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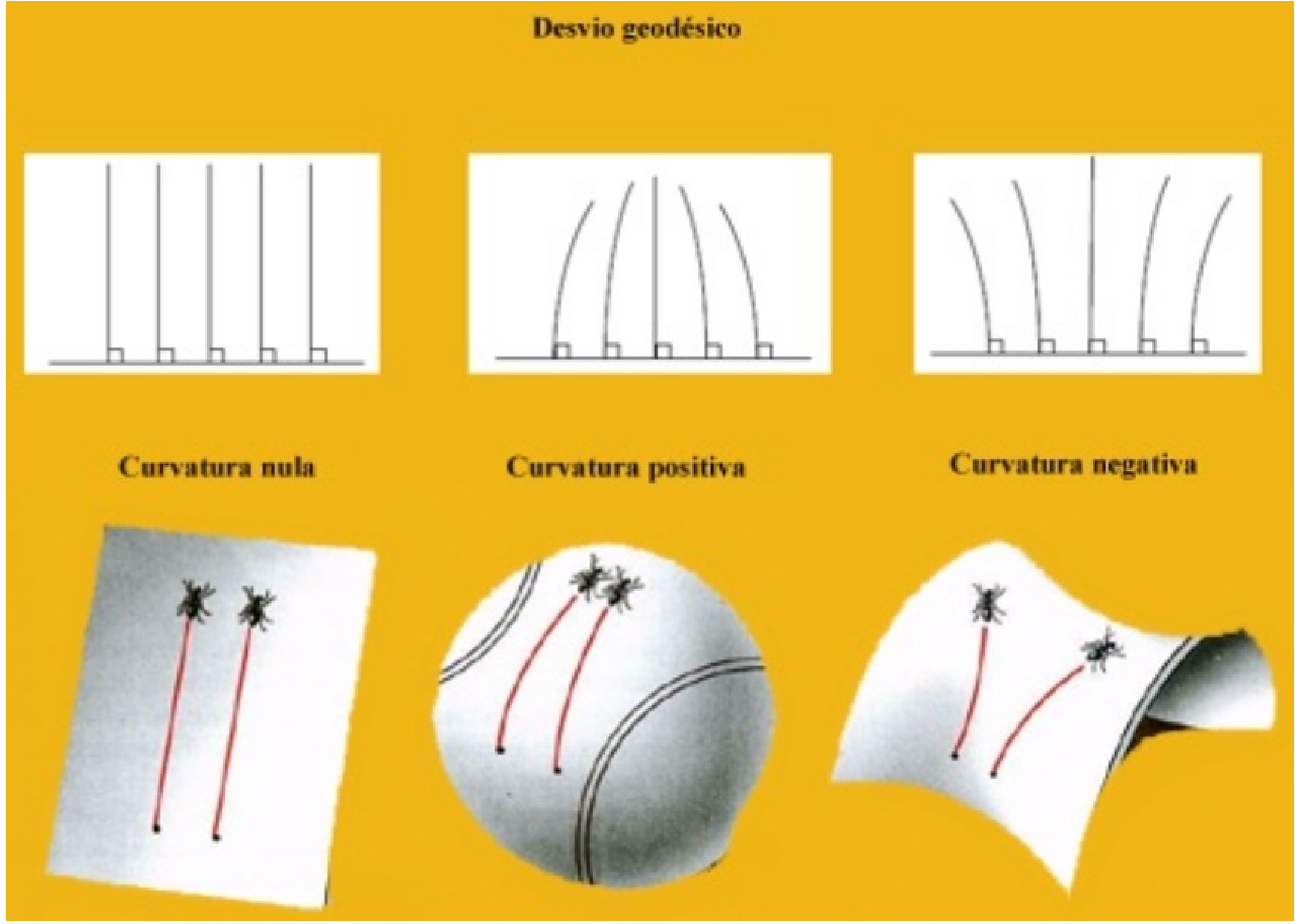
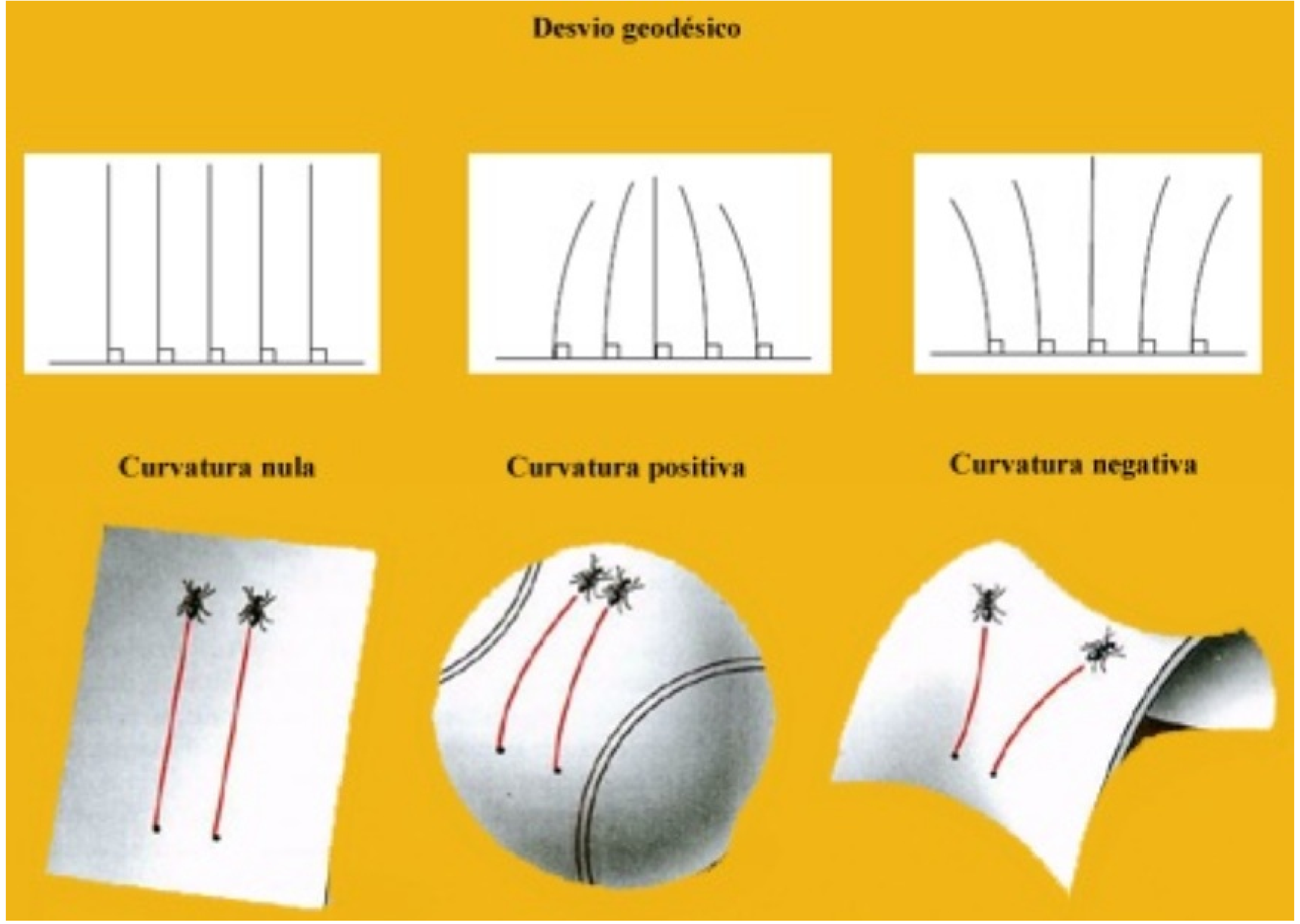
  (choice of) affine connection $\Rightarrow$ curvature
- (choice of) metric $\rightarrow$ Levi-Civita connection $\rightarrow$ curvature

  but curvature can be defined w/o metric, e.g. gauge theories

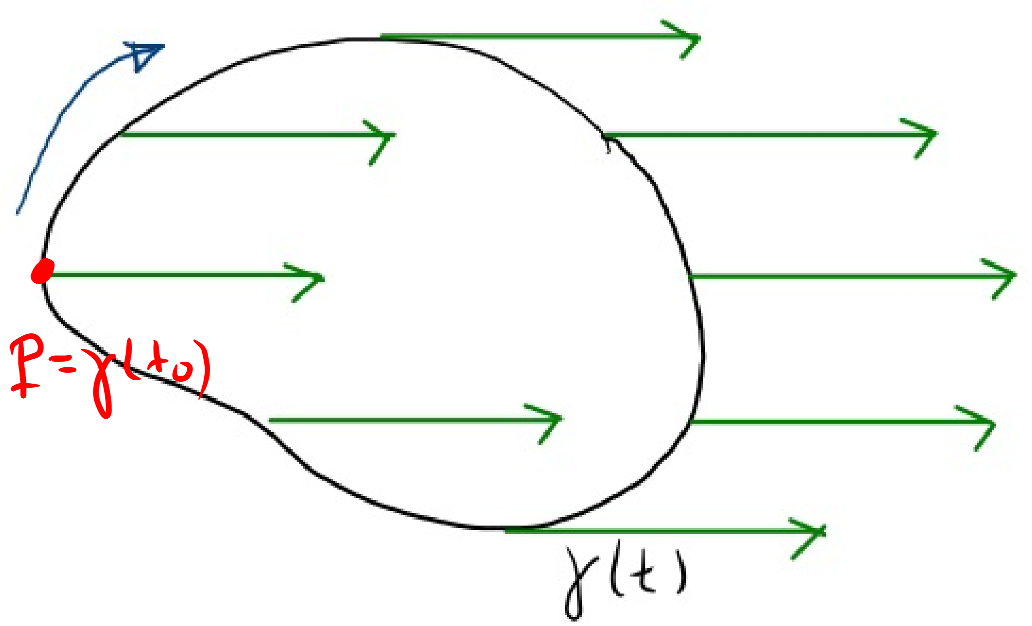


- In flat space, parallel geodesics remain parallel

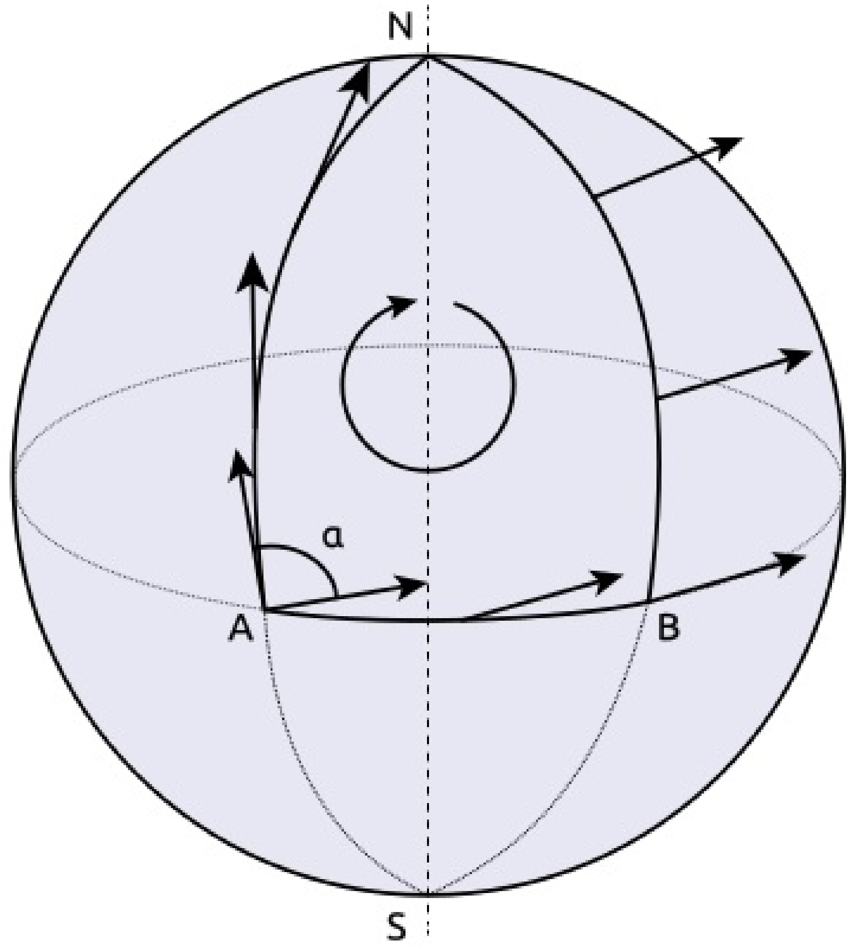


- In flat space, parallel geodesics remain parallel
- Curvature has the effect of making initially parallel geodesics to deviate

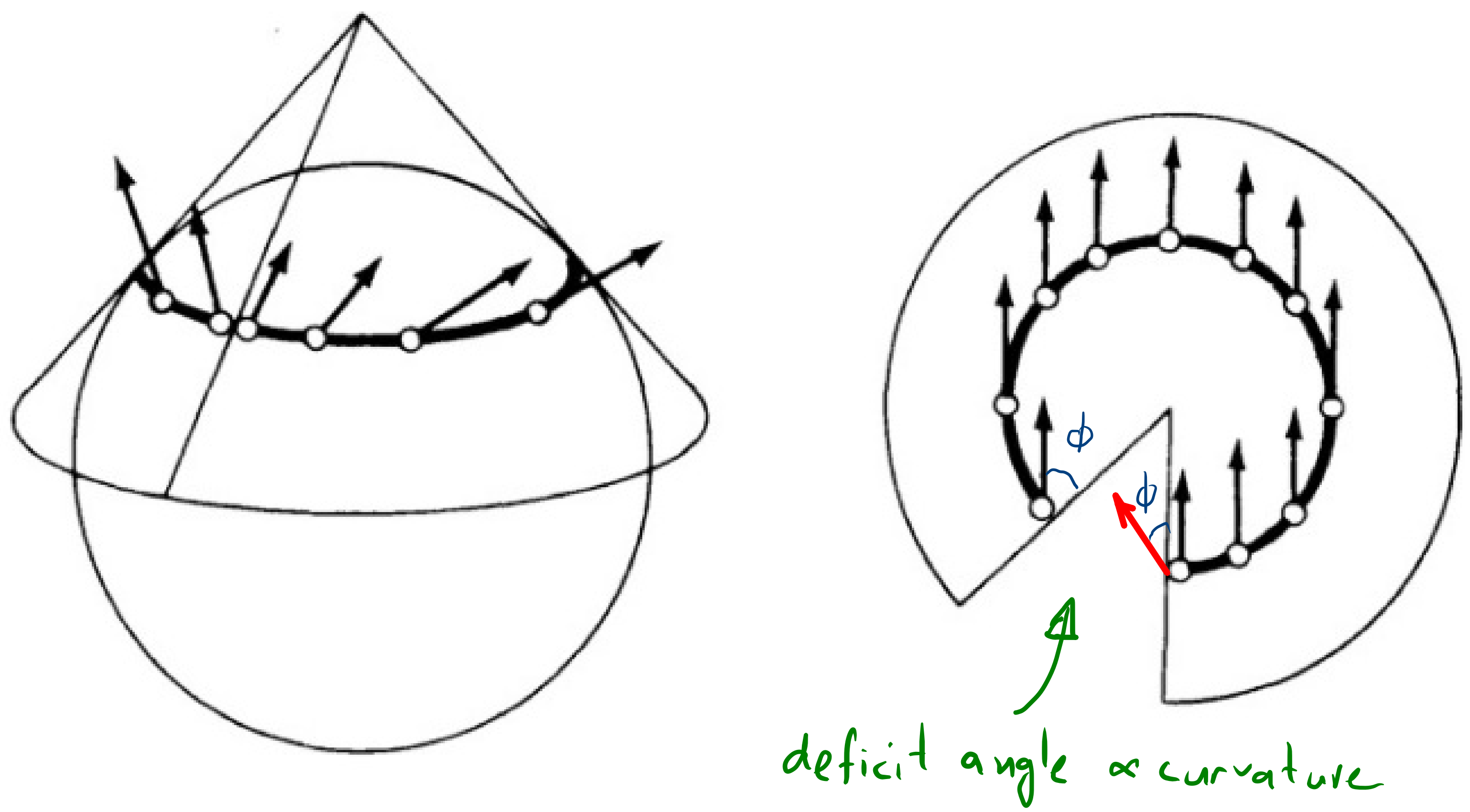
(relative acceleration) $\propto$ (curvature)



Flat Space Parallel Transport



Wikipedia

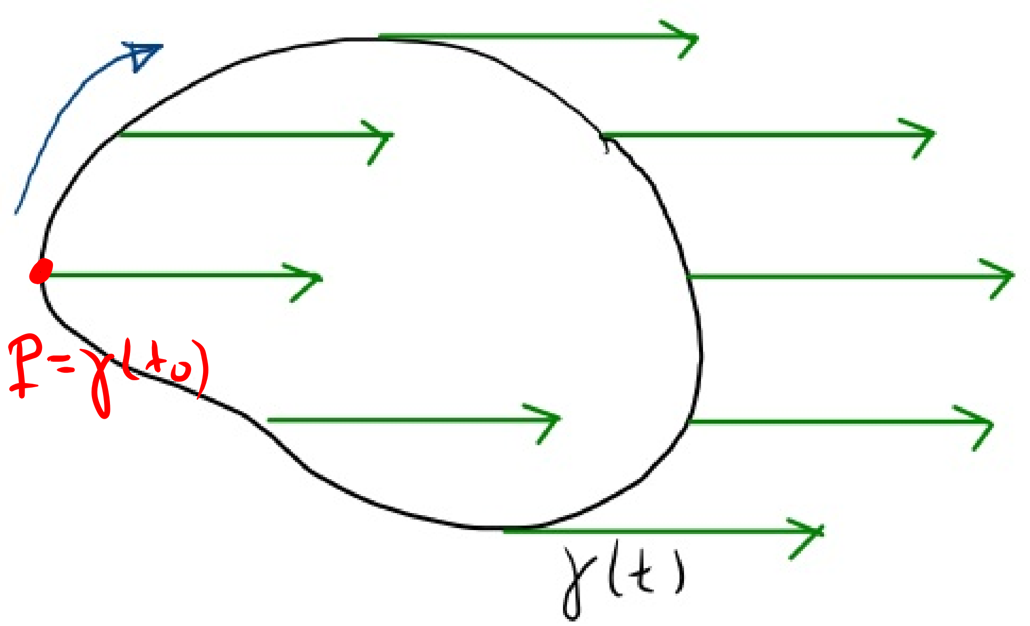


Vladimir I. Arnold, Mathematical Methods of Classical Mechanics (New York: Springer, 1989), 302, Fig. 231.

