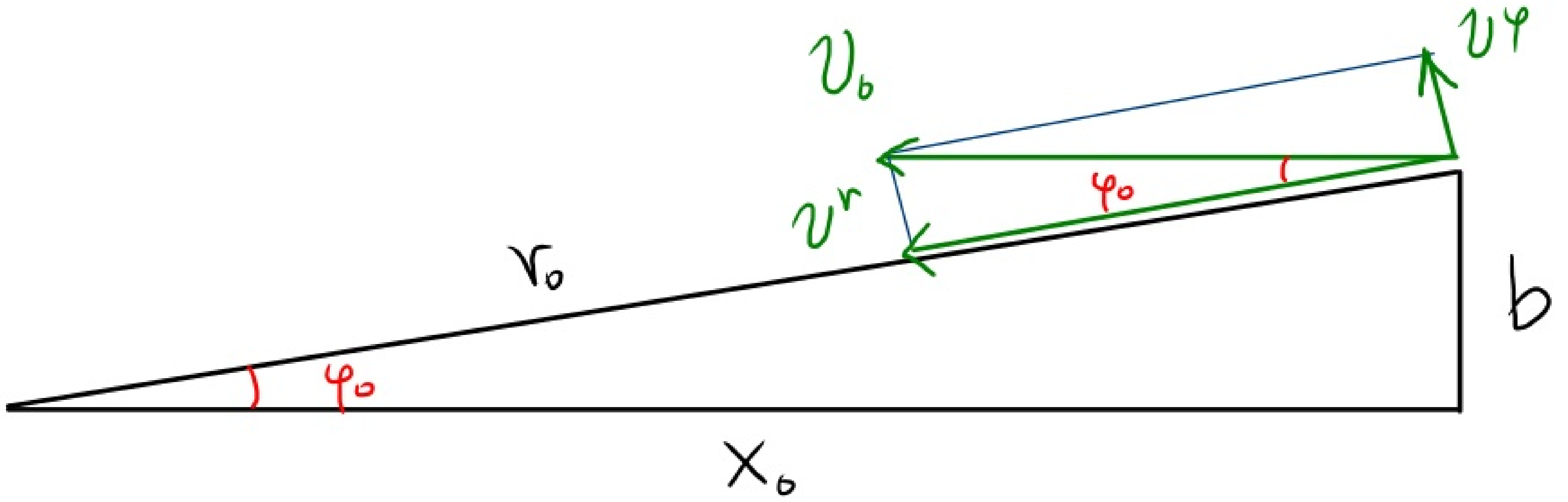
Scattering - Deflection of light

- emit a photon at $r = \infty$
  with $\vec{v} = -v_0 \hat{x} = -\hat{x} \quad (v_0 = 1)$



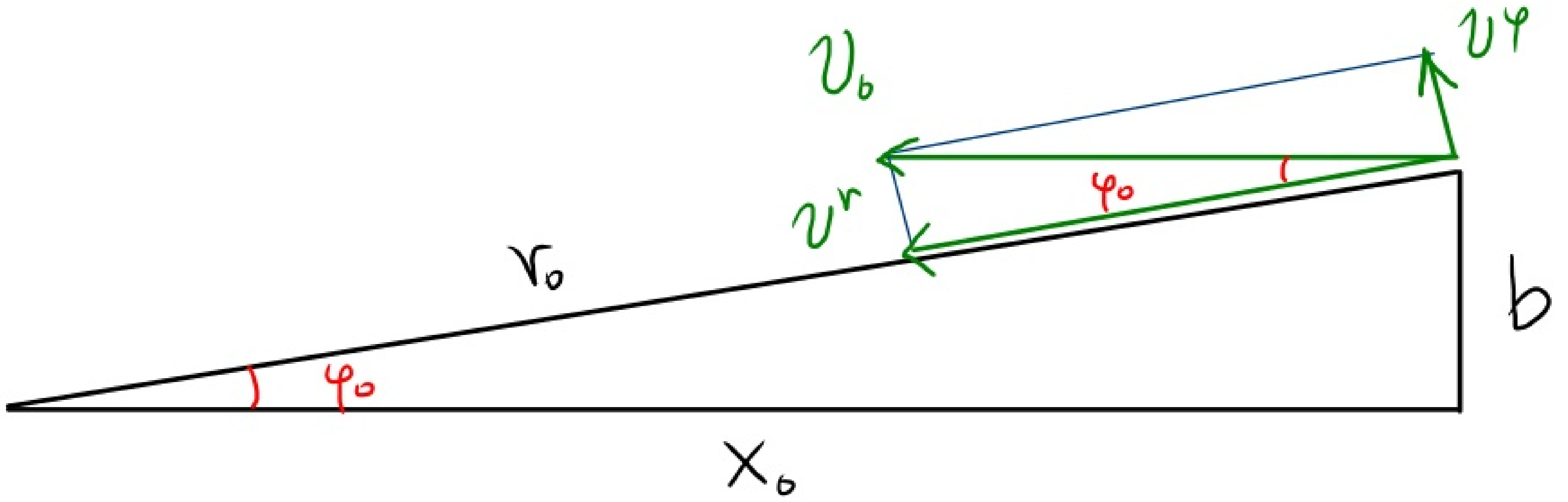
Scattering - Deflection of light

- emit a photon at $r = \infty$ with $\vec{v} = -v_0 \hat{x} = -\hat{x}$ $(v_0 = 1)$
- choose $x_0 \gg 2$, b impact parameter



Scattering - Deflection of light

- emit a photon at $r = \infty$ with $\vec{v} = -v_0 \hat{x} = -\hat{x}$ $(v_0 = 1)$
- choose $x_0 \gg 2$, b impact parameter



