- The Metric
  - Examples

The metric: $g_{\mu\nu}$, a choice of

- a $(0,2)$ tensor
- symmetric $g_{\mu\nu} = g_{\nu\mu}$
- non-degenerate

$$g \equiv \det g_{\mu\nu} < 0$$

$$g = g_0 \cdot g_1 \cdot g_2 \cdot g_3$$

number of negative evs
the signature s of the metric

- inverse $g^{\mu\nu}$ s.t.

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho}$$

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$$g \equiv \det g_{\mu\nu} < 0$$

$$g = g_0 \cdot g_1 \cdot g_2 \cdot g_3 \qquad g_0 < 0 \qquad s = 1$$

$$g_i > 0$$

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The metric: $g_{\mu\nu}$

Inner product: $U \cdot V = g_{\mu\nu} U^\mu V^\nu$

Norm
(spacetime) length: $\| U \|^2 = g_{\mu\nu} U^\mu U^\nu$

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$= g(U, V)$

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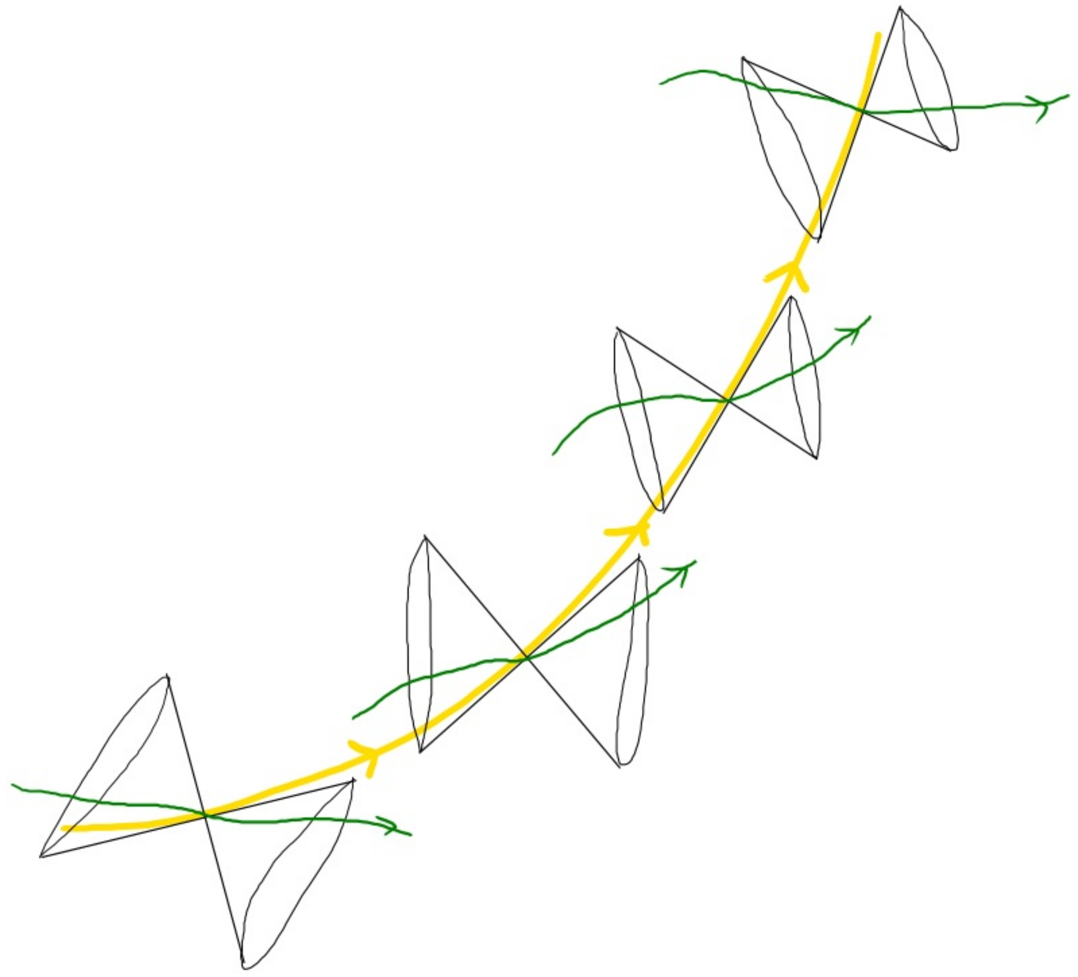
(spacetime) length: $\| U \|^2 = g_{\mu\nu} U^\mu U^\nu$

$= g(U, U)$

$\begin{cases} < 0 & \text{timelike} \\ = 0 & \text{null} \\ & \text{light like} \end{cases}$

> 0 spacelike



Nothing moves faster than light: particles move on causal curves: Everywher time/ligth like 4-velocities

<u>The metric:</u> $g_{\mu\nu}$

Index raising and lowering:

- duality $TM \leftrightarrow T^*M$
- depends on metric
- does not depend on basis

$$U_\mu = g_{\mu\nu} U^\nu$$

$$U^\mu = g^{\mu\nu} U_\nu$$

vector   1-form

The metric: $g_{\mu\nu}$

Index raising and lowering:
- duality $T^{(l,m)}\mathcal{M} \leftrightarrow T^{(l\pm1, m\mp1)}\mathcal{M}$
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- does not depend on basis

$$U_\mu = g_{\mu\nu} U^\nu$$

$$U^\mu = g^{\mu\nu} U_\nu$$

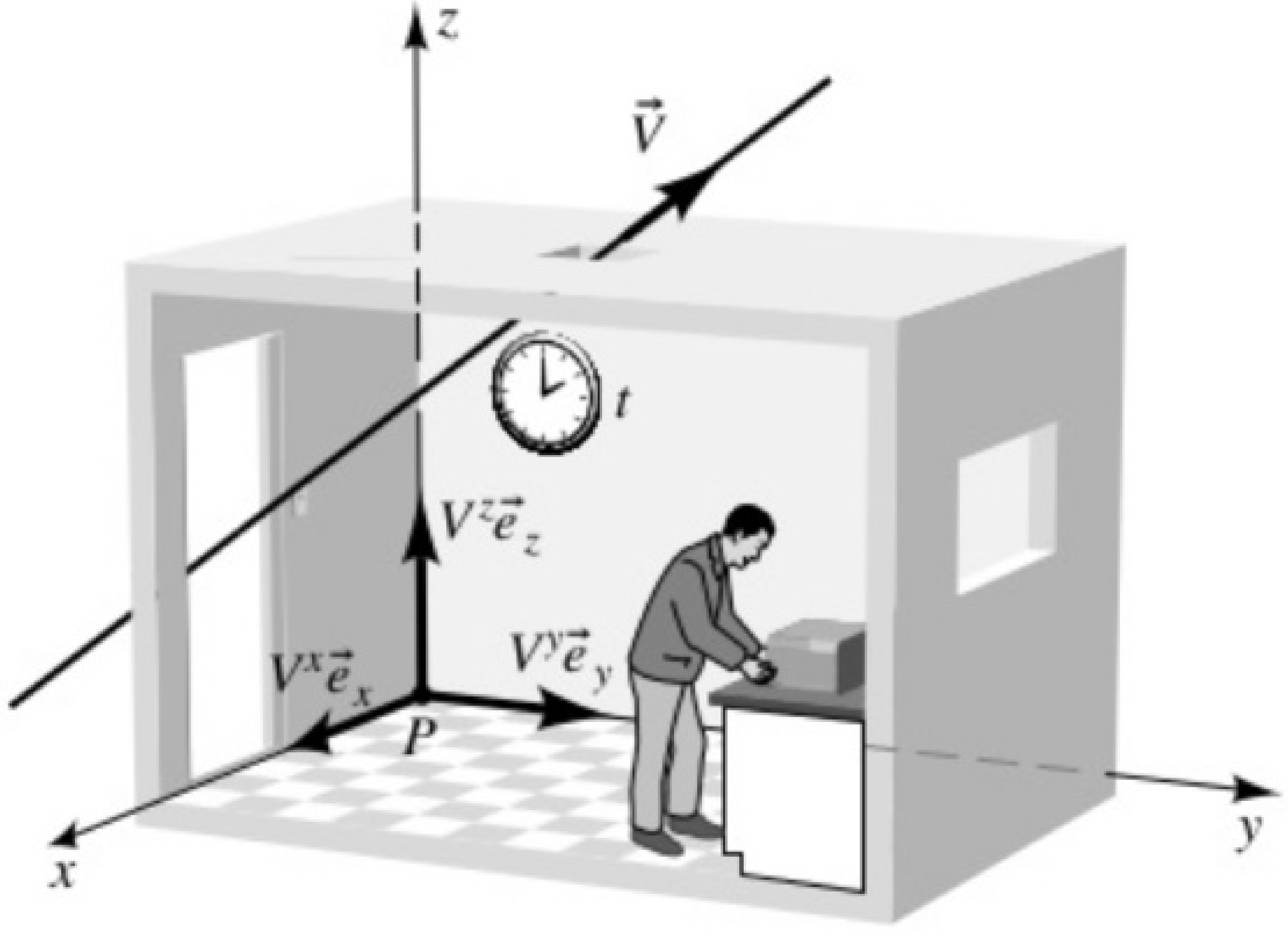
$$F_{\mu\nu} = g_{\mu\rho} F^\rho{}_\nu = g_{\mu\rho} g_{\nu\sigma} F^{\rho\sigma}$$

