- The Metric
- Causal Structure

The metric is an additional structure on a manifold

It provides:

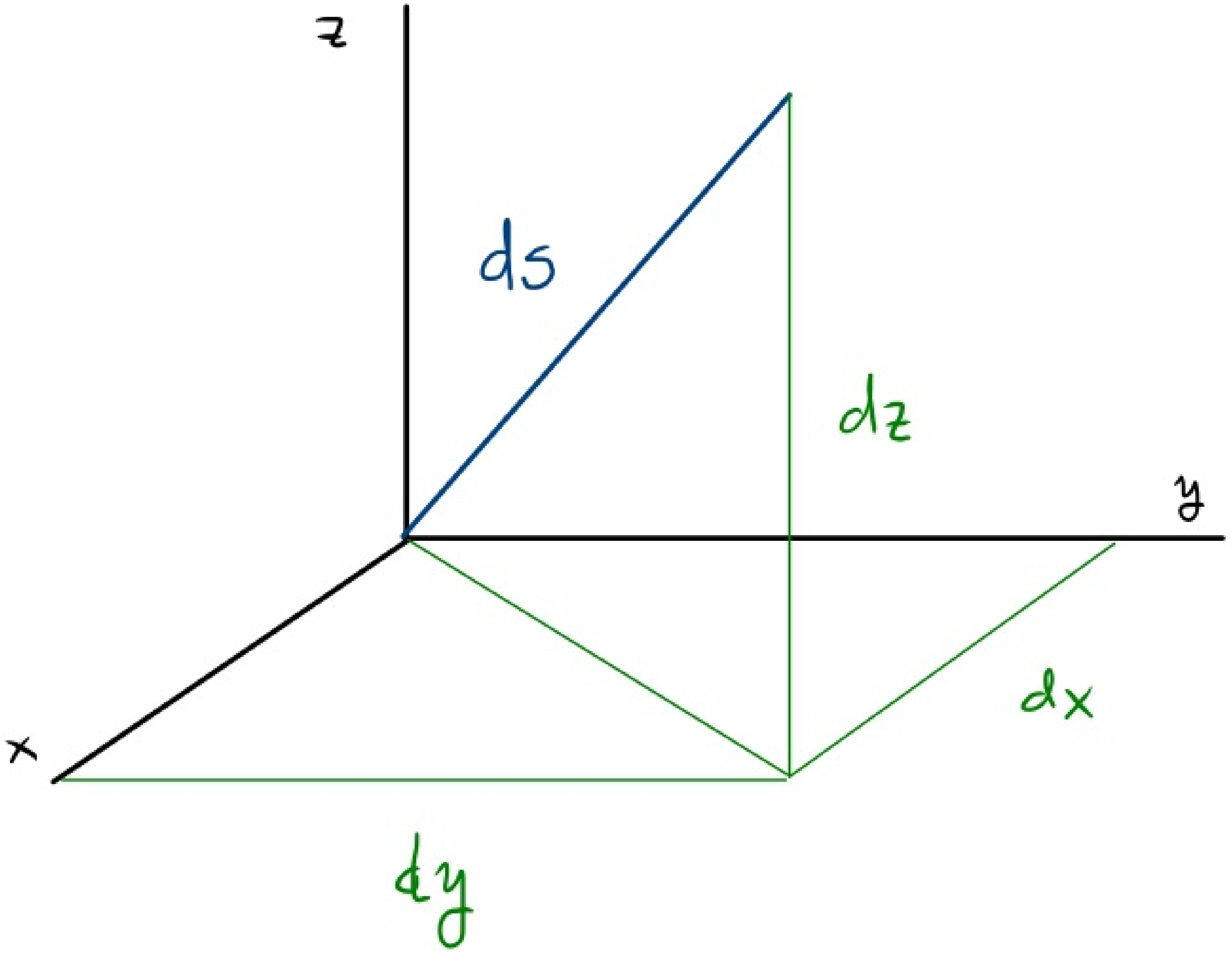
- locally inertial frames - where physics is easy...
- causality: future, past, max "speed of light"
- geometric, observer independent $\begin{cases} \text{path length} \\ \text{proper time} \end{cases}$
- parallel transport, geodesics: $\begin{cases} \text{locally "straightest" curves} \\ \text{" longest times} \end{cases}$

  geodesics: paths of the free!
- curvature = gravitation

- Line element: infinitesimal length

  - Euclidean $\mathbb{R}^3$:

    $$ds^2 = dx^2 + dy^2 + dz^2$$



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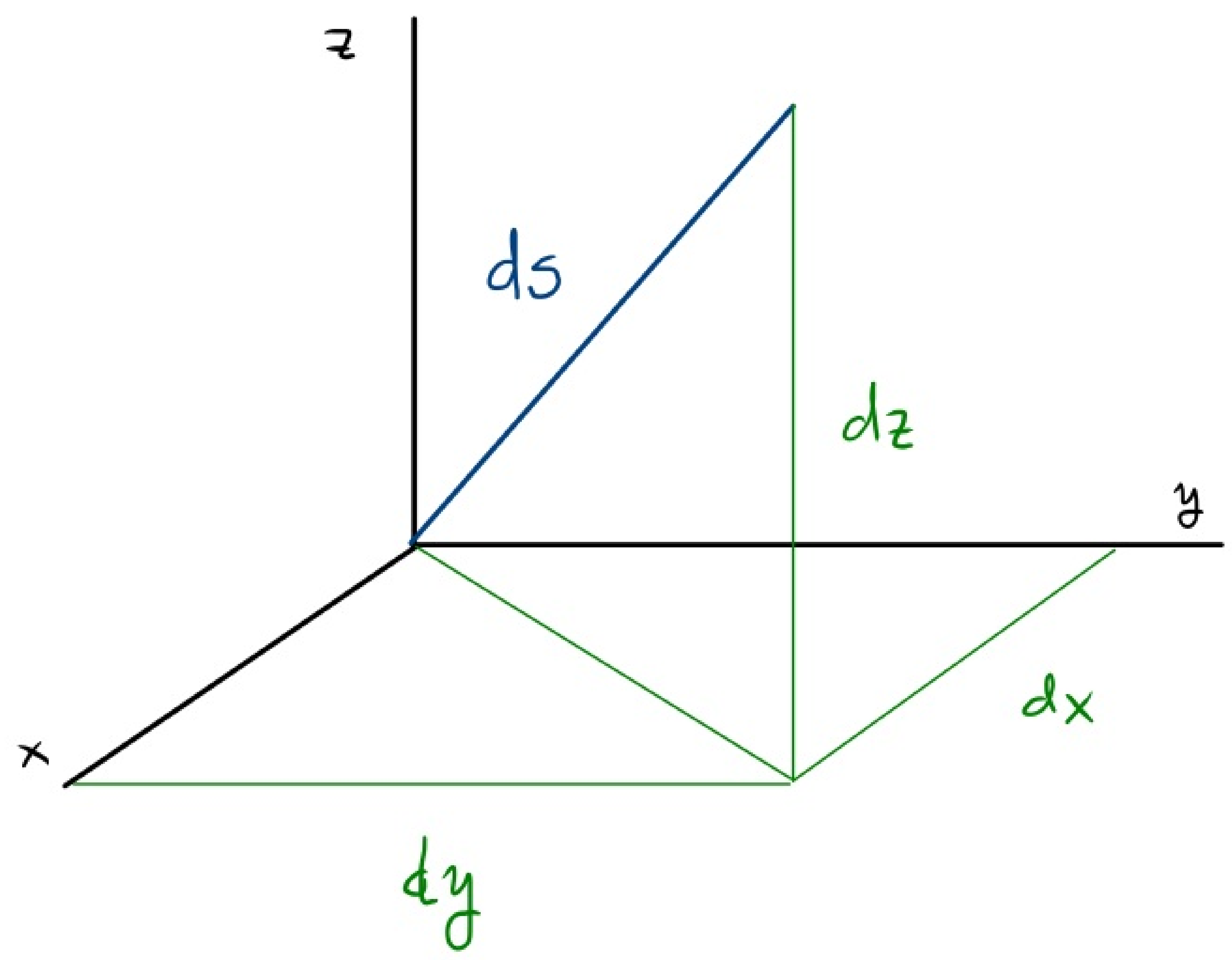

But also:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

$$= d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

$$= \dots$$

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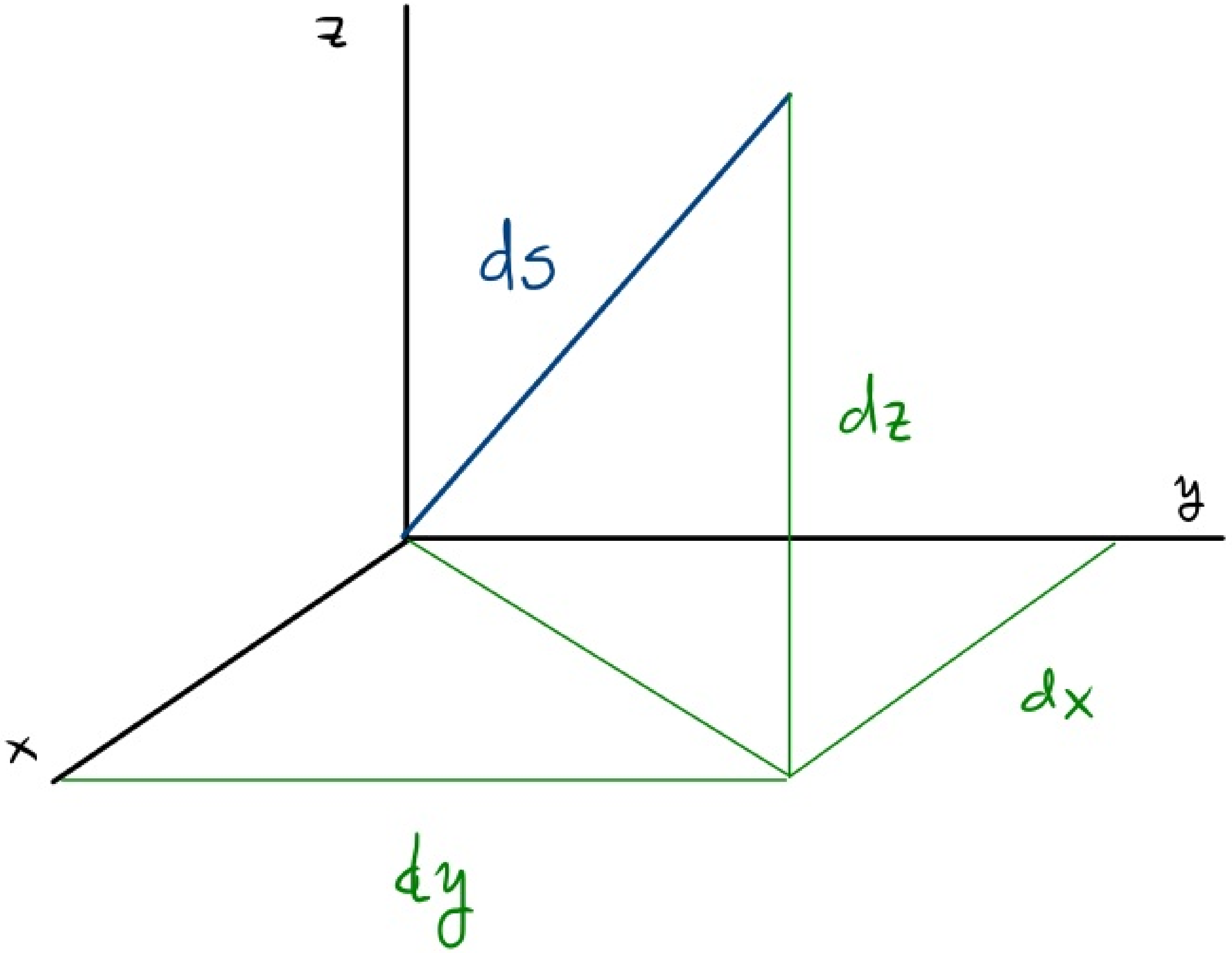

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In general:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

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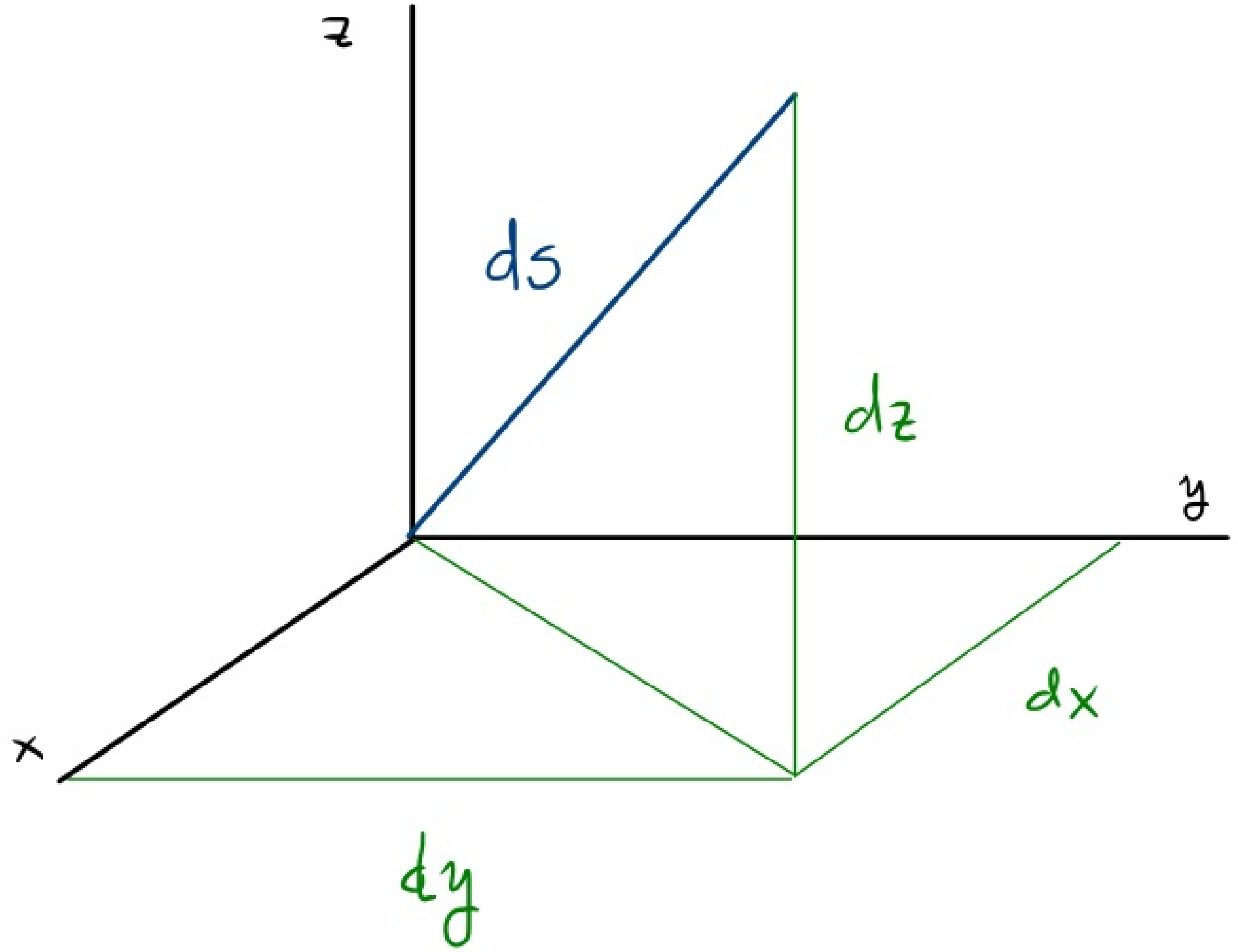

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In general:

symmetric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$g_{\mu\nu}$: a $(0,2)$ symmetric tensor

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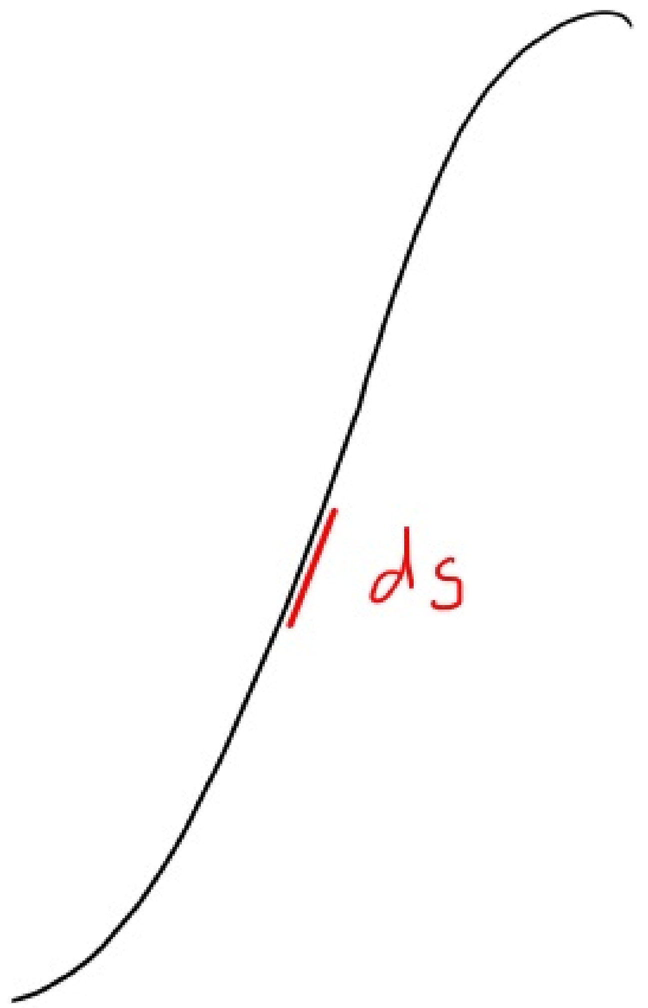