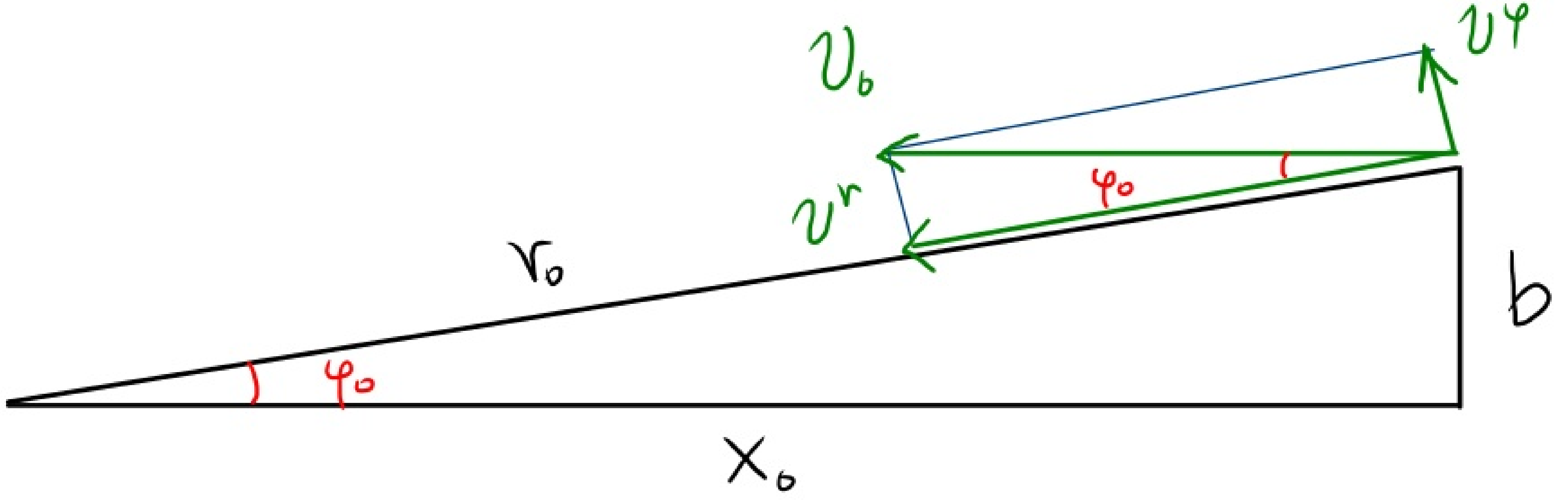
$$r_0 = (x_0^2 + b^2)^{1/2}$$

$$\phi_0 = \tan^{-1} \frac{b}{x_0}$$

$$v^r = -\underset{1}{\underset{\shortparallel}{v_0}} \cos\phi_0$$

Scattering - Deflection of light

- emit a photon at $r = \infty$ with $\vec{v} = -v_0 \hat{x} = -\hat{x}$ $(v_0 = 1)$
- choose $x_0 \gg 2$, b impact parameter



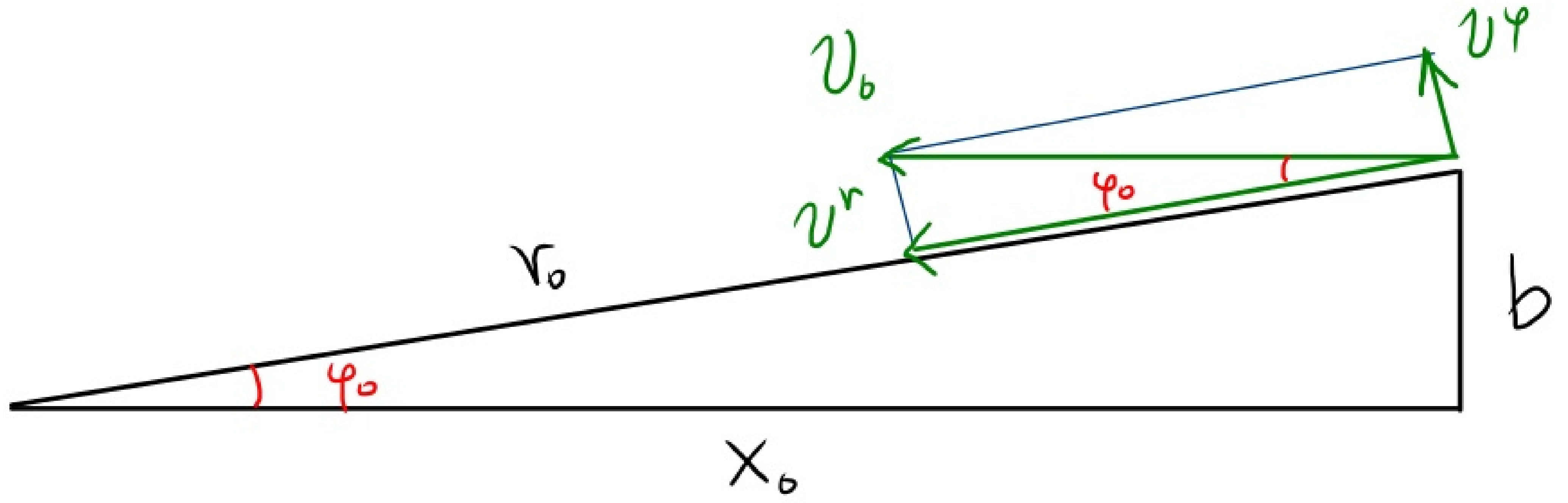
$$r_0 = \left(x_0^2 + b^2 \right)^{1/2}$$

$$\phi_0 = \tan^{-1} \frac{b}{x_0}$$

$$v^r = - \quad \cos\phi_0 = \frac{dr}{dt}(0) = \frac{dr/d\lambda}{dt/d\lambda}(0) = \frac{\dot{r}(0)}{e\left(1-\frac{2}{r_0}\right)^{-1}} \Rightarrow \dot{r}(0) = -e\left(1-\frac{2}{r_0}\right)^{-1}\cos\phi_0$$