Cone with metric dx^2+dy^2 on plane

* Flat Space: Parallel Transport of vector along closed curve leaves vector invariant at P
* Curved Space: " " " " $V \to V + \delta V$ at P

$$\delta V \propto (\text{curvature})$$



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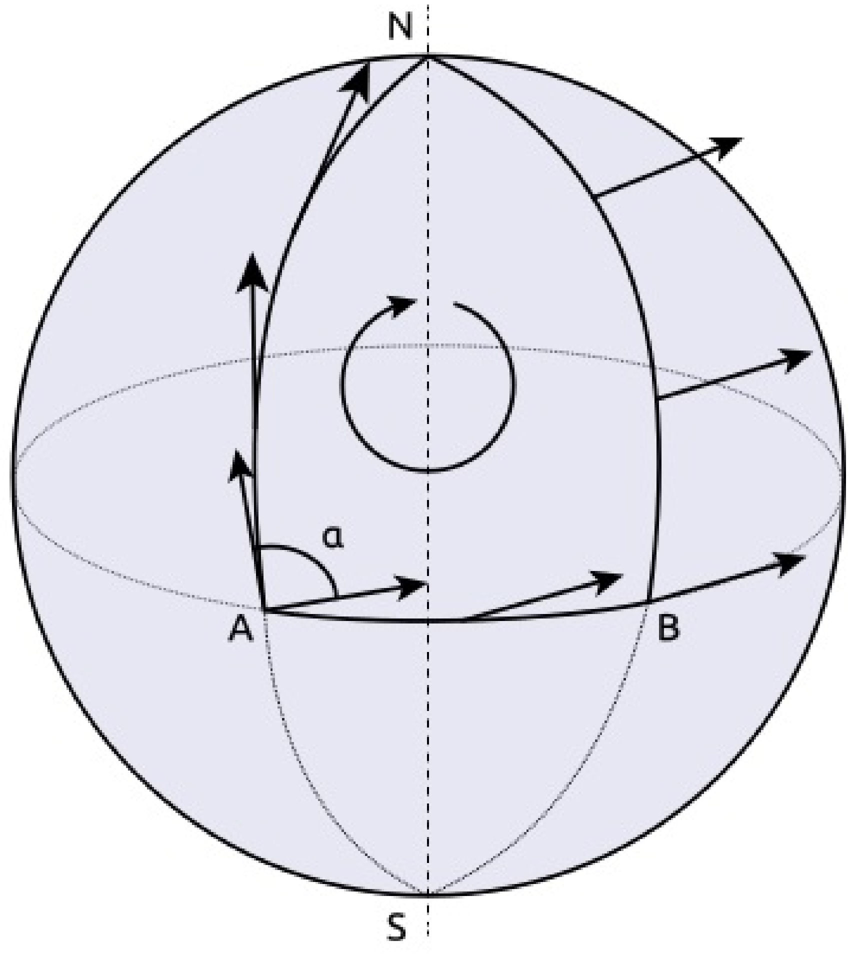

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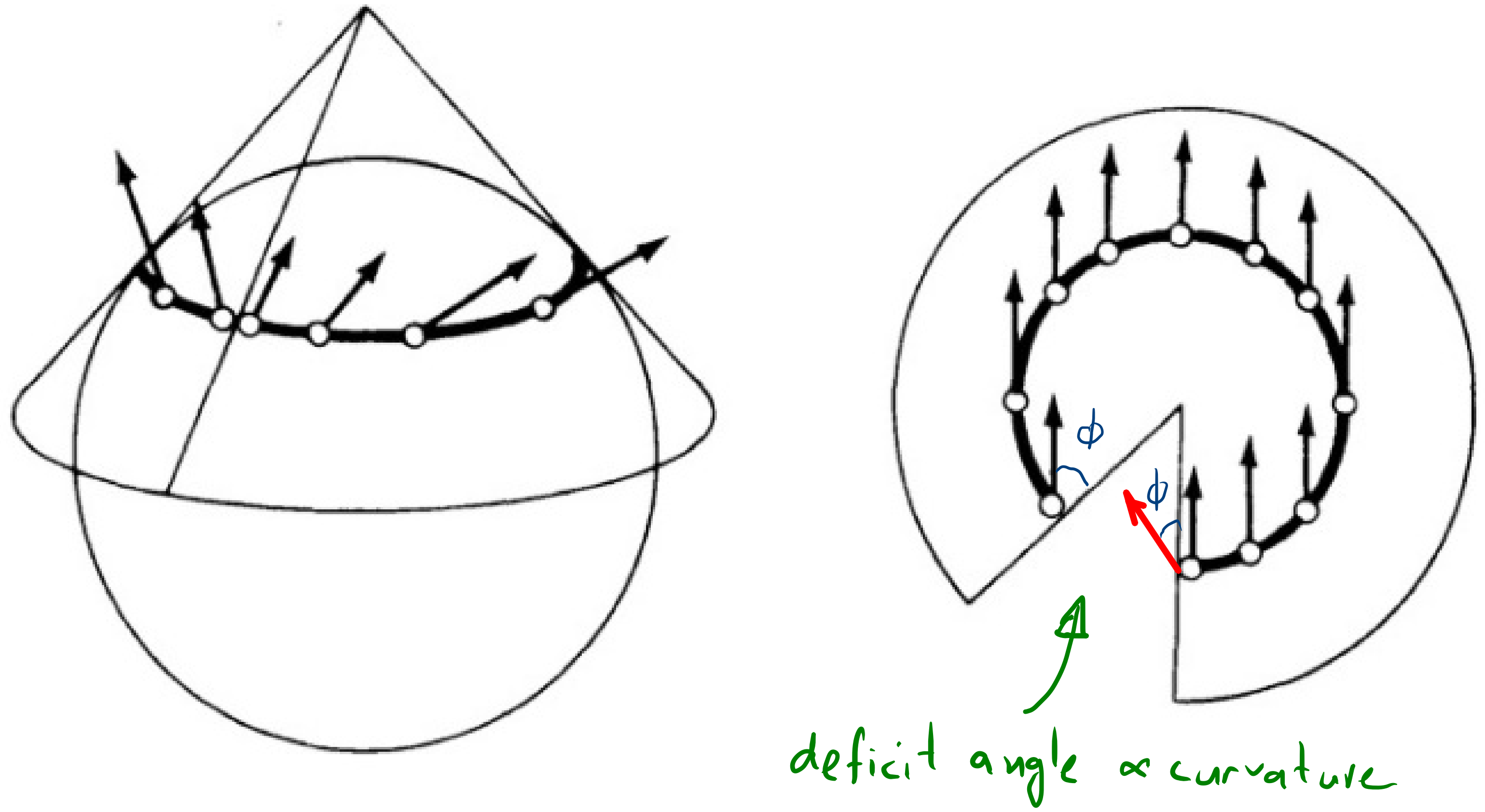

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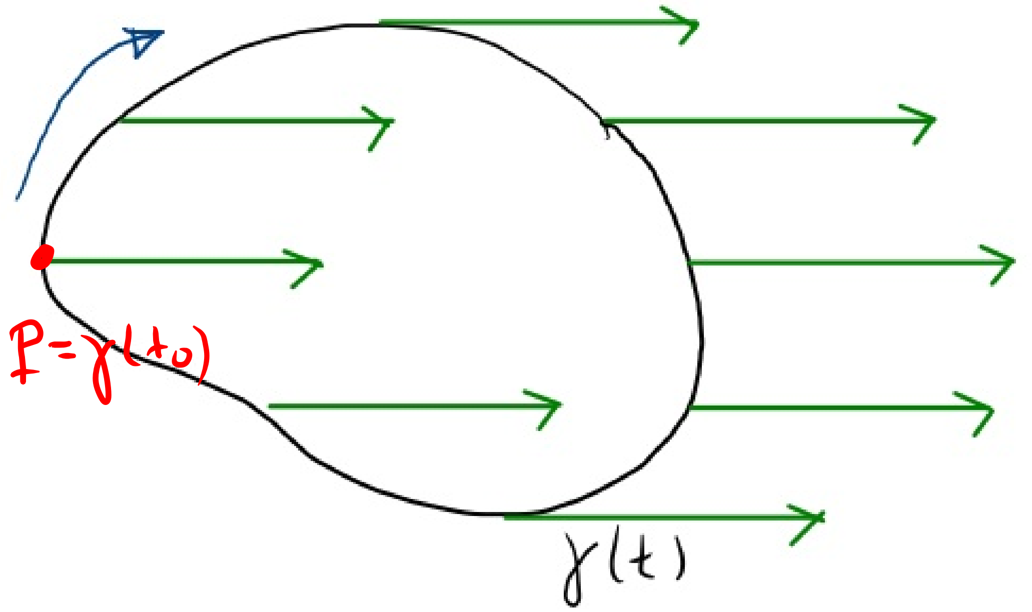
- measures deviation from flatness



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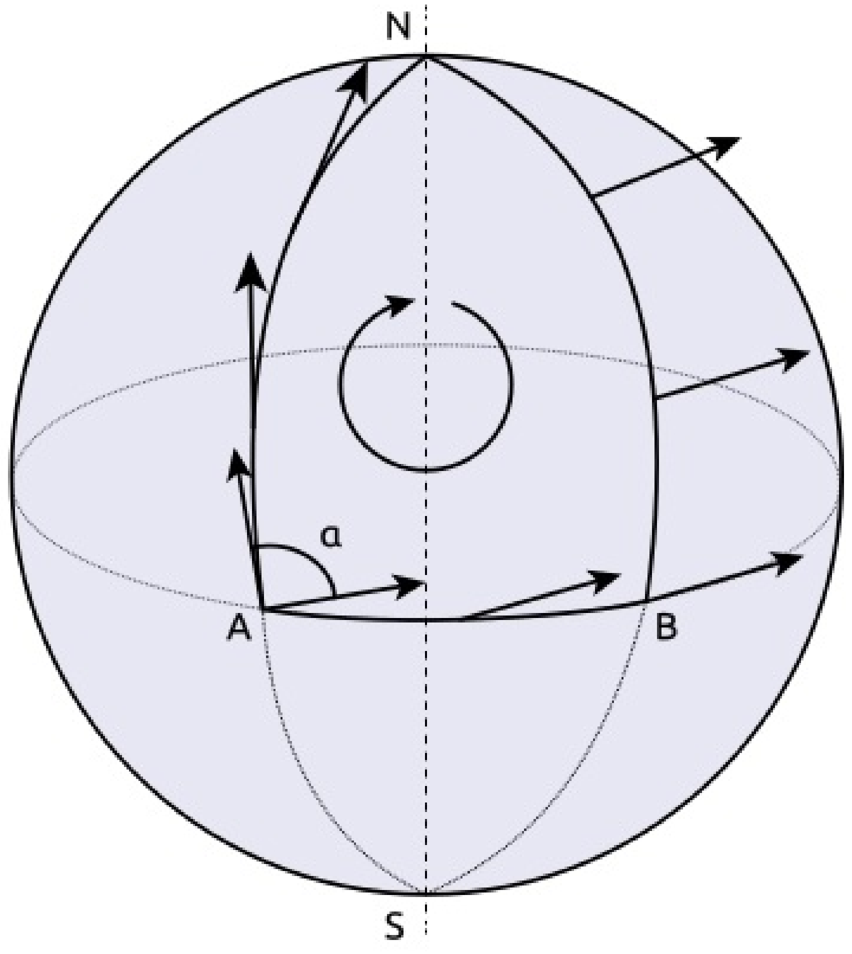

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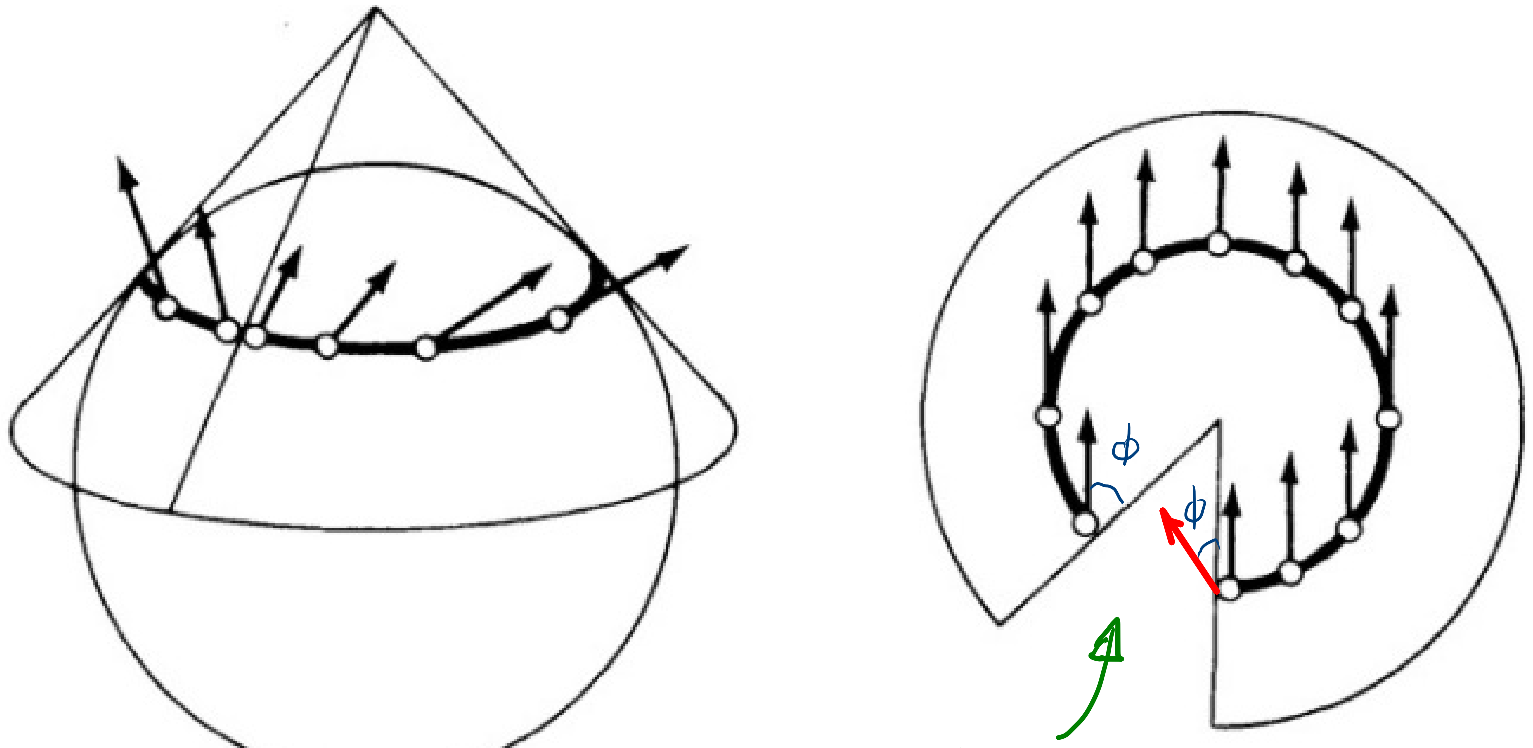

deficit angle $\propto$ curvature

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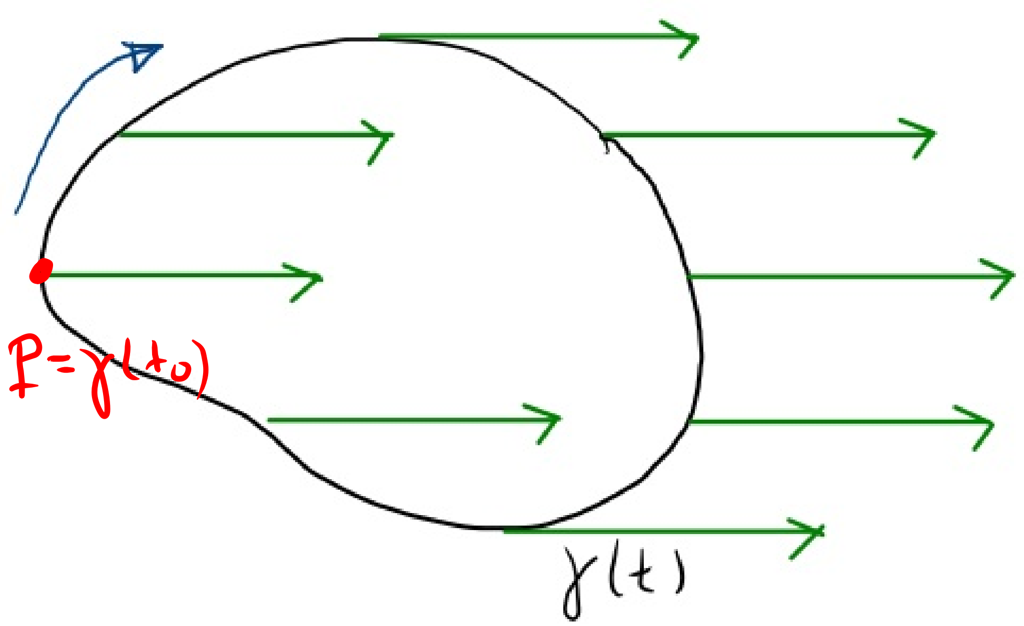
- measures deviation from flatness
- intrinsic notion, geometric property



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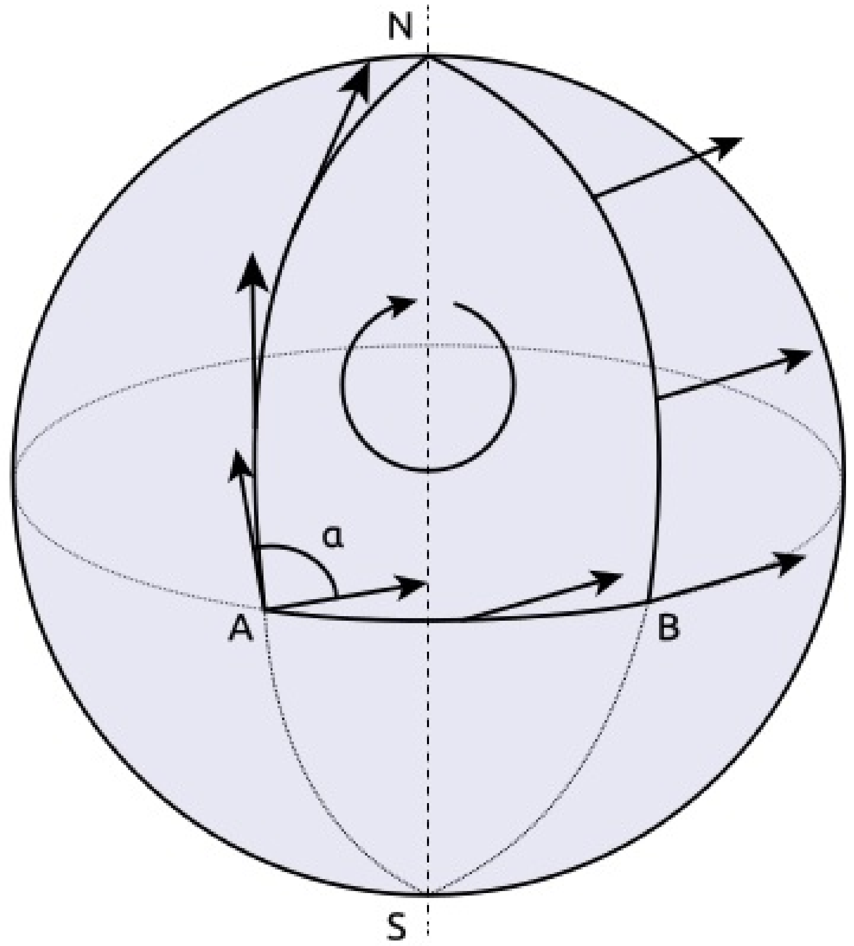

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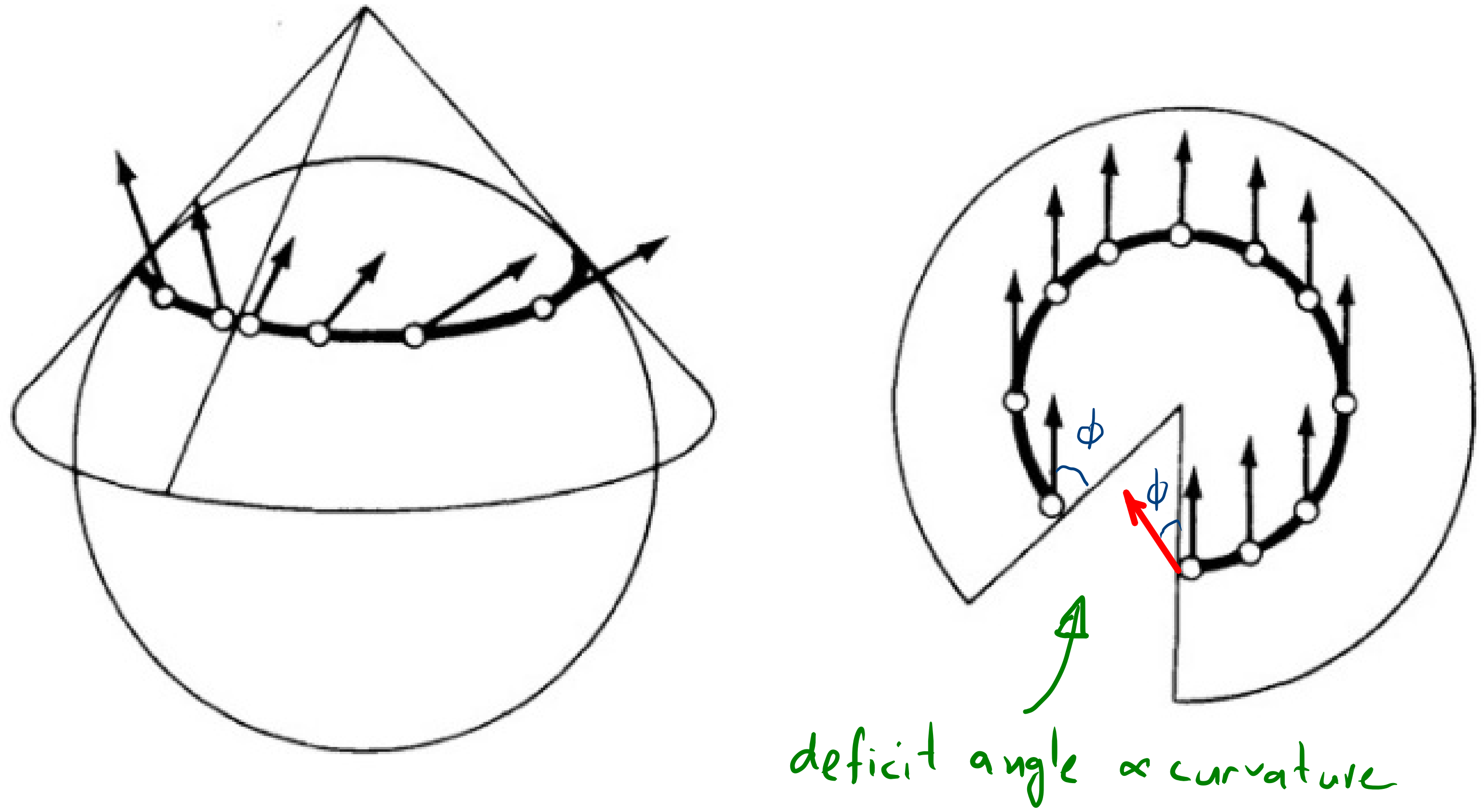