- Length of a finite curve:

$$s = \int ds = \int \left(dx^2 + dy^2 + dz^2\right)^{1/2} \equiv \int dt \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right\}^{1/2}$$

$\hookrightarrow$ line integral on curve $\gamma(t)$

$\hookrightarrow$ $x(t), y(t), z(t)$ must be given

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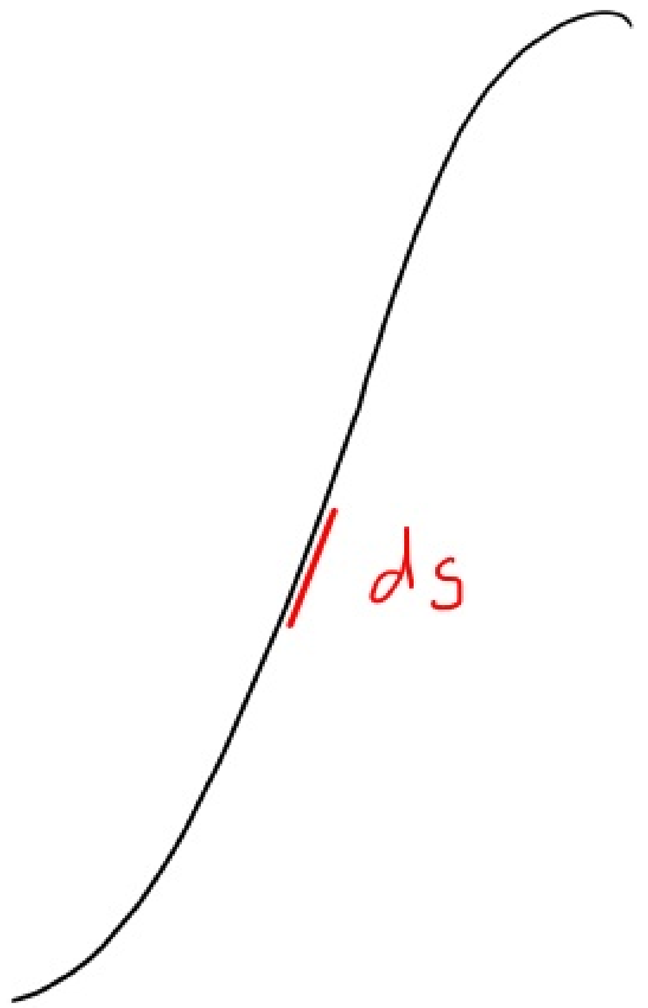
$$= \int \left\{ g_{\mu\nu}\, dx^\mu dx^\nu \right\}^{1/2} = \int dt \left\{ g_{\mu\nu}\, \frac{dx^\mu}{dt}\, \frac{dx^\nu}{dt} \right\}^{1/2}$$



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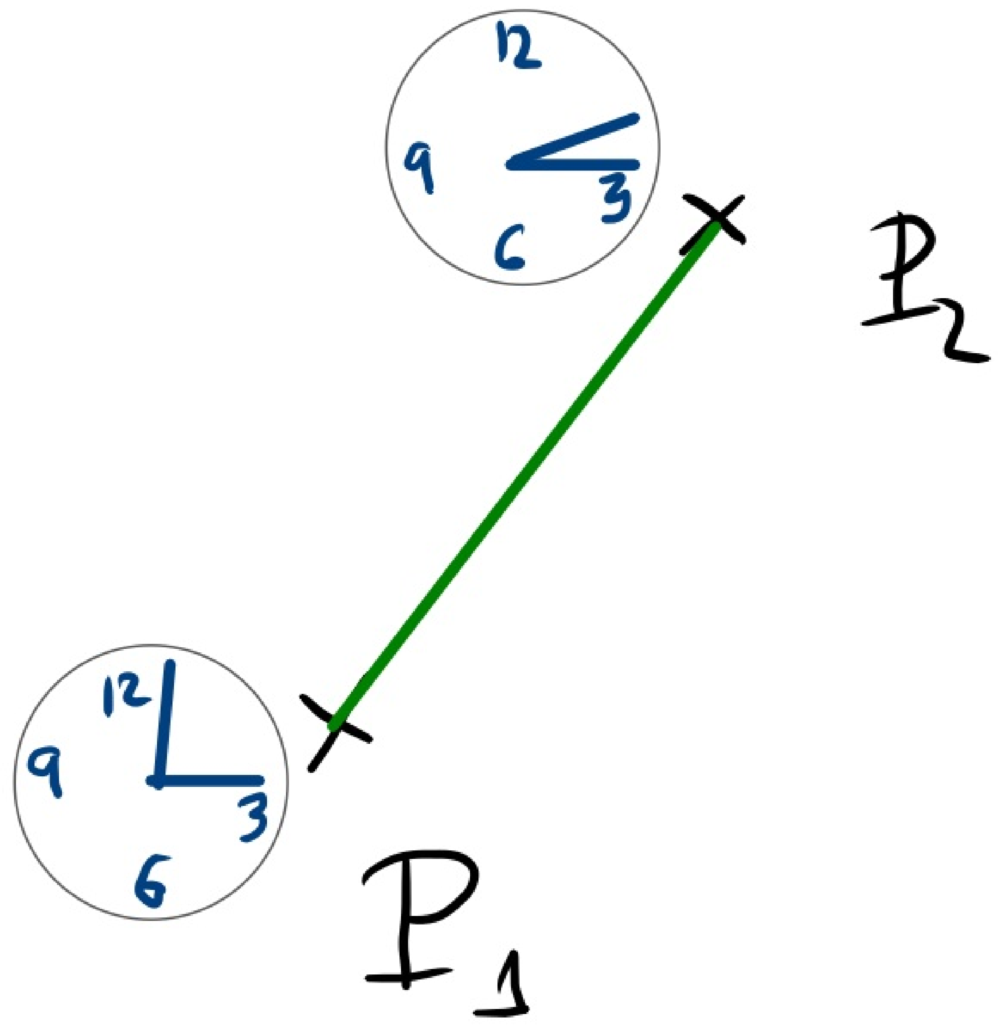
- Minkowski metric on $\mathbb{R}^4$: spacetime of events!

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$= (\text{spacetime distance of events})^2$$

- worldline of observer: chooses coordinates w.r.t. which she stays @ same place:

$$dx = dy = dz = 0$$



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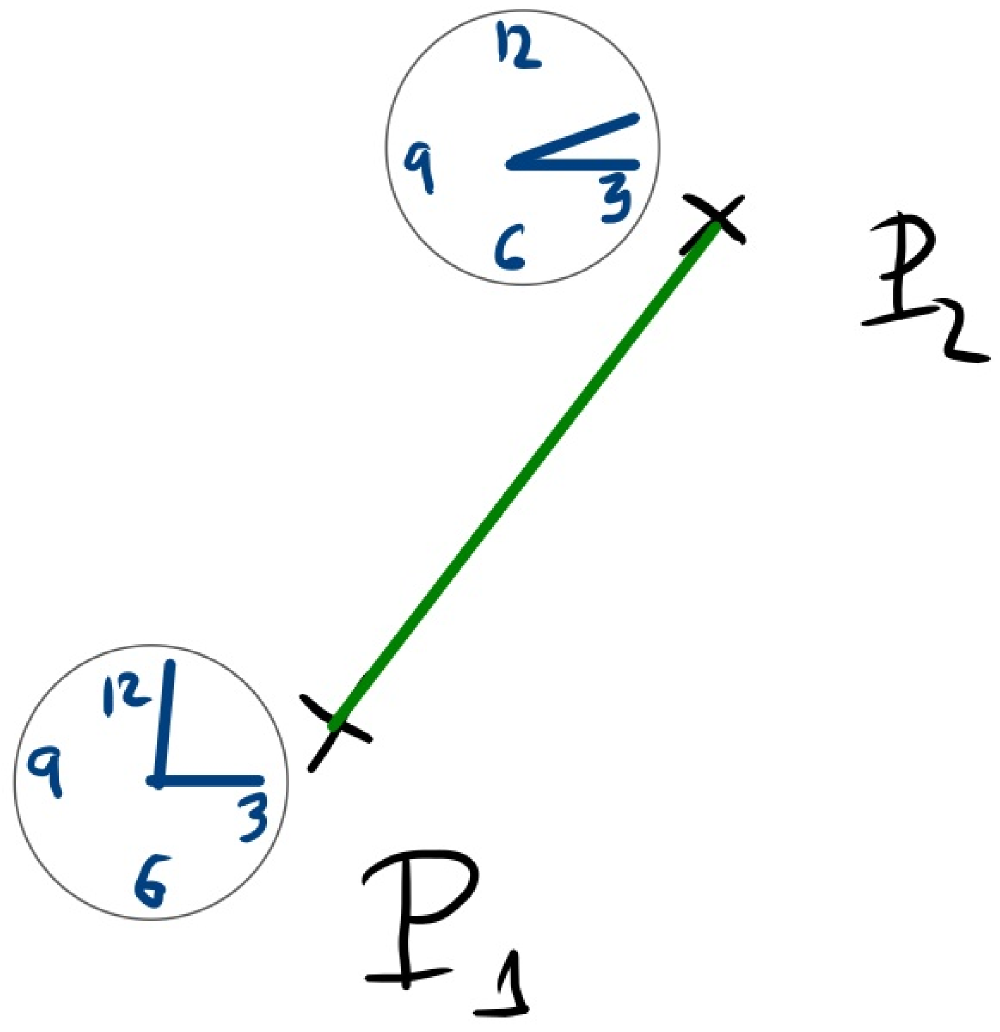


– worldline of observer: chooses coordinates w.r.t. which she stays @ same place:

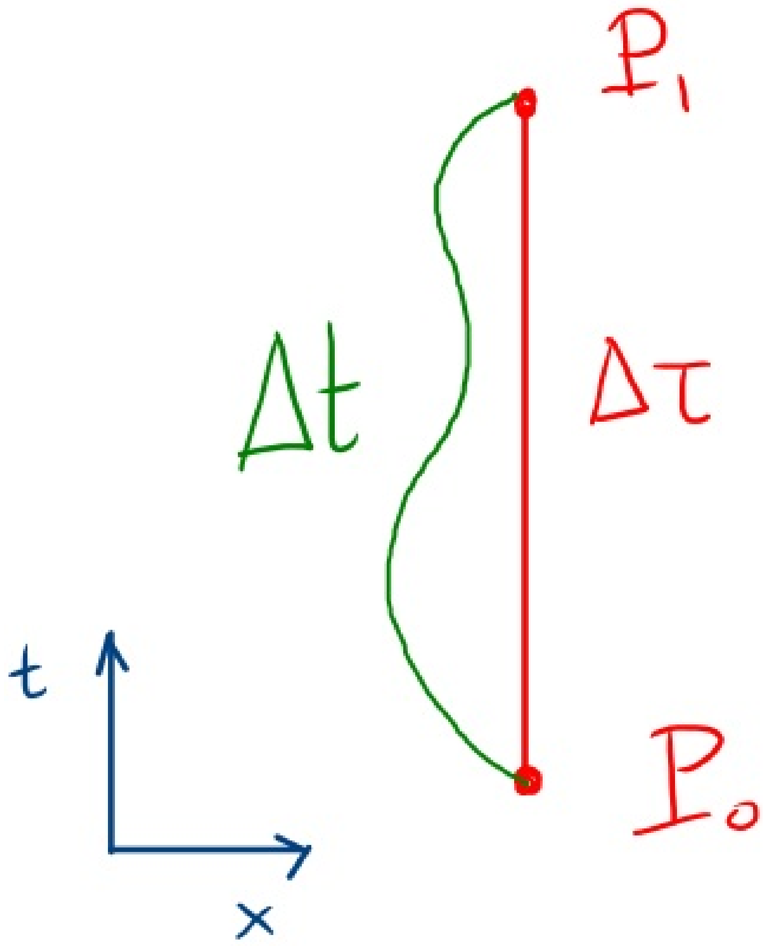
$$dx = dy = dz = 0$$

$$\Rightarrow ds^2 = -d\tau^2$$

her clock is ticking @ $d\tau$-rate
her proper time

- Proper time:
  - a geometric concept, independent of coordinates
  - it is the longest time between two events ("twin paradox")

$\rightsquigarrow$ the free age faster than anyone else meeting @ same events (the price of laziness...)



$$\Delta t < \Delta\tau$$

(disclaimer: this simple picture may become more complicated in the case of large curvature and/or strange topology for events that are far from each other, far compared to the scale of length introduced by the radius of curvature. Consult with your family's physicist)

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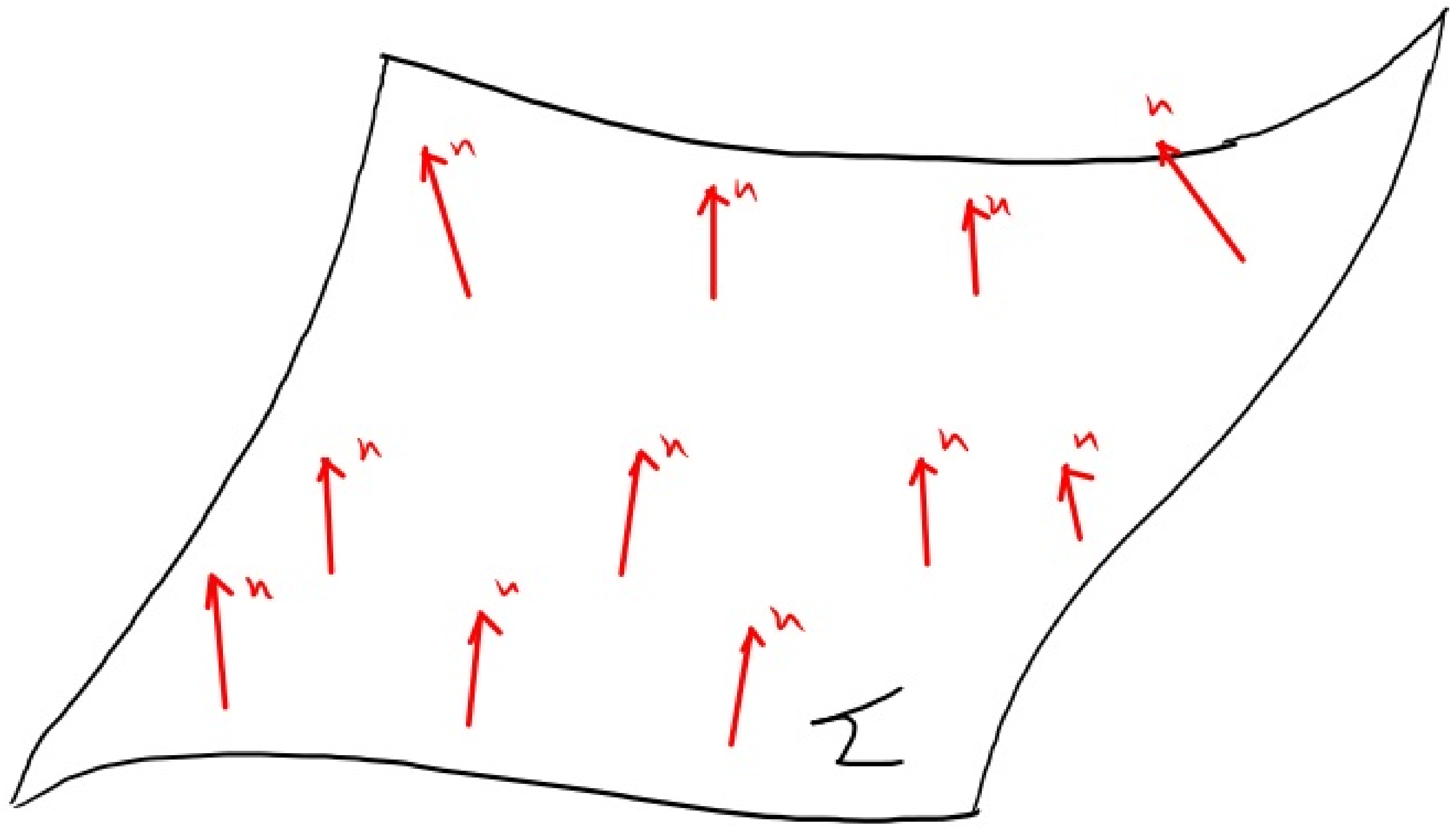

- Space:

  Defined by events for which $dt = 0$

  - the "simultaneous events"
  - observer dependent
  - a hypersurface w/ metric $d\Sigma^2 = dx^2 + dy^2 + dz^2 = g_{ij}\, dx^i dx^j \quad i,j = 1,2,3$
  - in GR, any "spacelike" surface can be "space". There may not be a natural way to choose one, or a global "constant" time defined on it.

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  (iii)' $g_p(U,U) \geq 0 \quad \forall\, U \in T_pM$ and $g(U,U) = 0 \Leftrightarrow U = 0$
* Theorem: every manifold has a Riemannian metric (comes from local diffeomorphicity to $\mathbb{R}^n$)

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* There are many metrics on a manifold!
We choose the one to do geometry with...
Nature makes a dynamical choice: In GR, solution to Einstein Eqs.

