Preliminaries:

slope1: defines forward light cone, extending from slope1 to slope2.

We make sure that slope1 → $0 \leq \theta 1 \leq \pi$

slope2 → $0 \leq \theta 2 - \theta 1$

So you have to make sure the slopes are entered in the correct order in order to mark the timelike separated events

In[ ]:=
```
lightCone[x0_, y0_, len_, slope1_, slope2_, color_] := Module[
   {x1, y1, x2, y2, x3, y3, x4, y4, θ1, θ2, θ, cone, l},
   l = Abs[len];
   If[slope1 > 0,
    θ1 = ArcTan[slope1],
    θ1 = ArcTan[slope1] + π
   ]; (* ArcTan gives  -π/2 ≤ θ ≤ π/2*)
   If[slope2 > 0,
    θ2 = ArcTan[slope2],
    θ2 = ArcTan[slope2] + π
   ];
   If[θ2 < θ1, θ = θ2; θ2 = θ1; θ1 = θ];
   x1 = x0 + l Cos[θ1]; y1 = y0 + l Sin[θ1];
   x2 = x0 + l Cos[θ2]; y2 = y0 + l Sin[θ2];
   x3 = x0 - l Cos[θ2]; y3 = y0 - l Sin[θ2];
   x4 = x0 - l Cos[θ1]; y4 = y0 - l Sin[θ1];
   cone = Polygon[{{x1, y1}, {x2, y2}, {x0, y0}, {x4, y4}, {x3, y3}, {x0, y0}}];
   (*Print["P1= (",x1,",",y1,")  P2= (",x2,",",y2,")"];*)
   Graphics[{color, cone}]
  ];
(*Show[{lightCone[0.,0.,1.,-1.,-4.5,Red],lightCone[2.,3.,1.,1.,3.6,Blue]}]*)
```

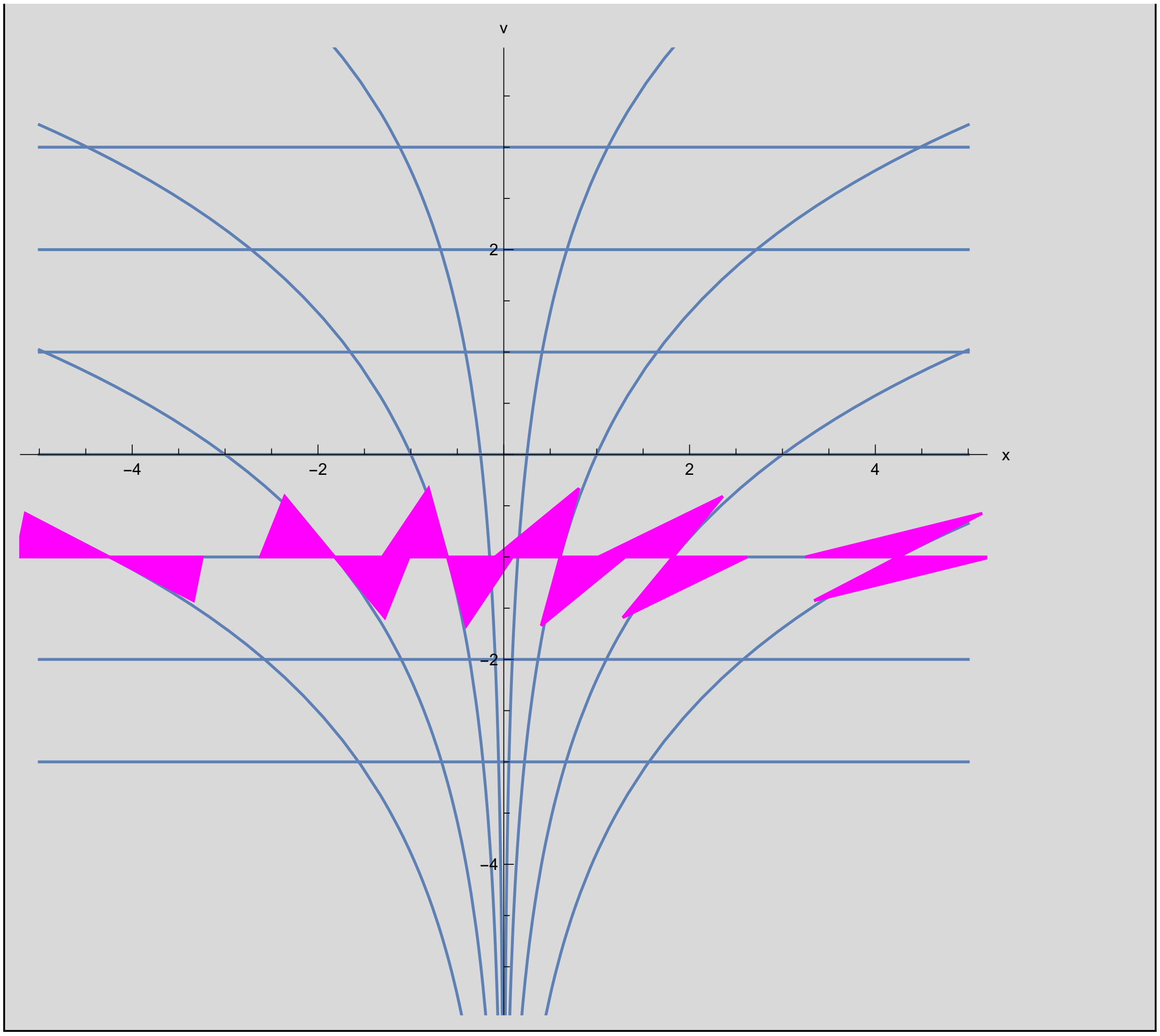
Problem 5

In[ ]:=
```
v1[x_, v0_] := v0;
v2[x_, x0_] := Log[(x/x0)^2];
dv1[x_] := 0;
dv2[x_] := 2/x;
```