Scattering - Deflection of light

- emit a photon at $r = \infty$ with $\vec{v} = -v_0 \hat{x} = -\hat{x}$ $\quad (v_0 = 1)$
- choose $x_0 \gg 2$, b impact parameter



$$r_0 = (x_0^2 + b^2)^{1/2}$$

$$\phi_0 = \tan^{-1} \frac{b}{x_0}$$

$$\dot{r}(0) = -e\left(1 - \frac{2}{r_0}\right)^{-1} \cos\phi_0$$

$$e^2 = [\dot{r}(0)]^2 + W_{eff}(r_0) \longrightarrow b = \frac{1}{e}$$

redefine b:
it is only appxtly
the impact parameter

Scattering - Deflection of light

- emit a photon at $r=\infty$ with $\vec{v} = -v_0\hat{x} = -\hat{x}$ $\quad (v_0 = 1)$
- choose $x_0 \gg 2$, b impact parameter

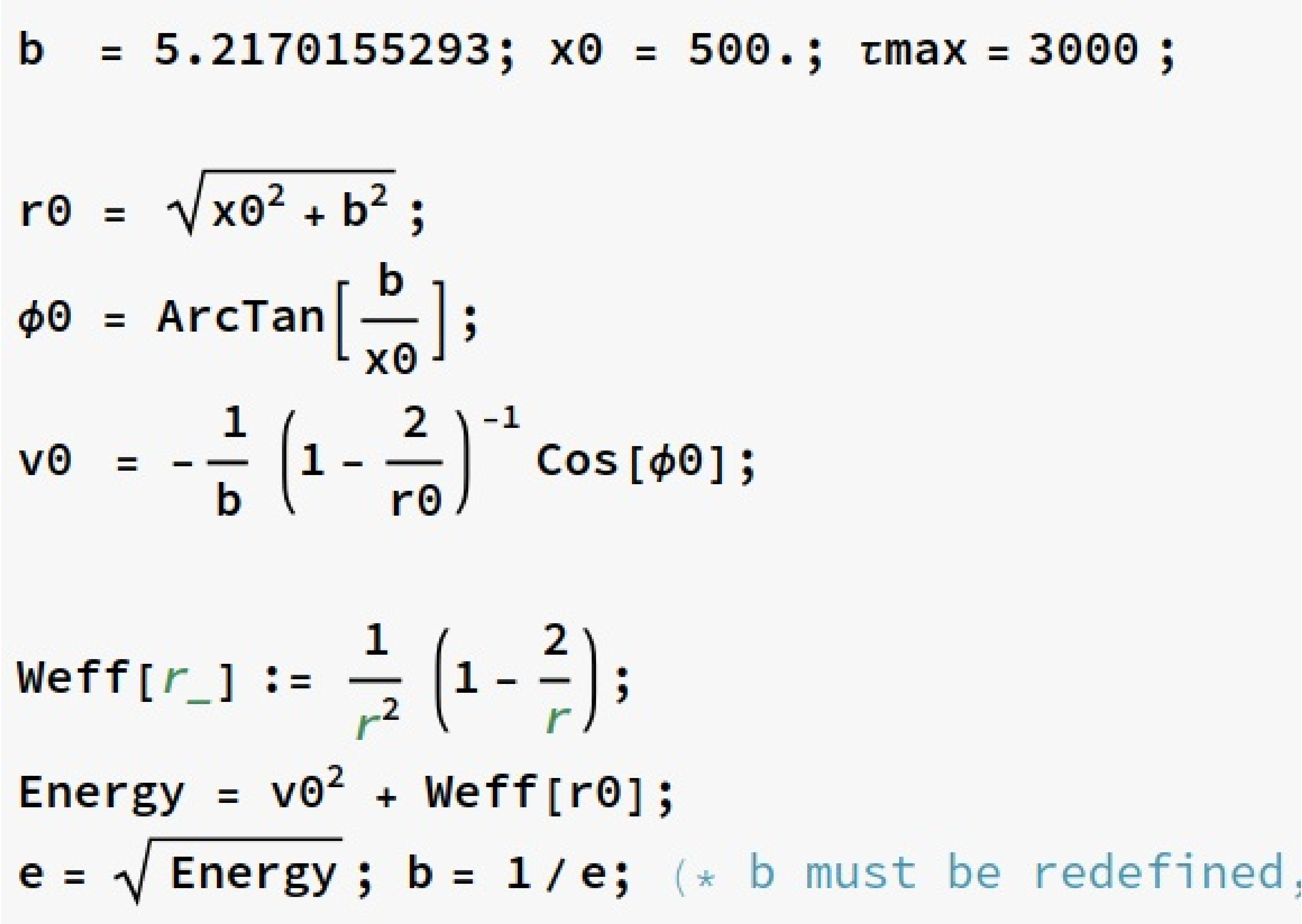
$$r_0 = (x_0^2 + b^2)^{1/2}$$

$$\phi_0 = \tan^{-1} \frac{b}{x_0}$$

$$\dot{r}(0) = -e\left(1-\frac{2}{r_0}\right)^{-1} \cos\phi_0$$

$$e^2 = [\dot{r}(0)]^2 + W_{eff}(r_0)$$

$$b = \frac{1}{e}$$



$$\mathtt{b\ \ =\ 5.2170155293;\ \ x0\ =\ 500.;\ \ \tau max = 3000\ ;}$$

$$\mathtt{r0\ =\ }\sqrt{\mathtt{x0}^2 + \mathtt{b}^2}\ ;$$

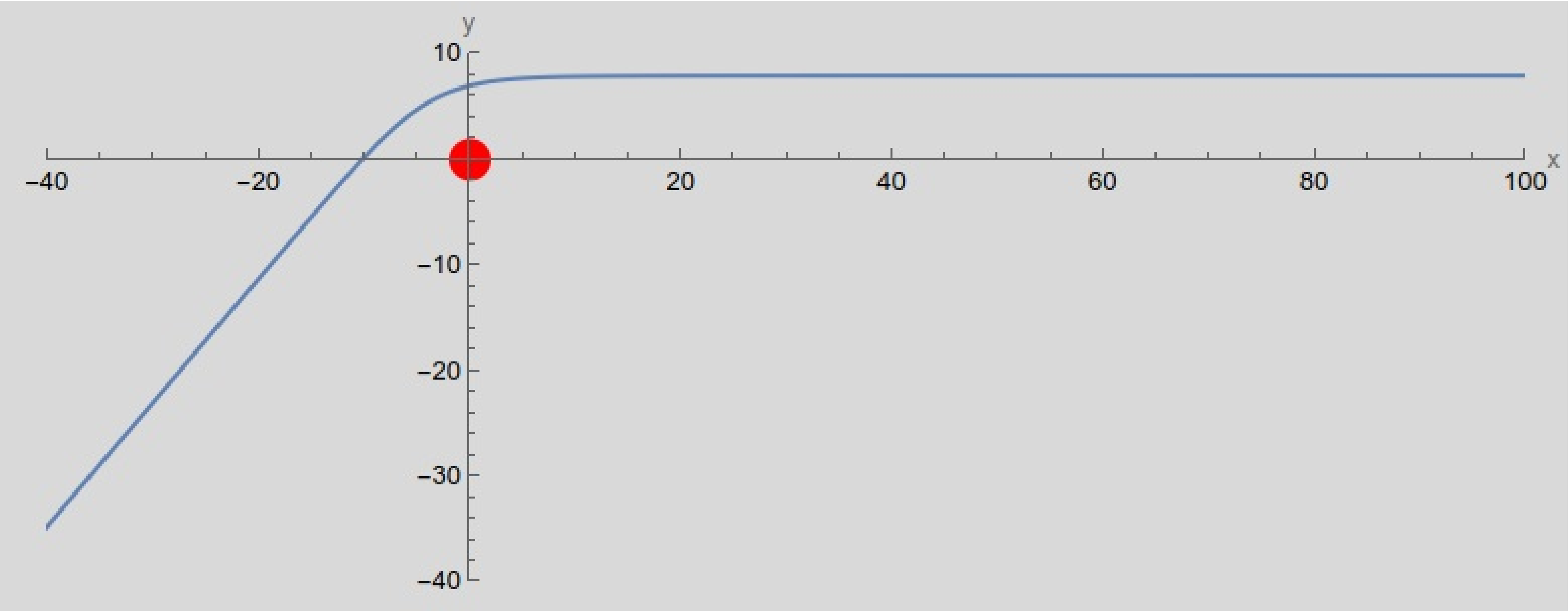
$$\mathtt{\phi 0\ =\ ArcTan}\left[\frac{\mathtt{b}}{\mathtt{x0}}\right];$$

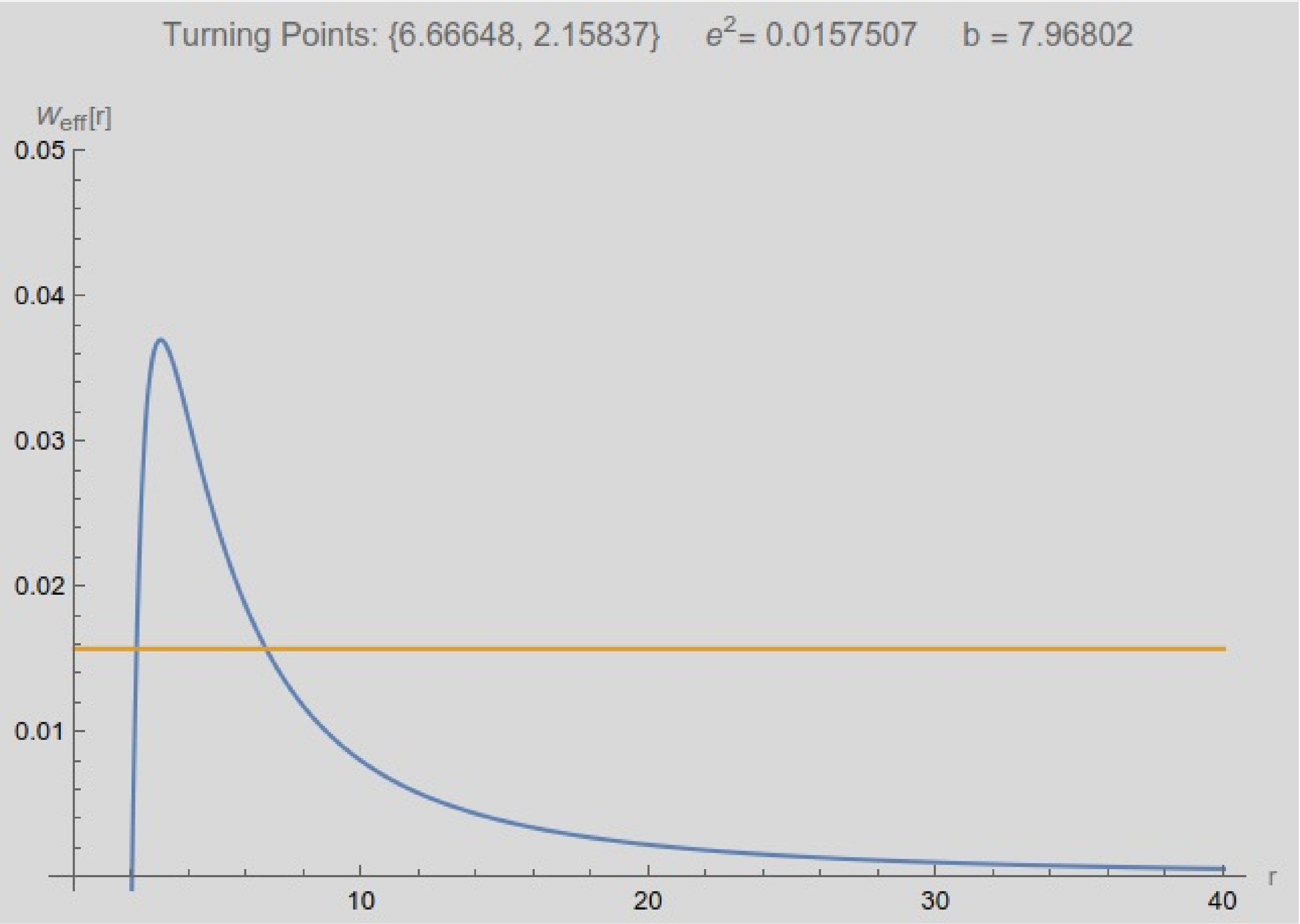
$$\mathtt{v0\ =\ -}\frac{1}{\mathtt{b}}\left(1-\frac{2}{\mathtt{r0}}\right)^{-1}\mathtt{Cos[\phi 0];}$$