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* (iii) $\Leftrightarrow$ the matrix $(g_{\mu\nu})$ is invertible

$\Leftrightarrow \det g \neq 0$ (exercise, or see video...)

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* we denote g^{-1} by $g^{\mu\nu}$, so that $g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}{}_{\nu}$

↳ also a symmetric tensor

Index Raising/Lowering

A metric (and its inverse) gives rise to an isomorphism between $T_p\mathcal{M}$ and $T_p^*\mathcal{M}$

- If $V \in T_p\mathcal{M}$, then $g(V, \cdot) \in T_p^*\mathcal{M}$

$\hookrightarrow$ a linear map from $T_p\mathcal{M} \to \mathbb{R}$

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  for simplicity $\tilde{V}_\mu \to V_\mu$

  & we call the operation "index lowering"

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- Similar maps between tensors of any rank. E.g.

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- Dual tensors do not depend on choice of basis/coordinates

Remember the duality between bases defined by $e^\alpha(e_\beta) = \delta^\alpha{}_\beta$? This duality is basis-dependent and does not give a dual tensor independent of the choice of basis

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row

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where $g_d = \mathrm{diag}(g_0, g_1, \dots, g_{n-1}) = \begin{pmatrix} g_0 & 0 & \cdots & 0 \\ 0 & g_1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & g_{n-1} \end{pmatrix}$

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Indeed: $g(\tilde{e}_\mu, \tilde{e}_\nu) = g(\Lambda^{\mu'}{}_\mu e_{\mu'}, \Lambda^{\nu'}{}_\nu e_{\nu'})$

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If s is the signature of the metric, and

$s=0$: the metric is Euclidean , $\Lambda \in O(n)$ orthogonal group

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$s=1$: the metric is Minkowskian $\Lambda \in O(1, n-1)$

" manifold " pseudo-Riemannian

$O(1, n-1)$: Lorentz group

Λ a Lorentz xfm

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~ coordinate bases may consist of orthogonal vectors, but not of constant length

e.g. $(\hat{r}, \hat{\theta}, \hat{\varphi})$ orthonormal

$(\partial_r, \partial_\theta, \partial_\varphi)$ are not!

$$g(\partial_r, \partial_r) = 1 \qquad g(\partial_\theta, \partial_\theta) = r^2 \qquad g(\partial_\varphi, \partial_\varphi) = r^2 \sin^2\theta$$

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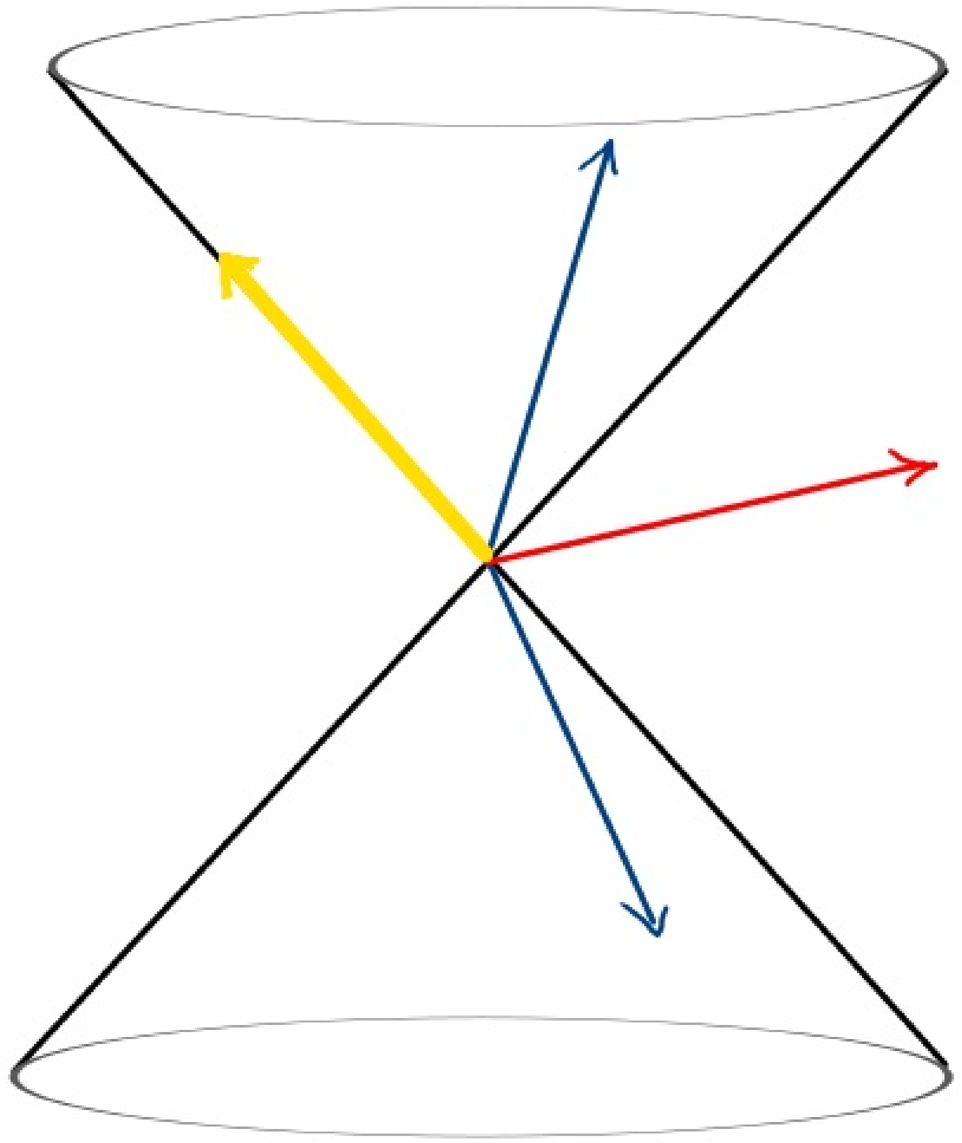

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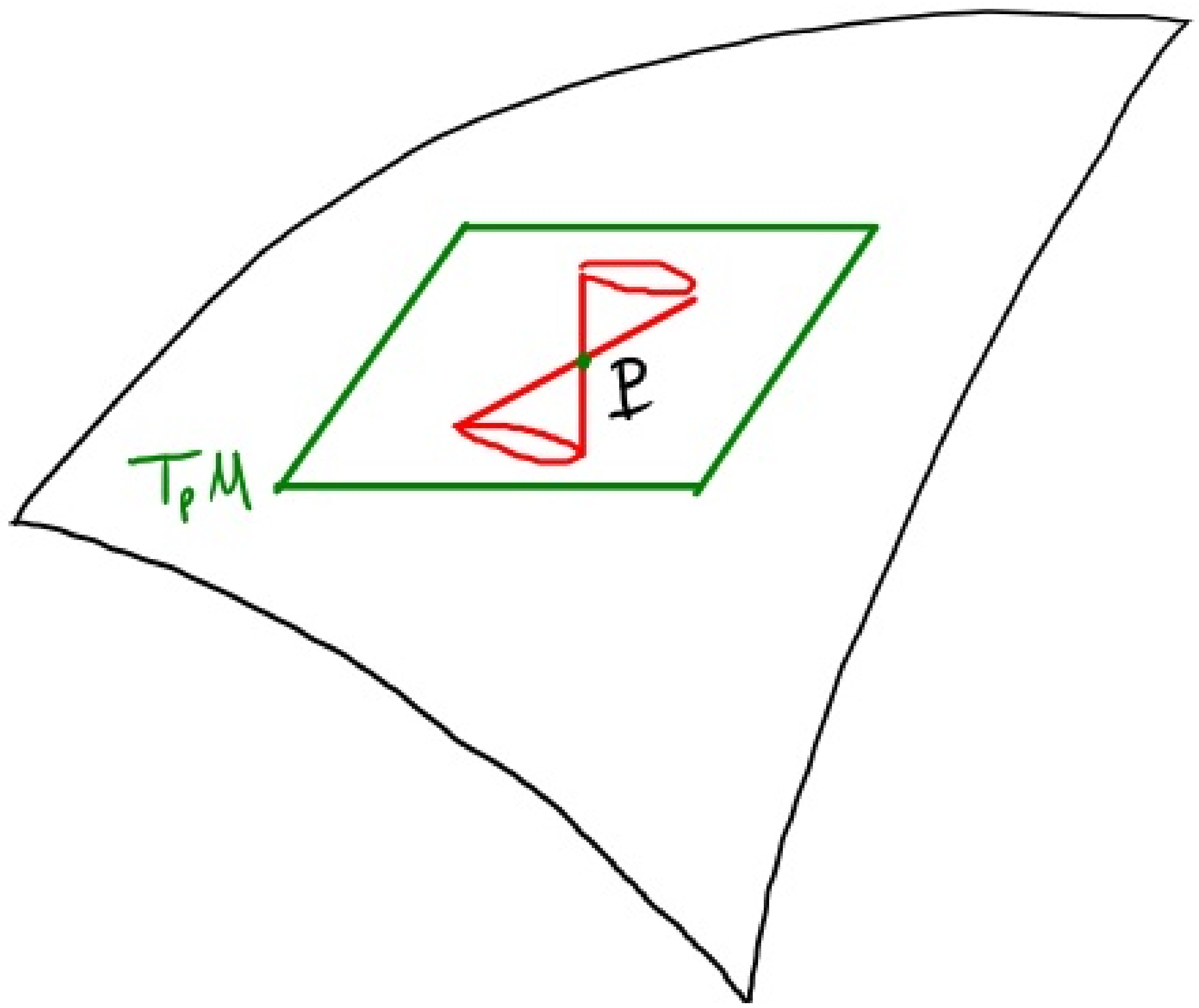

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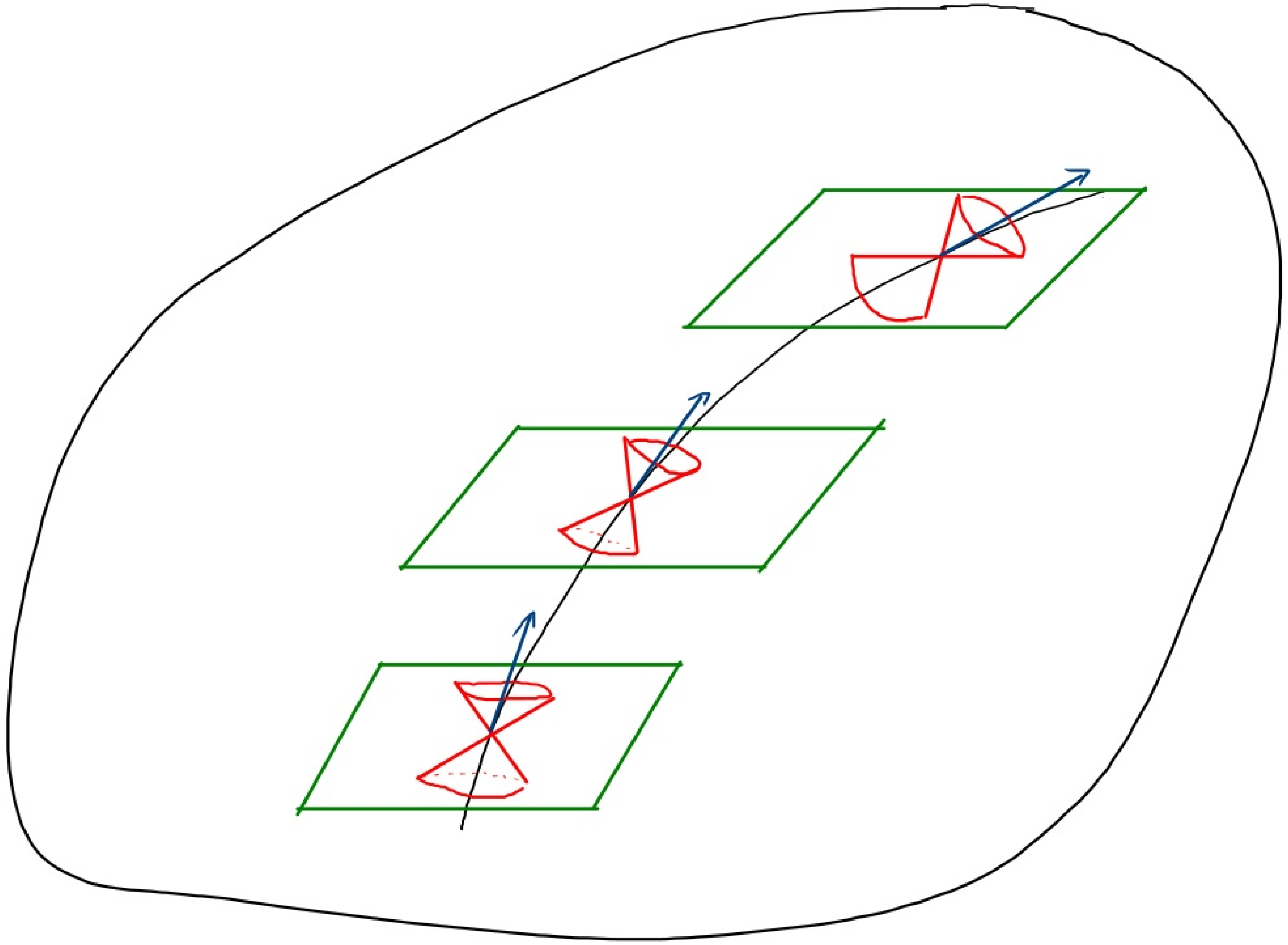

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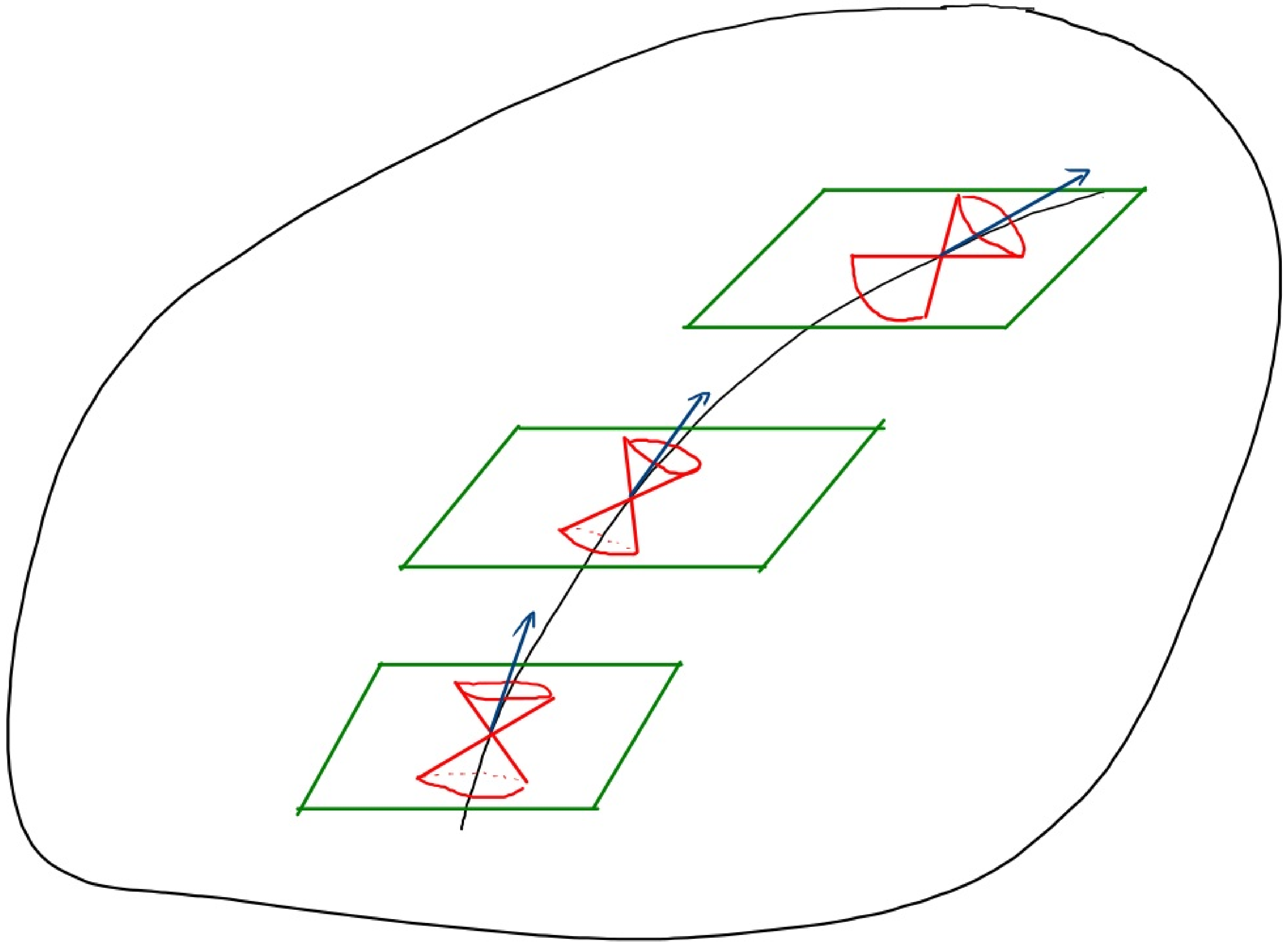