$$U \cdot V = g_{\mu\nu} U^\mu V^\nu = U_\mu V^\mu = U^\mu V_\mu$$

Locally Inertial Frames

We can always put the components of the metric in the form:

$$(g_{\alpha\beta}) = (\eta_{\alpha\beta}) = \mathrm{diag}(-1,1,1,1)$$



Hartle, Fig 7.6

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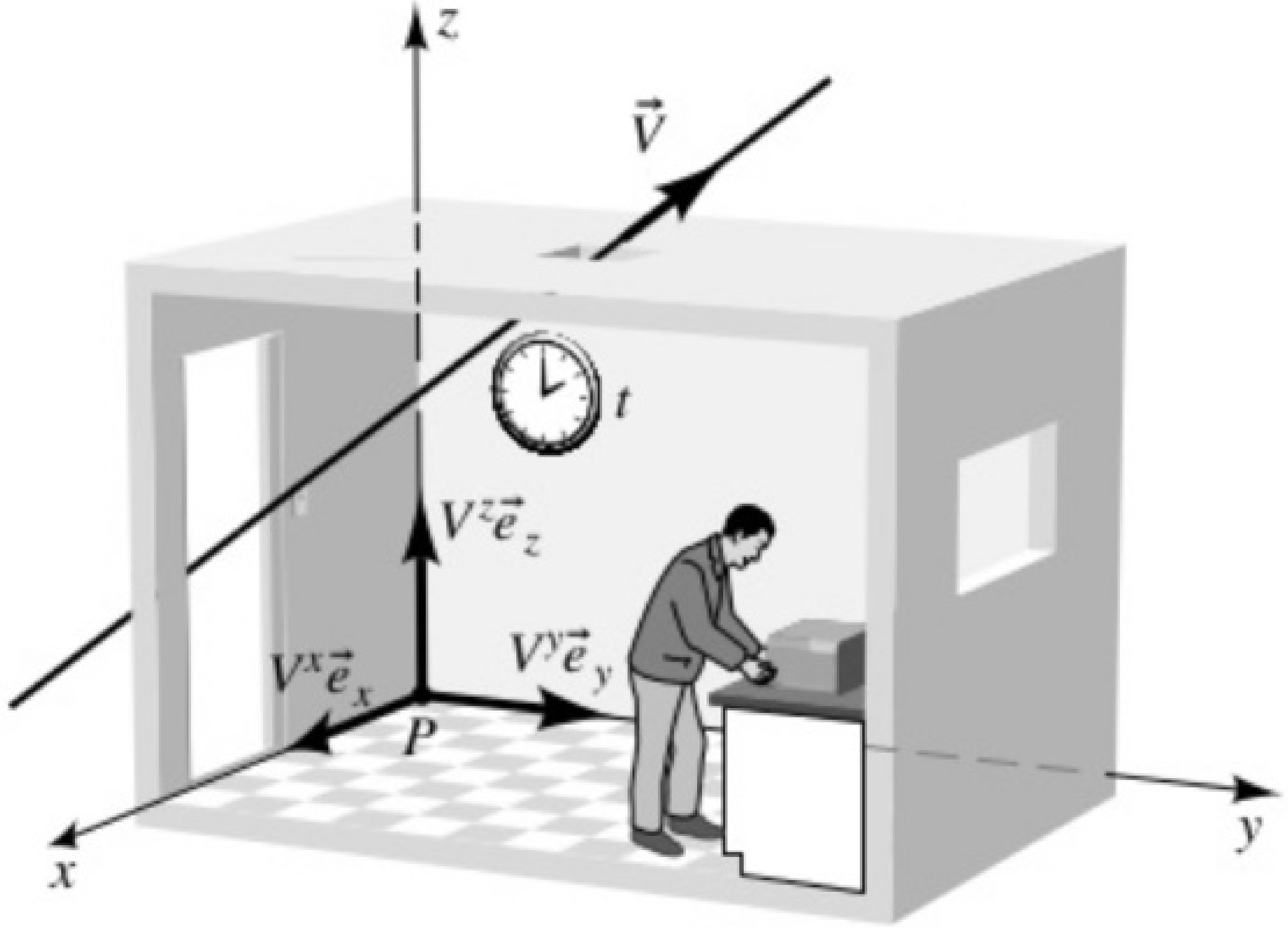

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The corresponding basis $\{e_\alpha\}$ is orthonormal:

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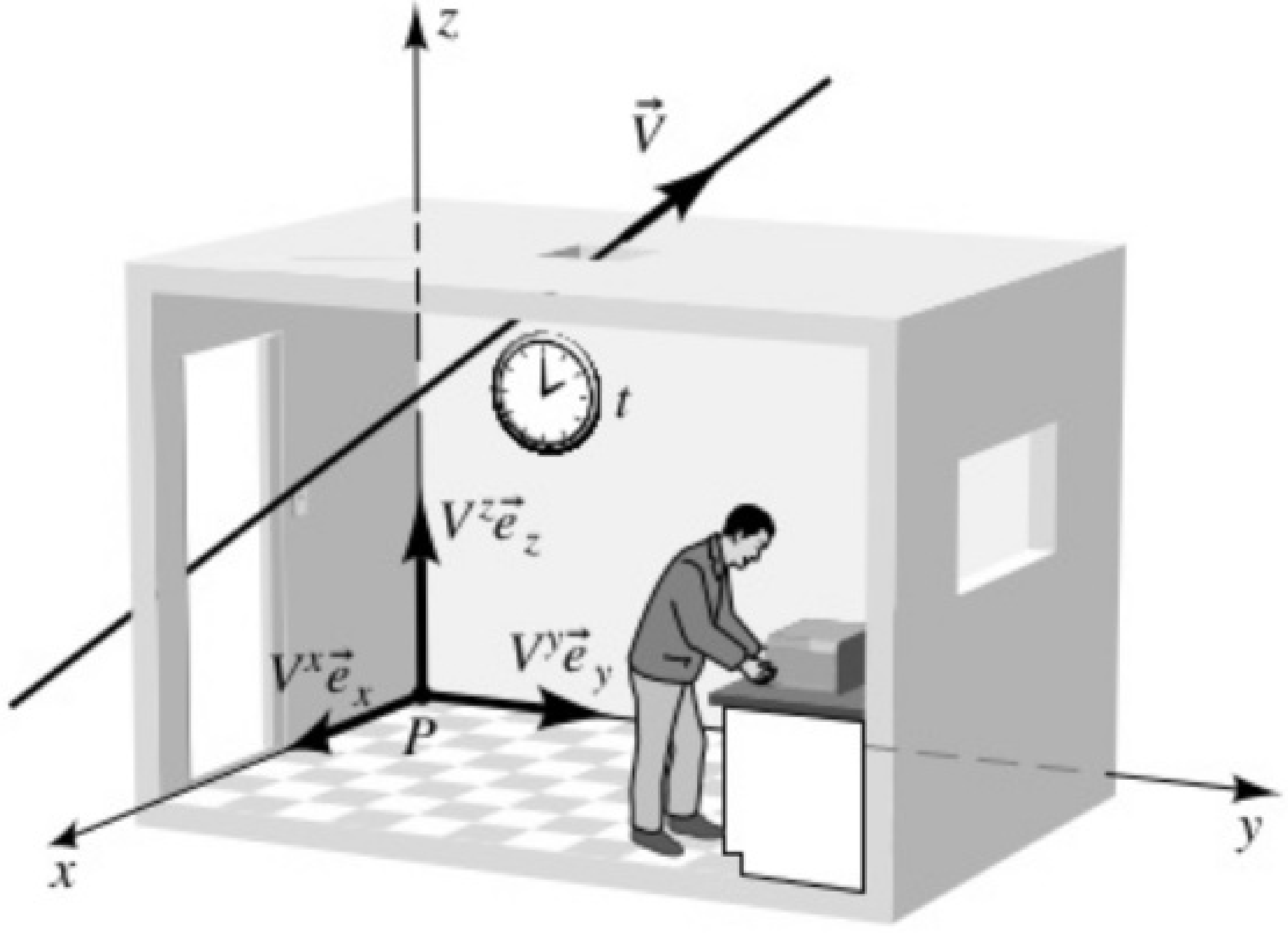