In[ ]:=
```
xmin = -5; xmax = 5; vmin = -5; vmax = 3.5;
v0s = {-3, -2, -1, 0, 1, 2, 3};
x0s = {0.25, 1, 3, 7};
g1 = Plot[Table[v1[x, v0], {v0, v0s}], {x, xmin, xmax}];
g2 = Plot[Table[v2[x, x0], {x0, x0s}], {x, xmin, xmax}];
vp = -1; xp = x /. FindRoot[v2[x, 7] == vp, {x, 4}];
l1 = lightCone[xp, vp, 1, dv1[xp], dv2[xp], Magenta];
vp = -1; xp = x /. FindRoot[v2[x, 3] == vp, {x, 2}];
l2 = lightCone[xp, vp, 0.8, dv1[xp], dv2[xp], Magenta];
vp = -1; xp = x /. FindRoot[v2[x, 1] == vp, {x, 0.5}];
l3 = lightCone[xp, vp, 0.7, dv1[xp], dv2[xp], Magenta];
vp = -1; xp = x /. FindRoot[v2[x, 1] == vp, {x, -0.5}];
l4 = lightCone[xp, vp, 0.7, dv1[xp], dv2[xp], Magenta];
vp = -1; xp = x /. FindRoot[v2[x, 3] == vp, {x, -2}];
l5 = lightCone[xp, vp, 0.8, dv1[xp], dv2[xp], Magenta];
vp = -1; xp = x /. FindRoot[v2[x, 7] == vp, {x, -4}];
l6 = lightCone[xp, vp, 1, dv1[xp], dv2[xp], Magenta];
Print["(x,v)= (", xp, ",", vp, ")"];
Show[g1, g2, l1, l2, l3, l4, l5, l6,
 PlotRange → {{xmin, xmax}, {vmin, vmax}},
 AspectRatio → 1, Axes → True, AxesLabel → {"x", "v"}]
```

(x,v)= (-4.24571,-1)

Out[ ]=



Problem 18

In[ ]:=

```
Integrate[1/Sqrt[1 - 2 M/r], r, Assumptions → M > 0 && r > 2 M]
```

Out[ ]=

$$r \sqrt{\frac{-2 M + r}{r}} + 2 M \operatorname{ArcTanh}\left[\sqrt{\frac{-2 M + r}{r}}\right]$$

In[ ]:= $\text{Integrate}\left[\frac{1}{\sqrt{1-2\frac{M}{r}}}, \{r, 2M, 3M\}, \text{Assumptions} \to M > 0 \;\&\&\; r > 2M\right]$

Out[ ]= $M\left(\sqrt{3} + 2\,\text{ArcCoth}\left[\sqrt{3}\right]\right)$

In[ ]:= N[%]

Out[ ]= 3.04901 M

In[ ]:= $\text{i1} = \text{Integrate}\left[\frac{r^2}{\sqrt{1-2\frac{M}{r}}}, r, \text{Assumptions} \to M > 0 \;\&\&\; r > 2M\right]$

Out[ ]= $$\frac{\sqrt{\frac{r^5}{-2M+r}}\left(\sqrt{r}\left(-30M^3 + 5M^2 r + M r^2 + 2r^3\right) + 30M^3\sqrt{-2M+r}\,\text{ArcTanh}\left[\frac{\sqrt{r}}{\sqrt{-2M+r}}\right]\right)}{6\,r^{5/2}}$$

In[ ]:= $\text{Integrate}\left[\frac{r^2}{\sqrt{1-2\frac{M}{r}}}, \{r, 2M, 3M\}, \text{Assumptions} \to M > 0 \;\&\&\; r > 2M\right]$

Out[ ]= $\frac{1}{2}M^3\left(16\sqrt{3} + \text{Log}\left[362 + 209\sqrt{3}\right]\right)$

In[ ]:= N[%]

Out[ ]= $17.1488\,M^3$

In[ ]:= 4. π %

Out[ ]= $215.498\,M^3$

Problem 20

In[ ]:= $\text{Integrate}\left[\sqrt{2M}\,(r-2M)^{-1/2}, r, \text{Assumptions} \to M > 0 \;\&\&\; r > 2M\right]$

Out[ ]= $$\frac{2\sqrt{2}\,M}{\sqrt{\frac{M}{-2M+r}}}$$

In[ ]:=

```
RevolutionPlot3D[2 √(2 (r - 2)), {r, 2.1, 5}, Axes → False, Boxed → False]
```

Out[ ]=