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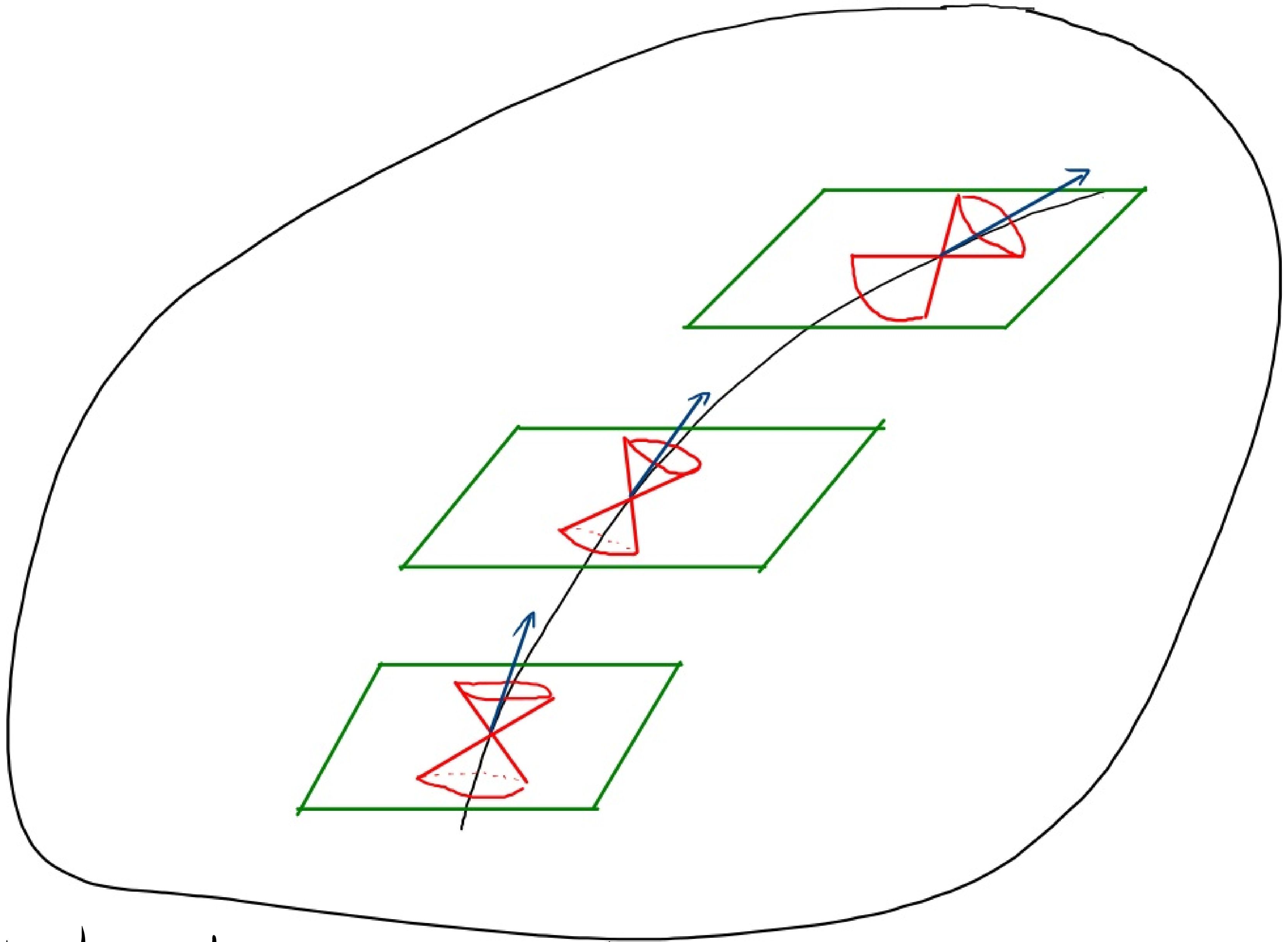

- massive particles move on worldlines w/ tangent vectors everywhere timelike
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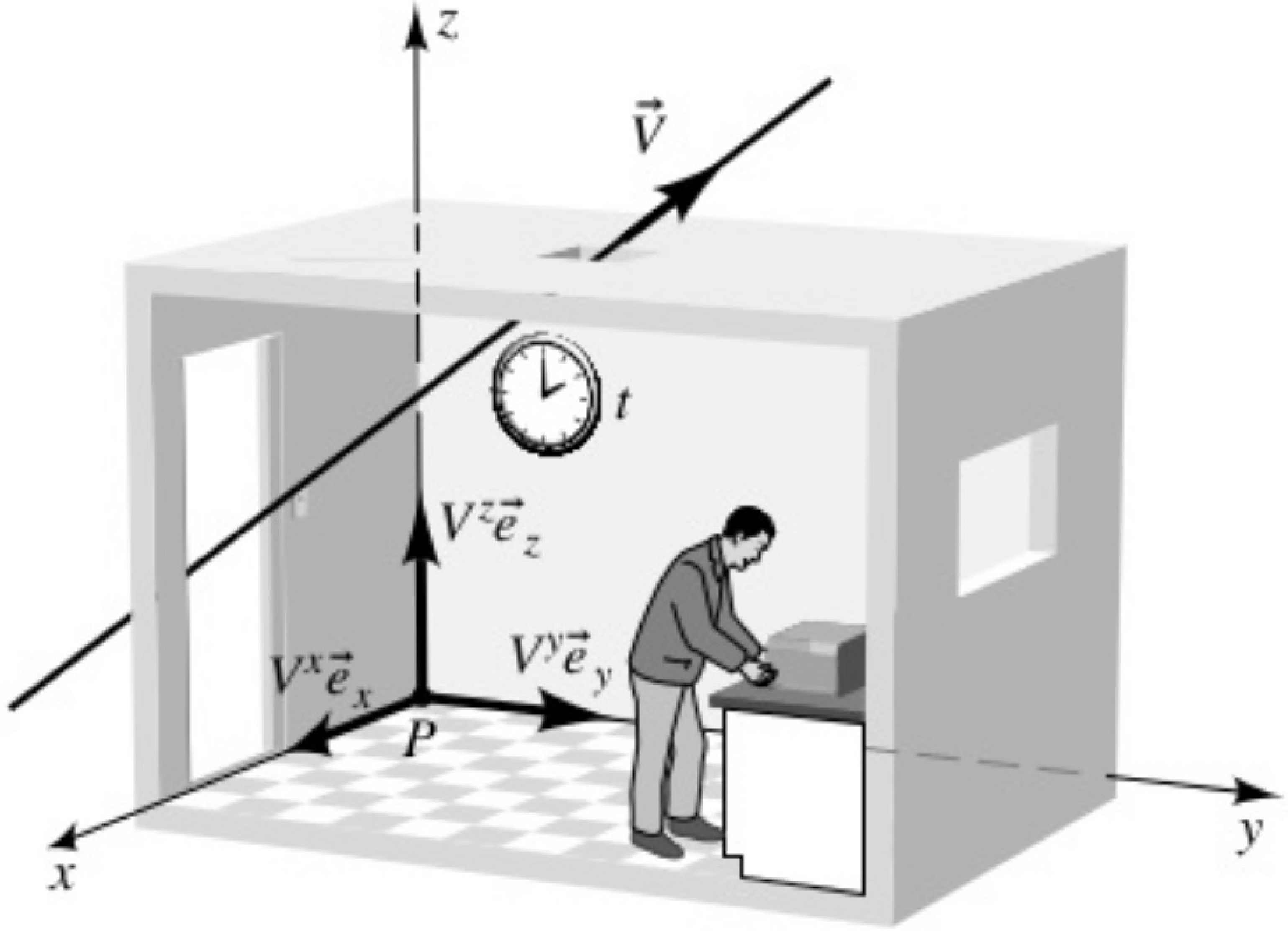
- there are no worldlines with tangent vectors that change category

Local Frames

Orthonormal bases define an "observer"



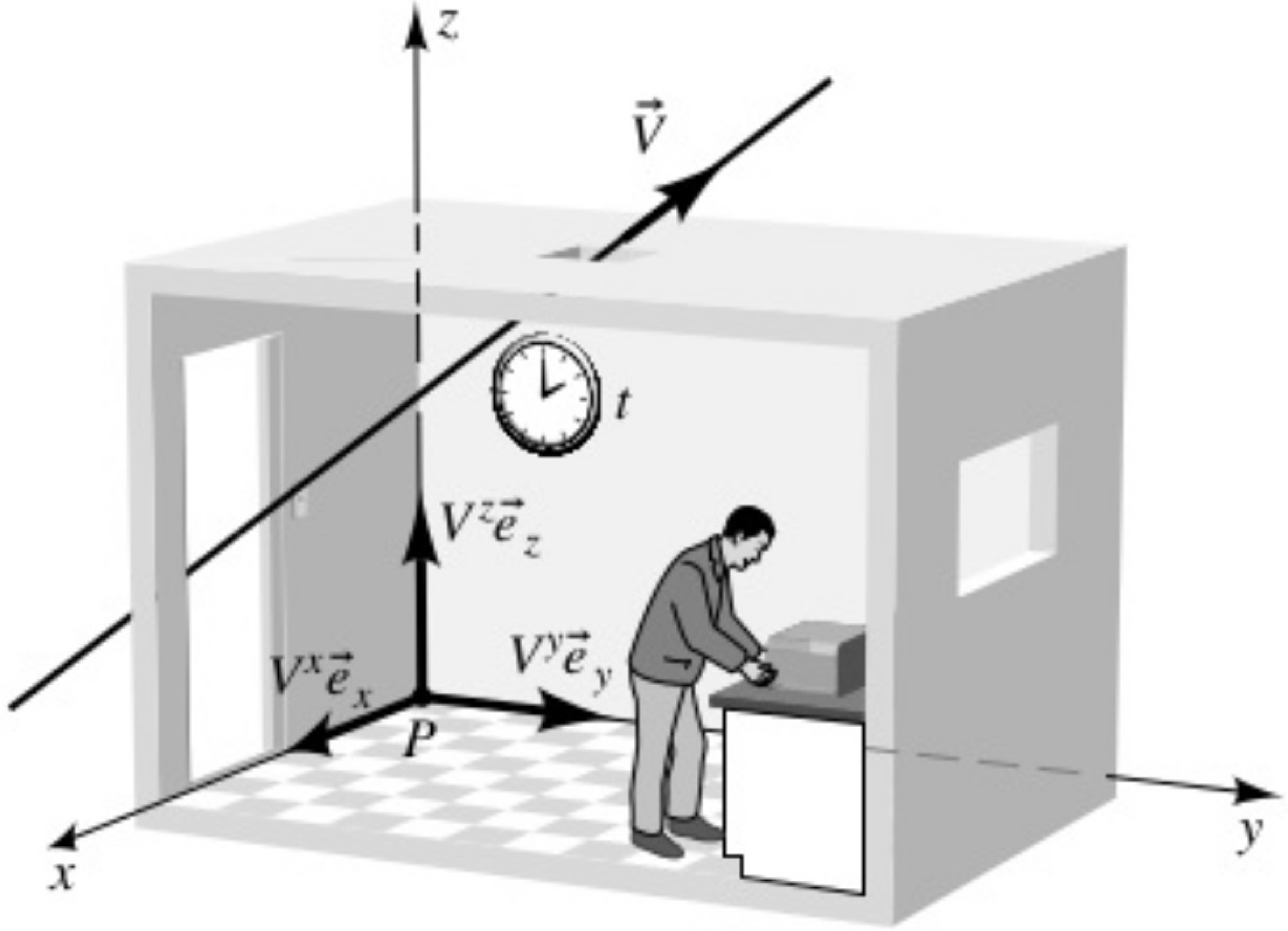
Hartle, Fig 7.6

Local Frames

Orthonormal bases define an "observer"

four-velocity of observer $u = e_0$

local cartesian axes $\{e_1, e_2, e_3\}$



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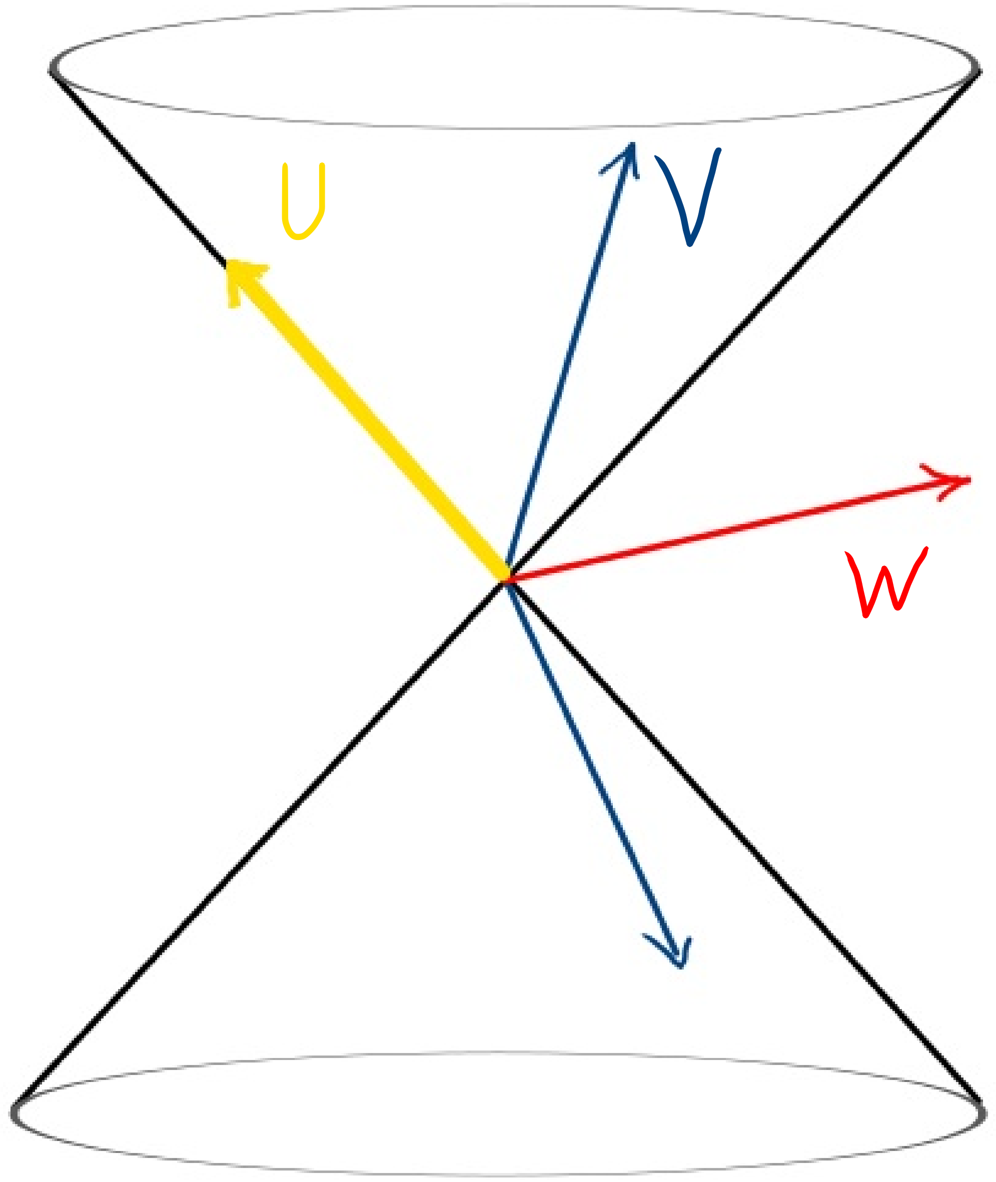

$\{e_0, e_1, e_2, e_3\}$ define the local light cone:

$g(V, V) < 0$ timelike

$g(U, U) = 0$ null / lightlike

$g(W, W) > 0$ spacelike

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Orthonormal bases define an "observer"