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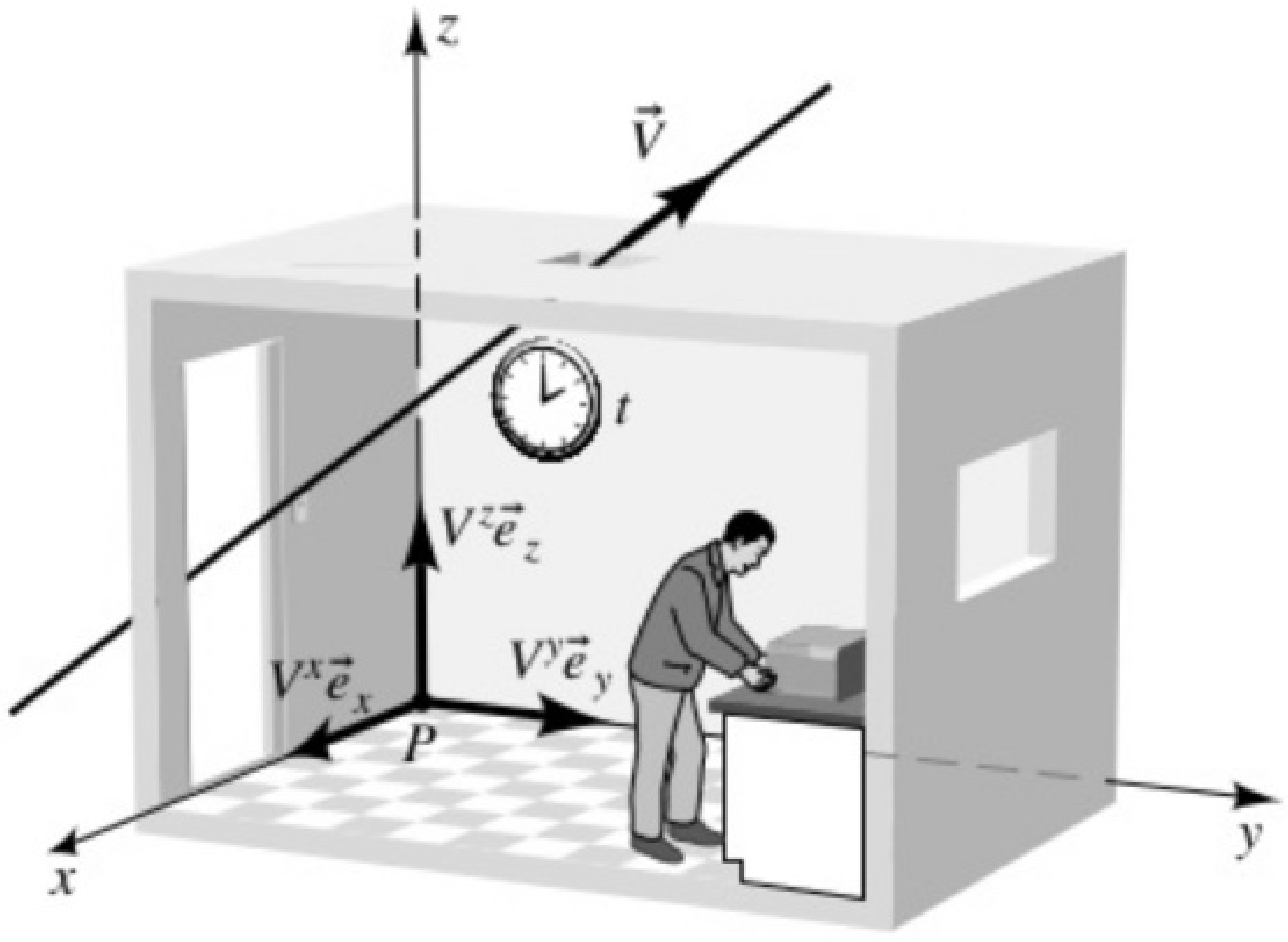
A Lorentz xfm $e_{\alpha'} = \Lambda_{\alpha'}{}^{\beta} e_\beta$, $\Lambda^T \eta \Lambda = \eta$, gives another orthonormal basis $\{e_{\alpha'}\}$

Locally Inertial Frames



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We can define smooth orthonormal vector fields bases, but they will not be coordinate bases

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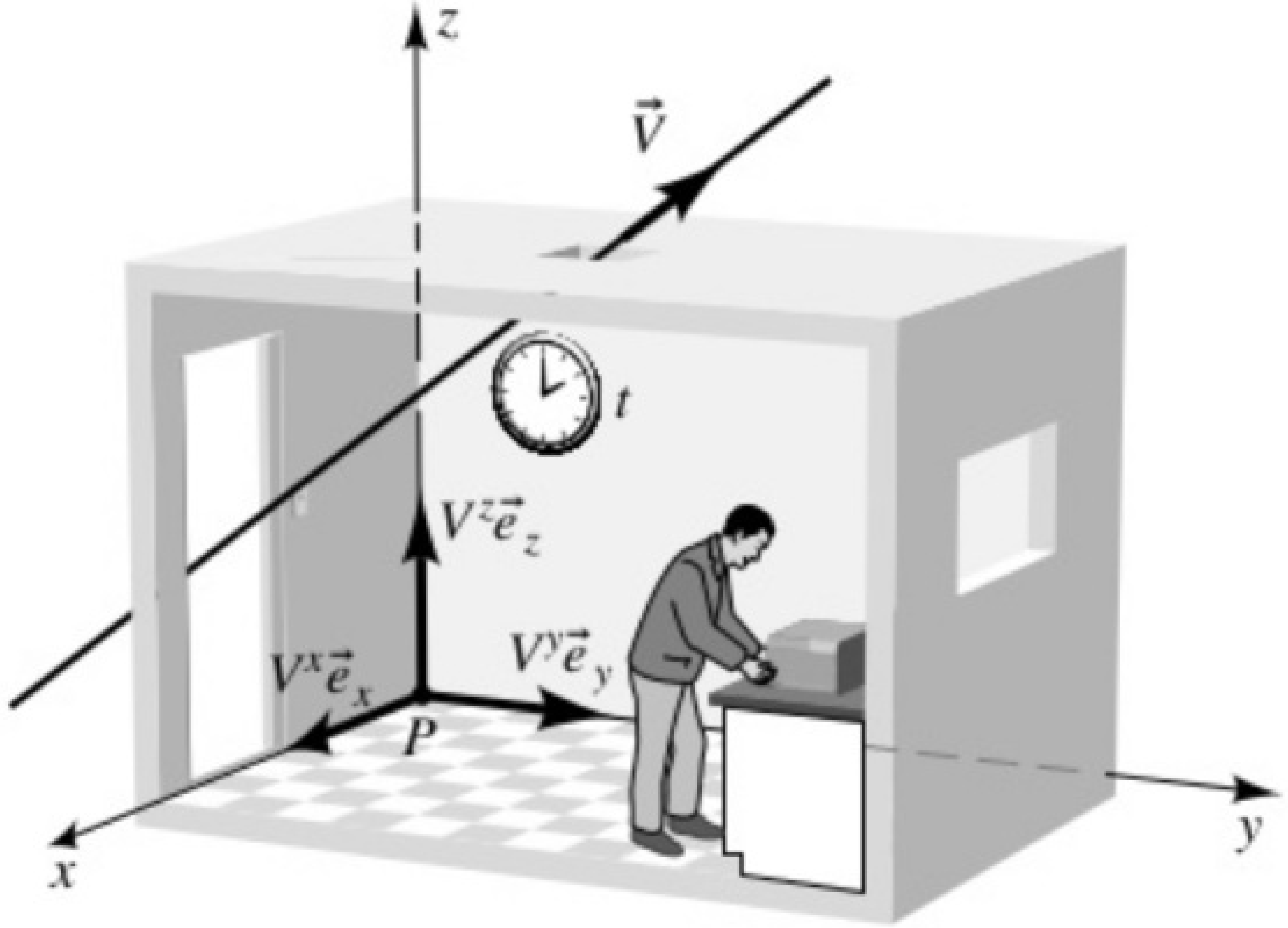