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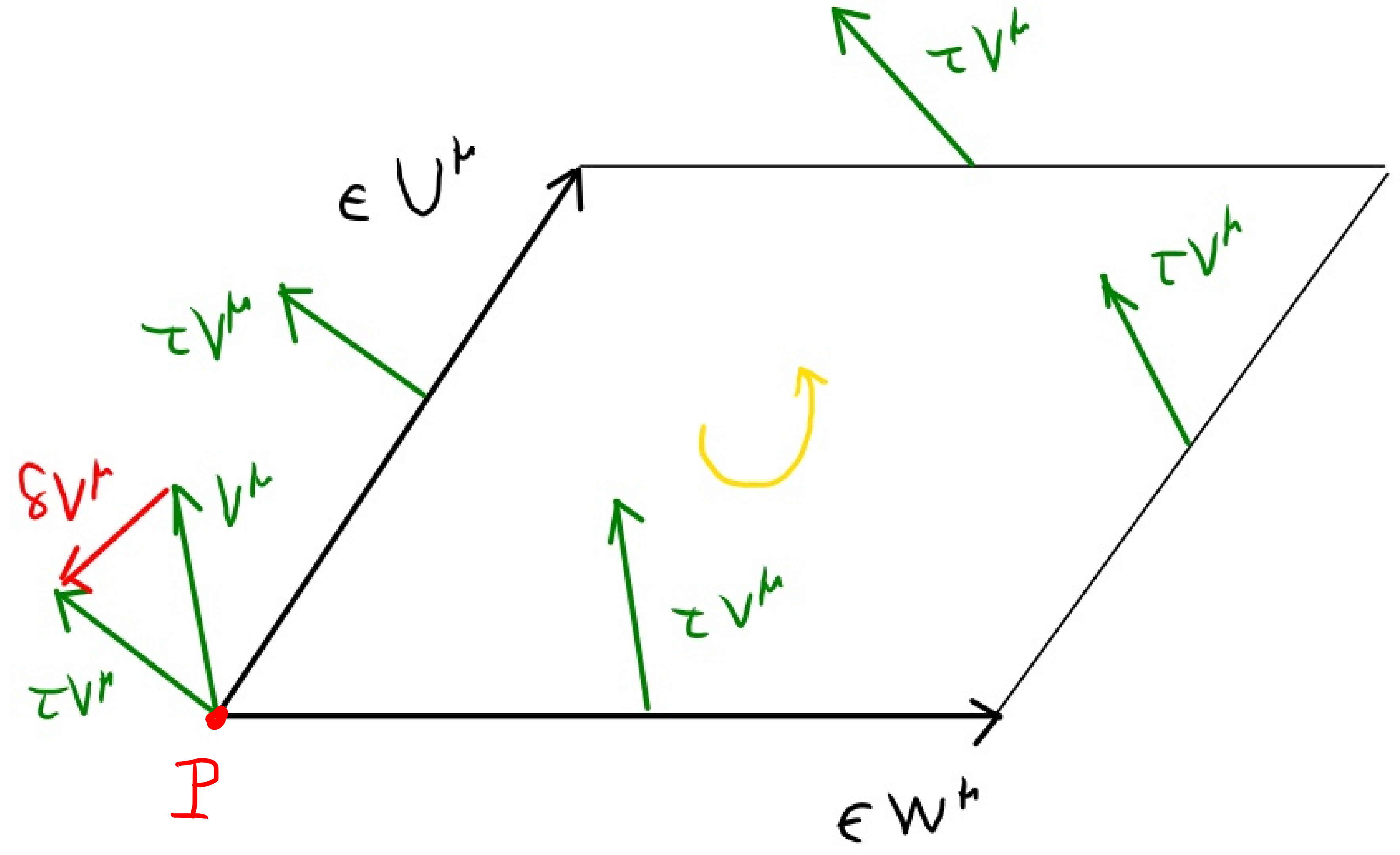
↳ global notion, shrink to get a local one

- shrink to infinitesimal curves

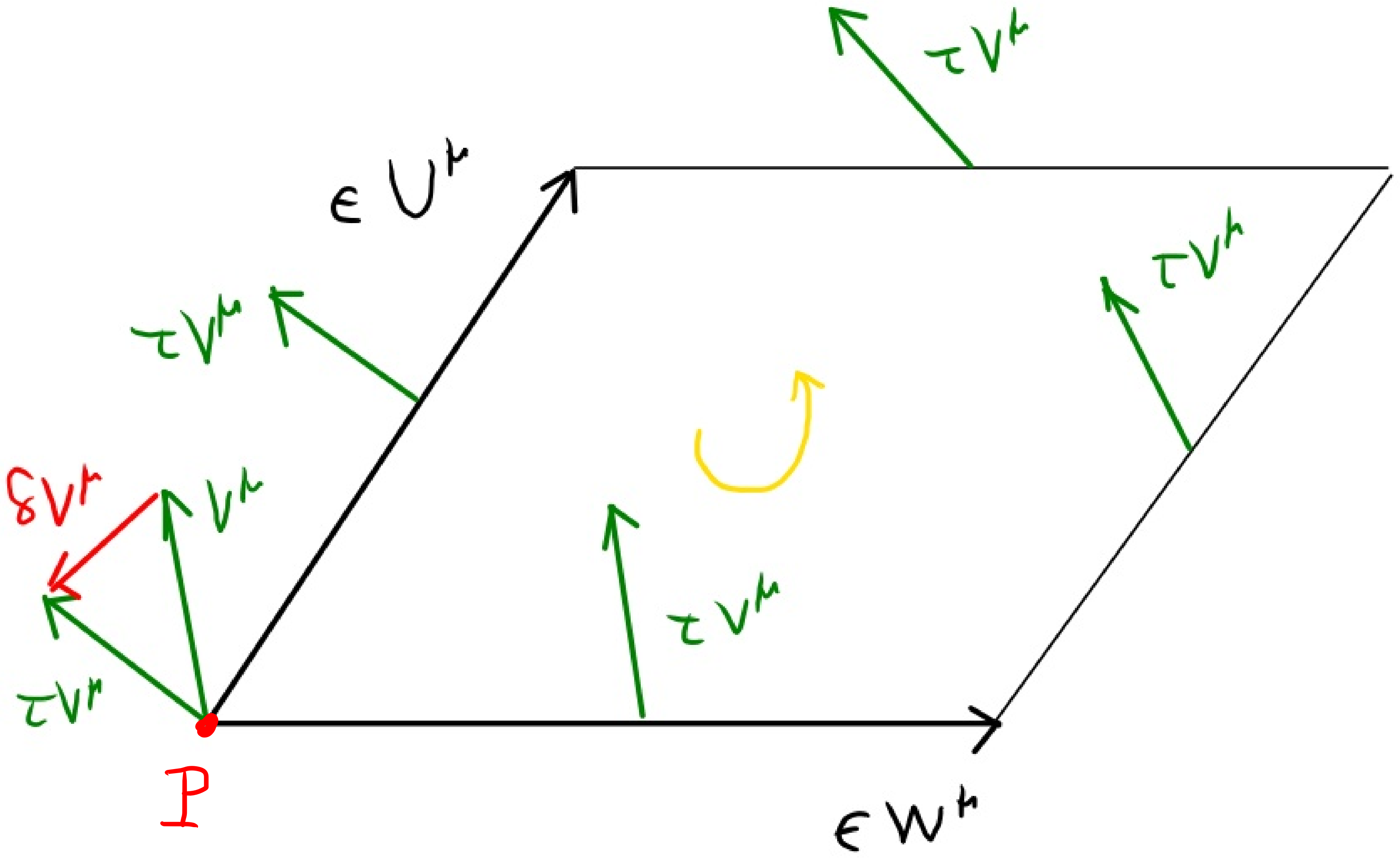
  - closed curve defined by ϵW^μ, ϵU^μ



- shrink to infinitesimal curves
  - closed curve defined by ϵW^μ, ϵU^μ
  - parallel transport $P \to P$

$$V^\mu \to \tau V^\mu = V^\mu + \delta V^\mu$$

$$\tau V^\mu = \Theta^\mu{}_\nu V^\nu \Rightarrow \delta V^\rho = R^\rho{}_\sigma V^\sigma$$



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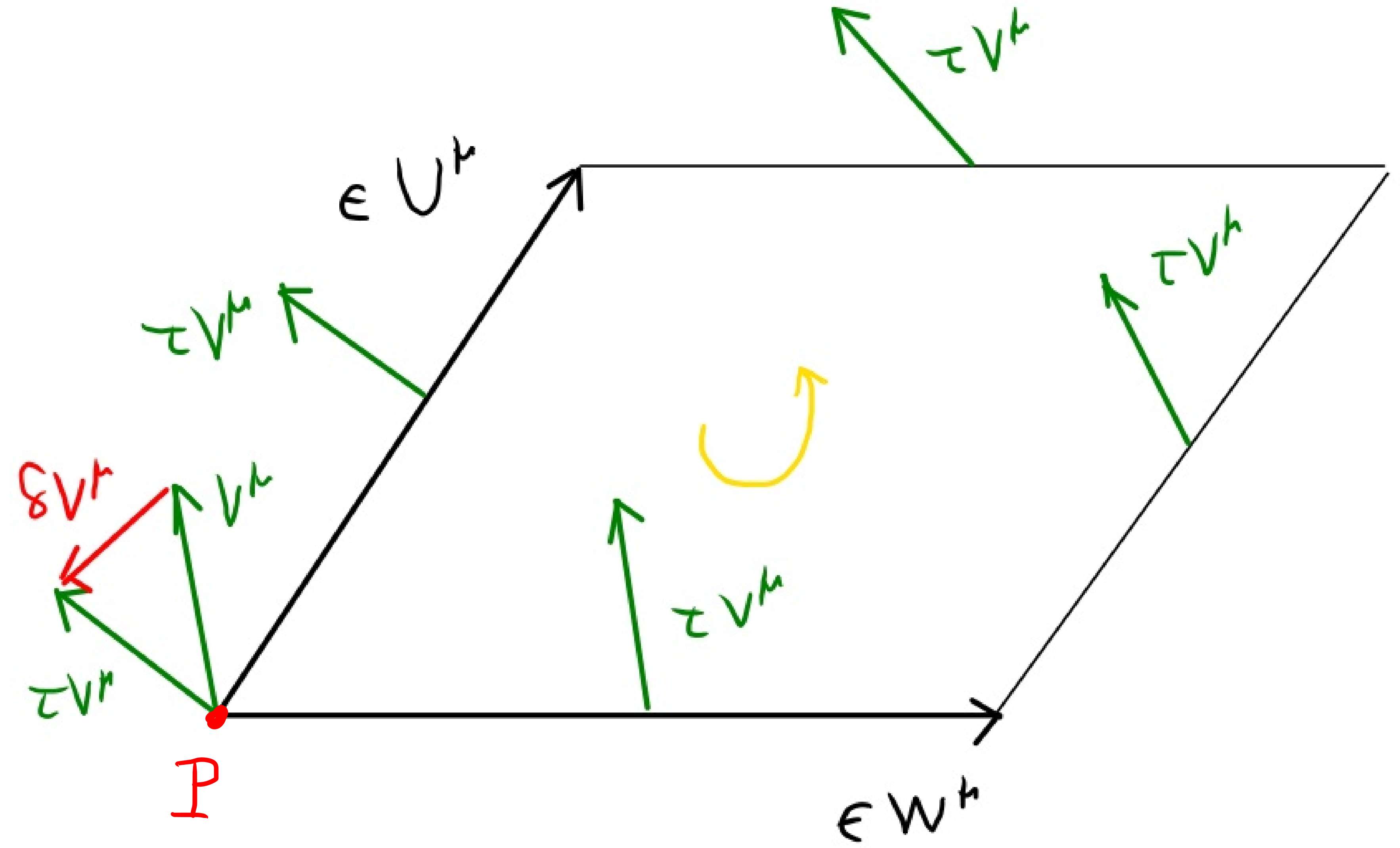

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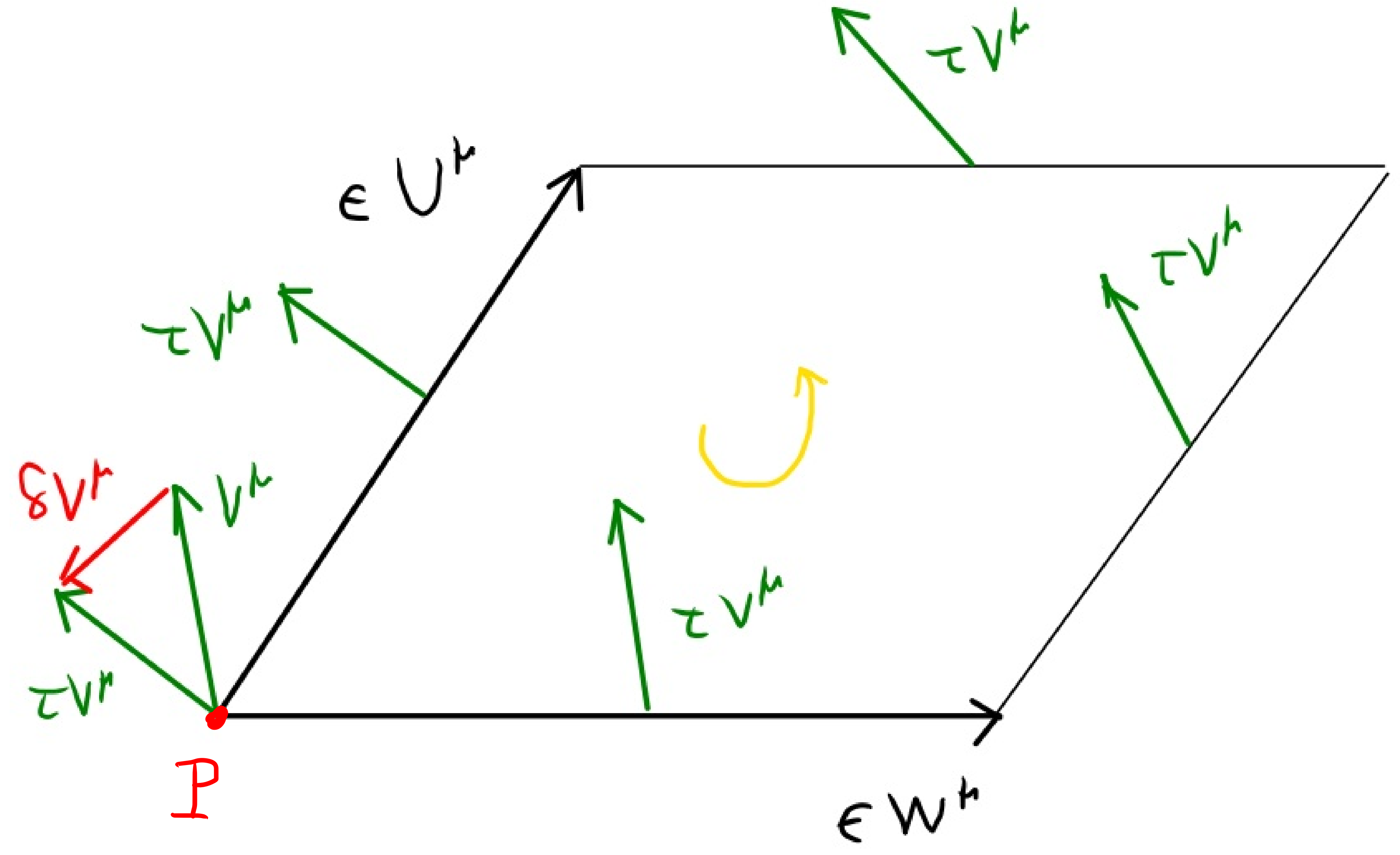
  - depends on W^μ, U^μ linearly

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma W^\mu U^\nu$$

- shrink to infinitesimal curves
  - $W^\mu \leftrightarrow U^\mu$ reverses direction of motion on curve: $\delta V \to -\delta V$

$$\Rightarrow R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$$



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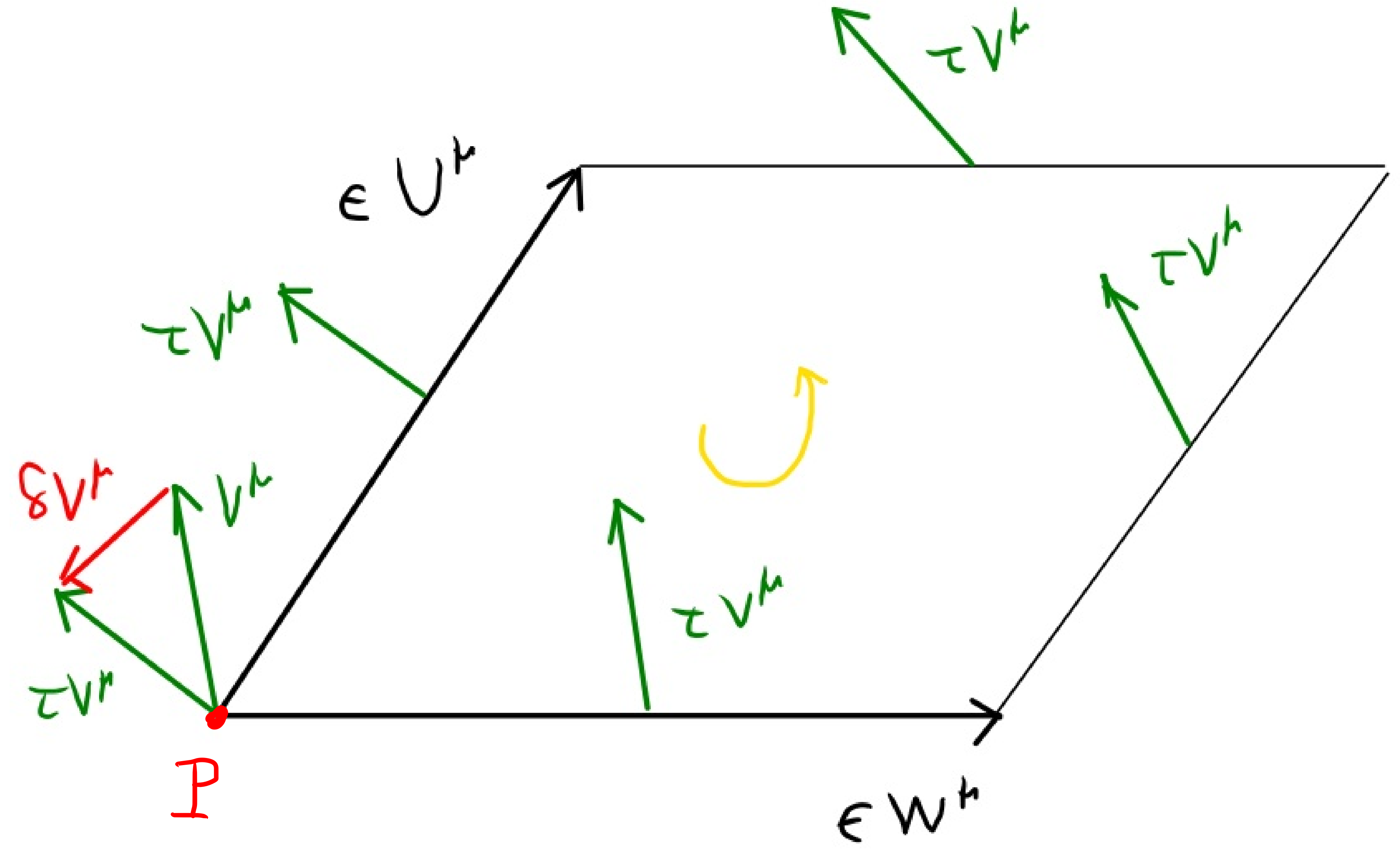

- $D_W V$: measures change of V along W relative to its parallel transport

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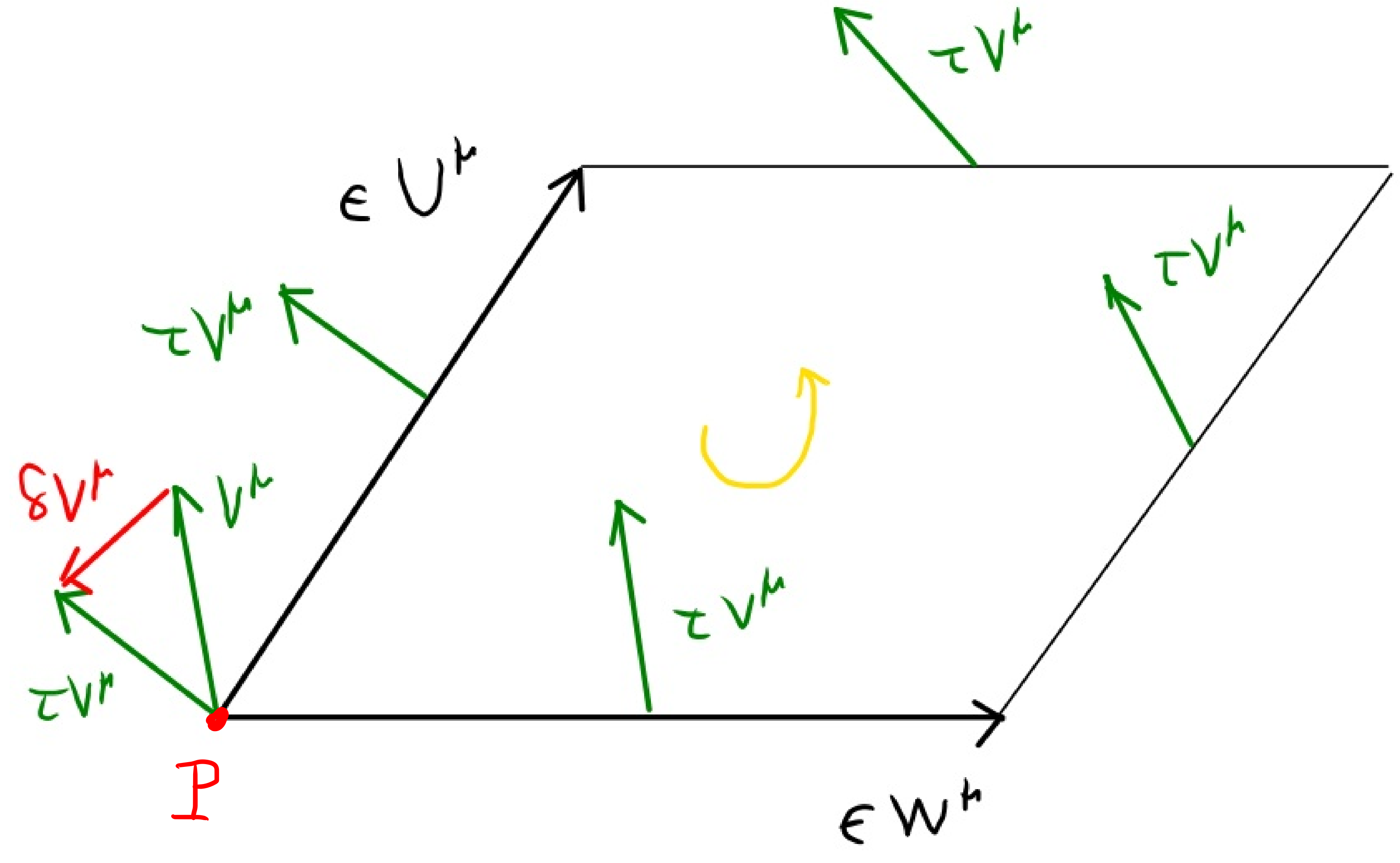