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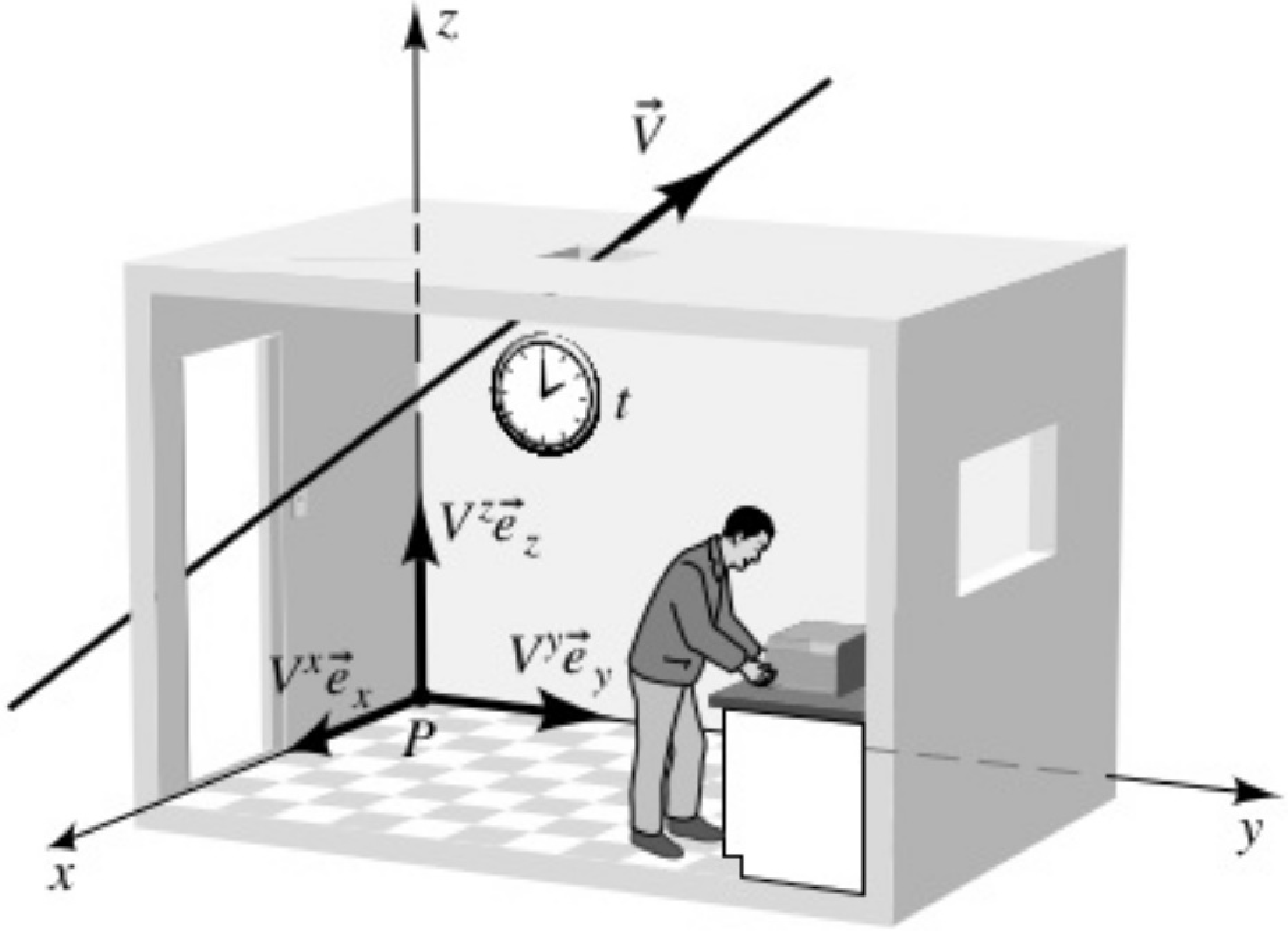
$u = (1, 0, 0, 0) = e_0$

Velocity of a passing by particle

$V = (\gamma, \gamma v, 0, 0) = \gamma e_0 + \gamma v e_1$

velocity

4-velocity



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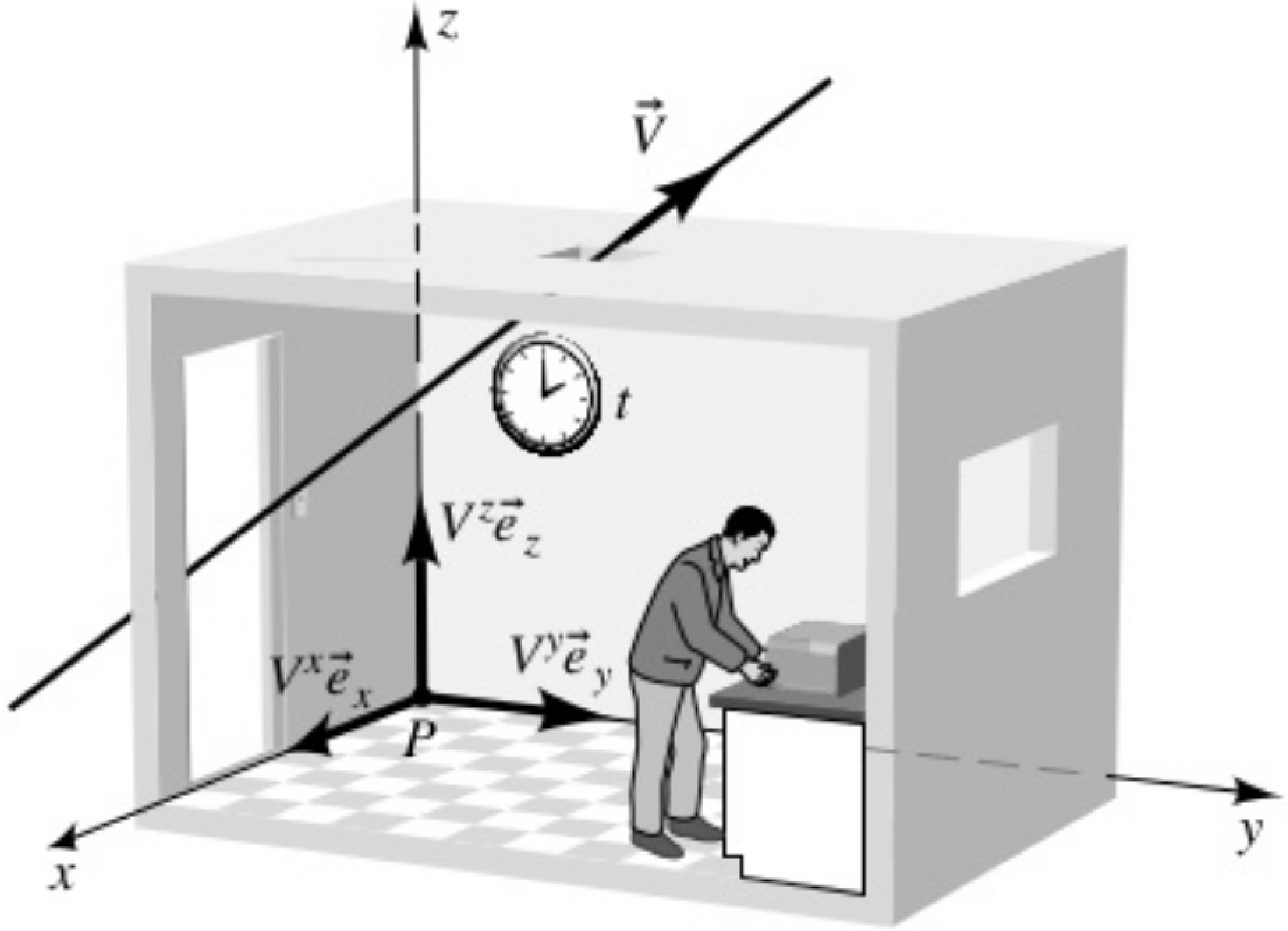

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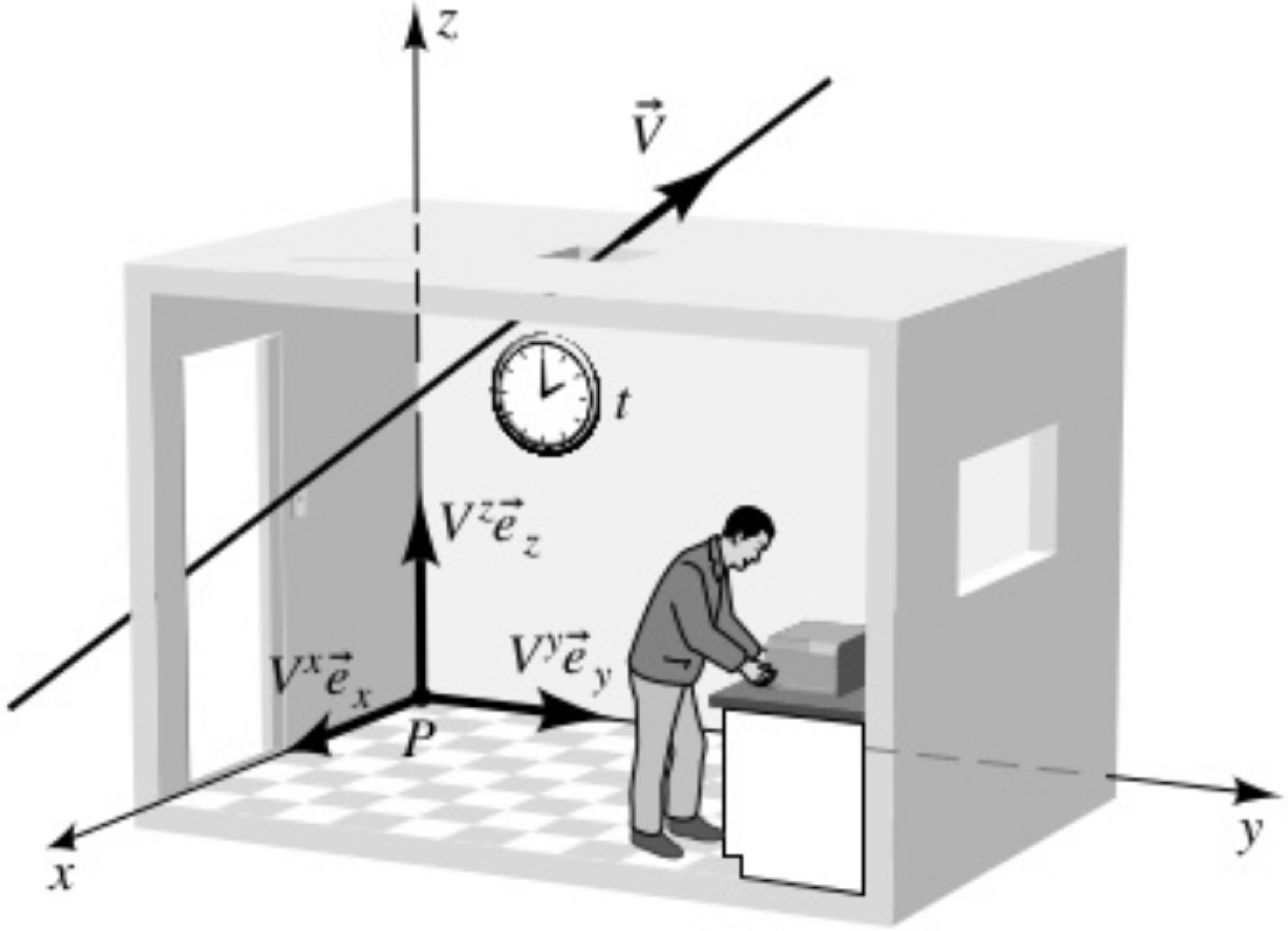

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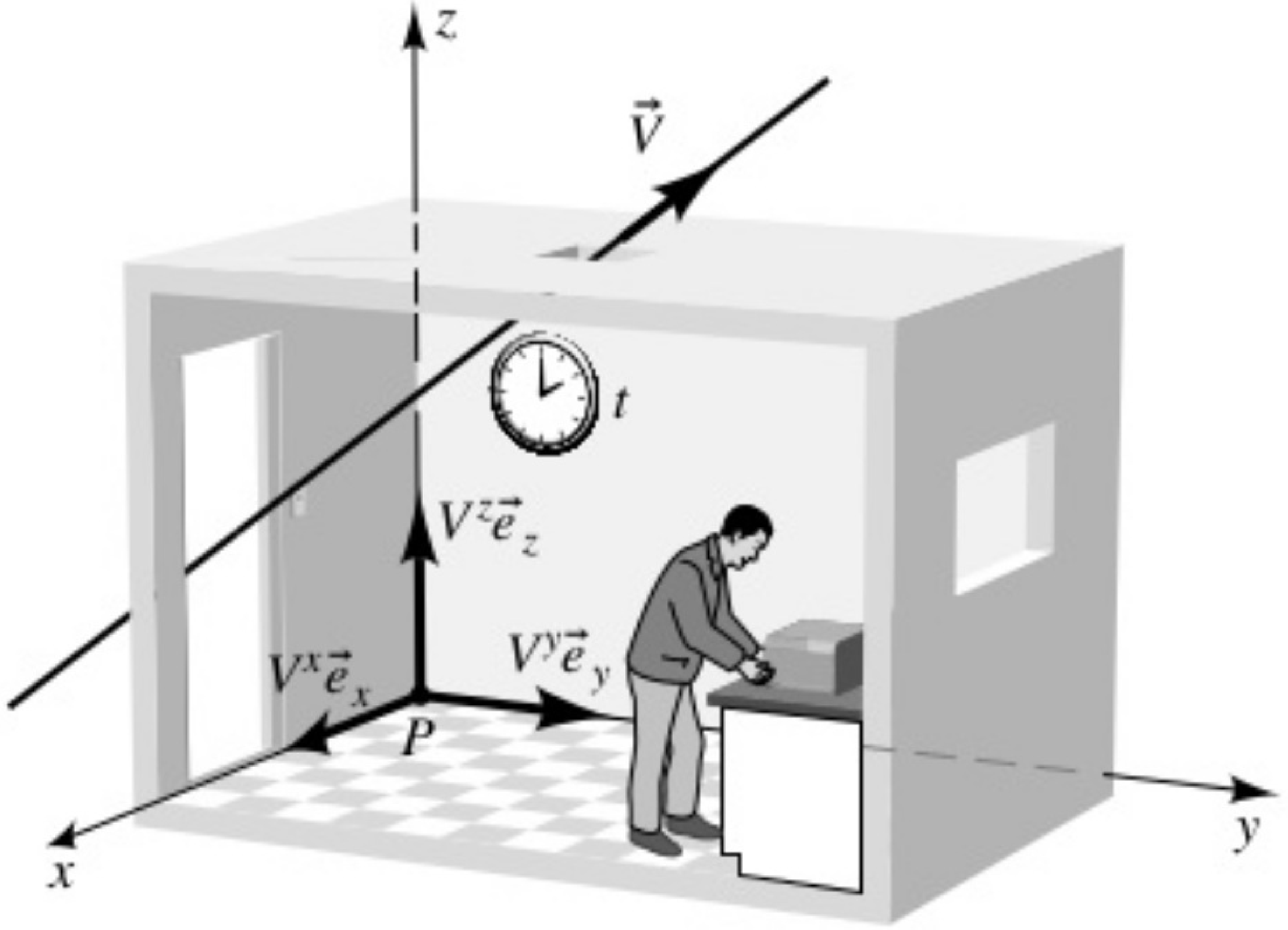

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coordinate independent formula,
gives relative speed in any coord. system

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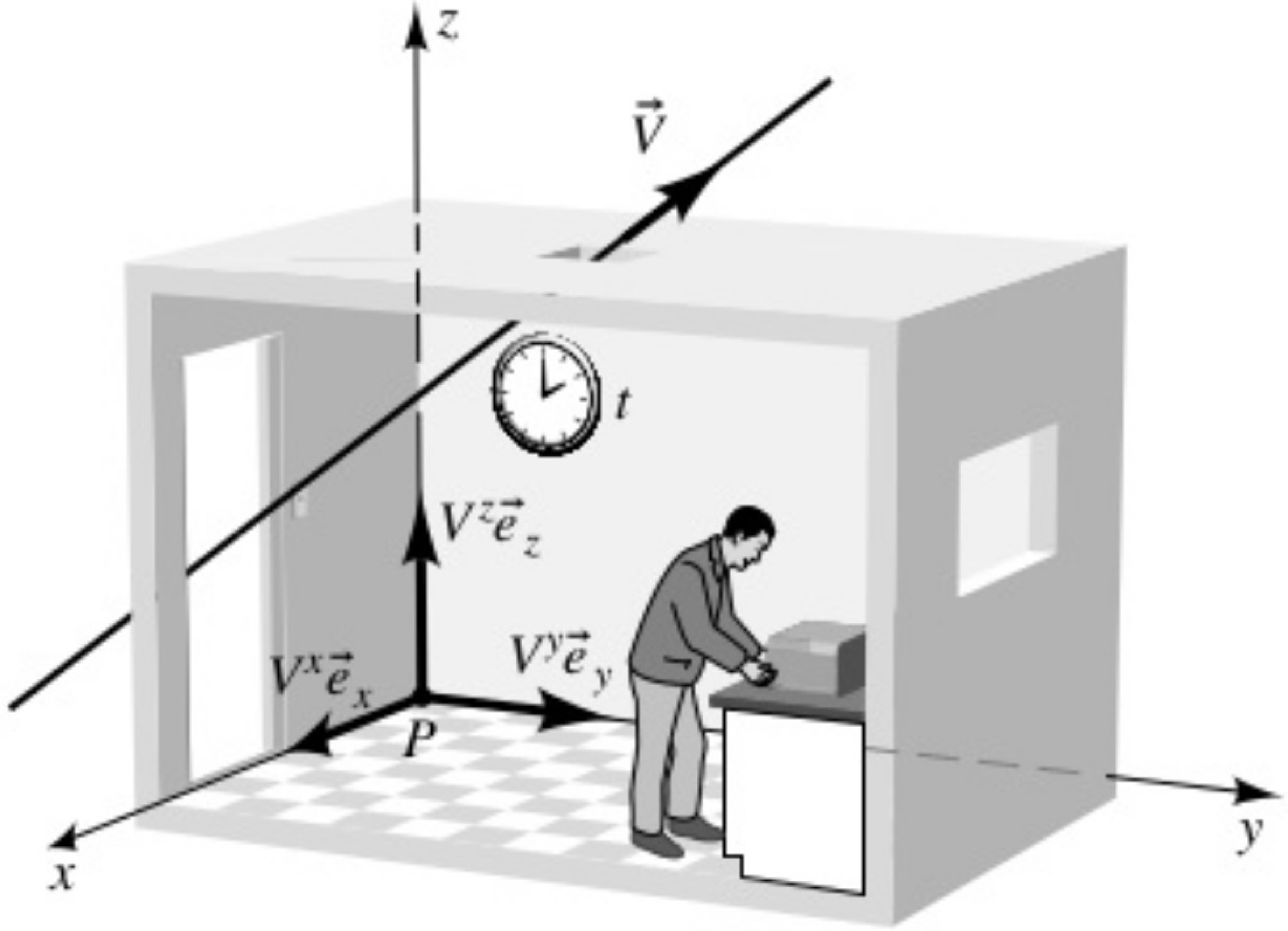
four-momentum of particle:

$p^\mu = (E, p^1, p^2, p^3)$



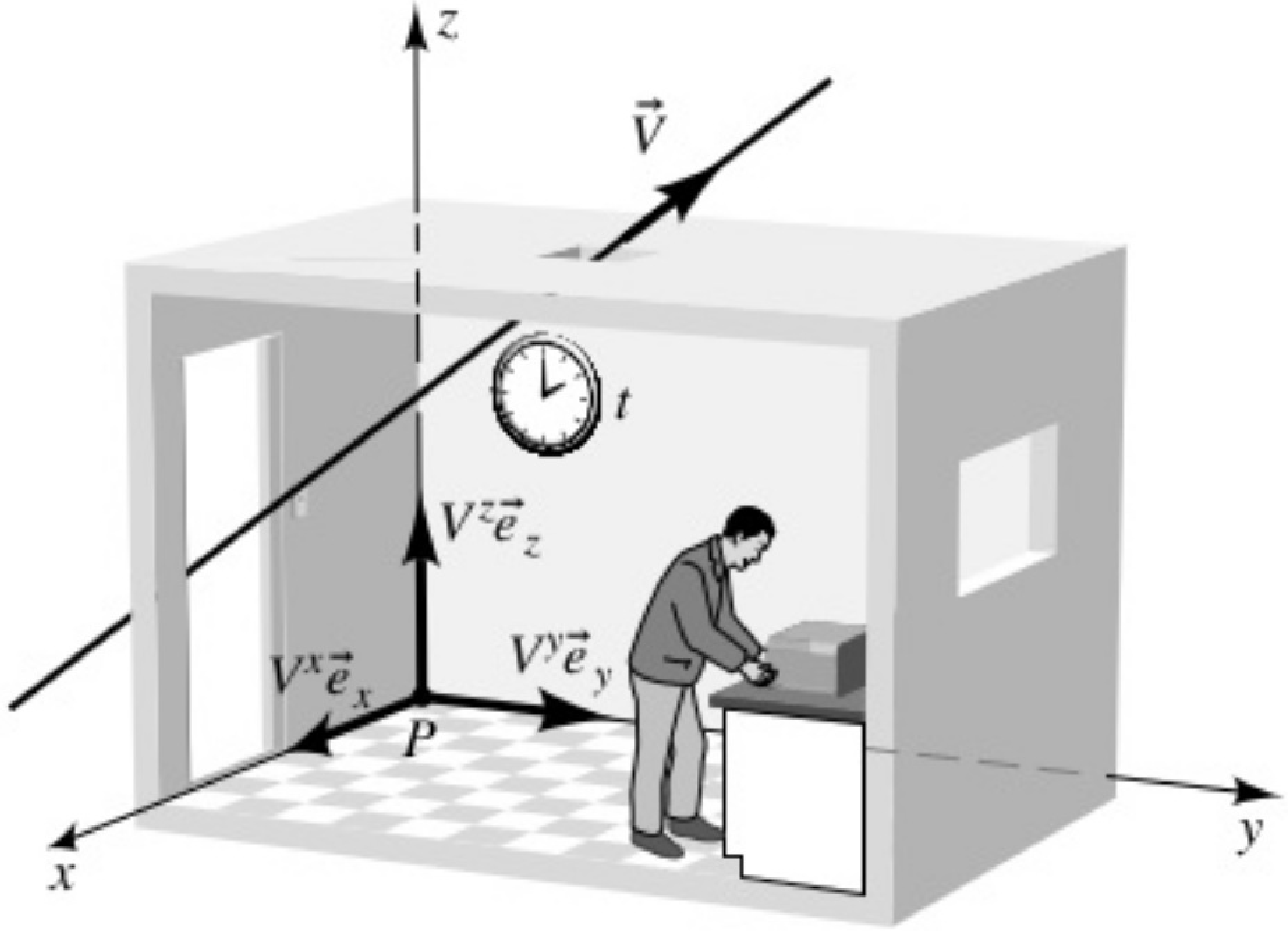
Hartle, Fig 7.6

Local Frames

Orthonormal bases define an "observer"

four-velocity of observer $\quad u = e_0$

$$u = (1, 0, 0, 0) = e_0$$



Hartle, Fig 7.6

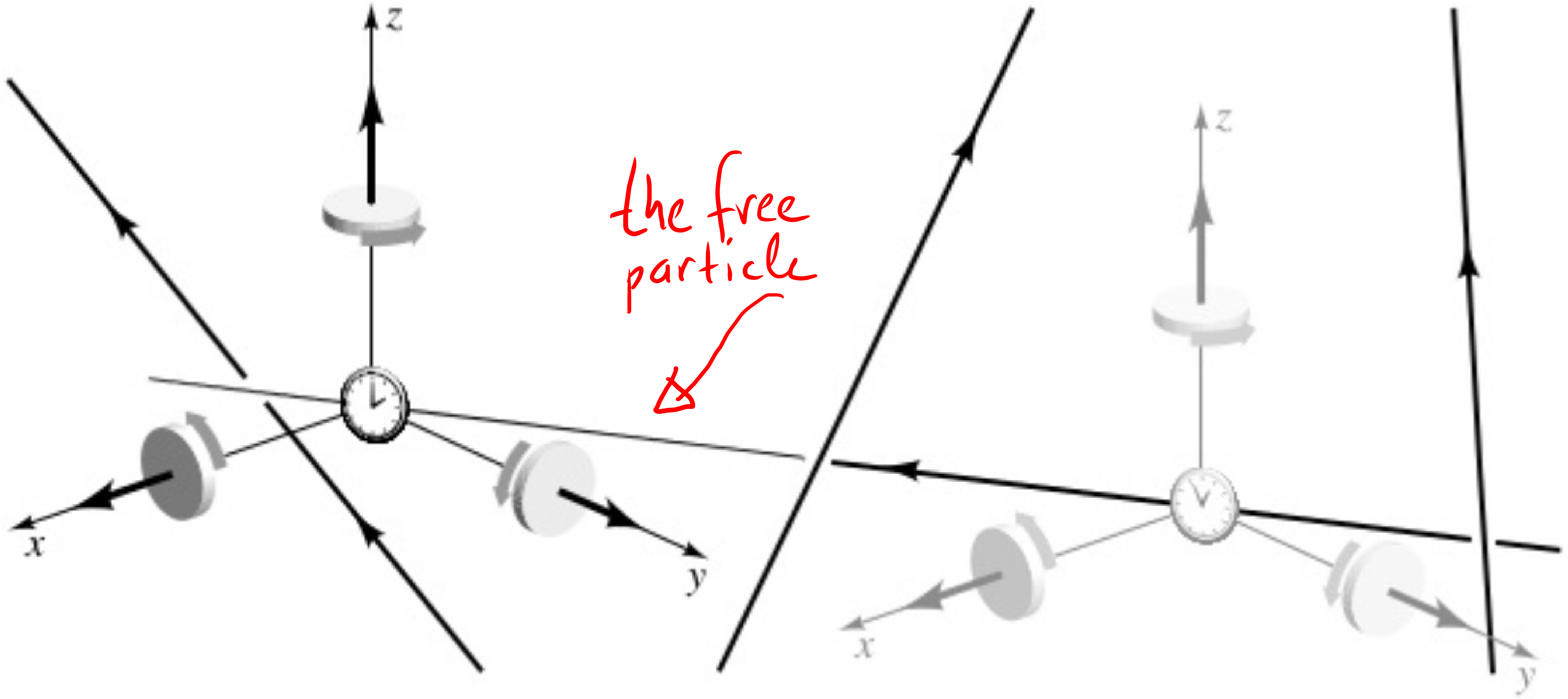
four-momentum of particle:

$$\left.\begin{aligned} p^\mu &= (E, p^1, p^2, p^3) \\ u^\mu &= (1, 0, 0, 0) \end{aligned}\right\} \Rightarrow \quad p_\mu u^\mu = -E$$

- a coordinate independent expression
- E and v easily defined for a local observer
  NOT for distant observers !!!

Local Inertial Observers

*Observers that observe free particles to move on straight lines @constant rate are the inertial observers

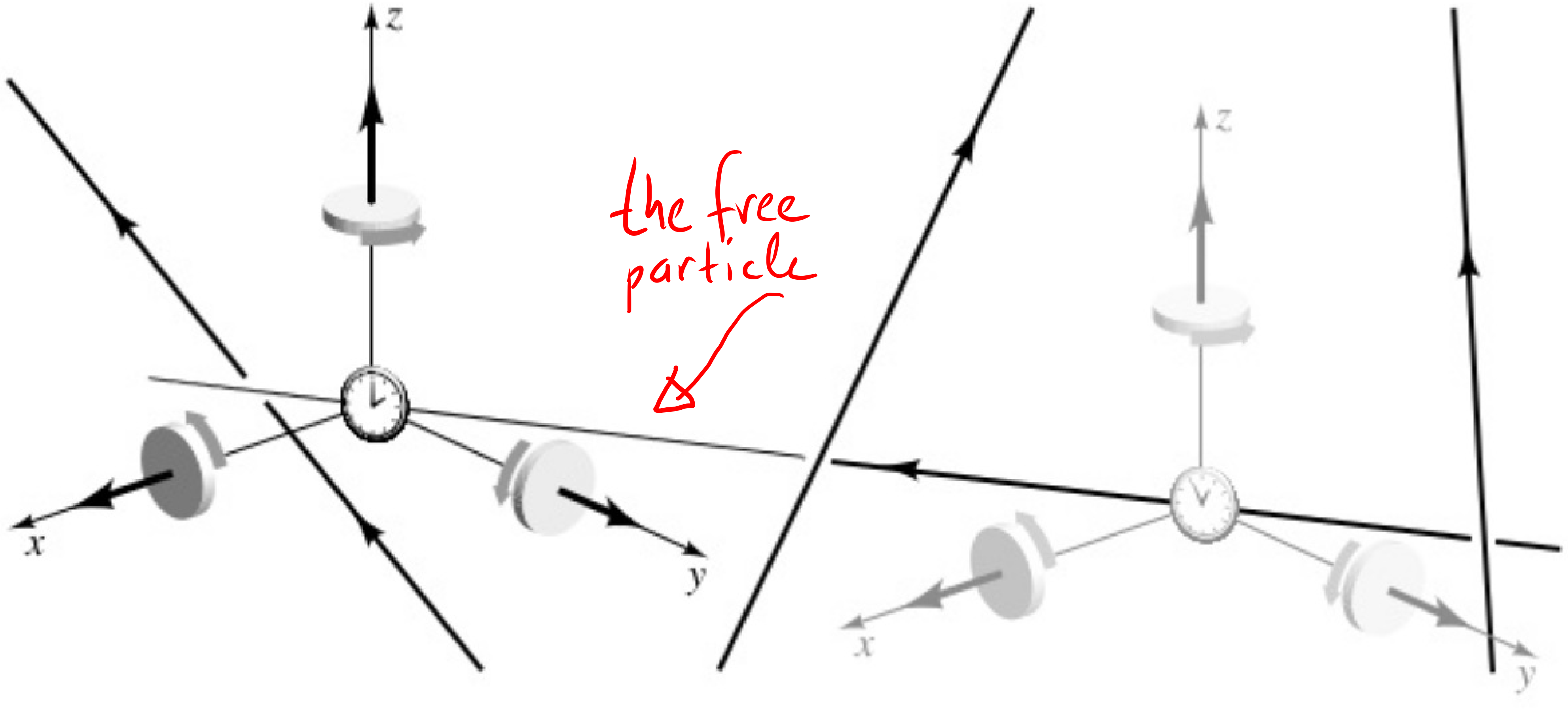


Hartle Fig 3.3

Local Inertial Observers

* Observers that observe free particles to move on straight lines @ constant rate are the inertial observers

⇝ freely "falling"



Hartle Fig 3.3

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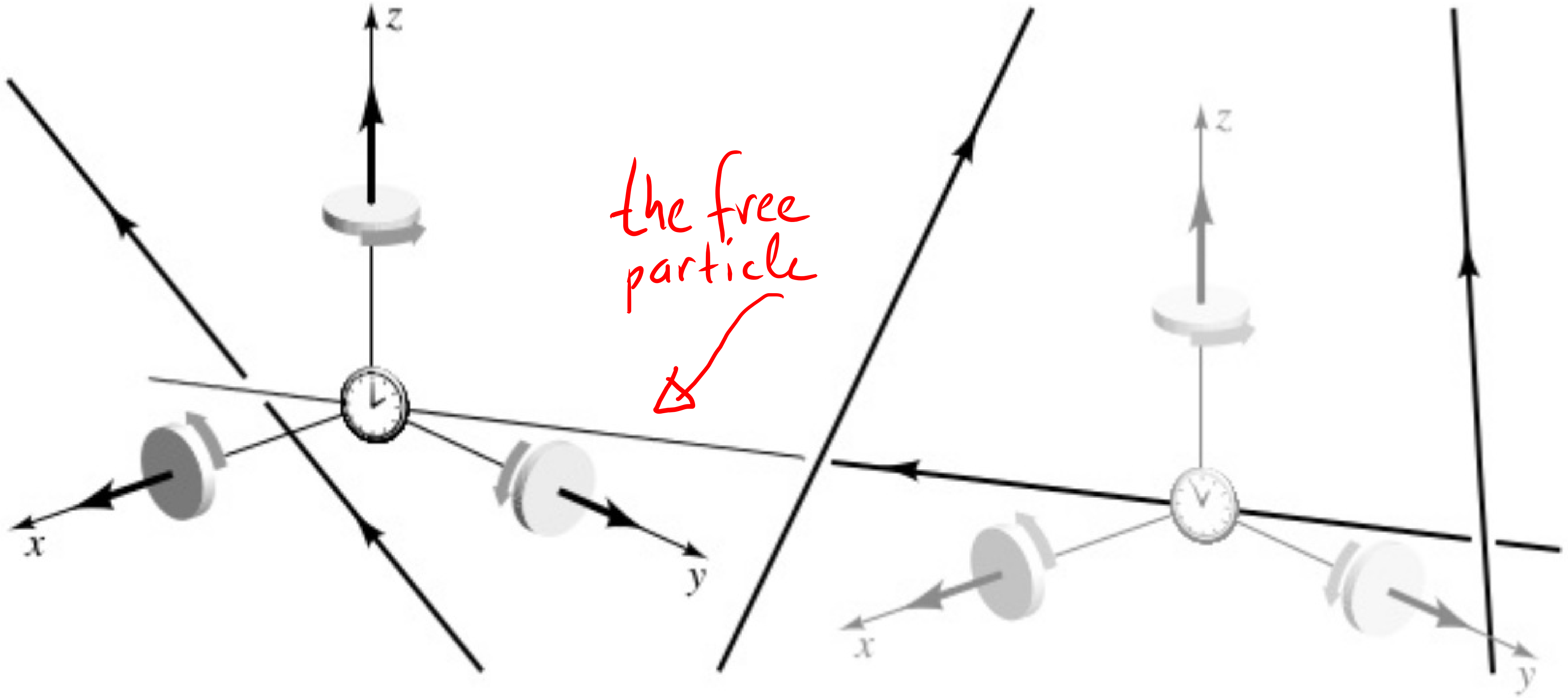

Hartle Fig 3.3

How to become one:

- follow a free massive particle and set origin of axes on it

Local Inertial Observers

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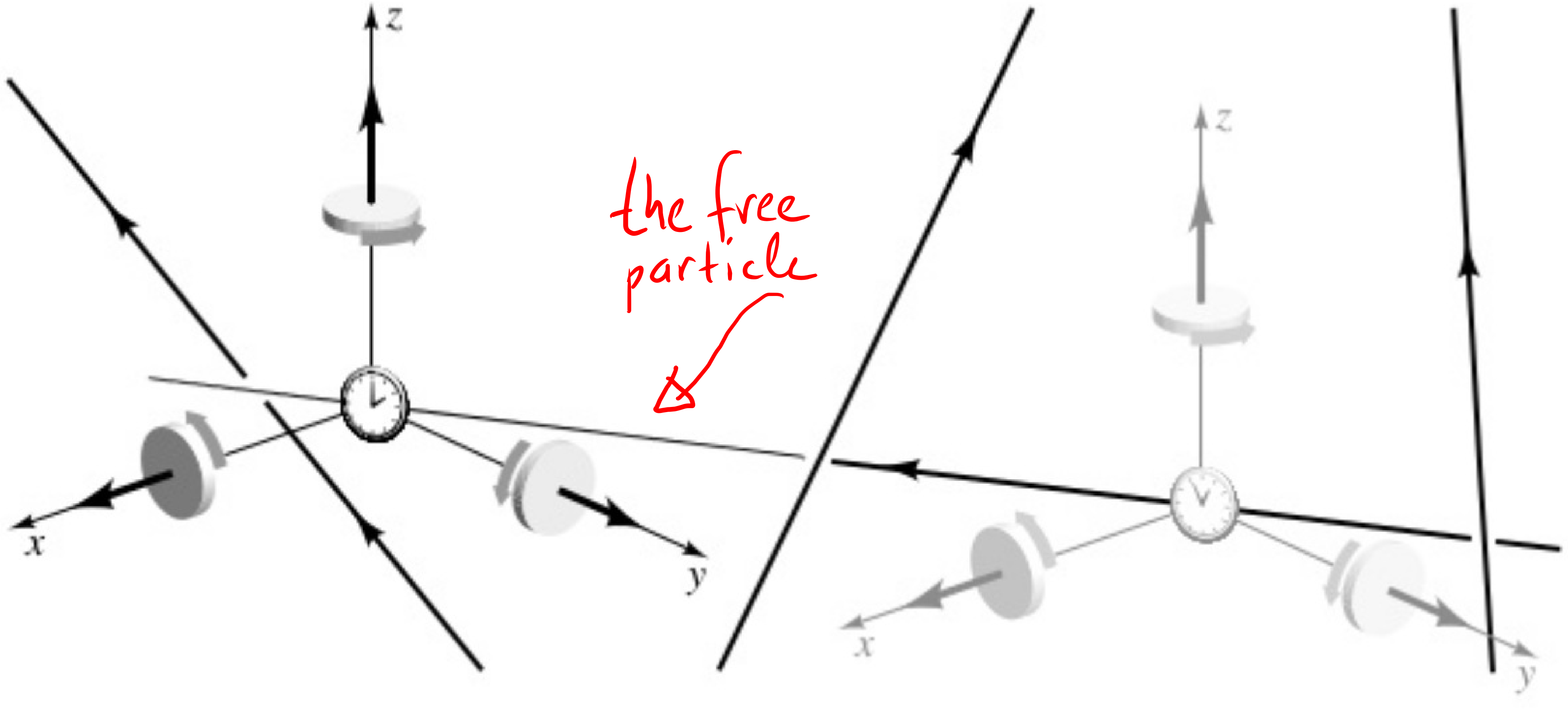
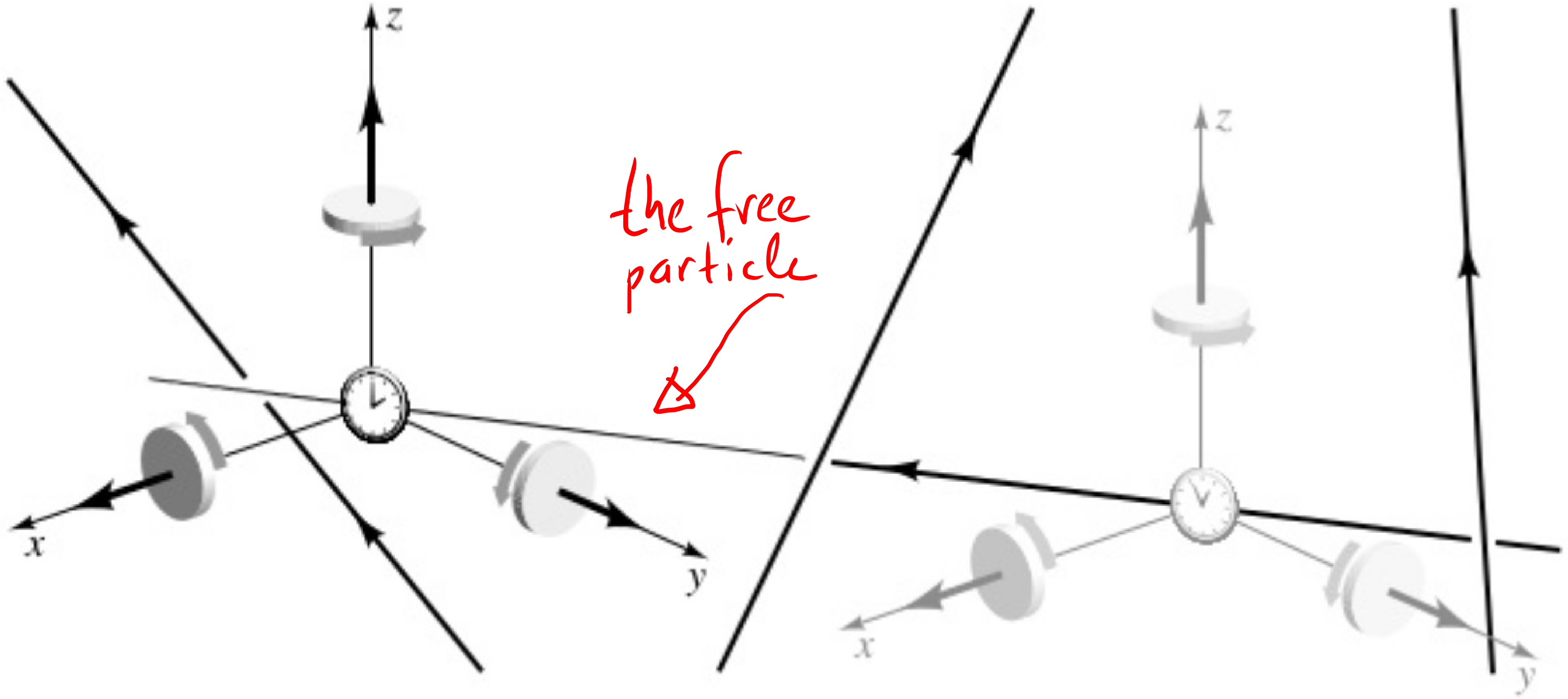


Hartle Fig 3.3

How to become one:

- follow a free massive particle and set origin of axes on it
- choose 3 perpendicular axes, set gyroscopes to spin in their direction

Local Inertial Observers



Hartle Fig 3.3

*Observers that observe free particles to move on straight lines @ constant rate are the inertial observers

⇝ freely "falling"

How to become one:

- follow a free massive particle and set origin of axes on it
- choose 3 perpendicular axes, set gyroscopes to spin in their direction
- let gyros spin freely, use them as Cartesian axes

$$\Rightarrow \quad g_{\mu\nu}\big|_0 = \eta_{\mu\nu} \quad \text{and} \quad \partial_\sigma g_{\mu\nu}\big|_0 = 0$$

Local Inertial Observers

*Observers that observe free particles to move on straight lines @ constant rate are the inertial observers

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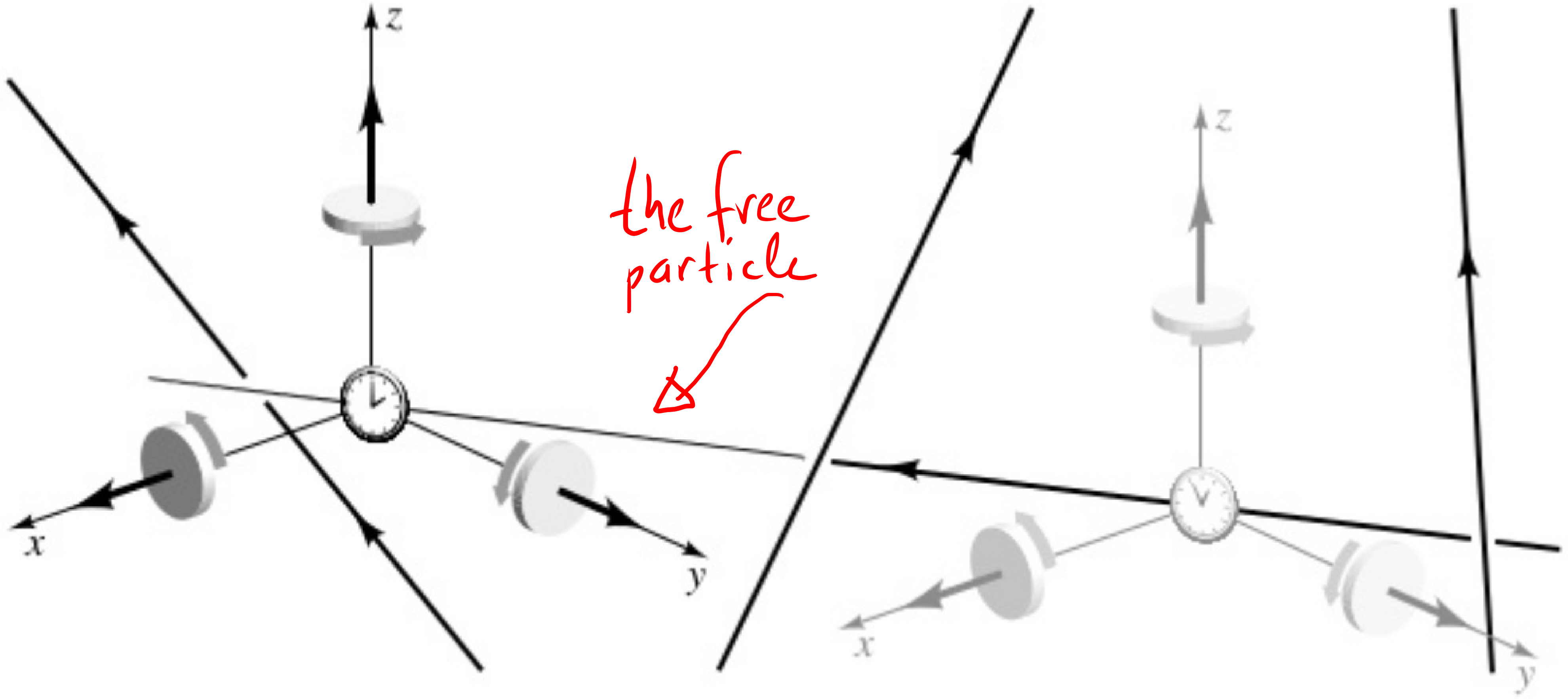

Hartle Fig 3.3

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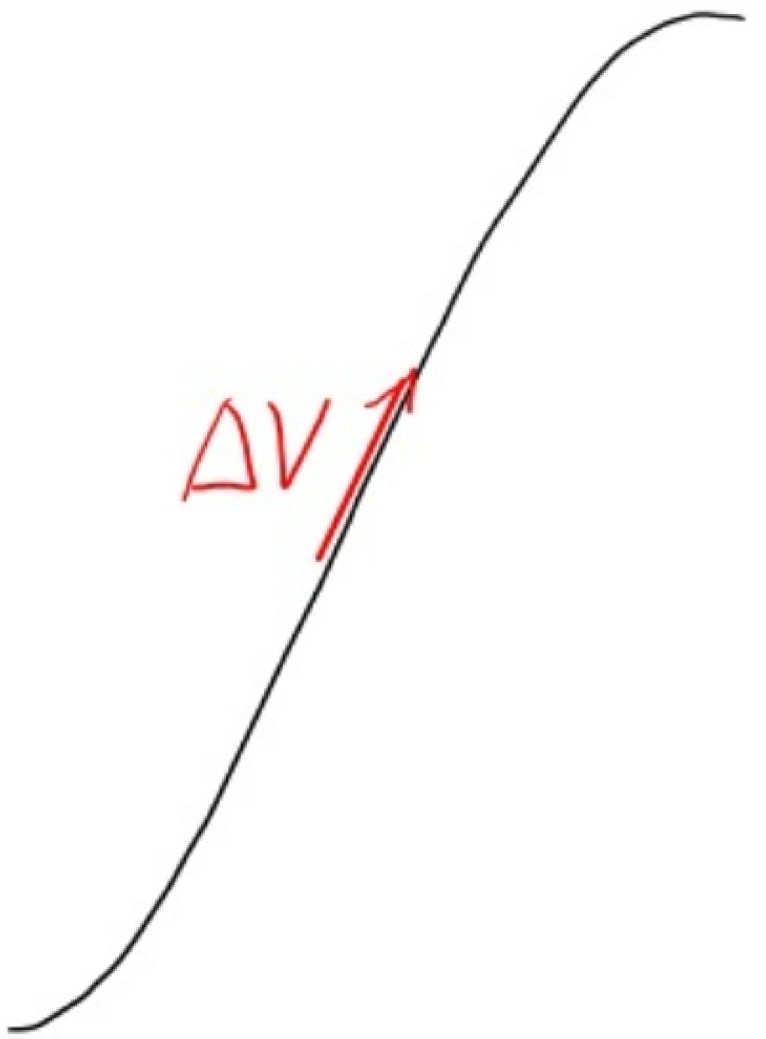
$\Rightarrow \quad g_{\mu\nu}|_0 = \eta_{\mu\nu} \quad$ and $\quad \partial_\sigma g_{\mu\nu}|_0 = 0$

Voilà: in a small enough region you can do SR physics!

Line element

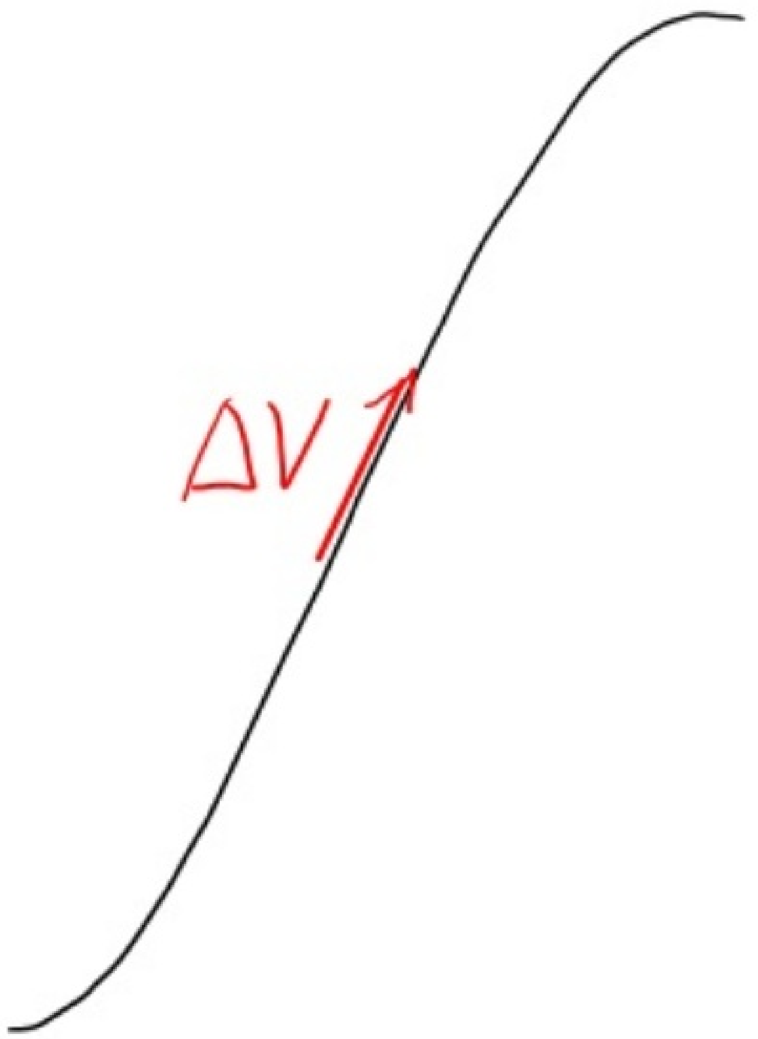
$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

$$\Delta V = \Delta x^\mu \partial_\mu$$



Line element

$$
\left.\begin{array}{l} g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu \\ \Delta V = \Delta x^\mu \partial_\mu \end{array}\right\} \Rightarrow
\begin{aligned} g(\Delta V, \Delta V) &= g(\Delta x^\mu \partial_\mu, \Delta x^\nu \partial_\nu) \\ &= g(\partial_\mu, \partial_\nu)\, \Delta x^\mu \Delta x^\nu \\ &= g_{\mu\nu}\, \Delta x^\mu\, \Delta x^\nu \end{aligned}
$$

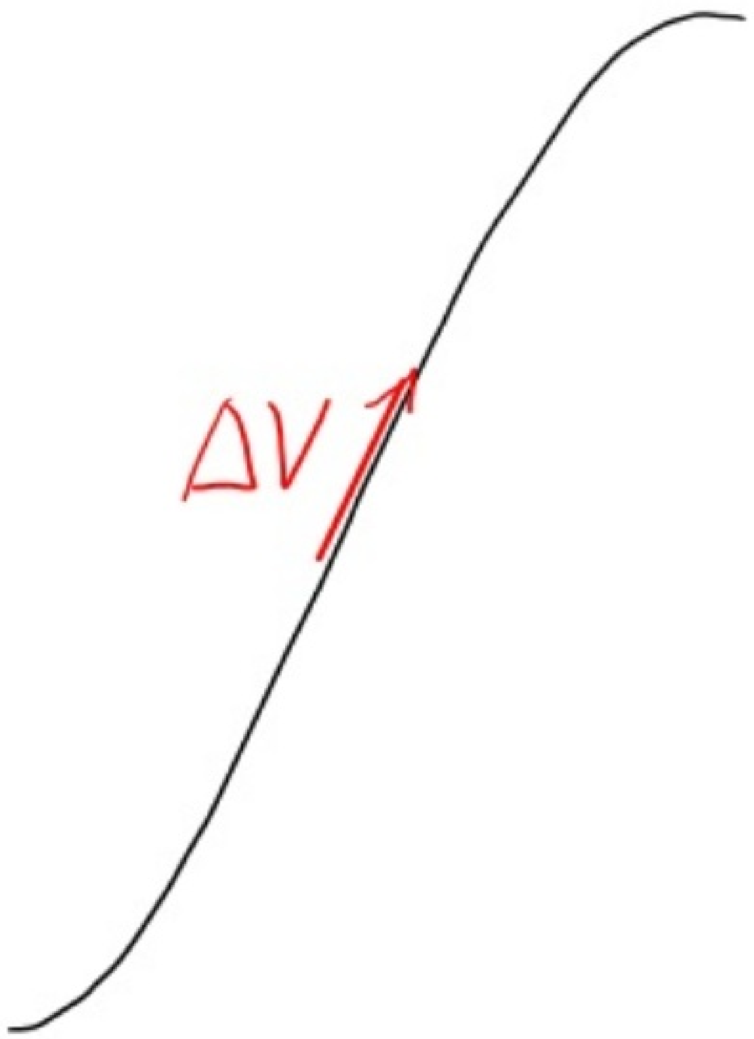


Line element

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

$$\Delta V = \Delta x^\mu \partial_\mu$$

$$\Rightarrow g(\Delta V, \Delta V) = g(\Delta x^\mu \partial_\mu, \Delta x^\nu \partial_\nu) = g(\partial_\mu, \partial_\nu)\, \Delta x^\mu \Delta x^\nu = g_{\mu\nu}\, \Delta x^\mu\, \Delta x^\nu$$