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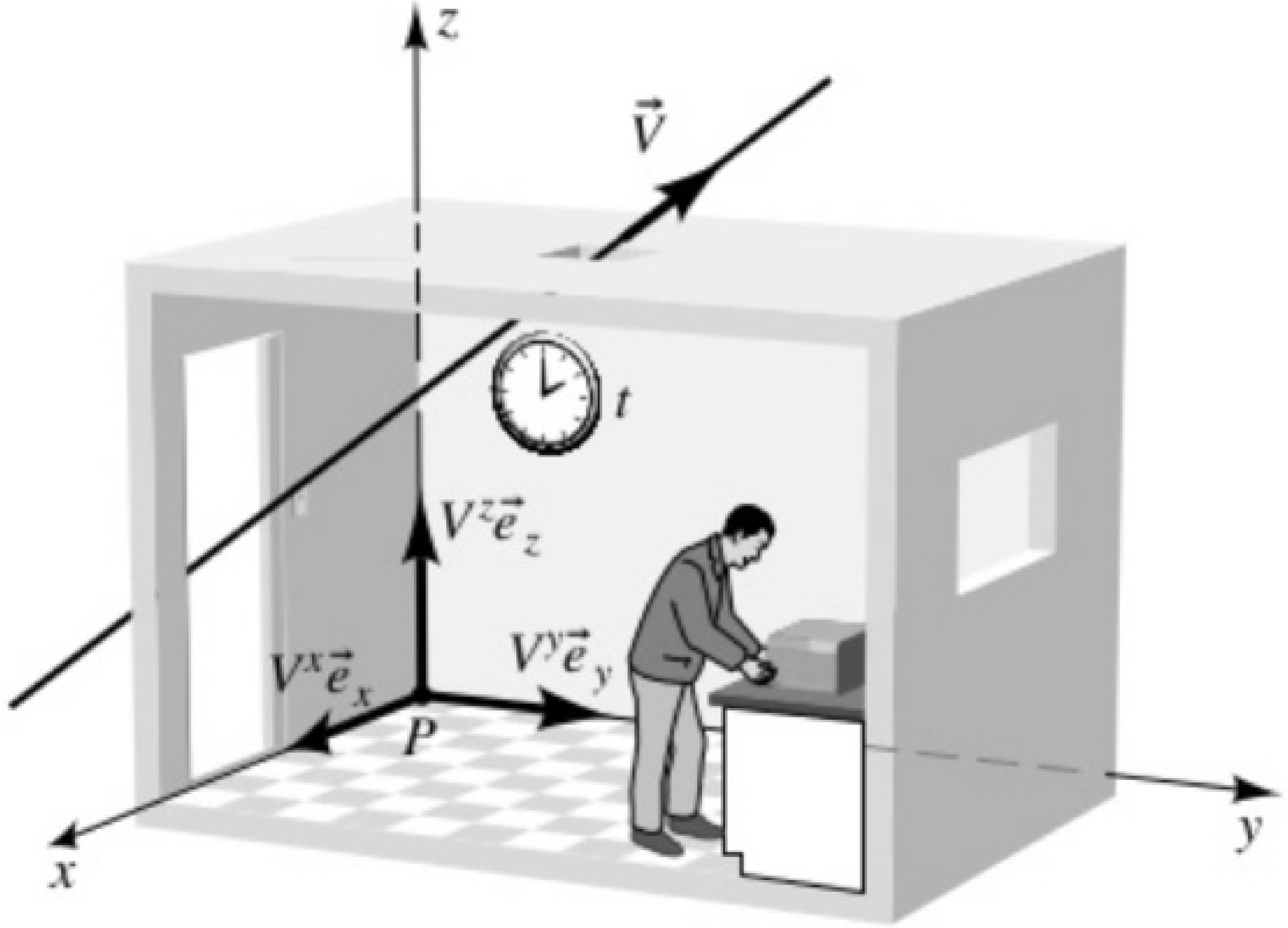
We can always find (non-unique) coordinate systems s.t. at one point P:

$$g_{\hat{\mu}\hat{\nu}}\big|_P = \eta_{\hat{\mu}\hat{\nu}} \quad , \quad \partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}} = 0 \quad , \quad \partial^2_{\hat{\rho}\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} \neq 0$$

Locally Inertial Frames



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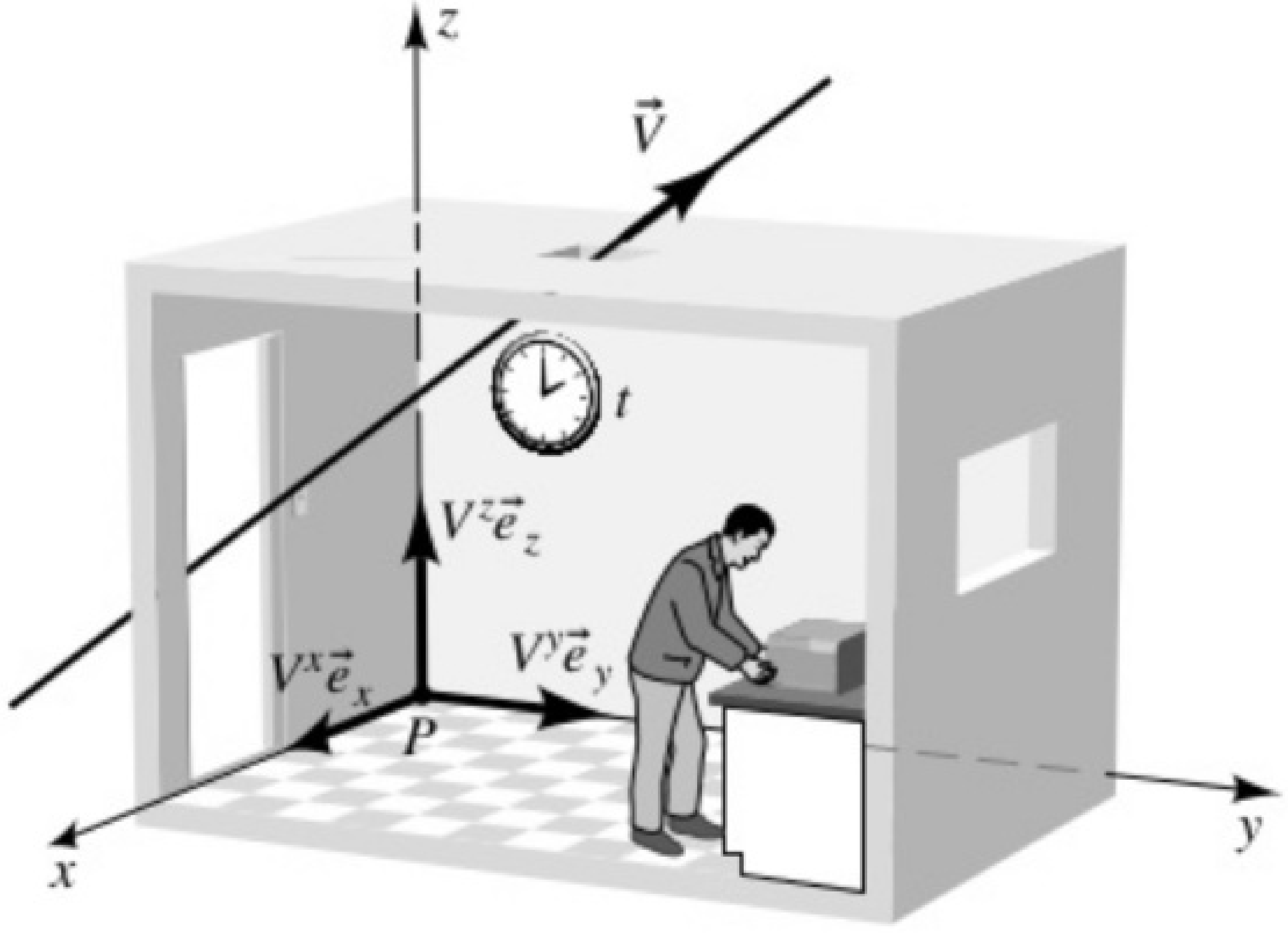
We can always find (non-unique) coordinate systems s.t. at one point P:

$$\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}} = 0 \qquad e_{\hat{\mu}} \cdot e_{\hat{\nu}} = \frac{\partial}{\partial x^{\hat{\mu}}} \cdot \frac{\partial}{\partial x^{\hat{\nu}}} = \eta_{\hat{\mu}\hat{\nu}}$$

a local inertial frame

Locally Inertial Frames

In a local inertial frame we do SR physics



Hartle, Fig 7.6

$$v^\mu = \frac{dx^\mu}{d\tau} = (\gamma, \gamma \vec{V})$$

$$u^\mu = e_0{}^\mu =$$

$= 4$-velocity of frame

$$v_\mu v^\mu = u_\mu u^\mu = -1$$

$$v_\mu u^\mu = -\gamma = -\frac{1}{\sqrt{1-V^2}}$$

$$V = \left(1 - \frac{1}{\gamma^2}\right)^{1/2} = \left(1 - \frac{1}{(u^\mu v_\mu)^2}\right)^{1/2}$$

$\longrightarrow$ basis/coordinate independent

Locally Inertial Frames

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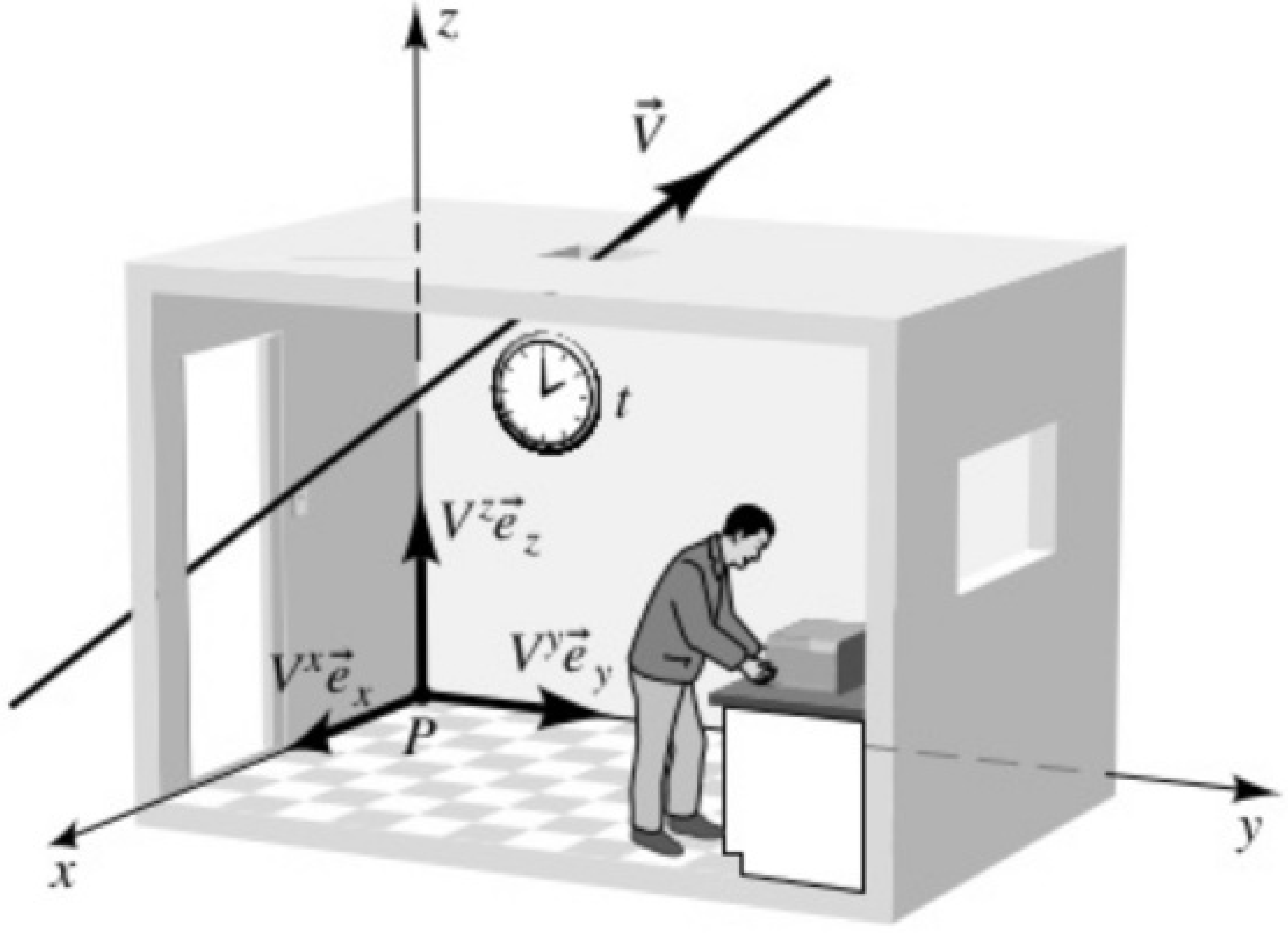

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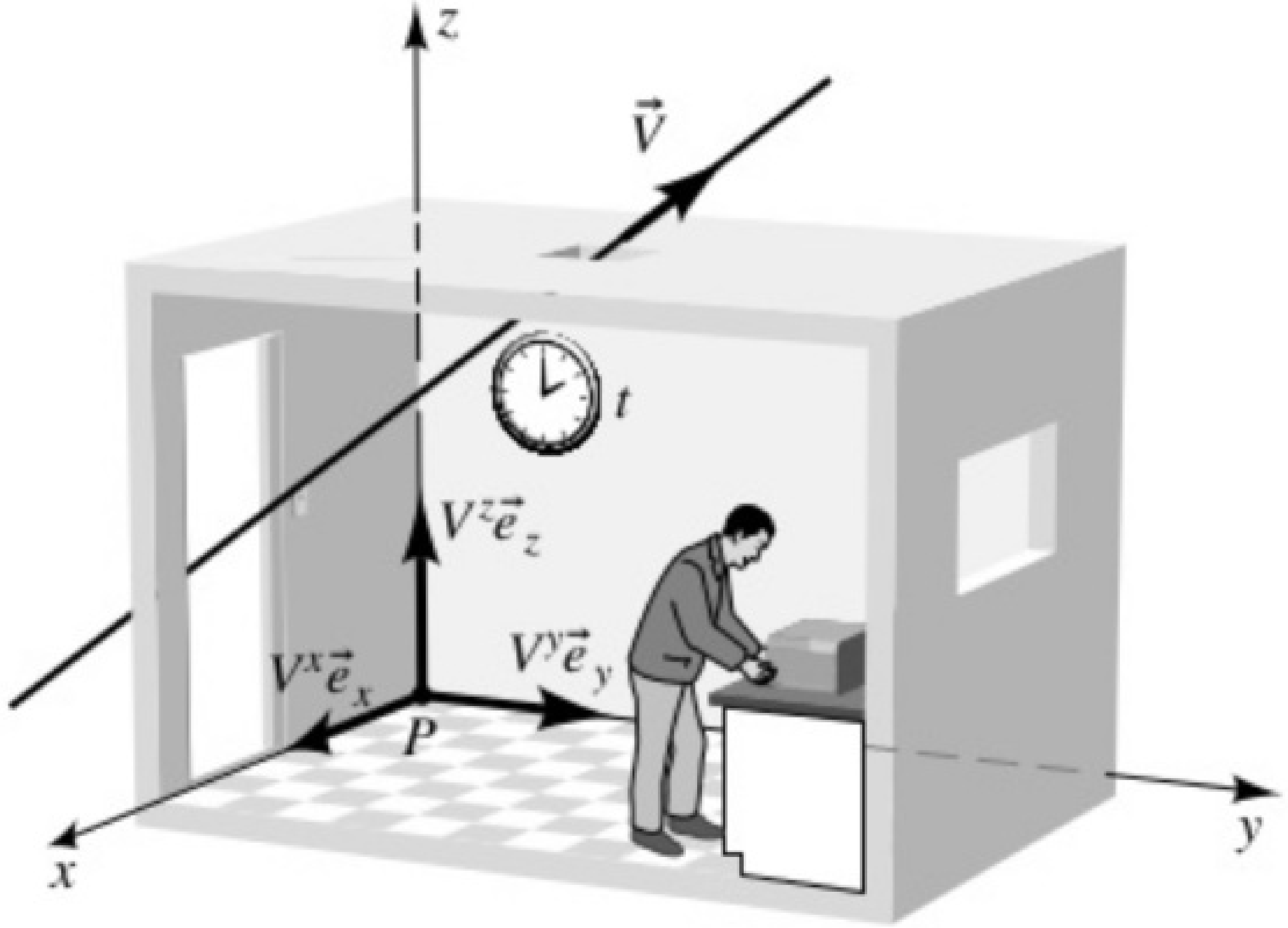
$$p^\mu = (E, p^1, p^2, p^3)$$