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  $\nabla_\nu V^\rho$: change of V^ρ along ∂_ν

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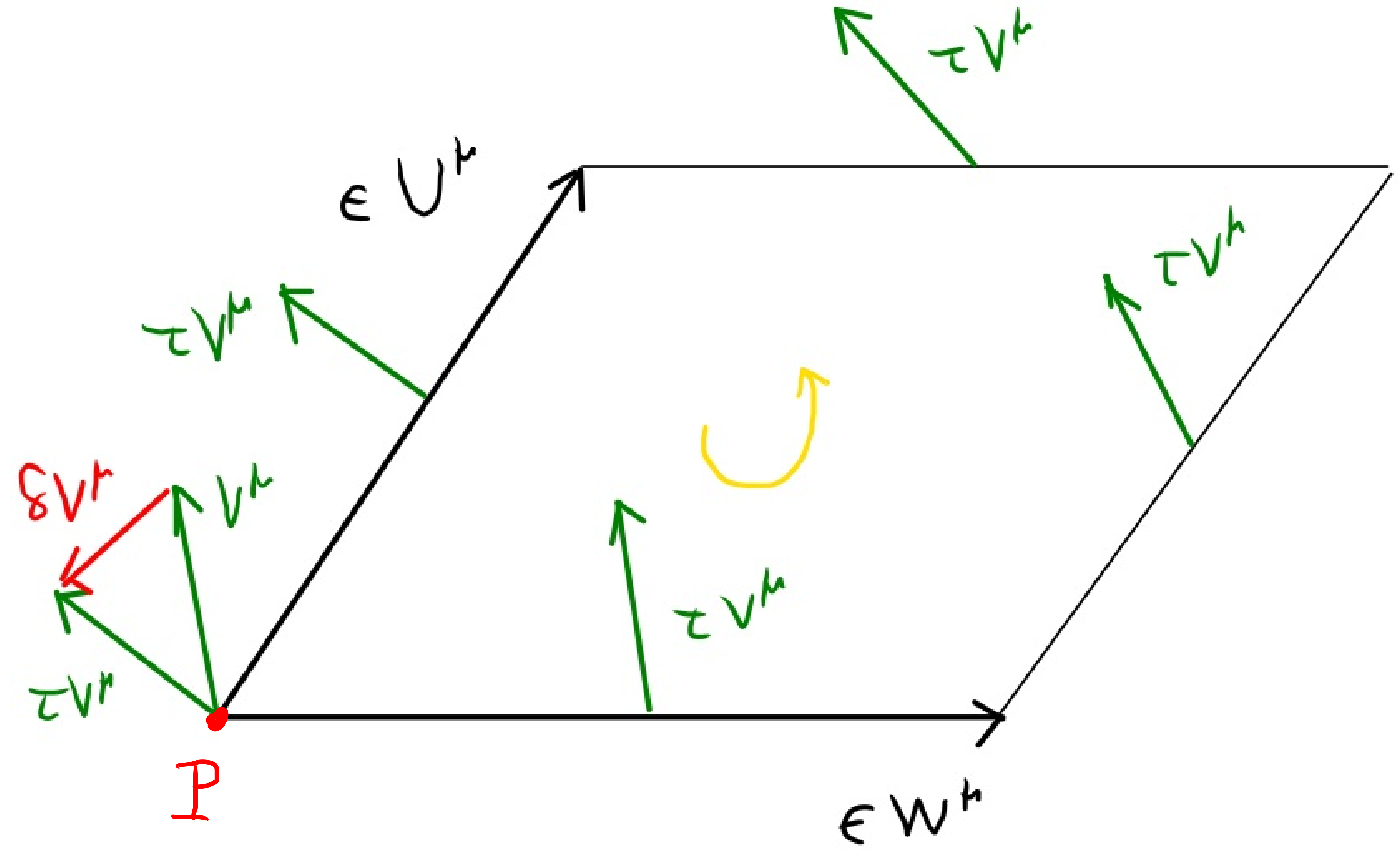

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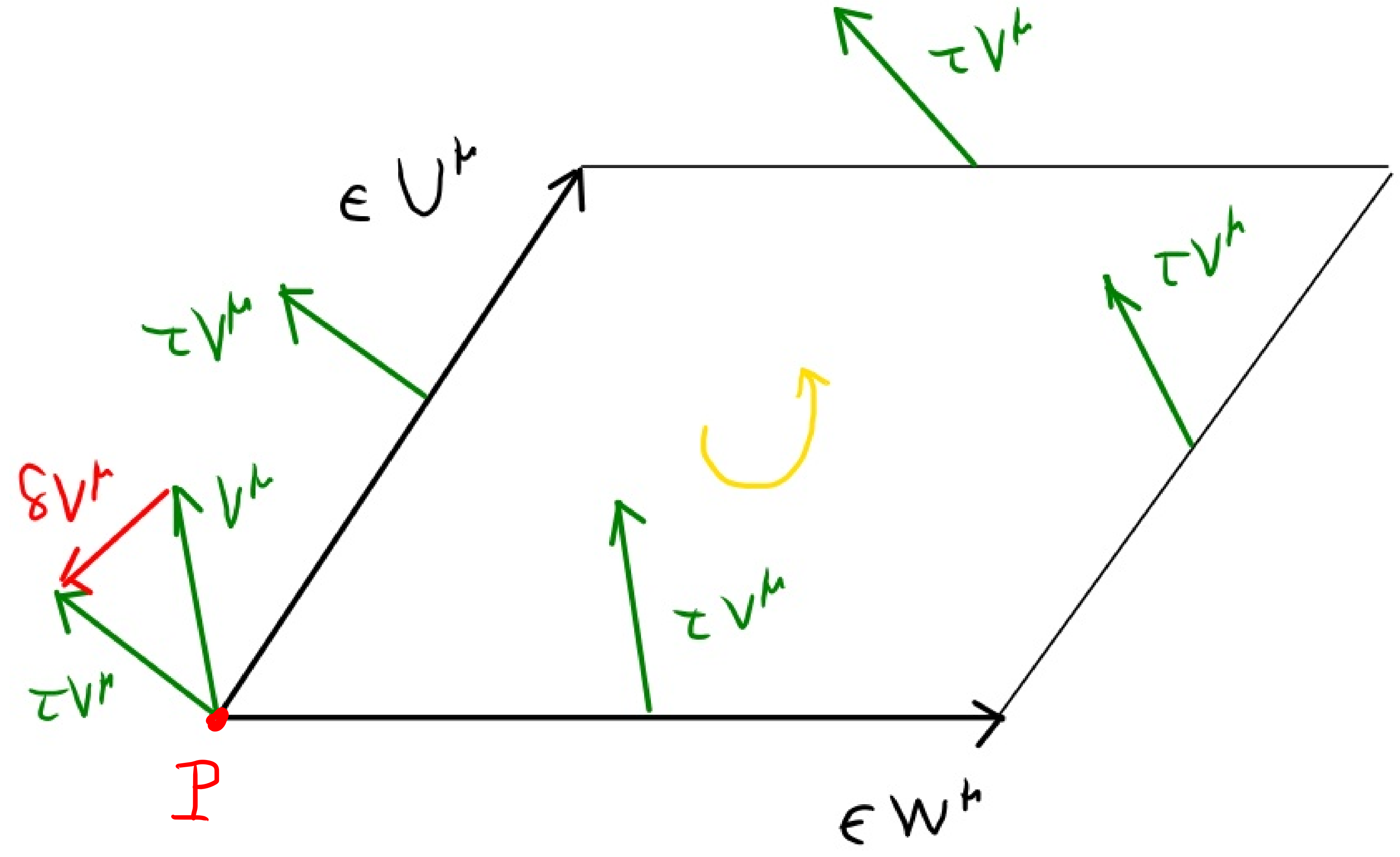
  $\nabla_\nu V^\rho$: change of V^ρ along ∂_ν

$\nabla_\mu \nabla_\nu V^\rho$: change along ∂_ν, then along ∂_μ

- shrink to infinitesimal curves

$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho$:
change along loop $\partial_\mu, \partial_\nu$



- $D_W V$: measures change of V along W relative to its parallel transport

$\nabla_\nu V^\rho$: change of V^ρ along ∂_ν

$\nabla_\mu \nabla_\nu V^\rho$: change along ∂_ν, then along ∂_μ

- ## Formal Definition

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda \qquad \text{(torsion free)}$$

$$[\nabla_\mu, \nabla_\nu] = \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu$$

- ## Formal Definition

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\underbrace{\lambda}\mu\nu} V^\lambda$$

component xfm indices

Careful: Placement of indices heavily author dependent!

- <u>Formal Definition</u>

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda$$

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$$\nabla_{\mu}\nabla_{\nu} V^{\rho} = \partial_{\mu} \underbrace{(\nabla_{\nu} V^{\rho})}_{\text{a (1,1) tensor}} - \Gamma^{\lambda}{}_{\underline{\mu\nu}} \nabla_{\lambda} V^{\rho} + \Gamma^{\rho}{}_{\mu\lambda} \nabla_{\nu} V^{\underline{\lambda}}$$

1st index   2nd index

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$$= \partial_\mu [\partial_\nu V^\rho + \Gamma^\rho{}_{\nu\lambda} V^\lambda]$$

$$- \Gamma^\lambda{}_{\mu\nu} [\partial_\lambda V^\rho + \Gamma^\rho{}_{\lambda\sigma} V^\sigma]$$

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$$= \partial_\mu \partial_\nu V^\rho + \partial_\mu \Gamma^\rho_{\nu\lambda} V^\lambda + \Gamma^\rho_{\nu\lambda} \partial_\mu V^\lambda$$

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$\lambda \leftrightarrow \sigma$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho = \partial_\mu \Gamma^\rho{}_{\nu\lambda} V^\lambda + \Gamma^\rho{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\lambda} V^\lambda - (\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}) [\partial_\lambda V^\rho + \Gamma^\rho{}_{\lambda\sigma} V^\sigma]$$

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- **Formal Definition**

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$$R_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu$$

- ## Formal Definition

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- $[\nabla_\mu, \nabla_\nu] V = R \quad {}_{\mu\nu} V$

- $[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$

  $\hookrightarrow$ torsion free $\Rightarrow$ depends on V, but not on ∂V

- $[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$

— obvious that $R^\rho{}_{\sigma\mu\nu} = - R^\rho{}_{\sigma\nu\mu}$ since $[\nabla_\mu, \nabla_\nu] \to [\nabla_\nu, \nabla_\mu] = -[\nabla_\mu, \nabla_\nu]$

- $[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$

_ obvious that $R^\rho{}_{\sigma\mu\nu} = - R^\rho{}_{\sigma\nu\mu}$ since $[\nabla_\mu, \nabla_\nu] \to [\nabla_\nu, \nabla_\mu] = -[\nabla_\mu, \nabla_\nu]$

- expression $R = \partial\Gamma + \Gamma\Gamma$ valid for any torsion free $\tilde{\nabla}$:

$$R^\rho{}_{\sigma\mu\nu} = \tilde{\nabla}_\mu C^\rho{}_{\nu\sigma} - \tilde{\nabla}_\nu C^\rho{}_{\mu\sigma} + C^\rho{}_{\mu\lambda} C^\lambda{}_{\nu\sigma} - C^\rho{}_{\nu\lambda} C^\lambda{}_{\mu\sigma}$$

- **Action of $[\nabla_\mu, \nabla_\nu]$ on one forms:**

The contraction $\omega_\mu V^\mu$ is a function, therefore

$$[\nabla_\mu, \nabla_\nu](\omega_\lambda V^\lambda) = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, \omega_\lambda V^\lambda = 0$$

torsion free condition!

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(torsion free condition)

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$$= (\nabla_\mu \nabla_\nu \omega_\lambda) V^\lambda + (\nabla_\nu \omega_\lambda)(\nabla_\mu V^\lambda) + (\nabla_\mu \omega_\lambda)(\nabla_\nu V^\lambda) + \omega_\lambda \nabla_\mu \nabla_\nu V^\lambda$$

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$\mu \leftrightarrow \nu$

$$\nabla_\nu \nabla_\mu (\omega_\lambda V^\lambda) = (\nabla_\nu \nabla_\mu \omega_\lambda) V^\lambda + (\nabla_\mu \omega_\lambda)(\nabla_\nu V^\lambda) + (\nabla_\nu \omega_\lambda)(\nabla_\mu V^\lambda) + \omega_\lambda \nabla_\nu \nabla_\mu V^\lambda$$

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$$\Rightarrow [\nabla_\mu, \nabla_\nu](\omega_\lambda V^\lambda) = (\underbrace{[\nabla_\mu, \nabla_\nu]\omega_\lambda}_{\text{we want this}}) V^\lambda + \omega_\lambda (\underbrace{[\nabla_\mu, \nabla_\nu]}_{\text{we know that...}} V^\lambda)$$

• Action of $[\nabla_\mu, \nabla_\nu]$ on one forms:

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$$= \Big\{ \underbrace{[\nabla_\mu, \nabla_\nu]\omega_\lambda + R^\sigma{}_{\lambda\mu\nu}\,\omega_\sigma}_{=\,0} \Big\} V^\lambda$$

$$\Rightarrow [\nabla_\mu, \nabla_\nu](\omega_\lambda V^\lambda) = \Big(\underbrace{[\nabla_\mu, \nabla_\nu]\omega_\lambda}_{\text{we want this}}\Big) V^\lambda + \omega_\lambda\Big(\underbrace{[\nabla_\mu, \nabla_\nu]V^\lambda}_{\text{we know that...}}\Big)$$

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• Action of $[\nabla_\mu, \nabla_\nu]$ on higher rank tensors:

$$[\nabla_\mu, \nabla_\nu] S^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = R^{\mu_1}{}_{\lambda\mu\nu} S^{\lambda\cdots\mu_k}{}_{\nu_1\cdots\nu_l} + \cdots + R^{\mu_k}{}_{\lambda\mu\nu} S^{\mu_1\cdots\lambda}{}_{\nu_1\cdots\nu_l}$$

$$- R^\lambda{}_{\nu_1\mu\nu} S^{\mu_1\cdots\mu_k}{}_{\lambda\cdots\nu_l} - \cdots - R^\lambda{}_{\nu_l\mu\nu} S^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\lambda}$$

Symmetries

- $R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}$

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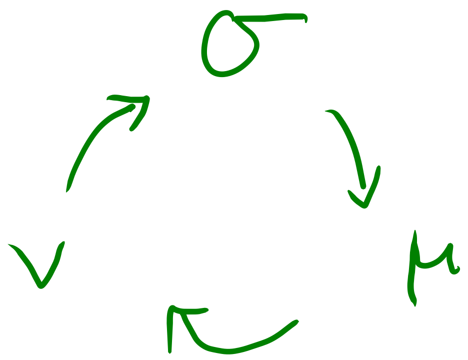

cyclic permutation

Symmetries

- $R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}$
- $R^{\rho}{}_{[\sigma\mu\nu]} = 0 \iff R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma} = 0$

If $\exists\, g_{\mu\nu}$ and ∇_μ its Christoffel/Levi-Civita connection ($\nabla g = 0$ + torsion free), then

$$R_{\rho\sigma\mu\nu} \equiv g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu}$$

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$\Rightarrow \dfrac{n^2(n^2-1)}{12}$ independent components

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$\Rightarrow \quad \dfrac{n^2(n^2-1)}{12}$ independent components

$n=2$	1	indep. component(s)
$n=3$	6	"
$n=4$	20	"

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Symmetries

1^{st} Bianchi identity

- $R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}$
- $R^{\rho}{}_{[\sigma\mu\nu]} = 0 \quad \Leftrightarrow \quad R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma} = 0$

$\Rightarrow \dfrac{n^2(n^2-1)}{12}$ independent components

\+ 2^{nd} Bianchi identity:

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \quad \Leftrightarrow$$

$$\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} = 0$$

- $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$
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Symmetries

constraints values at neighboring points!

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+ 2nd Bianchi identity:

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \Leftrightarrow$$

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

- $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$
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Independent contractions (assume Christoffel connections)

- Ricci tensor: $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \Rightarrow R_{\mu\nu} = R_{\nu\mu}$

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$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right) \qquad n>2$$

$$+ \frac{2}{(n-1)(n-2)} g_{\rho[\mu} g_{\nu]\sigma} R$$

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- Symmetries remain: $C_{[\rho\sigma][\mu\nu]} = C_{\rho\sigma\mu\nu}$, $C_{\rho\sigma\mu\nu} = C_{\mu\nu\rho\sigma}$, $C_{\rho[\sigma\mu\nu]} = 0$

Independent contractions

(assume Christoffel connections)

- trace free $C^{\lambda}{}_{\mu\lambda\nu} = 0$

(of course, we subtracted out $R_{\mu\nu}$ and R from $R^{\rho}{}_{\sigma\mu\nu}$)

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- trace free $C^{\lambda}{}_{\mu\lambda\nu} = 0$
- independent components: $\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2}$

$n \leq 3$ $C_{\rho\sigma\mu\nu} = 0$

$n = 4$ 10 independent comp.