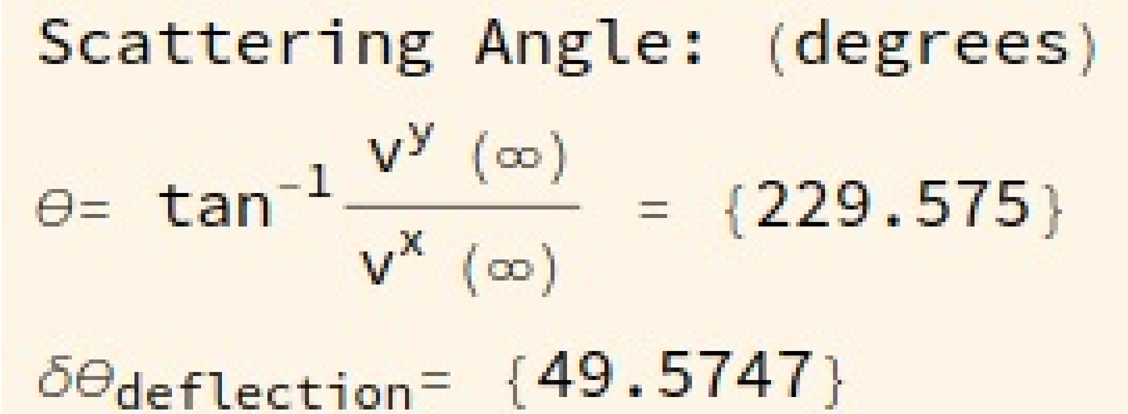
$$\mathtt{Weff[}r\_\mathtt{]\ :=\ }\frac{1}{r^2}\left(1-\frac{2}{r}\right);$$

$$\mathtt{Energy\ =\ v0}^2\mathtt{\ +\ Weff[r0];}$$

$$\mathtt{e\ =\ }\sqrt{\mathtt{Energy}}\ ;\ \mathtt{b\ =\ 1/e;}\ \ \mathtt{(*\ b\ must\ be\ redefined}$$