$$p^\mu p_\mu = -m^2 = -E^2 + p^2$$

$$p^\mu u_\mu = -E$$

Locally Inertial Frames

Photons: $E = \hbar \omega \quad \vec{p} = \hbar \vec{k} \quad (p^i = \hbar k^i)$



Hartle, Fig 7.6

$p^\mu u_\mu = -\hbar\omega \Rightarrow$

$$\omega = -\frac{1}{\hbar} p^\mu u_\mu$$

We may choose affine parameter λ, s.t.

$$p^\mu = U^\mu = \frac{dx^\mu}{d\lambda}$$

$$p^\mu = (E, p^1, p^2, p^3)$$

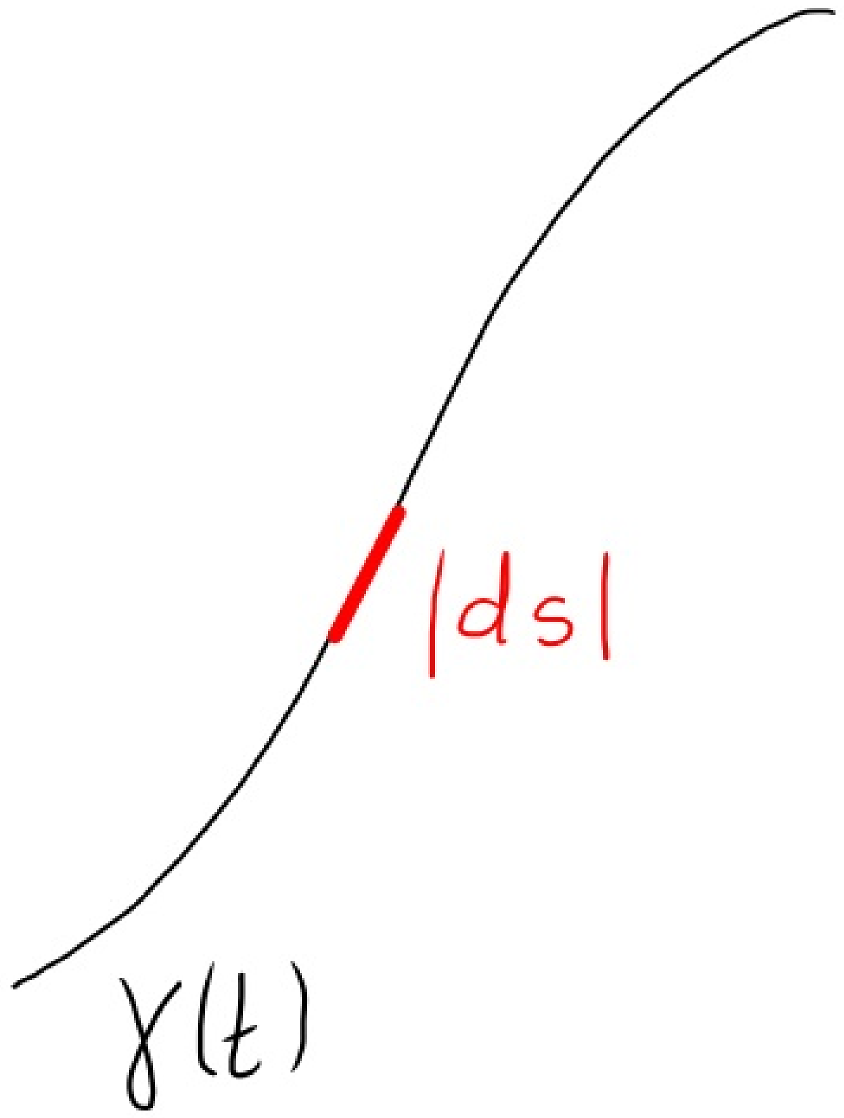
$$p^\mu p_\mu = -m^2 = -E^2 + p^2$$

$$p^\mu u_\mu = -E$$

Line element

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$|ds| = \left| g_{\mu\nu} \frac{dx^\mu}{dt} \cdot \frac{dx^\nu}{dt} \right|^{1/2} \cdot dt = |\,\|V\|\,|\; dt$$



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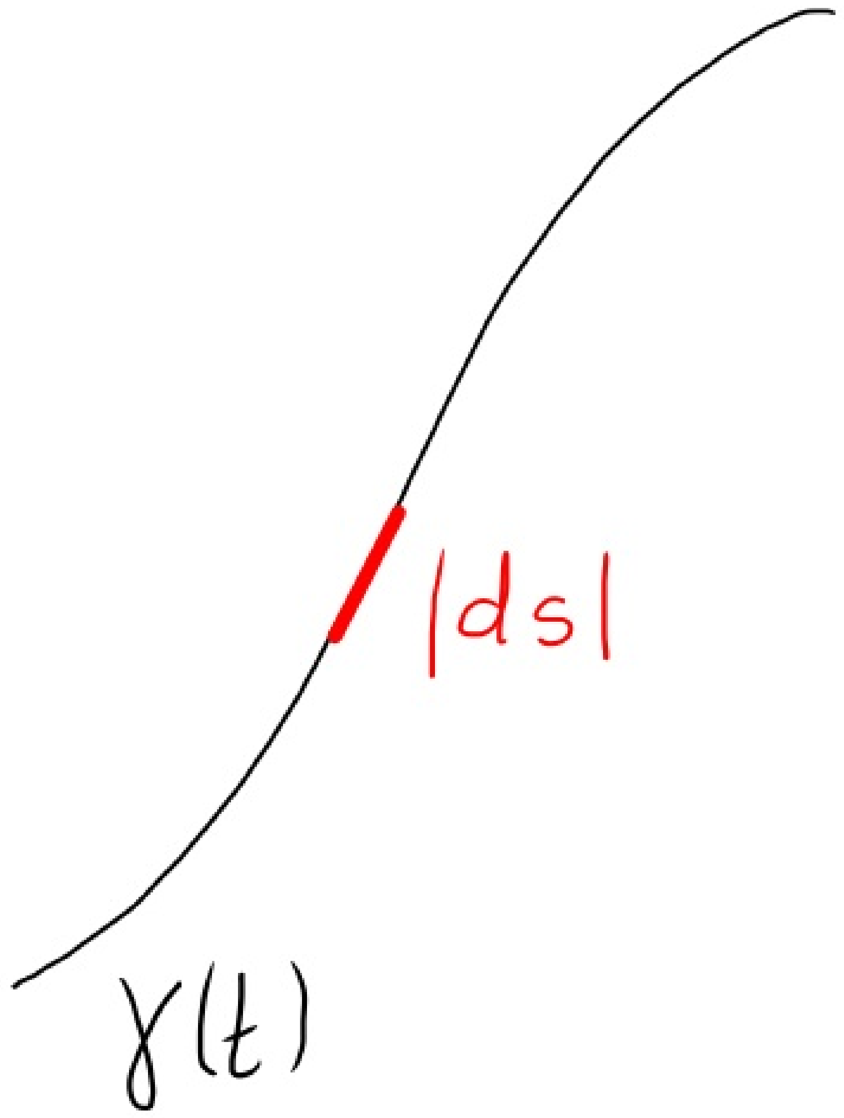

$$|ds| = \left| g_{\mu\nu} \frac{dx^\mu}{dt} \cdot \frac{dx^\nu}{dt} \right|^{1/2} \cdot dt = |\|V\|| \, dt$$