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↳ e.g. gravitational waves

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$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

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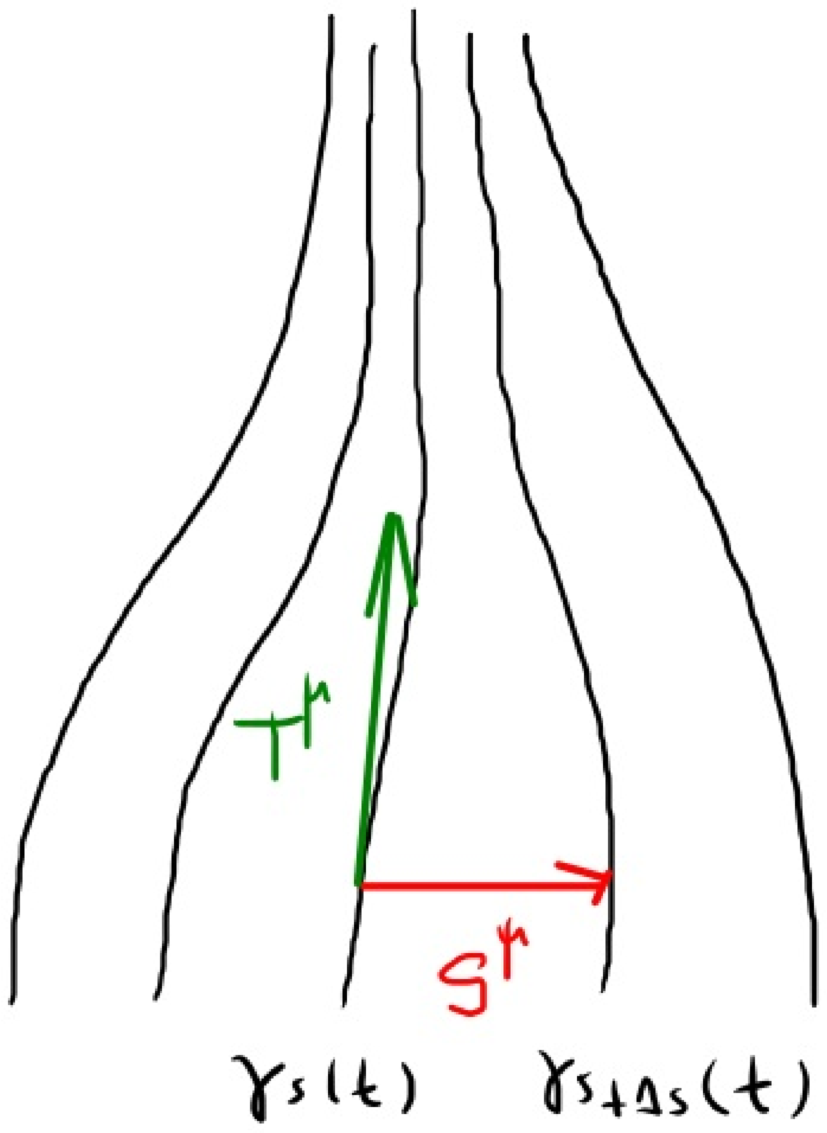
Geodesic Deviation

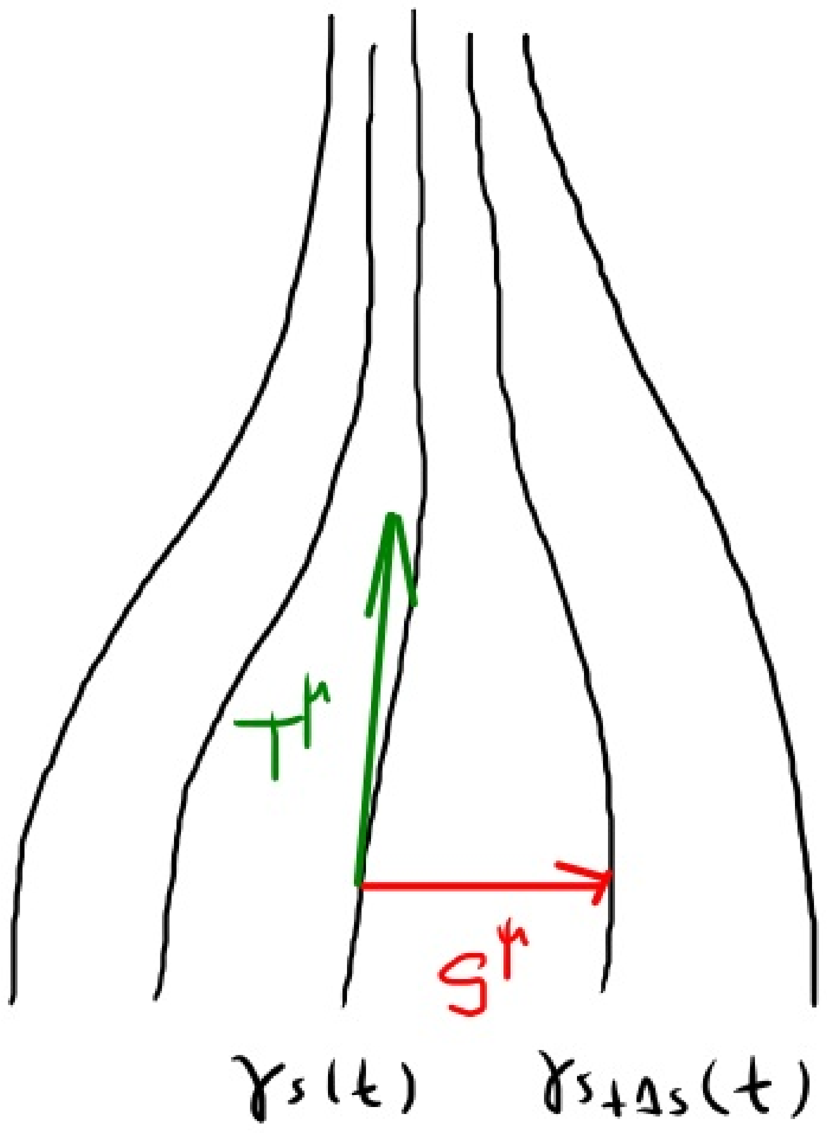
- Consider a one-parameter family of geodesics $\gamma_s(t)$

  t: affine parameter

  $s \in \mathbb{R}$



Geodesic Deviation



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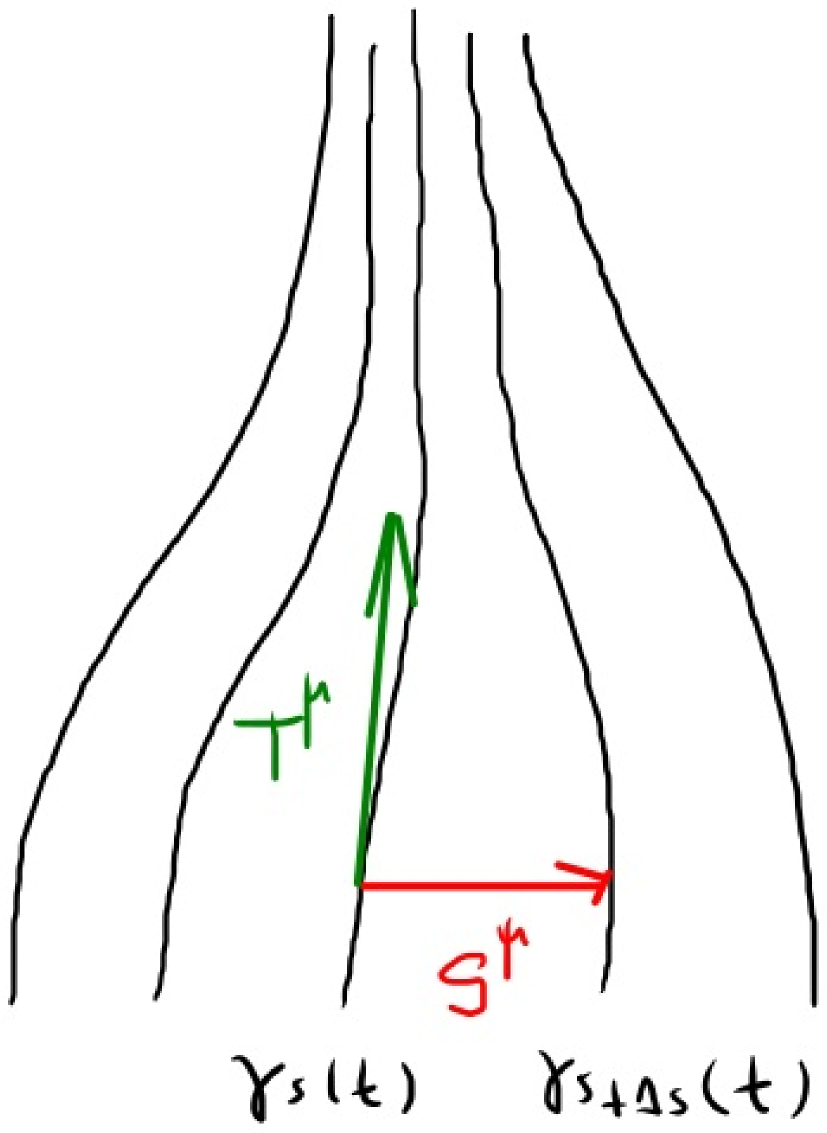

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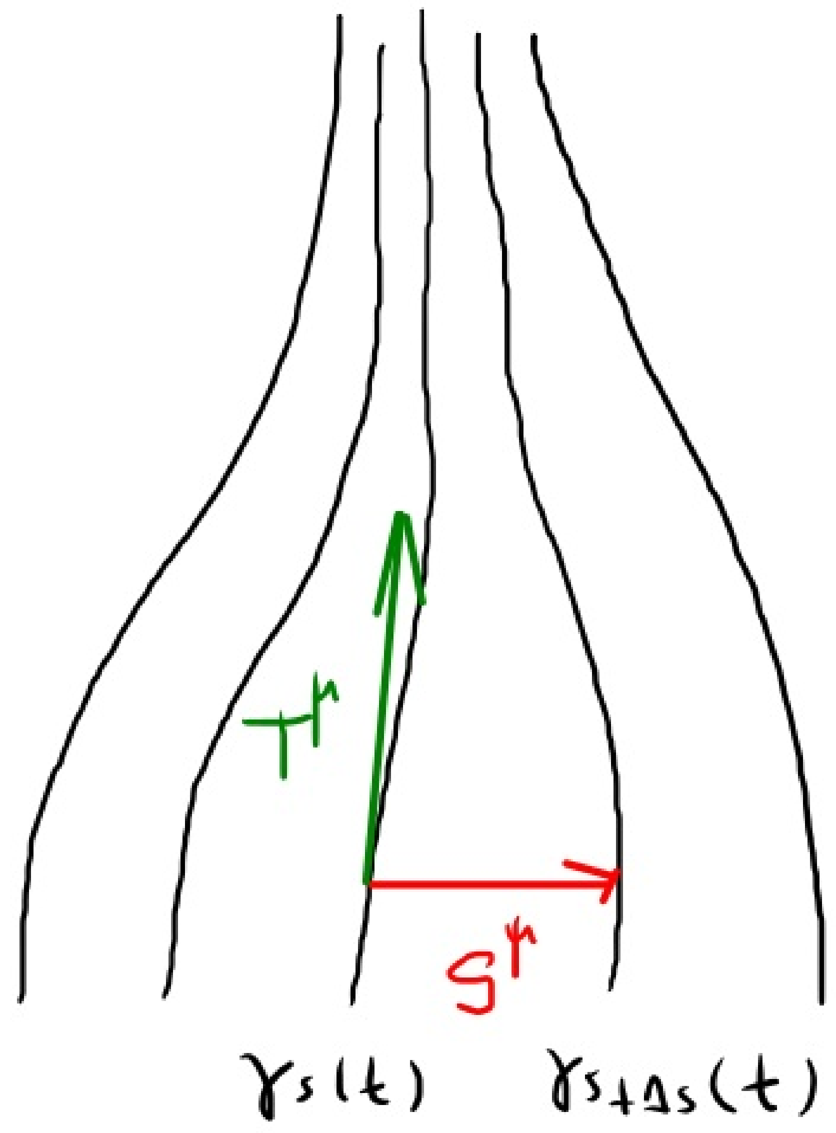

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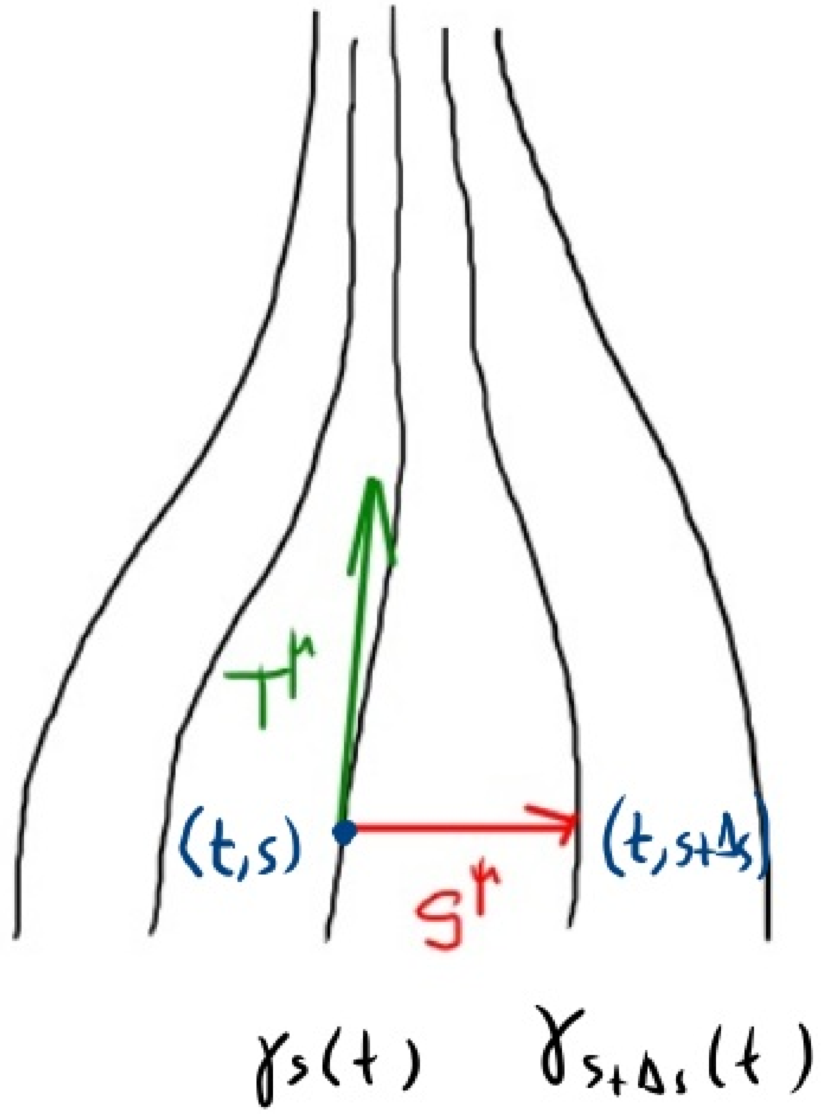
$$\Rightarrow \quad T^\mu = \partial_t \quad \text{tangent vectors, s.t. } T^\nu \nabla_\nu T^\mu = 0$$

because tangent to geodesics!

Geodesic Deviation



- Consider a one-parameter family of geodesics $\gamma_s(t)$

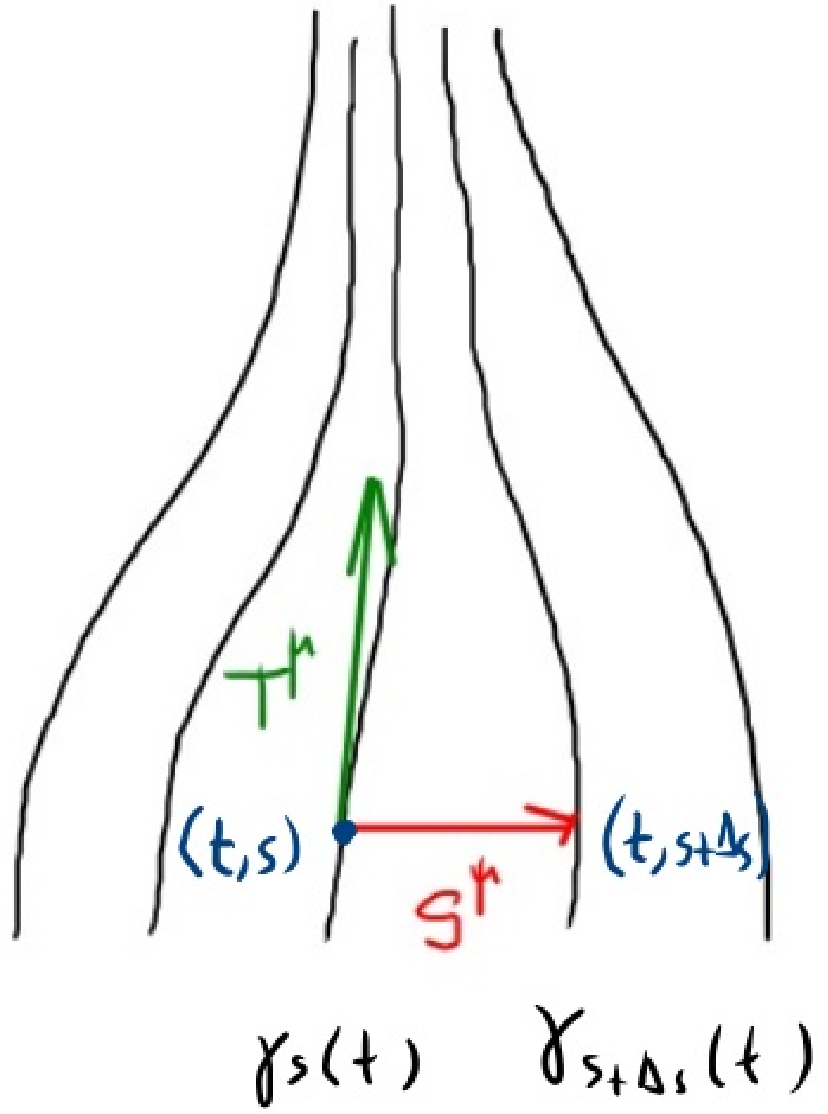
  t: affine parameter

  $s \in \mathbb{R}$

- consider a small enough open set where they don't cross
- assume they form a 2-d surface w/coordinates (s,t)

$\Rightarrow$
- $T^\mu = \partial_t$ tangent vectors, s.t. $T^\nu \nabla_\nu T^\mu = 0$
- $S^\mu = \partial_s$ deviation vectors: point to $\gamma_{s+\Delta s}(t)$ at same t

Geodesic Deviation



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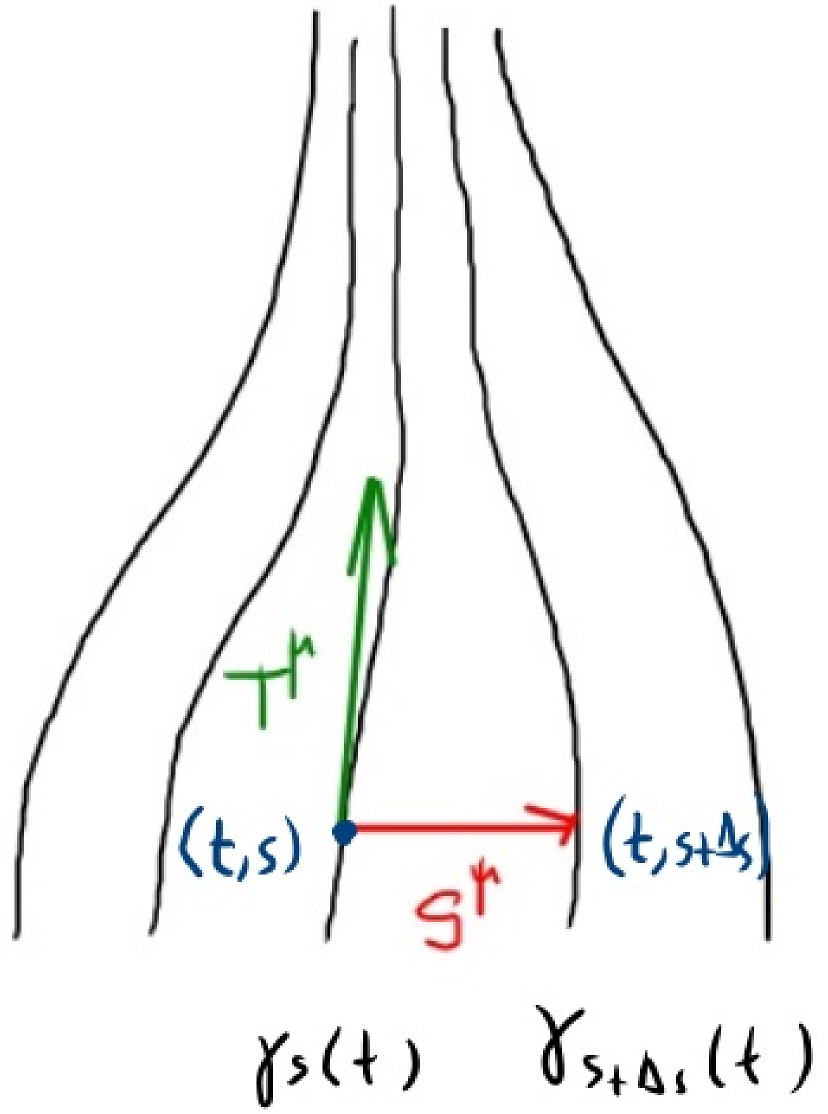
- $[S,T]^\mu = 0 \Leftrightarrow S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu$ $\leftarrow$ coordinate vectors condition

Geodesic Deviation

$D_T S^\mu$ = "relative velocity"



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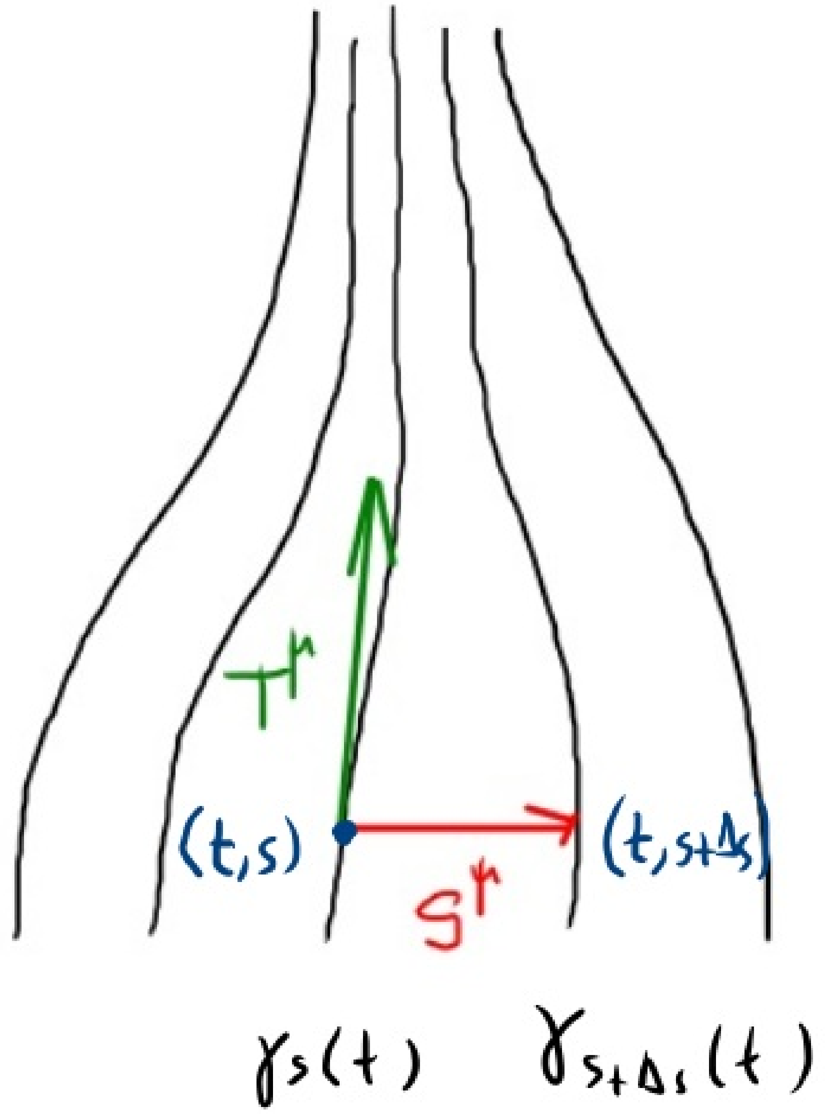
Geodesic Deviation

$$D_T S^\mu = \text{"relative velocity"}$$

$$= T^\nu \nabla_\nu S^\mu \overset{(2)}{=} (\nabla_\nu T^\mu) S^\nu$$



$\Rightarrow$ • $T^\mu = \partial_t$ tangent vectors, s.t. $T^\nu \nabla_\nu T^\mu = 0$ (1)