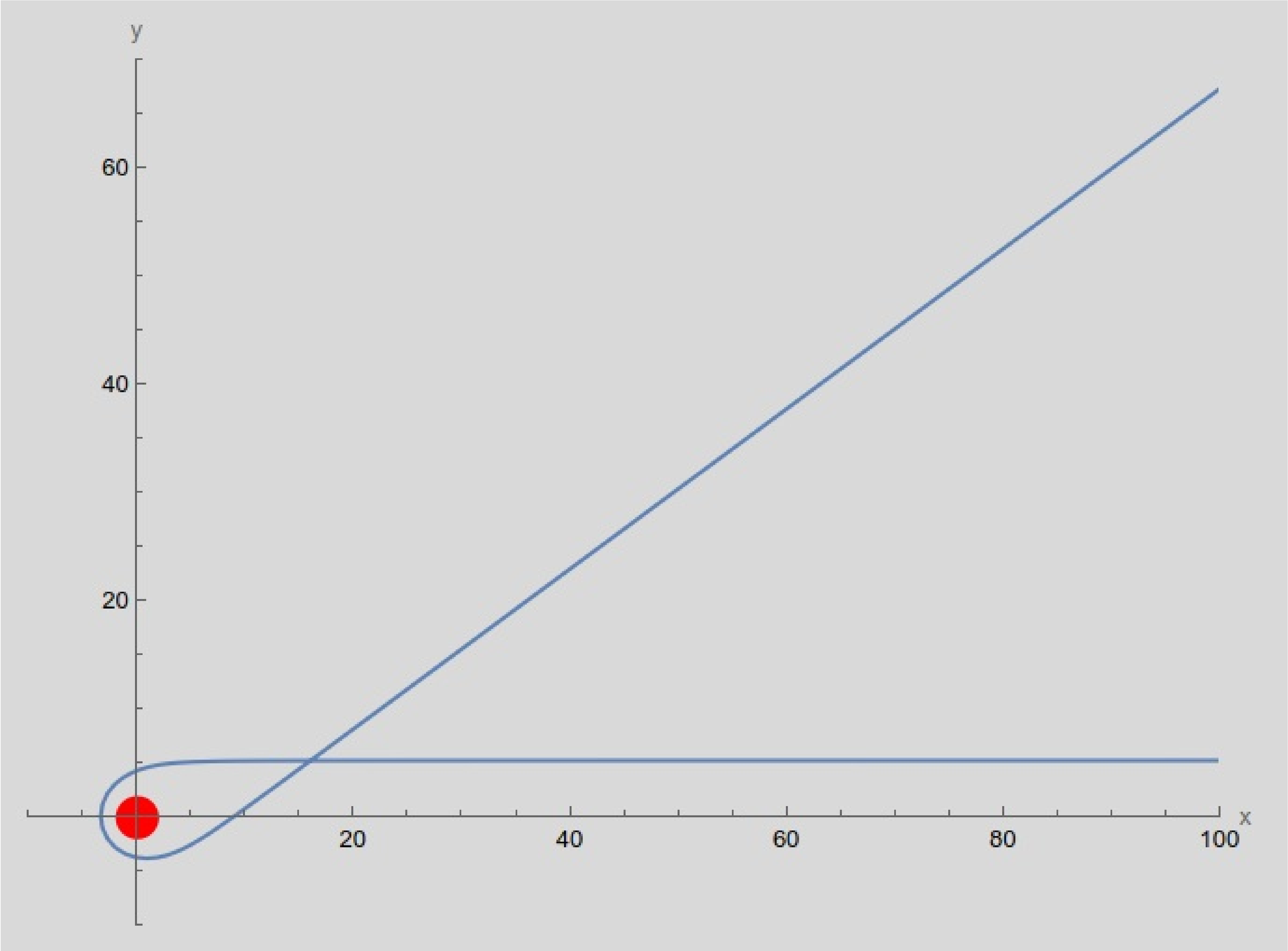


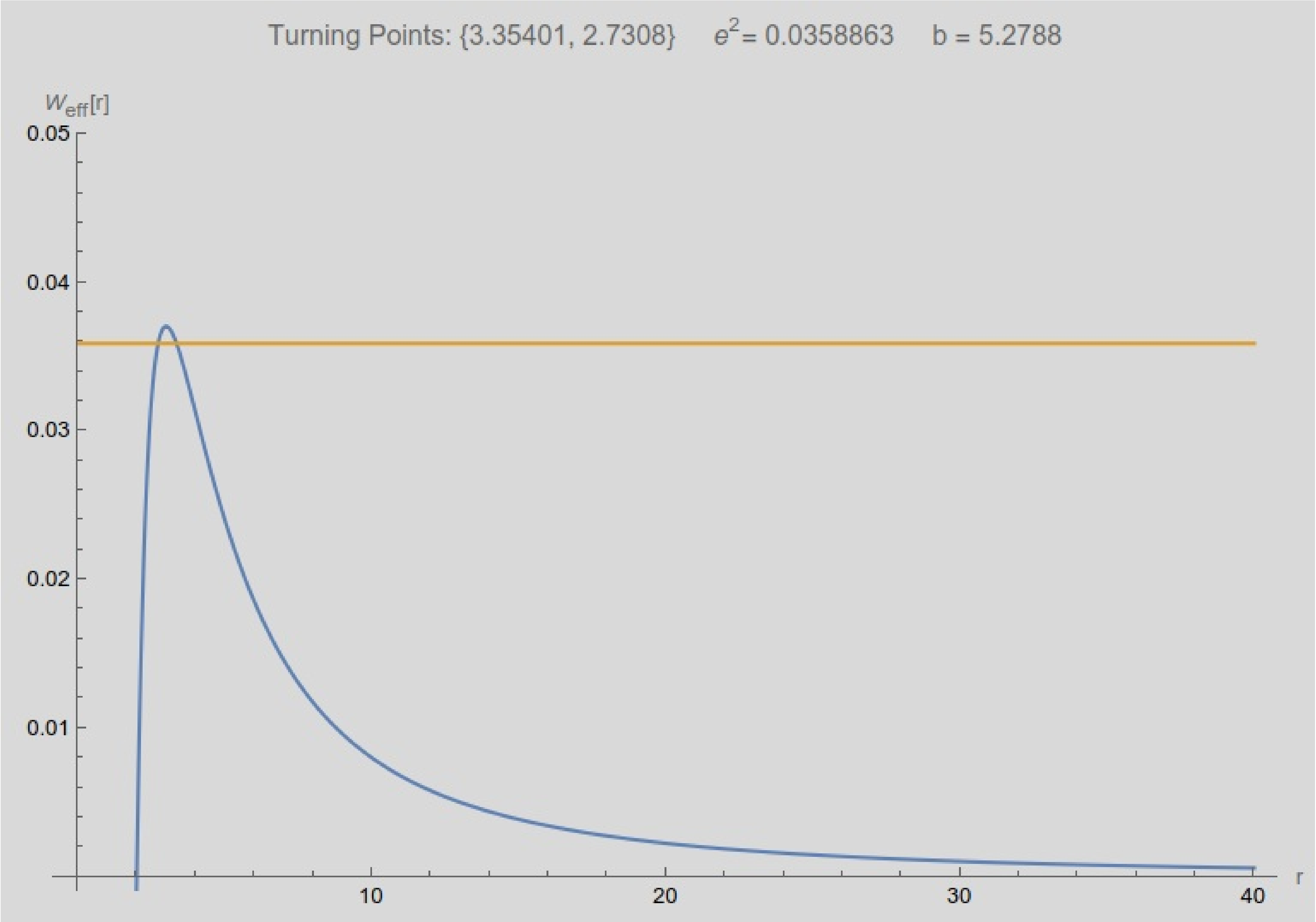

Turning Points: {6.66648, 2.15837} e^2= 0.0157507   b = 7.96802

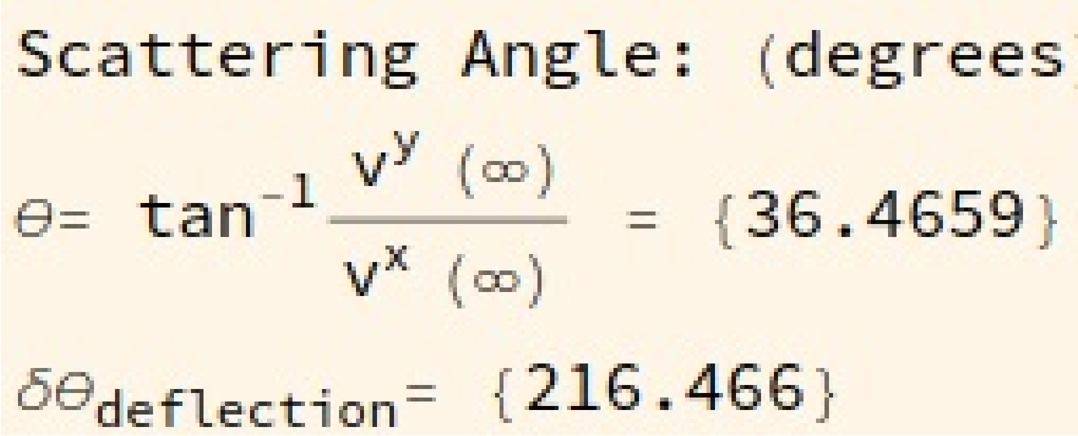

Scattering Angle: (degrees)