For timelike curves $(m>0)$

$$d\tau^2 = -ds^2 = |g_{\mu\nu} dx^\mu dx^\nu| \qquad \text{particle's proper time}$$

Line element

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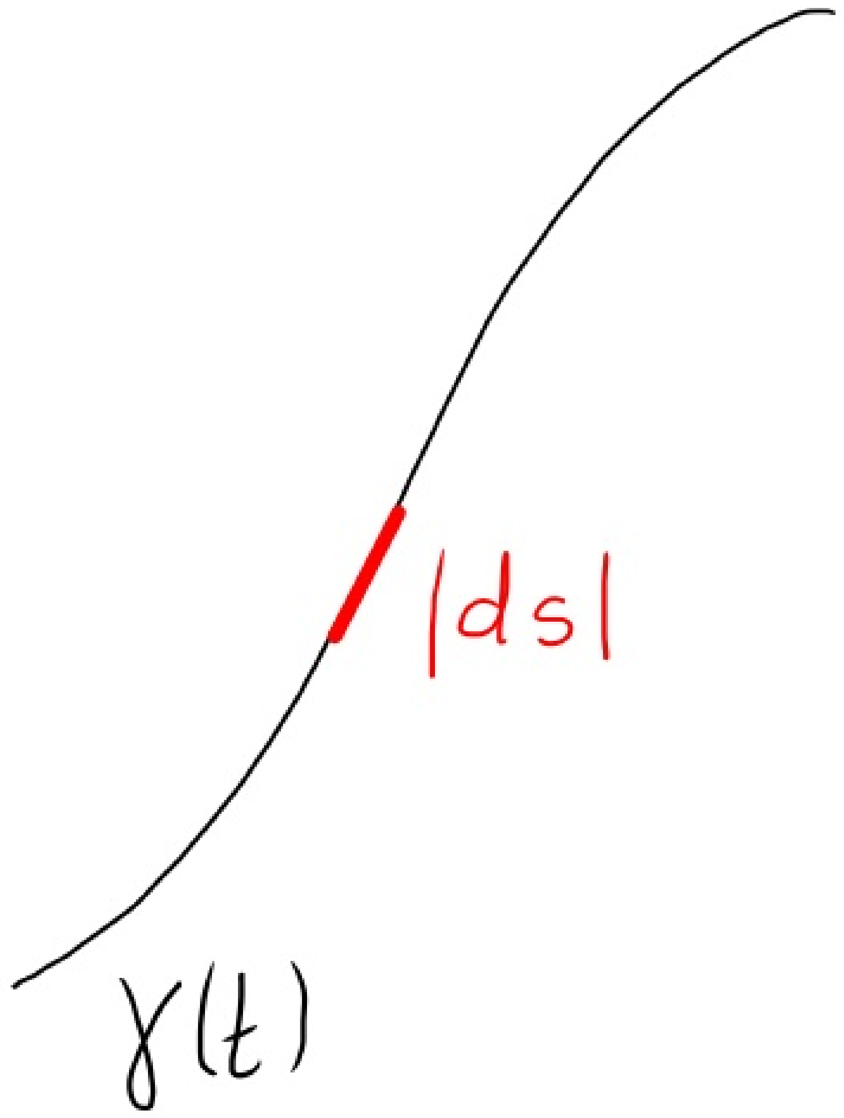

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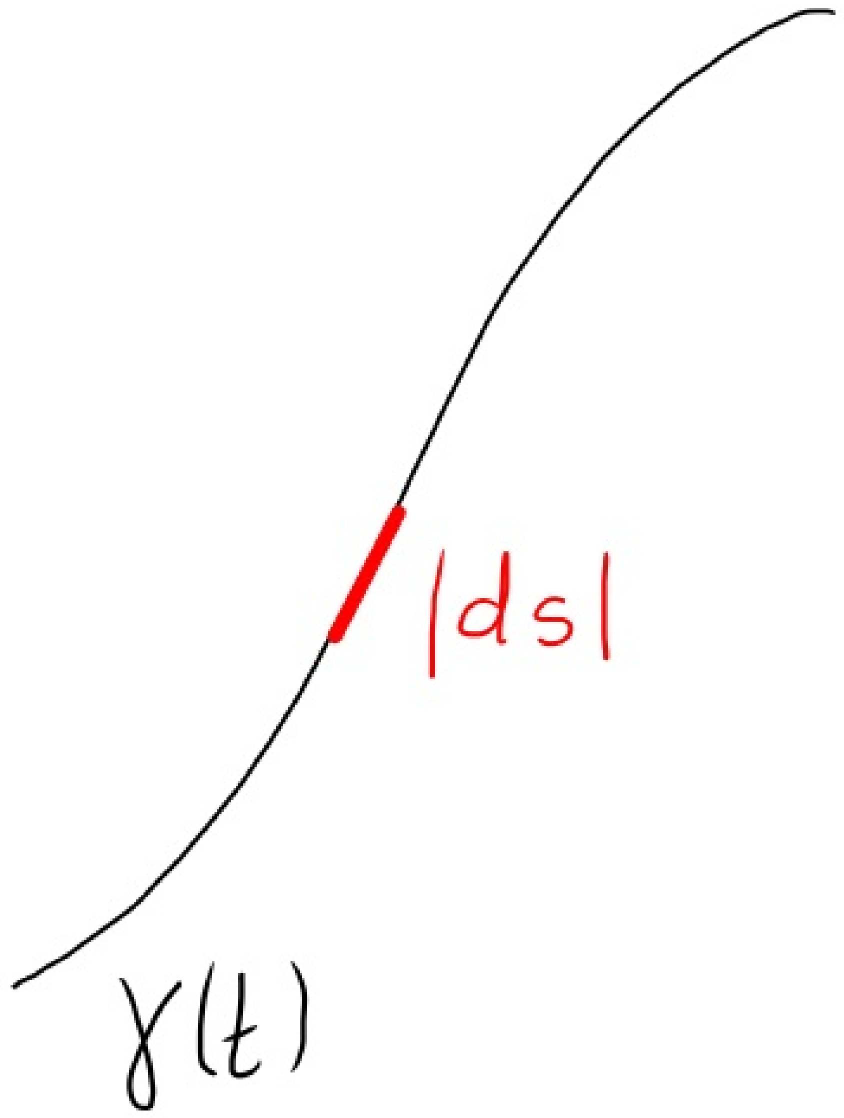
$$d\tau^2 = -ds^2 = |g_{\mu\nu} dx^\mu dx^\nu| \qquad \text{particle's proper time}$$

$$\tau_{AB} = \int_A^B d\tau = \int_A^B |g_{\mu\nu} dx^\mu dx^\nu|^{1/2} = \int_A^B dt \left| g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right|^{1/2}$$

Line element

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$V^\mu = \frac{dx^\mu}{dt}$$



$V^\mu V_\mu < 0$ everywhere: timelike curve

$V^\mu V_\mu = 0$ " lightlike "

$V^\mu V_\mu > 0$ " spacelike "