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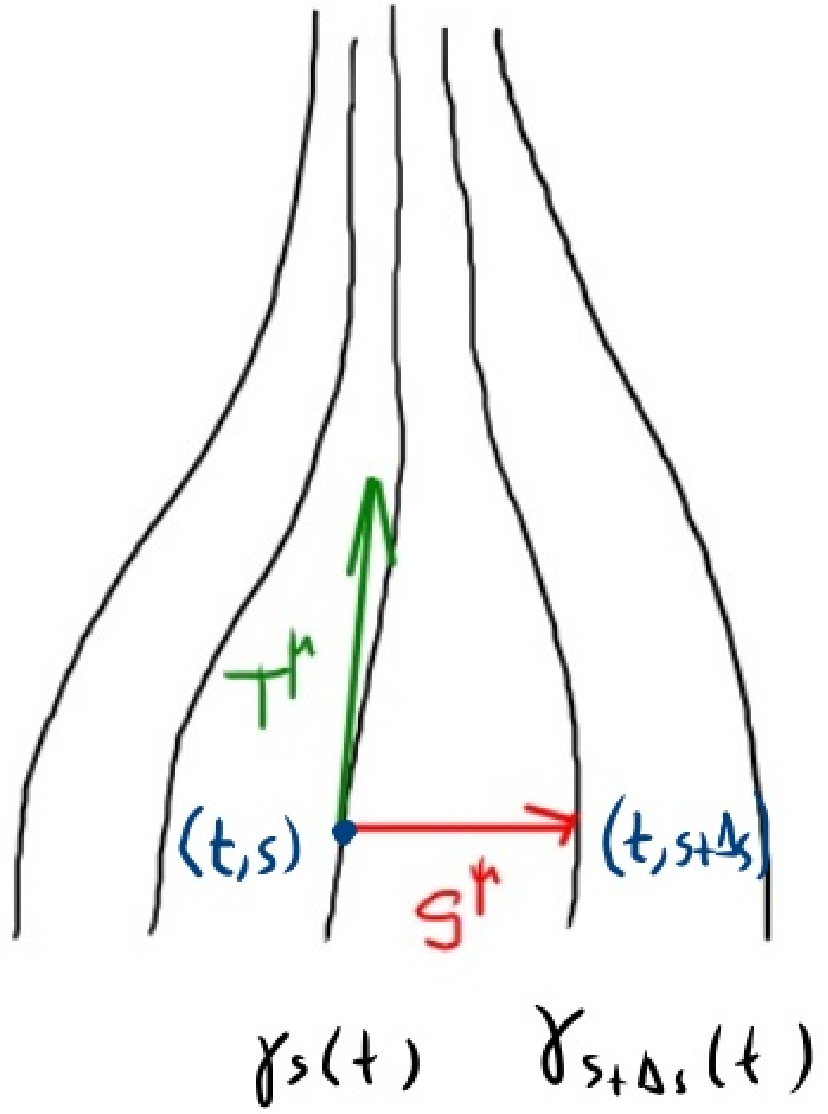
Geodesic Deviation



$$D_T S^\mu = \text{“ relative velocity”}$$

$$= T^\nu \nabla_\nu S^\mu \overset{(2)}{=} (\nabla_\nu T^\mu)\, S^\nu$$

$$\equiv B^\mu{}_\nu\, S^\nu \quad , \quad B^\mu{}_\nu \equiv \nabla_\nu T^\mu$$

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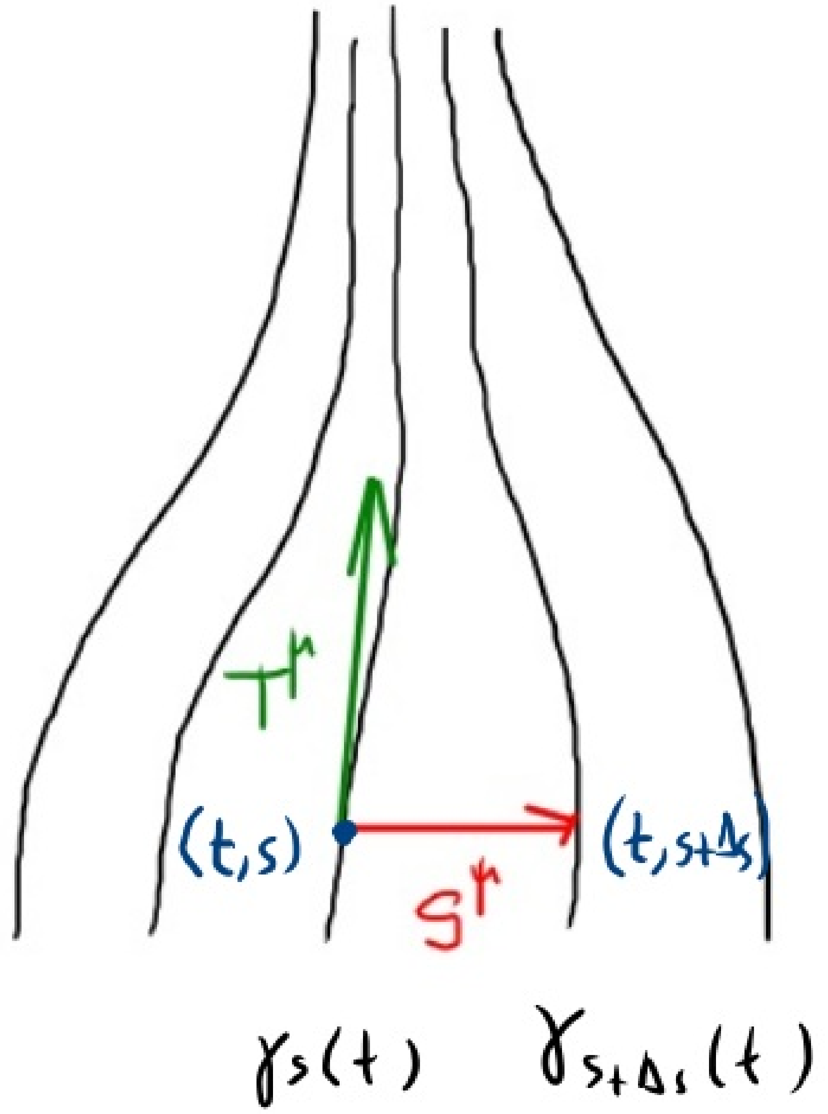

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$$\equiv B^\mu{}_\nu S^\nu \,, \quad B^\mu{}_\nu \equiv \nabla_\nu T^\mu$$

- linear xfm of S^μ
- failure of S^μ to be parallel transported $\Rightarrow$ (failure of neighboring geodesics to remain parallel)

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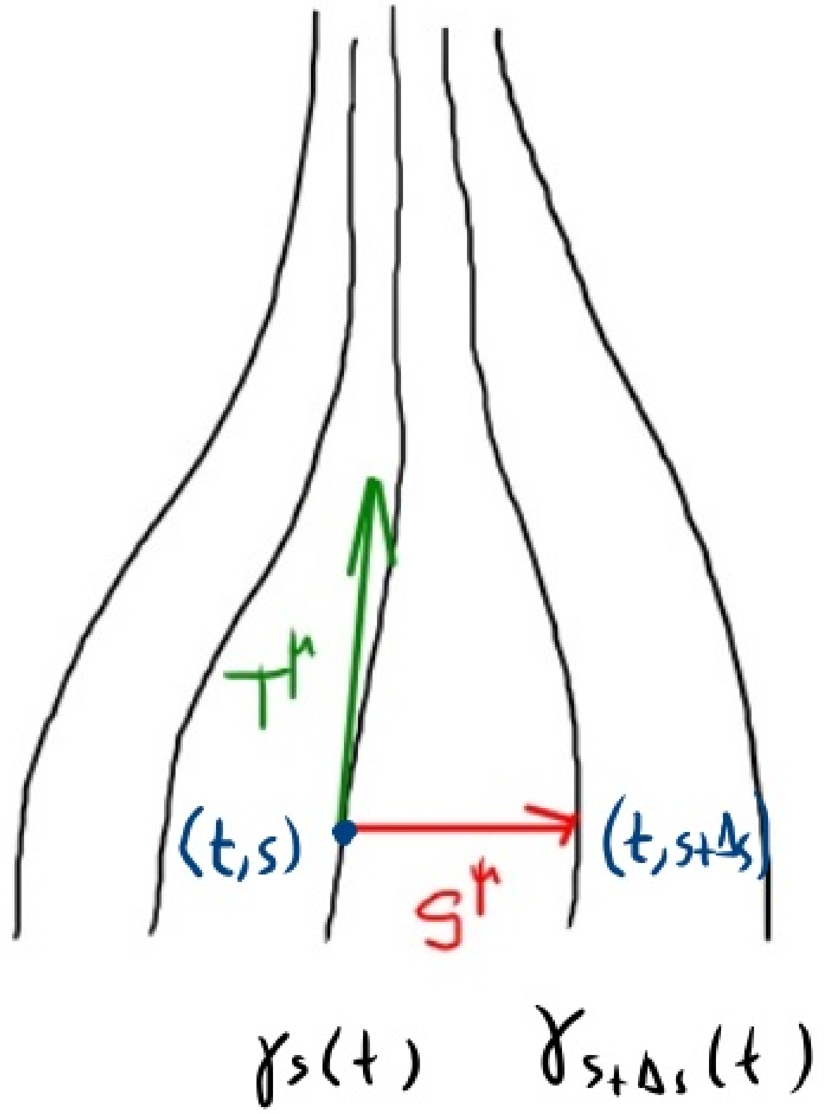
Geodesic Deviation

- relative velocity: $V^\mu \equiv D_T S^\mu = T^\rho \nabla_\rho S^\mu$
- relative acceleration: $A^\mu \equiv D_T V^\mu = T^\rho \nabla_\rho V^\mu$



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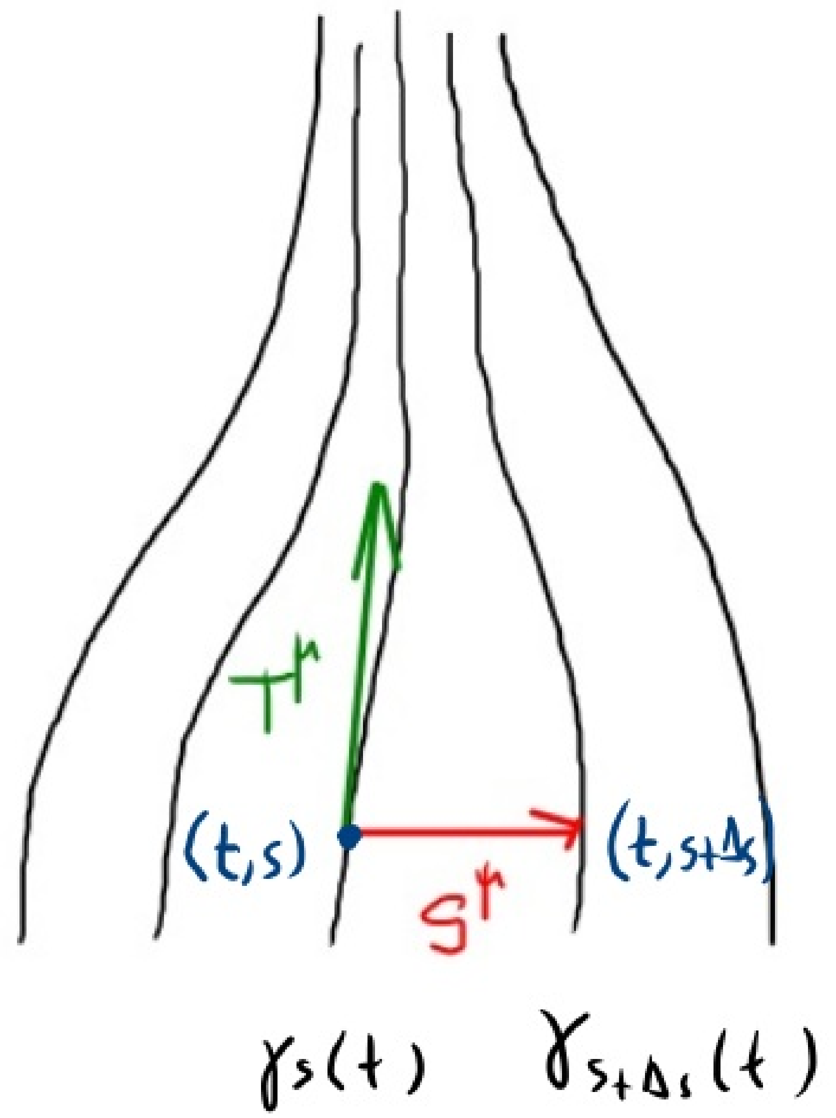

$$T^\nu \nabla_\nu T^\mu = 0 \quad (1)$$

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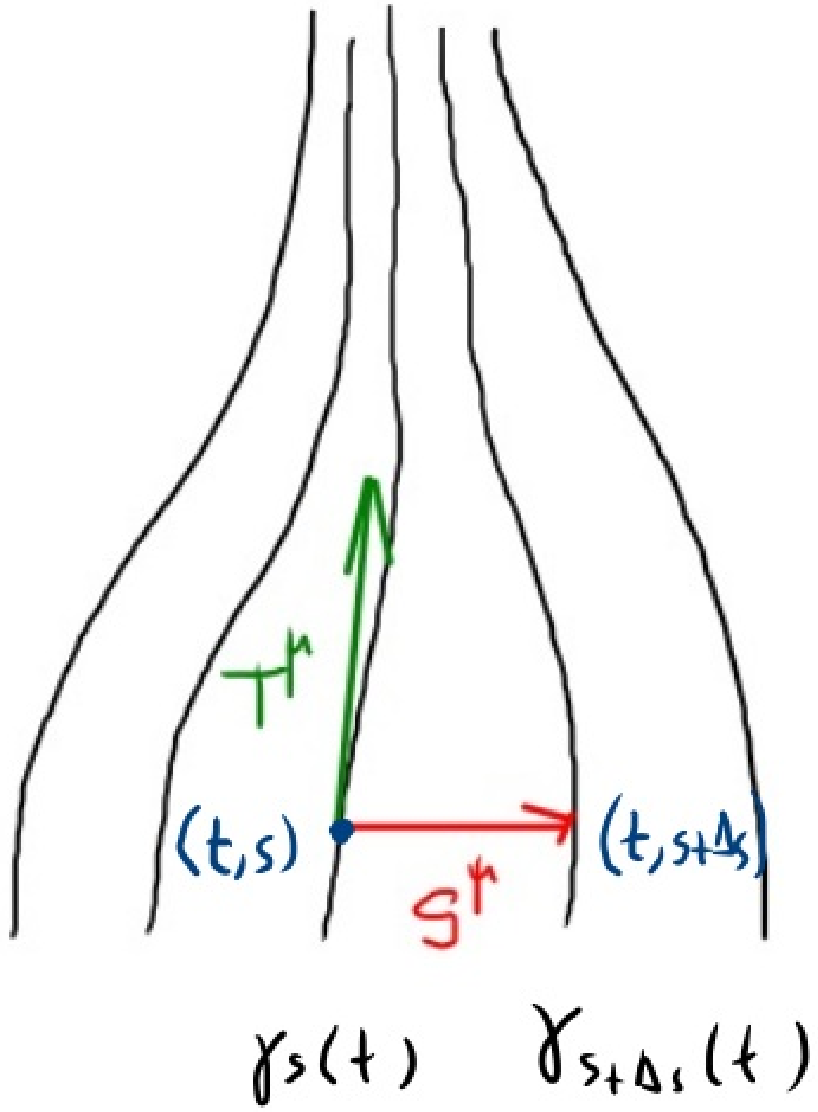

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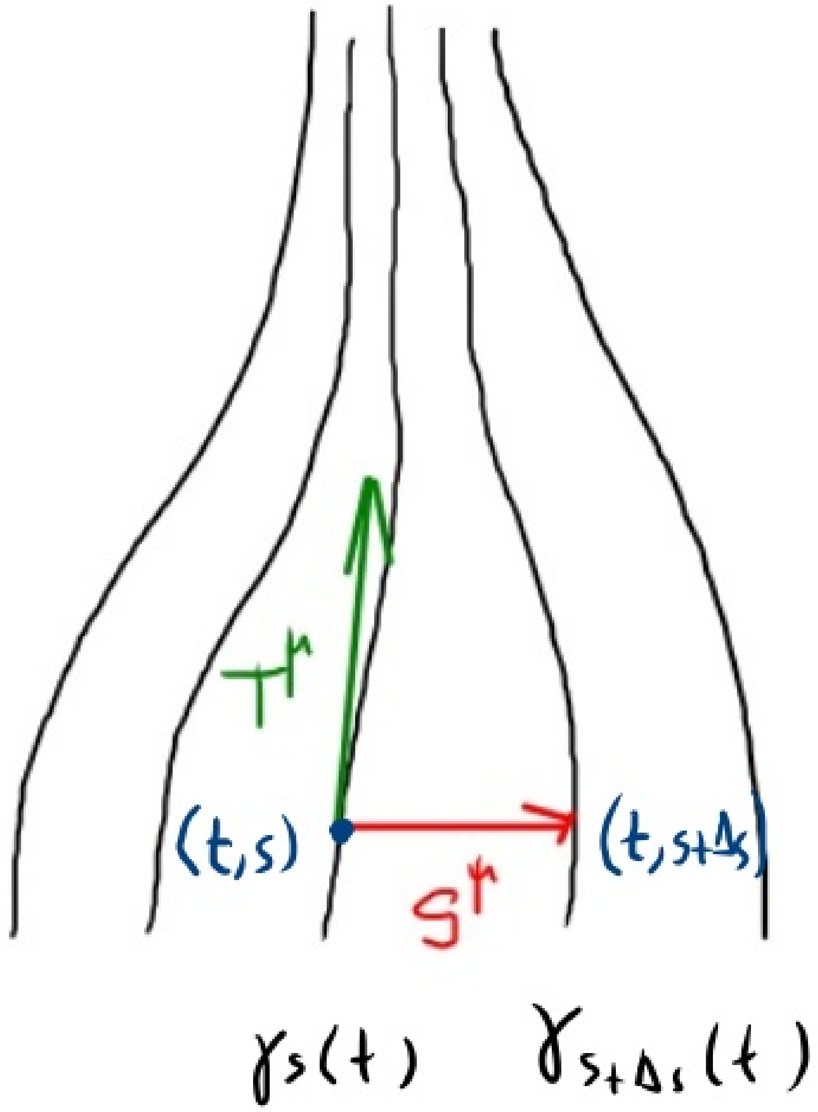
$$\overset{(2)}{=} T^\rho \nabla_\rho \left(S^\sigma \nabla_\sigma T^\mu \right)$$

$$= \left(T^\rho \nabla_\rho S^\sigma \right) \nabla_\sigma T^\mu + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$