$$\theta = \tan^{-1}\frac{v^{y}\ (\infty)}{v^{x}\ (\infty)} = \{229.575\}$$

$$\delta\theta_{\mathrm{deflection}} = \{49.5747\}$$

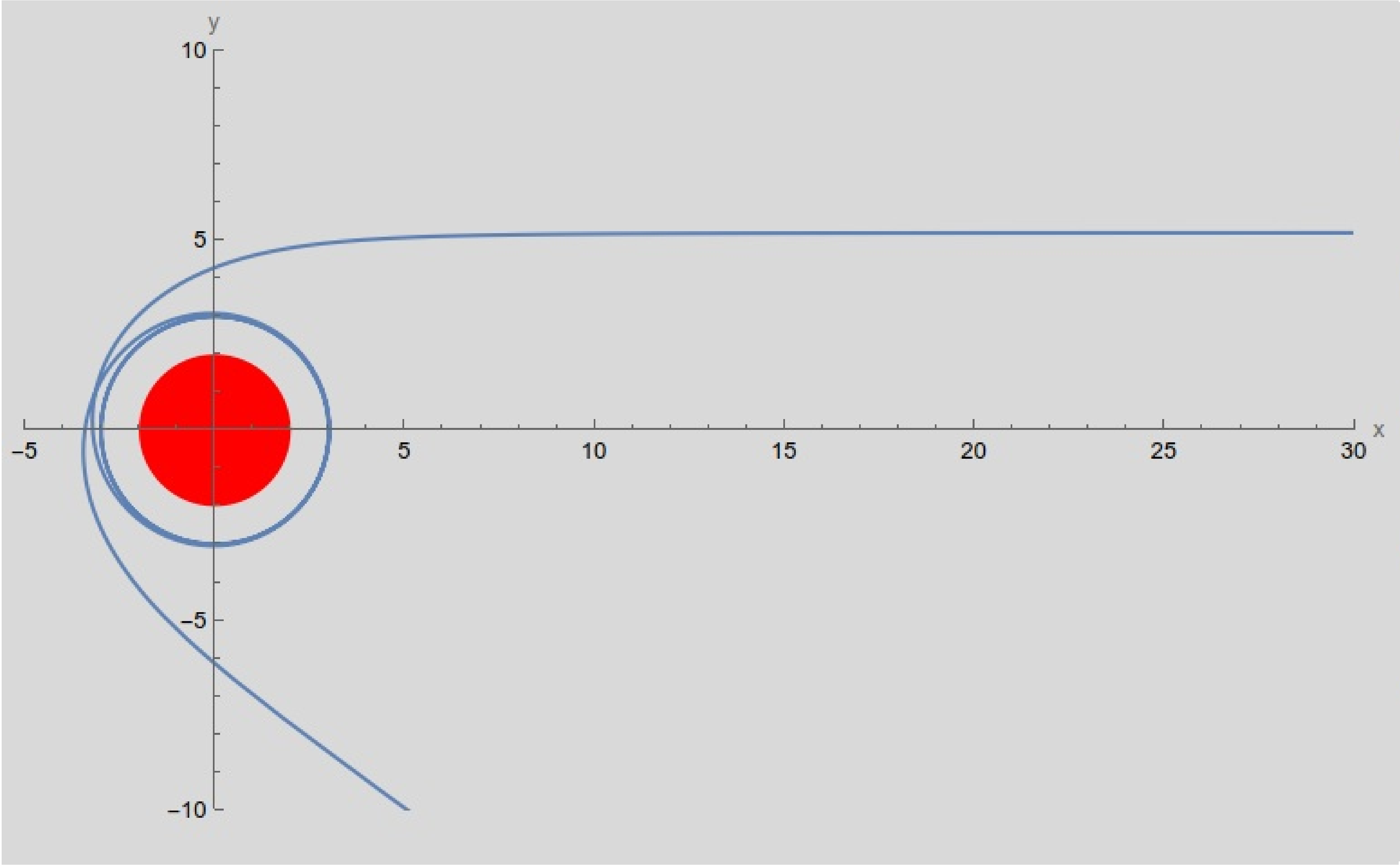


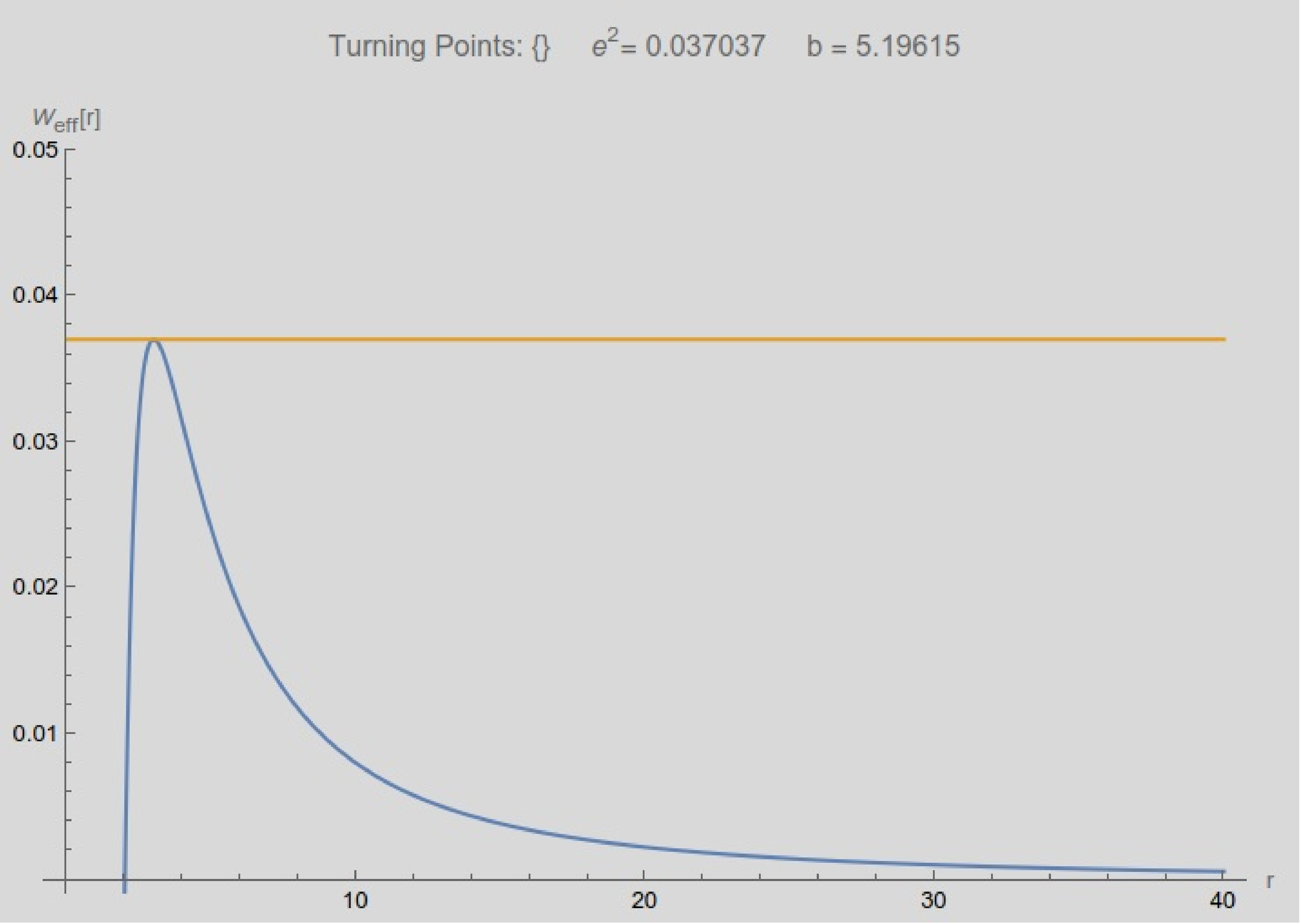

Turning Points: {3.35401, 2.7308} e^2= 0.0358863   b = 5.2788

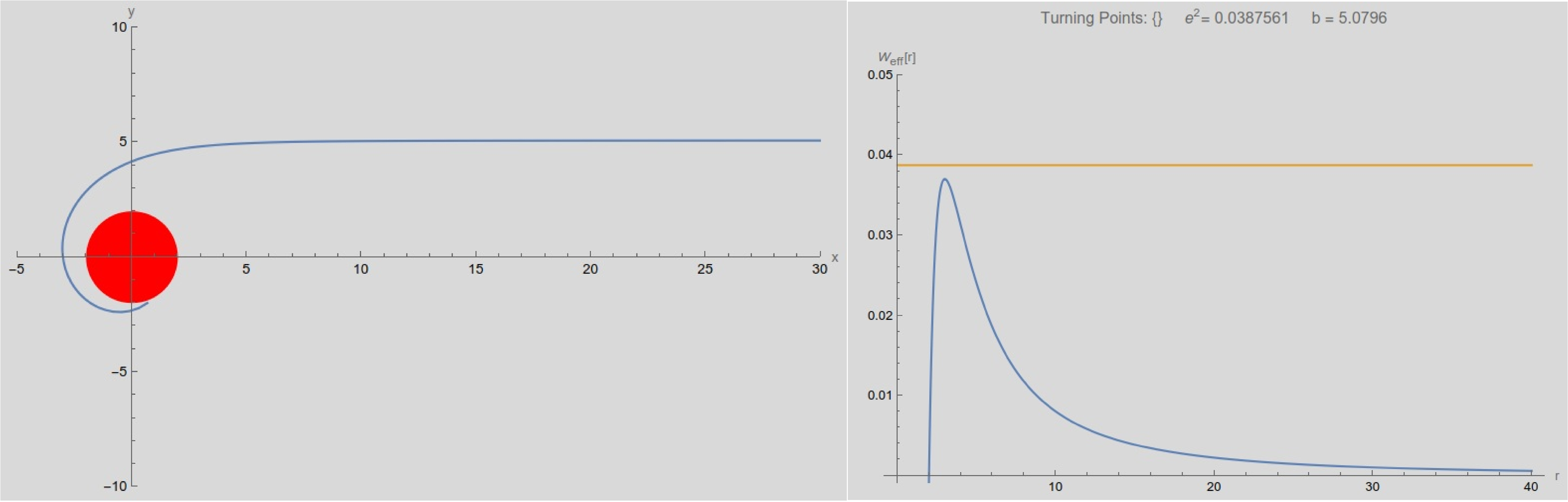

Scattering Angle: (degrees

$$\theta = \tan^{-1}\frac{v^{y}(\infty)}{v^{x}(\infty)} = \{36.4659\}$$

$$\delta\theta_{\text{deflection}} = \{216.466\}$$


y
10
5
−5
−10
x
−5
5
10
15
20
25
30


Turning Points: {} e^2= 0.037037 b = 5.19615
$W_{\mathrm{eff}}[r]$
0.05
0.04
0.03
0.02
0.01
r
10
20
30
40



Turning Points: {} e^2= 0.0387561 b = 5.0796