We write $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

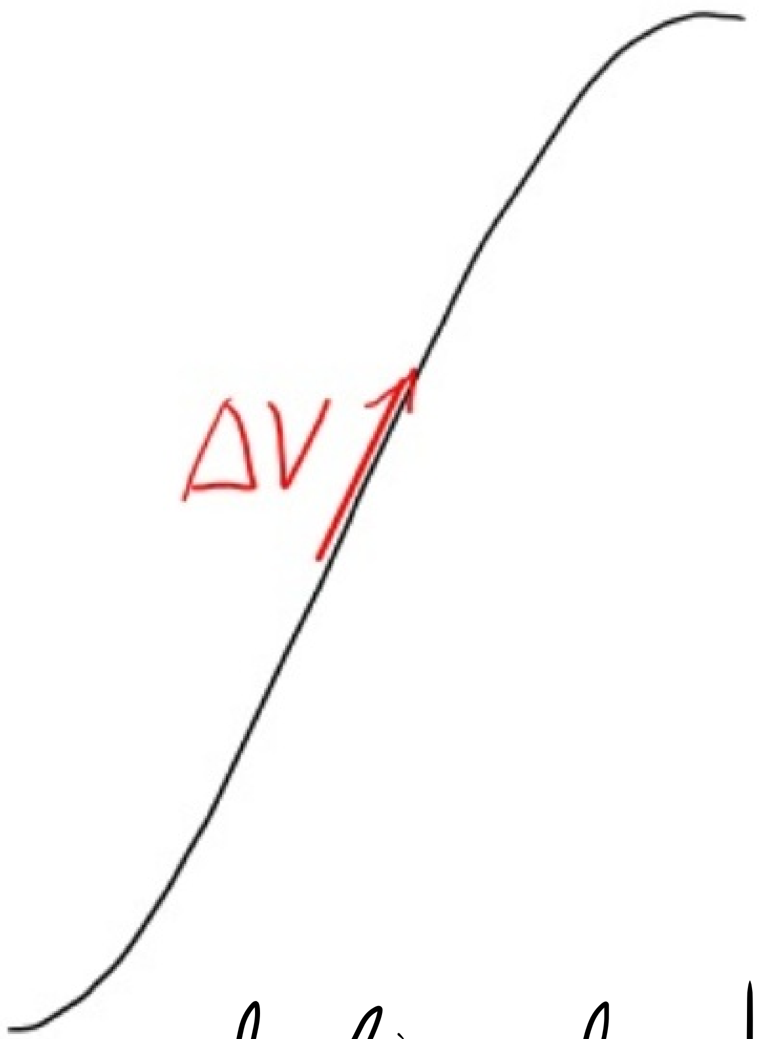
$\hookrightarrow$ In bibliography this could mean $g \equiv ds^2$, $dx^\mu dx^\nu = dx^\mu \otimes dx^\nu \neq dx^\nu dx^\mu$

Line element

$$\left.\begin{array}{l} g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu \\ \Delta V = \Delta x^\mu \partial_\mu \end{array}\right\} \Rightarrow g(\Delta V, \Delta V) = g(\Delta x^\mu \partial_\mu, \Delta x^\nu \partial_\nu)$$

$$= g(\partial_\mu, \partial_\nu)\, \Delta x^\mu \Delta x^\nu$$

$$= g_{\mu\nu}\, \Delta x^\mu\, \Delta x^\nu$$



We write $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$, and we use it as infinitesimal line element

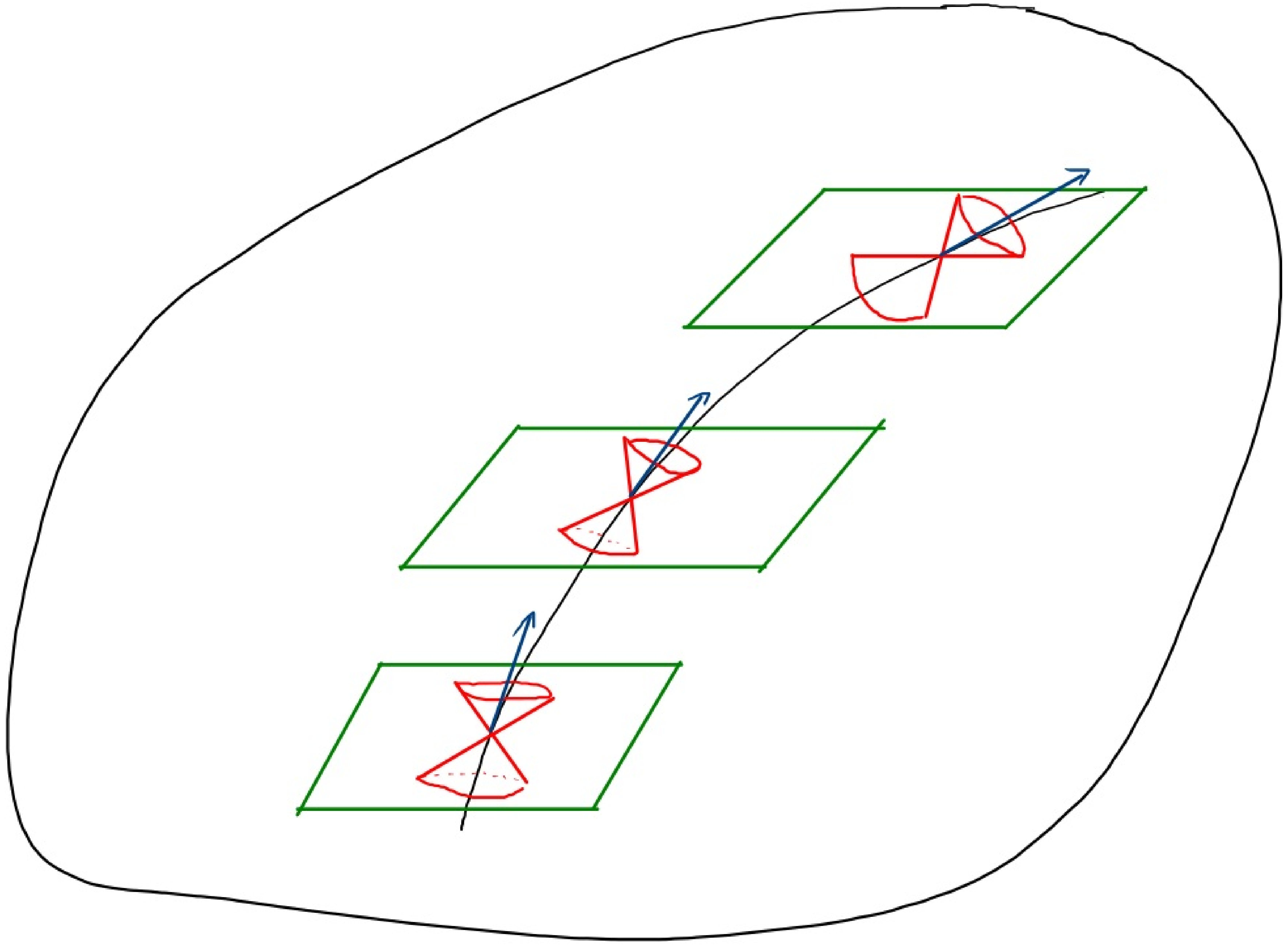
$$S_{AB} = \int_A^B ds = \int_A^B \left\{ |g_{\mu\nu}\, dx^\mu dx^\nu| \right\}^{1/2} \equiv \int_{t_A}^{t_B} dt \left\{ \left| g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right| \right\}^{1/2}$$

We focus on 3 types of curves:

$ds^2 < 0$ everywhere $\to$ timelike curves

$ds^2 = 0$ " $\to$ null/lightlike "

$ds^2 > 0$ " $\to$ spacelike "



$\Rightarrow$ tangent vector V is of the same type at each point ($g(V,V)$ does not change sign)

tangent vectors are the 4-velocities of particles $(ds^2 \leq 0)$

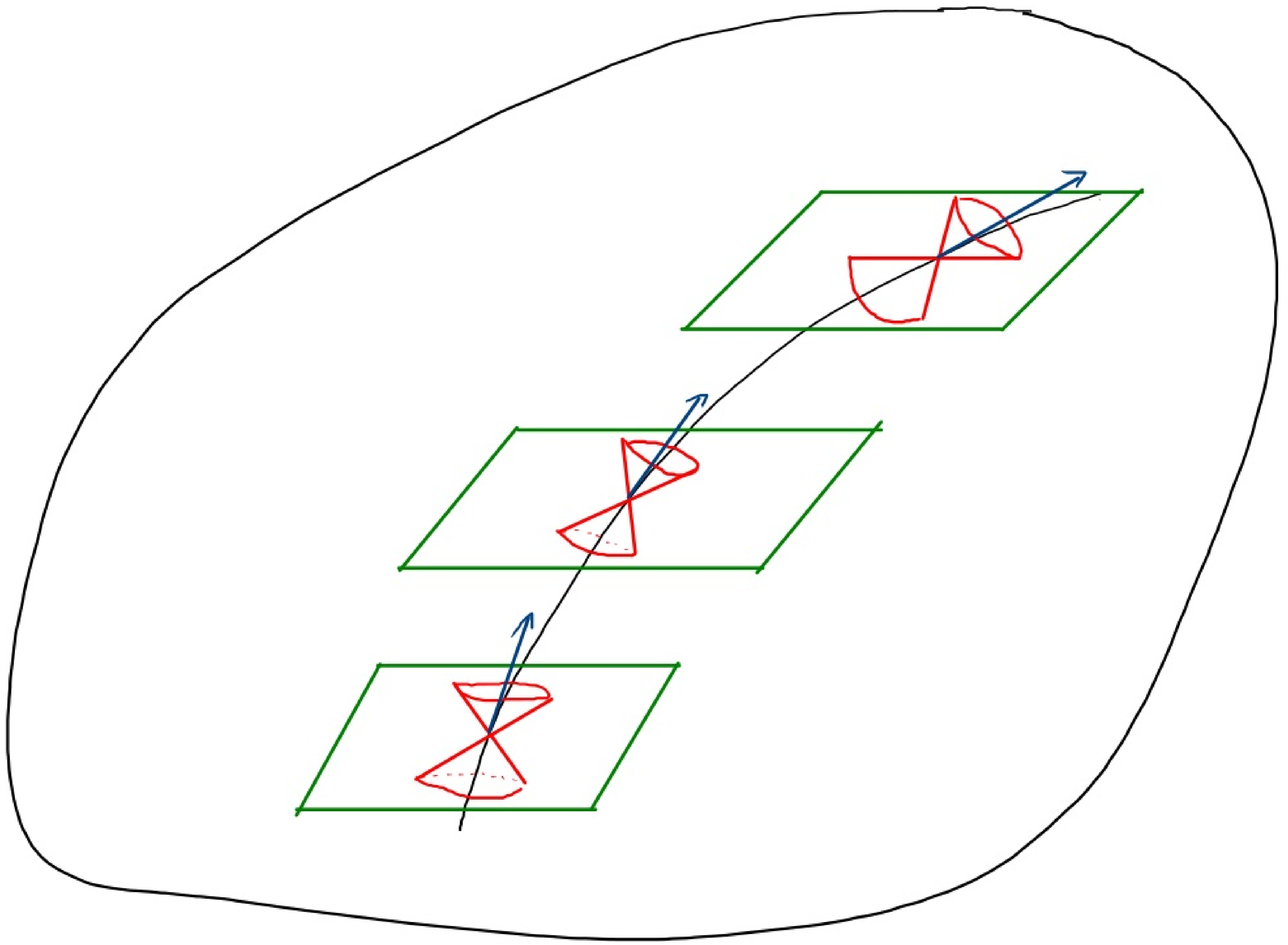
$$V^\mu = \frac{dx^\mu}{dt}$$

We focus on 3 types of curves:

$ds^2 < 0$ everywhere $\rightarrow$ timelike curves

$ds^2 = 0$ " $\rightarrow$ null/lightlike "

$ds^2 > 0$ " $\rightarrow$ spacelike "



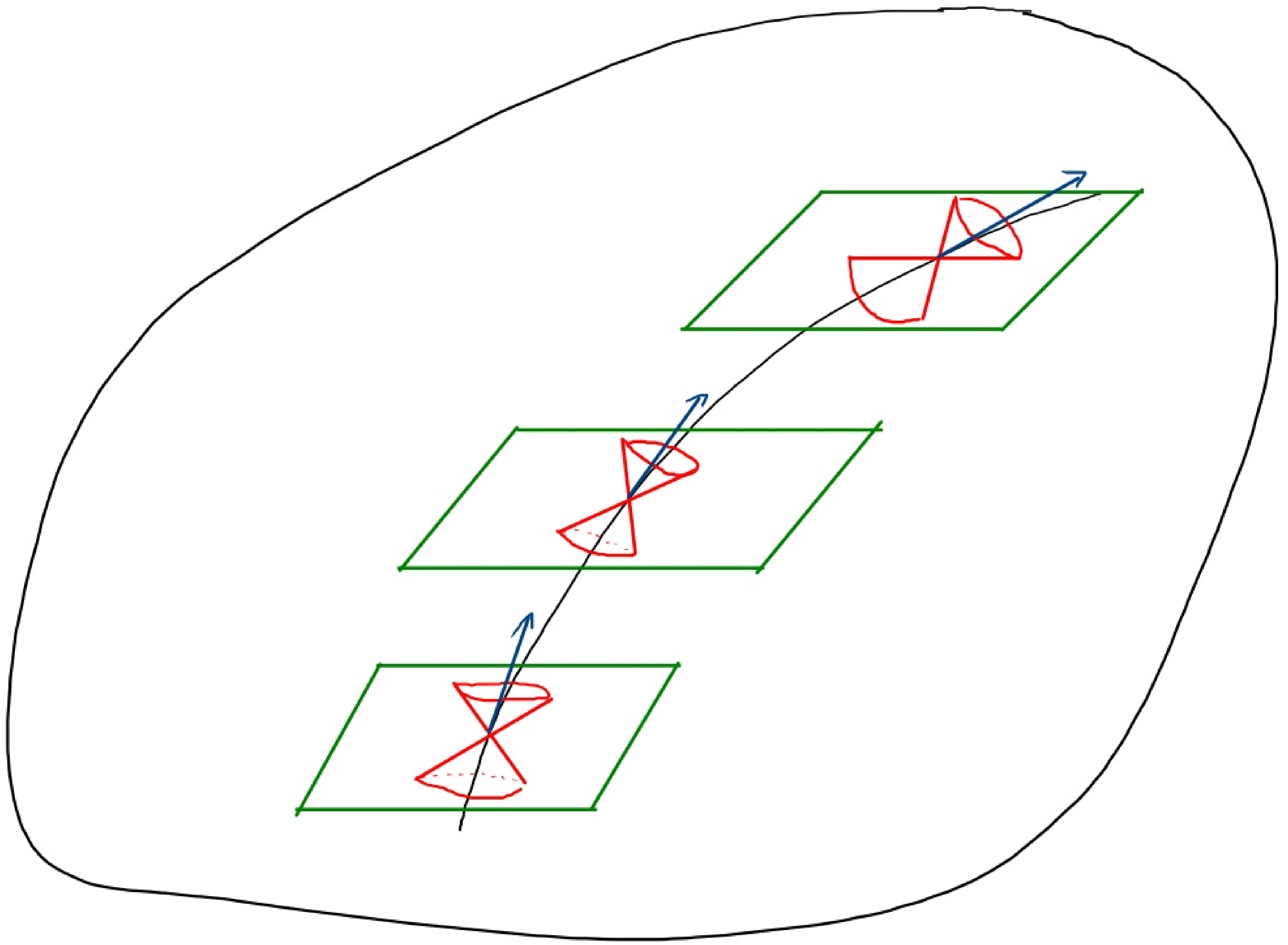
* Timelike curves are worldlines of observers and massive particles
* Null curves are worldlines of massless particles
* Causal curves: any event on curve can influence/be influenced by any other event on curve (timelike or null)

We focus on 3 types of curves:

$ds^2 < 0$ everywhere $\rightarrow$ timelike curves

$ds^2 = 0$ " $\rightarrow$ null/lightlike "

$ds^2 > 0$ " $\rightarrow$ space like "



* Light always travels in a direction on the local light cone

Not "exceeding speed of light" a local concept: particles always move in a direction within or on the local light cone

$\leadsto$ distances between faraway particles can increase at a rate > 1 !