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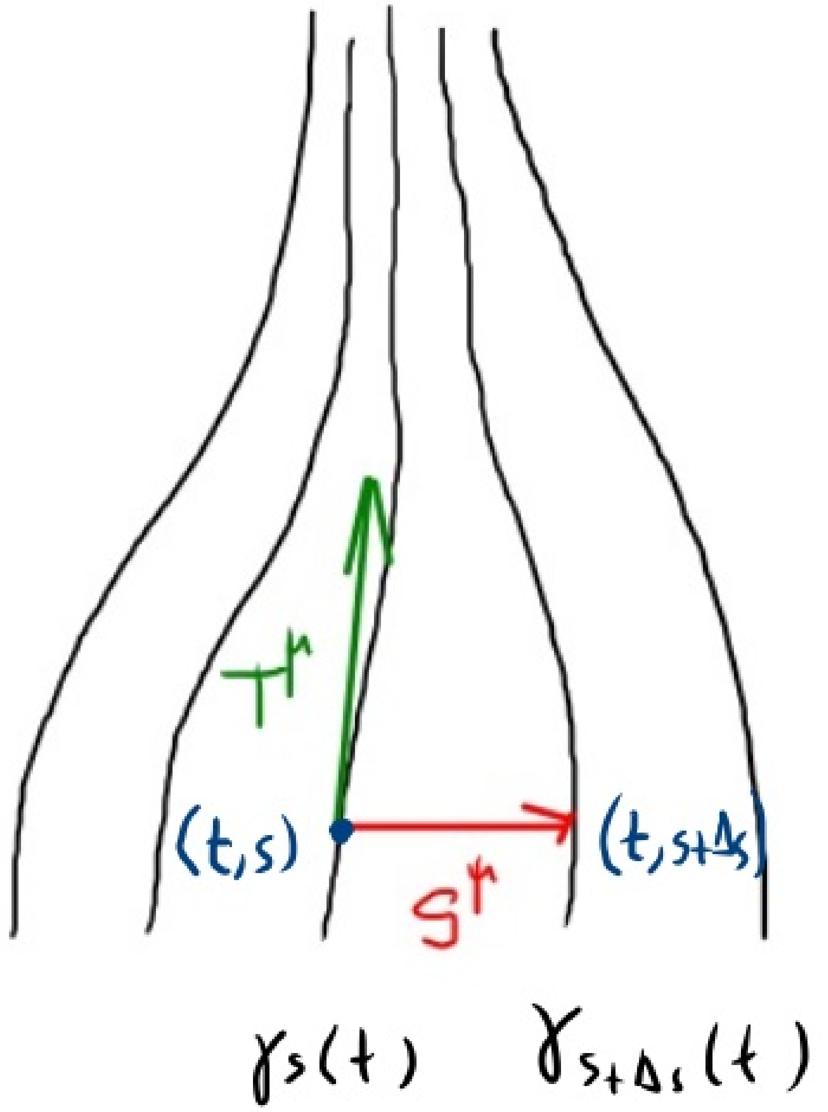

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$$\Downarrow (2)$$

$$= (S^\rho \nabla_\rho T^\sigma) \nabla_\sigma T^\mu + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu{}_{\nu\rho\sigma} T^\nu)$$

Geodesic Deviation

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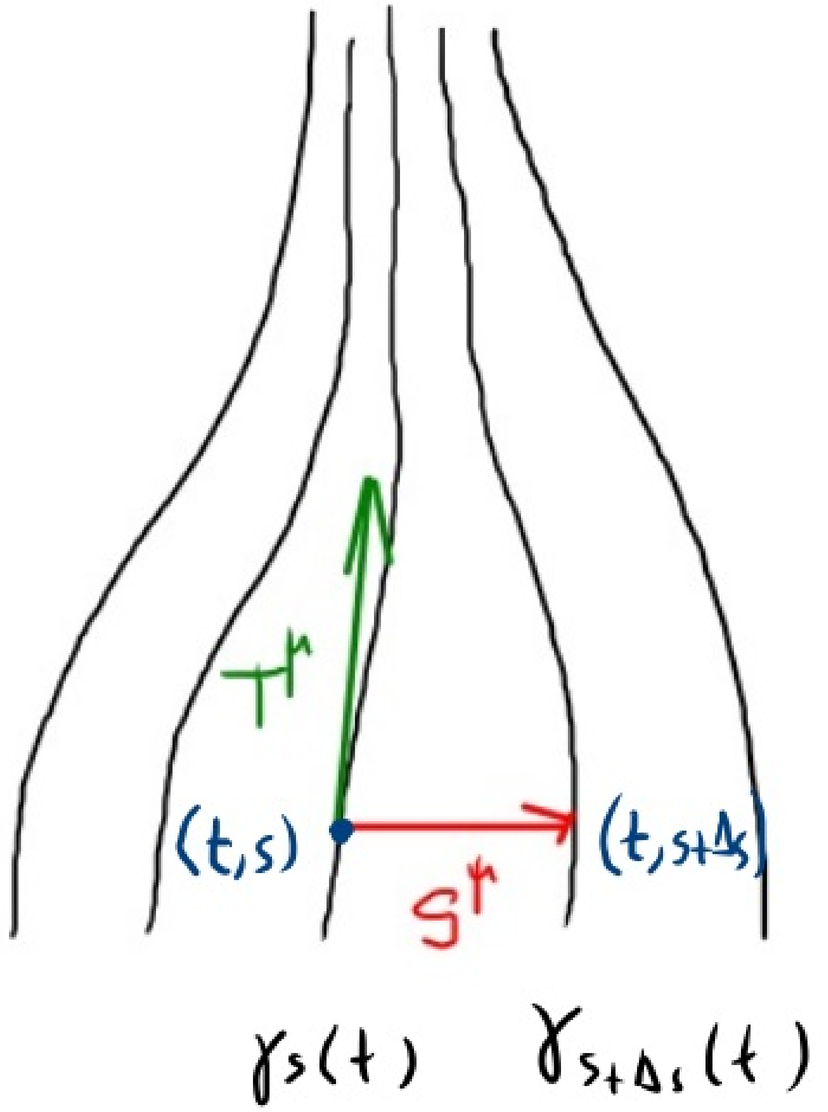
$$= (S^\rho \nabla_\rho T^\sigma) \nabla_\sigma T^\mu + \underbrace{T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu}_{S^\sigma \{ \nabla_\sigma [T^\rho \nabla_\rho T^\mu] \} - S^\sigma \{ \nabla_\sigma T^\rho \nabla_\rho T^\mu \}} + R^\mu{}_{\nu\rho\sigma} T^\nu)$$



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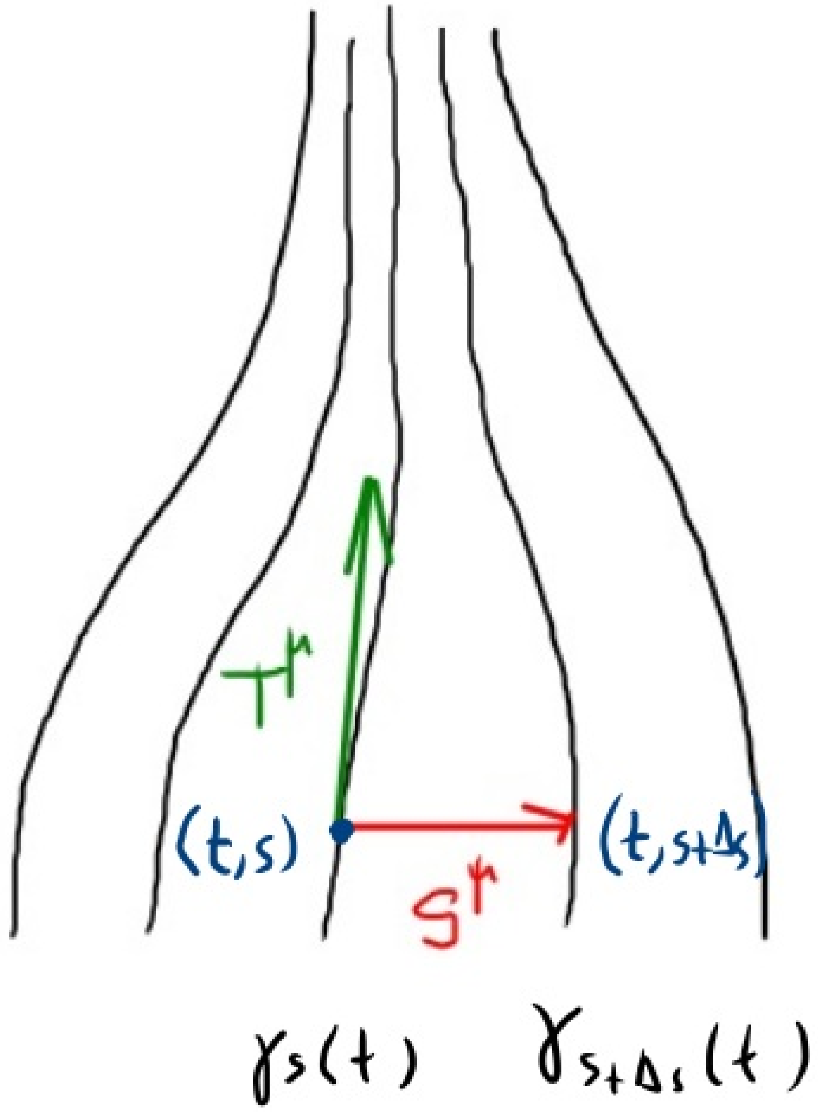

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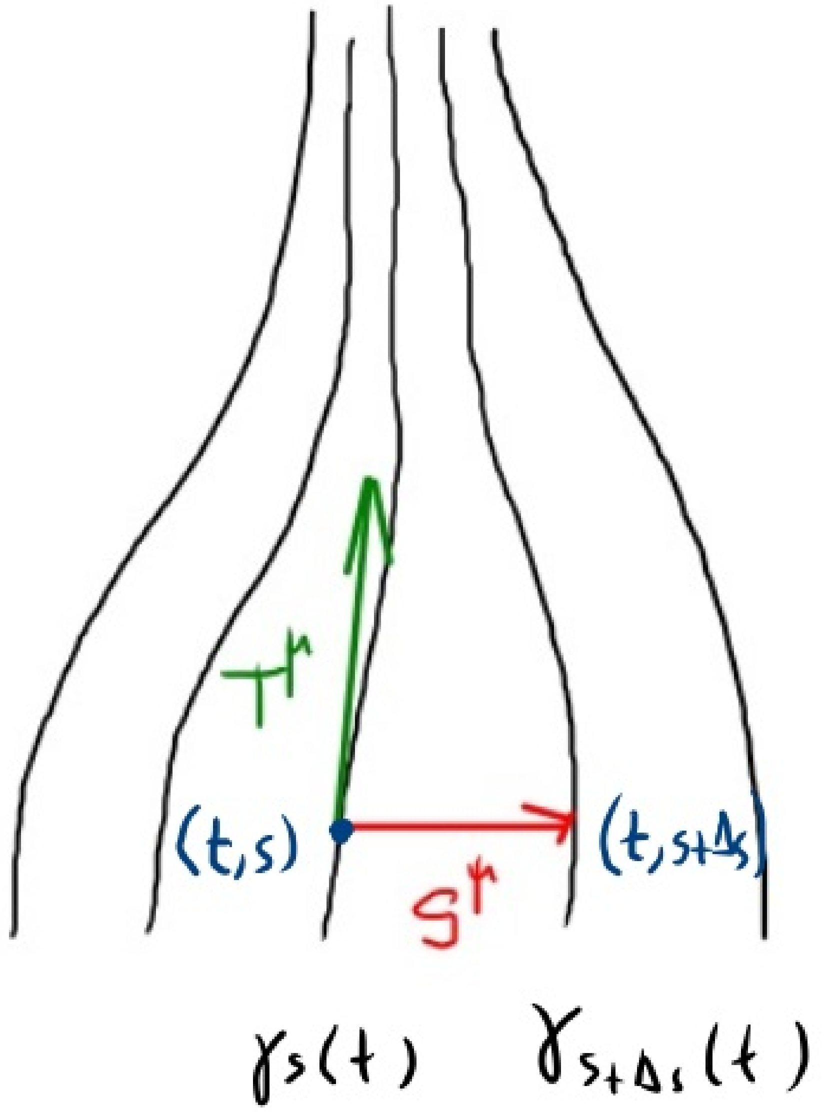
parallel transported, eq. (1)

Geodesic Deviation

$$\Rightarrow A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$



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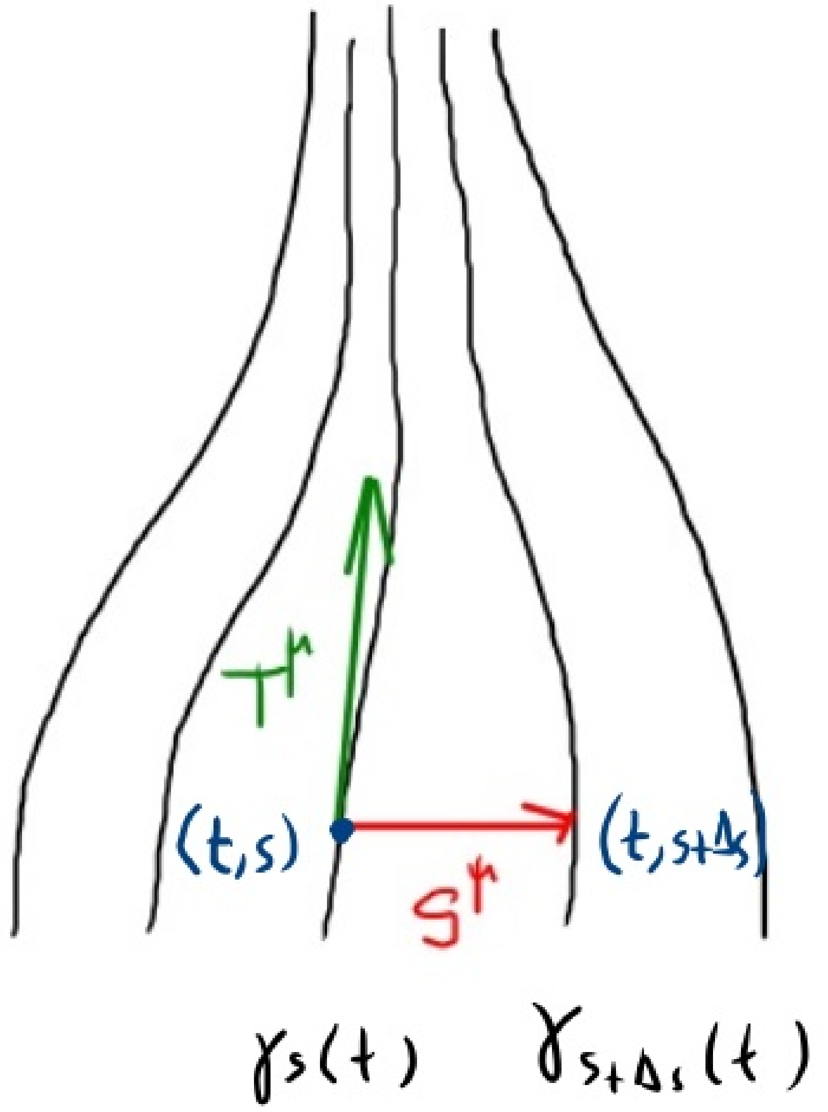
parallel transported, eq (1)

Geodesic Deviation

$$\Rightarrow A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

geodesic deviation equation

$$\begin{pmatrix} \text{relative} \\ \text{acceleration} \end{pmatrix} \propto R$$



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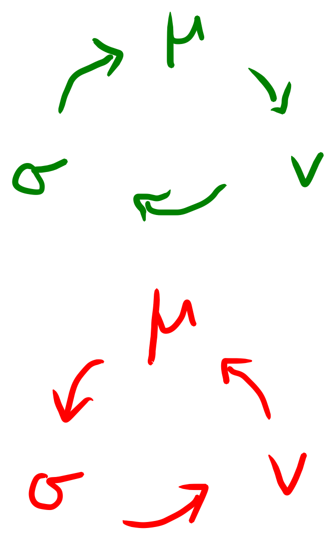
parallel transported, eq. (1)

Exercise: Prove $R^{\rho}{}_{[\sigma\mu\nu]} = 0$ (torsion free)

Exercise: Prove $R^{\rho}{}_{[\sigma\mu\nu]} = 0$

$$R^{\rho}{}_{[\sigma\mu\nu]} = 0 \Leftrightarrow \frac{1}{3!}\Big(R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma} - R^{\rho}{}_{\sigma\nu\mu} - R^{\rho}{}_{\mu\sigma\nu} - R^{\rho}{}_{\nu\mu\sigma} \Big) = 0$$



Exercise: Prove $R^{\rho}{}_{[\sigma\mu\nu]} = 0$

$$R^{\rho}{}_{[\sigma\mu\nu]} = 0 \Leftrightarrow \cancel{\frac{1}{3!}}\Big(R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma}$$

$$\left(\text{use } R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}\right) \quad - R^{\rho}{}_{\sigma\nu\mu} - R^{\rho}{}_{\mu\sigma\nu} - R^{\rho}{}_{\nu\mu\sigma}\Big) = 0$$

$$\Leftrightarrow R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R_{\mu\nu\sigma} = 0$$

Exercise: Prove $R^\rho{}_{[\sigma\mu\nu]} = 0$

$$R^\rho{}_{[\sigma\mu\nu]} = 0 \Leftrightarrow \left(R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R^\rho{}_{\mu\nu\sigma} - R^\rho{}_{\sigma\nu\mu} - R^\rho{}_{\mu\sigma\nu} - R^\rho{}_{\nu\mu\sigma} \right) = 0$$

$$\Leftrightarrow R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R_{\mu\nu\sigma} = 0$$

Choose inertial frame: $\Gamma^\mu{}_{\nu\rho} = 0$ at P (torsion free)

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma}$$

$$R^\rho{}_{\nu\sigma\mu} = \partial_\sigma \Gamma^\rho{}_{\mu\nu} - \partial_\mu \Gamma^\rho{}_{\sigma\nu}$$

$$R^\rho{}_{\mu\nu\sigma} = \partial_\nu \Gamma^\rho{}_{\sigma\mu} - \partial_\sigma \Gamma^\rho{}_{\nu\mu}$$

Exercise: Prove $R^{\rho}{}_{[\sigma\mu\nu]} = 0$

$$R^{\rho}{}_{[\sigma\mu\nu]} = 0 \Leftrightarrow \Big(R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma}$$

$$- R^{\rho}{}_{\sigma\nu\mu} - R^{\rho}{}_{\mu\sigma\nu} - R^{\rho}{}_{\nu\mu\sigma} \Big) = 0$$

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$$R^{\rho}{}_{\mu\nu\sigma} = \partial_\nu \cancel{\Gamma^{\rho}{}_{\sigma\mu}} - \partial_\sigma \cancel{\Gamma^{\rho}{}_{\nu\mu}} +$$

Exercise: Prove $R^{\rho}{}_{[\sigma\mu\nu]} = 0$

$$R^{\rho}{}_{[\sigma\mu\nu]} = 0 \Leftrightarrow \Big(R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma} - R^{\rho}{}_{\sigma\nu\mu} - R^{\rho}{}_{\mu\sigma\nu} - R^{\rho}{}_{\nu\mu\sigma} \Big) = 0$$

$$\Leftrightarrow R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R_{\mu\nu\sigma} = 0$$

Choose inertial frame: $\Gamma^{\mu}{}_{\nu\rho} = 0$ at P (torsion free)

$$\left.\begin{aligned}
R^{\rho}{}_{\sigma\mu\nu} &= \partial_{\mu}\cancel{\Gamma^{\rho}{}_{\nu\sigma}} - \partial_{\nu}\cancel{\Gamma^{\rho}{}_{\mu\sigma}} + \\
R^{\rho}{}_{\nu\sigma\mu} &= \partial_{\sigma}\cancel{\Gamma^{\rho}{}_{\mu\nu}} - \partial_{\mu}\cancel{\Gamma^{\rho}{}_{\sigma\nu}} + \\
R^{\rho}{}_{\mu\nu\sigma} &= \partial_{\nu}\cancel{\Gamma^{\rho}{}_{\sigma\mu}} - \partial_{\sigma}\cancel{\Gamma^{\rho}{}_{\nu\mu}} +
\end{aligned}\right\} = 0$$

If a tensor is 0 at one frame, it is 0 at all frames!

Exercise: Prove $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ (Christoffel connection $\nabla g = 0$ + torsion free)

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$$0 = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \right) g_{\rho\sigma} \qquad \text{(metric compatibility)}$$

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$$0 = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, g_{\rho\sigma}$$

$$= - R^\lambda{}_{\rho\mu\nu}\, g_{\lambda\sigma} - R^\lambda{}_{\sigma\mu\nu}\, g_{\rho\lambda}$$

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$$= -R_{\sigma\rho\mu\nu} - R_{\rho\sigma\mu\nu}$$

$$\Rightarrow \quad R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$

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$$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow R_{\rho\mu\nu\sigma} + R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} = 0$$

Exercise: Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

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$$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow R_{\rho\mu\nu\sigma} + \cancel{R_{\rho\sigma\mu\nu}} + R_{\rho\nu\sigma\mu} = 0$$

$$R^{\sigma}{}_{[\rho\mu\nu]} = 0 \Rightarrow \cancel{R_{\sigma\rho\mu\nu}} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0 \quad \oplus$$

$$\cancel{R_{\rho\mu\nu\sigma}} + \underline{\underline{R_{\rho\nu\sigma\mu}}} + \cancel{\cancel{R_{\sigma\nu\rho\mu}}} + \underline{R_{\sigma\mu\nu\rho}} = 0 \quad (1)$$

$$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \Rightarrow \cancel{R_{\mu\nu\sigma\rho}} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$$

$$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \Rightarrow R_{\nu\sigma\rho\mu} + \cancel{R_{\nu\mu\sigma\rho}} + R_{\nu\rho\mu\sigma} = 0 \quad \oplus$$

$$\cancel{R_{\mu\rho\nu\sigma}} + \underline{R_{\mu\sigma\rho\nu}} + \cancel{\cancel{R_{\nu\sigma\rho\mu}}} + \underline{\underline{R_{\nu\rho\mu\sigma}}} = 0 \quad (2)$$

Exercise: Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$(1)+(2) \Rightarrow \quad 2R_{\rho\nu\sigma\mu} + 2R_{\sigma\mu\nu\rho} = 0$

$$\cancel{R_{\rho\mu\nu\sigma}} + R_{\rho\nu\sigma\mu} + \cancel{R_{\sigma\nu\rho\mu}} + R_{\sigma\mu\nu\rho} = 0 \quad (1)$$

$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \Rightarrow \cancel{R_{\mu\nu\sigma\rho}} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$

$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \Rightarrow R_{\nu\sigma\rho\mu} + \cancel{R_{\nu\mu\sigma\rho}} + R_{\nu\rho\mu\sigma} = 0 \quad \oplus$

$$\cancel{R_{\mu\rho\nu\sigma}} + R_{\mu\sigma\rho\nu} + \cancel{R_{\nu\sigma\rho\mu}} + R_{\nu\rho\mu\sigma} = 0 \quad (2)$$

Exercise: Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$$(1)+(2) \Rightarrow \cancel{2}\, R_{\rho\nu\sigma\mu} + \cancel{2}\, R_{\sigma\mu\nu\rho} = 0 \Rightarrow R_{\rho\nu\sigma\mu} = R_{\sigma\mu\rho\nu}$$

$$\cancel{R_{\rho\mu\nu\sigma}} + R_{\rho\nu\sigma\mu} + \cancel{R_{\sigma\nu\rho\mu}} + R_{\sigma\mu\nu\rho} = 0 \quad (1)$$

$$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \Rightarrow \cancel{R_{\mu\nu\sigma\rho}} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$$

$$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \Rightarrow R_{\nu\sigma\rho\mu} + \cancel{R_{\nu\mu\sigma\rho}} + R_{\nu\rho\mu\sigma} = 0 \quad \oplus$$

$$\cancel{R_{\mu\rho\nu\sigma}} + R_{\mu\sigma\rho\nu} + \cancel{R_{\nu\sigma\rho\mu}} + R_{\nu\rho\mu\sigma} = 0 \quad (2)$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \Leftrightarrow \frac{1}{3!}\Big(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu}$$

$$- \nabla_\lambda R_{\sigma\rho\mu\nu} - \nabla_\rho R_{\lambda\sigma\mu\nu} - \nabla_\sigma R_{\rho\lambda\mu\nu} \Big) = 0$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \Leftrightarrow \cancel{\frac{1}{3!}}\Big(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu}$$
$$- \nabla_\lambda R_{\sigma\rho\mu\nu} - \nabla_\rho R_{\lambda\sigma\mu\nu} - \nabla_\sigma R_{\rho\lambda\mu\nu} \Big) = 0$$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$\begin{tikzcd} & \rho \arrow[dr] & \\ \lambda \arrow[ur] & & \sigma \arrow[ll] \end{tikzcd}$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$\nabla_\mu \quad \nabla_\nu \quad \nabla_\sigma$$

$$\nabla_\sigma \quad \nabla_\mu \quad \nabla_\nu$$

$$\nabla_\nu \quad \nabla_\sigma \quad \nabla_\mu$$

$$\begin{tikzcd} & \nu \arrow[dr] & \\ \mu \arrow[ur] & & \sigma \arrow[ll] \end{tikzcd}$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[\nabla_\mu, \nabla_\nu] \nabla_\sigma$$

$$[\nabla_\sigma, \nabla_\mu] \nabla_\nu$$

$$[\nabla_\nu, \nabla_\sigma] \nabla_\mu$$

<u>Exercise:</u> Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma]$

$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu]$

$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu] \nabla_\sigma - \nabla_\sigma [\nabla_\mu, \nabla_\nu]$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu]$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$$

<u>Exercise:</u> Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Longleftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$

$$= \nabla_\mu \nabla_\nu \nabla_\sigma - \nabla_\nu \nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu \nabla_\nu + \nabla_\sigma \nabla_\nu \nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu]$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\iff \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$

$$= \nabla_\mu \nabla_\nu \nabla_\sigma - \nabla_\nu \nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu \nabla_\nu + \nabla_\sigma \nabla_\nu \nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] = [\nabla_\sigma, \nabla_\mu]\nabla_\nu - \nabla_\nu[\nabla_\sigma, \nabla_\mu]$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Longleftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$

$$= \nabla_\mu\nabla_\nu\nabla_\sigma - \nabla_\nu\nabla_\mu\nabla_\sigma - \nabla_\sigma\nabla_\mu\nabla_\nu + \nabla_\sigma\nabla_\nu\nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] = [\nabla_\sigma, \nabla_\mu]\nabla_\nu - \nabla_\nu[\nabla_\sigma, \nabla_\mu]$$

$$= \nabla_\sigma\nabla_\mu\nabla_\nu - \nabla_\mu\nabla_\sigma\nabla_\nu - \nabla_\nu\nabla_\sigma\nabla_\mu + \nabla_\nu\nabla_\mu\nabla_\sigma$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$
$$= \nabla_\mu \nabla_\nu \nabla_\sigma - \nabla_\nu \nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu \nabla_\nu + \nabla_\sigma \nabla_\nu \nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] = [\nabla_\sigma, \nabla_\mu]\nabla_\nu - \nabla_\nu[\nabla_\sigma, \nabla_\mu]$$
$$= \nabla_\sigma \nabla_\mu \nabla_\nu - \nabla_\mu \nabla_\sigma \nabla_\nu - \nabla_\nu \nabla_\sigma \nabla_\mu + \nabla_\nu \nabla_\mu \nabla_\sigma$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] = [\nabla_\nu, \nabla_\sigma]\nabla_\mu - \nabla_\mu[\nabla_\nu, \nabla_\sigma]$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$

$$= \nabla_\mu\nabla_\nu\nabla_\sigma - \nabla_\nu\nabla_\mu\nabla_\sigma - \nabla_\sigma\nabla_\mu\nabla_\nu + \nabla_\sigma\nabla_\nu\nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] = [\nabla_\sigma, \nabla_\mu]\nabla_\nu - \nabla_\nu[\nabla_\sigma, \nabla_\mu]$$

$$= \nabla_\sigma\nabla_\mu\nabla_\nu - \nabla_\mu\nabla_\sigma\nabla_\nu - \nabla_\nu\nabla_\sigma\nabla_\mu + \nabla_\nu\nabla_\mu\nabla_\sigma$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] = [\nabla_\nu, \nabla_\sigma]\nabla_\mu - \nabla_\mu[\nabla_\nu, \nabla_\sigma]$$

$$= \nabla_\nu\nabla_\sigma\nabla_\mu - \nabla_\sigma\nabla_\nu\nabla_\mu - \nabla_\mu\nabla_\nu\nabla_\sigma + \nabla_\mu\nabla_\sigma\nabla_\nu$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \quad \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] =$$

$$= \nabla_\mu \cancel{\nabla_\nu} \nabla_\sigma - \nabla_\nu \cancel{\nabla_\mu} \nabla_\sigma - \nabla_\sigma \cancel{\nabla_\mu} \nabla_\nu + \nabla_\sigma \cancel{\nabla_\nu} \nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] =$$

$$= \nabla_\sigma \cancel{\nabla_\mu} \nabla_\nu - \nabla_\mu \cancel{\nabla_\sigma} \nabla_\nu - \nabla_\nu \cancel{\nabla_\sigma} \nabla_\mu + \nabla_\nu \cancel{\nabla_\mu} \nabla_\sigma$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] =$$

$$= \nabla_\nu \cancel{\nabla_\sigma} \nabla_\mu - \nabla_\sigma \cancel{\nabla_\nu} \nabla_\mu - \nabla_\mu \cancel{\nabla_\nu} \nabla_\sigma + \nabla_\mu \cancel{\nabla_\sigma} \nabla_\nu$$

$+$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] + [[\nabla_\sigma, \nabla_\mu], \nabla_\nu] + [[\nabla_\nu, \nabla_\sigma], \nabla_\mu] = 0$$

Jacobi Identity

Exercise:

Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu,\nabla_\nu],\nabla_\sigma]V^\rho = [\nabla_\mu,\nabla_\nu]\nabla_\sigma V^\rho - \nabla_\sigma [\nabla_\mu,\nabla_\nu]V^\rho$$

$$= \overbrace{-R^\lambda{}_{\sigma\mu\nu}\nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu}\nabla_\sigma V^\lambda}^{(1,1)\ \text{tensor}} - \nabla_\sigma(R^\rho{}_{\lambda\mu\nu}V^\lambda)$$

<u>Exercise:</u> Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = [\nabla_\mu, \nabla_\nu] \nabla_\sigma V^\rho - \nabla_\sigma [\nabla_\mu, \nabla_\nu] V^\rho$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda - \nabla_\sigma (R^\rho{}_{\lambda\mu\nu} V^\lambda)$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda - (\nabla_\sigma R^\rho{}_{\lambda\mu\nu}) V^\lambda - R^\rho{}_{\lambda\mu\nu} (\nabla_\sigma V^\lambda)$$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = [\nabla_\mu, \nabla_\nu] \nabla_\sigma V^\rho - \nabla_\sigma [\nabla_\mu, \nabla_\nu] V^\rho$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda - \nabla_\sigma (R^\rho{}_{\lambda\mu\nu} V^\lambda)$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + \cancel{R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda} - (\nabla_\sigma R^\rho{}_{\lambda\mu\nu}) V^\lambda - \cancel{R^\rho{}_{\lambda\mu\nu} (\nabla_\sigma V^\lambda)}$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla\ ,\ \nabla], \nabla\ ] V^\rho = -R^\lambda \quad \nabla_\lambda V^\rho - \nabla\ R^\rho{}_\lambda \quad V^\lambda$$

$$[[\nabla\ ,\ \nabla], \nabla\ ] V^\rho = -R^\lambda \quad \nabla_\lambda V^\rho - \nabla\ R^\rho{}_\lambda \quad V^\lambda$$

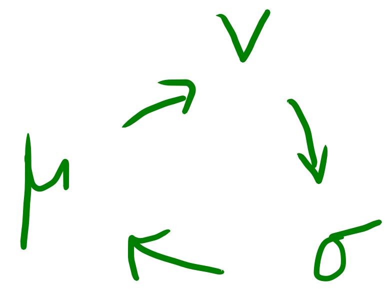
$$\begin{tikzcd} \mu \arrow[r] & \nu \arrow[d] \\ & \sigma \arrow[ul] \end{tikzcd}$$

Exercise:

Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$



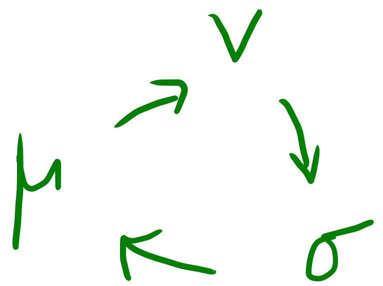
$$[[\nabla, \nabla], \nabla] V^\rho = -R^\lambda \quad \nabla_\lambda V^\rho - \nabla \; R^\rho{}_\lambda \; V^\lambda$$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda$$



Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda$$

$+$

$$0 = -0 \, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda$$

use $R^\lambda{}_{[\mu\nu\sigma]} = 0$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda$$

$+$

$$0 = -0 \, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda$$

$$\Rightarrow \nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\nu R_{\rho\lambda\sigma\mu} + \nabla_\mu R_{\rho\lambda\nu\sigma} = 0$$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda \qquad +$$

$$0 = -0\, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda$$

$$\Rightarrow \nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\nu R_{\rho\lambda\sigma\mu} + \nabla_\mu R_{\rho\lambda\nu\sigma} = 0$$

$$\Rightarrow \nabla_\sigma R_{\mu\nu\rho\lambda} + \nabla_\nu R_{\sigma\mu\rho\lambda} + \nabla_\mu R_{\nu\sigma\rho\lambda} = 0$$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda$$

$+$

$$0 = -0 \, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda$$

$$\Rightarrow \nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\nu R_{\rho\lambda\sigma\mu} + \nabla_\mu R_{\rho\lambda\nu\sigma} = 0$$

$$\Rightarrow \underline{\nabla_\sigma R_{\mu\nu}}{}_{\rho\lambda} + \underline{\nabla_\nu R_{\sigma\mu}}{}_{\rho\lambda} + \underline{\nabla_\mu R_{\nu\sigma}}{}_{\rho\lambda} = 0$$

<u>Exercise:</u> Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda$$

$+$

$$0 = -0 \, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda$$

$$\Rightarrow \nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\nu R_{\rho\lambda\sigma\mu} + \nabla_\mu R_{\rho\lambda\nu\sigma} = 0$$

$$\Rightarrow \underline{\nabla_\sigma R_{\mu\nu}}{}_{\rho\lambda} + \underline{\nabla_\nu R_{\sigma\mu}}{}_{\rho\lambda} + \underline{\nabla_\mu R_{\nu\sigma}}{}_{\rho\lambda} = 0 \Rightarrow \nabla_{[\sigma} R_{\mu\nu]\rho\lambda} = 0$$

<u>Exercise</u>: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma}$$

<u>Exercise</u>: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma} \rightarrow$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2}$$

Exercise: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma} \rightarrow$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2} \longrightarrow \frac{n(n-1)}{2} \cdot \frac{n(n-1)}{2} = \left[\frac{n(n-1)}{2}\right]^2$$

Exercise: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma} \rightarrow$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2} \longrightarrow \frac{n(n-1)}{2}\cdot\frac{n(n-1)}{2} = \left[\frac{n(n-1)}{2}\right]^2$$

$$R_{\mu[\nu\rho\sigma]} = 0$$

Exercise: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma} \rightarrow$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2} \rightarrow \frac{n(n-1)}{2} \cdot \frac{n(n-1)}{2} = \left[\frac{n(n-1)}{2}\right]^2$$

$$R_{\mu[\nu\rho\sigma]} = 0$$

$$n \qquad \frac{n(n-1)(n-2)}{3!}$$

3-combination of n objects

Exercise: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma} \rightarrow$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2} \rightarrow \frac{n(n-1)}{2} \cdot \frac{n(n-1)}{2} = \left[\frac{n(n-1)}{2}\right]^2$$

$$R_{\mu[\nu\rho\sigma]} = 0$$

$$n \qquad \frac{n(n-1)(n-2)}{3!} \rightarrow \frac{n^2(n-1)(n-2)}{6}$$ conditions

Exercise: Count independent components of Riemann (Levi-Civita connection)

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$$n \qquad \frac{n(n-1)(n-2)}{3!} \rightarrow \frac{n^2(n-1)(n-2)}{6} \quad \text{conditions}$$

$$\frac{n^2(n-1)^2}{4} - \frac{n^2(n-1)(n-2)}{6} = \frac{n^2(n-1)}{2}\left[\frac{n-1}{2} - \frac{n-2}{3}\right] = \frac{n^2(n-1)}{2}\frac{(n+1)}{6} = \frac{n^2(n^2-1)}{12}$$

Exercise: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma} \rightarrow$$

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$$\frac{n^2(n-1)^2}{4} - \frac{n^2(n-1)(n-2)}{6} = \frac{n^2(n-1)}{2}\left[\frac{n-1}{2} - \frac{n-2}{3}\right] = \frac{n^2(n-1)}{2}\frac{(n+1)}{6} = \frac{n^2(n^2-1)}{12}$$

other symmetries

$$R_{[\mu\nu\rho\sigma]} = 0$$

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

not independent!

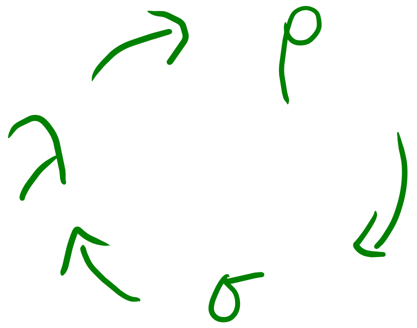
Exercise: Prove that the Bianchi identities imply $\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \neq 0$

Corrolary: - we can't have $R_{\mu\nu} = 8\pi T_{\mu\nu}$

- $\nabla^\mu G_{\rho\mu} = 0$ and we may have $G_{\mu\nu} = 8\pi T_{\mu\nu}$

Exercise: Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2}\nabla_{\rho} R \neq 0$

Bianchi id: $\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} = 0$



<u>Exercise:</u> Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2}\nabla_{\rho} R \neq 0$

Bianchi id: $\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} = 0 \Rightarrow$

$$0 = g^{\nu\sigma} g^{\mu\lambda} \left(\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} \right)$$

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$$= g^{\mu\lambda} \nabla_{\lambda} \left(g^{\nu\sigma} R_{\rho\sigma\mu\nu} \right) + g^{\nu\sigma} \nabla_{\sigma} \left(g^{\mu\lambda} R_{\lambda\rho\mu\nu} \right) + \nabla_{\rho} \left(g^{\nu\sigma} g^{\mu\lambda} R_{\sigma\lambda\mu\nu} \right)$$

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$(-)\cdot(-) = (+)$

$\downarrow$

$g^{\nu\sigma} R_{\sigma\rho\nu\mu}$

$\downarrow$

$R^{\nu}{}_{\rho\nu\mu}$

$\downarrow$

$R_{\rho\mu}$

<u>Exercise:</u> Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R \neq 0$

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$g^{\nu\sigma} R_{\rho\sigma\mu\nu} \downarrow g^{\nu\sigma} R_{\sigma\rho\nu\mu} \downarrow R^{\nu}{}_{\rho\nu\mu} \downarrow R_{\rho\mu}$

$g^{\mu\lambda} R_{\lambda\rho\mu\nu} \downarrow R^{\mu}{}_{\rho\mu\nu} \downarrow R_{\rho\nu}$

Exercise: Prove that the Bianchi identities imply $\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \neq 0$

Bianchi id: $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0 \Rightarrow$

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$g^{\nu\sigma} R_{\sigma\rho\nu\mu} \rightarrow R^\nu{}_{\rho\nu\mu} \rightarrow R_{\rho\mu}$

$R^\mu{}_{\rho\mu\nu} \rightarrow R_{\rho\nu}$

$g^{\mu\lambda} R^\nu{}_{\sigma\lambda\mu\nu} \xrightarrow{(-)} -g^{\mu\lambda} R^\nu{}_{\sigma\lambda\nu\mu} \rightarrow -g^{\mu\lambda} R_{\lambda\mu} \rightarrow -R$

Exercise: Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \neq 0$

Bianchi id: $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0 \Rightarrow$

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$$= g^{\mu\lambda} \nabla_\lambda R_{\rho\mu} + g^{\nu\sigma} \nabla_\sigma R_{\rho\nu} - \nabla_\rho R$$

$\downarrow$

$R_{\rho\nu}$

$\downarrow$

$R_{\rho\mu}$

$\downarrow$

$-R$

Exercise: Prove that the Bianchi identities imply $\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \neq 0$

Bianchi id: $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0 \Rightarrow$

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$$= g^{\mu\lambda} \nabla_\lambda R_{\rho\mu} + g^{\nu\sigma} \nabla_\sigma R_{\rho\nu} - \nabla_\rho R$$

$$= \nabla^\mu R_{\rho\mu} + \nabla^\nu R_{\rho\nu} - \nabla_\rho R$$

$$= 2 \nabla^\mu R_{\rho\mu} - \nabla_\rho R$$

Exercise: Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2}\nabla_{\rho} R \neq 0$

Bianchi id: $\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} = 0 \Rightarrow$

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$$= g^{\mu\lambda} \nabla_{\lambda} \left(g^{\nu\sigma} R_{\rho\sigma\mu\nu} \right) + g^{\nu\sigma} \nabla_{\sigma} \left(g^{\mu\lambda} R_{\lambda\rho\mu\nu} \right) + \nabla_{\rho} \left(g^{\nu\sigma} g^{\mu\lambda} R_{\sigma\lambda\mu\nu} \right)$$

$$= g^{\mu\lambda} \nabla_{\lambda} R_{\rho\mu} + g^{\nu\sigma} \nabla_{\sigma} R_{\rho\nu} - \nabla_{\rho} R$$

$$= \nabla^{\mu} R_{\rho\mu} + \nabla^{\nu} R_{\rho\nu} - \nabla_{\rho} R$$

$$= 2 \nabla^{\mu} R_{\rho\mu} - \nabla_{\rho} R \Rightarrow \nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R$$

__Exercise:__ Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2}\nabla_{\rho} R \neq 0$

Then $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow$

$$\nabla^{\mu} G_{\mu\nu} = \nabla^{\mu} R_{\mu\nu} - \nabla^{\mu}\left(\frac{1}{2} g_{\mu\nu} R\right)$$

<u>Exercise:</u> Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R \neq 0$

Then $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow$

$$\nabla^{\mu} G_{\mu\nu} = \nabla^{\mu} R_{\mu\nu} - \nabla^{\mu}\left(\frac{1}{2} g_{\mu\nu} R\right)$$

$$= \frac{1}{2} \nabla_{\nu} R - \frac{1}{2} g_{\mu\nu} \nabla^{\mu} R$$

<u>Exercise:</u> Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2}\nabla_{\rho} R \neq 0$

Then $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow$

$$\nabla^{\mu} G_{\mu\nu} = \nabla^{\mu} R_{\mu\nu} - \nabla^{\mu}\left(\frac{1}{2} g_{\mu\nu} R\right)$$

$$= \frac{1}{2}\nabla_{\nu} R - \frac{1}{2} g_{\mu\nu} \nabla^{\mu} R$$

$$= \frac{1}{2}\nabla_{\nu} R - \frac{1}{2}\nabla_{\nu} R = 0$$