Line element

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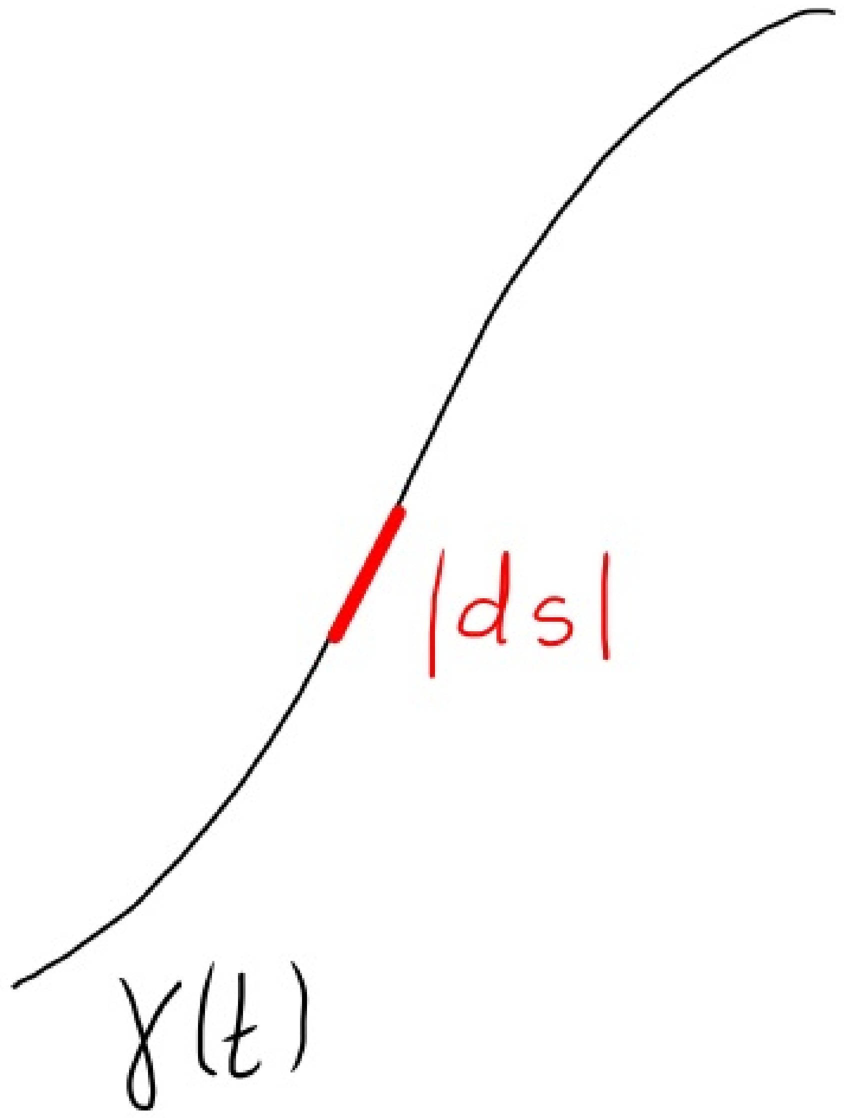
$$V^\mu = \frac{dx^\mu}{dt}$$

timelike curves: worldlines of $m > 0$

lightlike curves: worldlines of $m = 0$



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Examples Hartle 7.3

$$ds^2 = -x^2 dt^2 + dx^2 \quad , \quad x>0$$

$\partial_t \cdot \partial_t = -x^2 < 0$ everywhere timelike

$\partial_x \cdot \partial_x = +1 > 0$ " spacelike

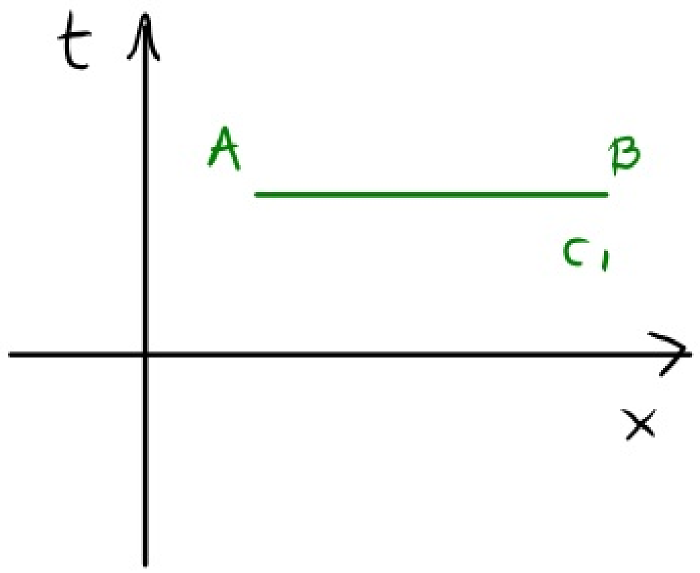
$$\partial_t \cdot \partial_t \equiv g(\partial_t, \partial_t) \equiv g_{00}$$

$$\partial_x \cdot \partial_x \equiv g(\partial_x, \partial_x) = g_{11}$$

Examples Hartle 7.3

$$ds^2 = -x^2\,dt^2 + dx^2 \quad , \quad x > 0$$

$$C_1: \quad ds^2 = dx^2 \Rightarrow S_{AB} = \int_A^B ds = \int_{x_A}^{x_B} dx = x_B - x_A$$



Examples Hartle 7.3

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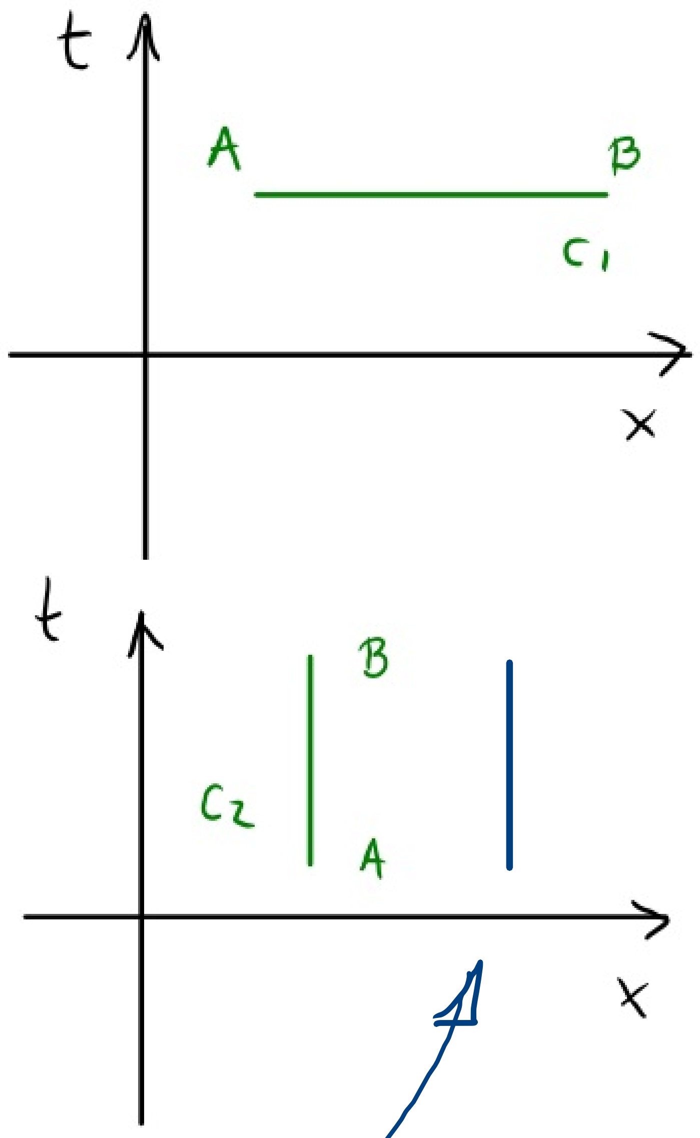
$$C_2: \quad ds^2 = -d\tau^2 = -x_A^2 dt^2 \Rightarrow \tau_{AB} = \int_A^B d\tau = \int_{t_A}^{t_B} x_A \, dt = x_A (t_B - t_A)$$



the spacetime length depends on x_A

different length than AB

Examples Hartle 7.3

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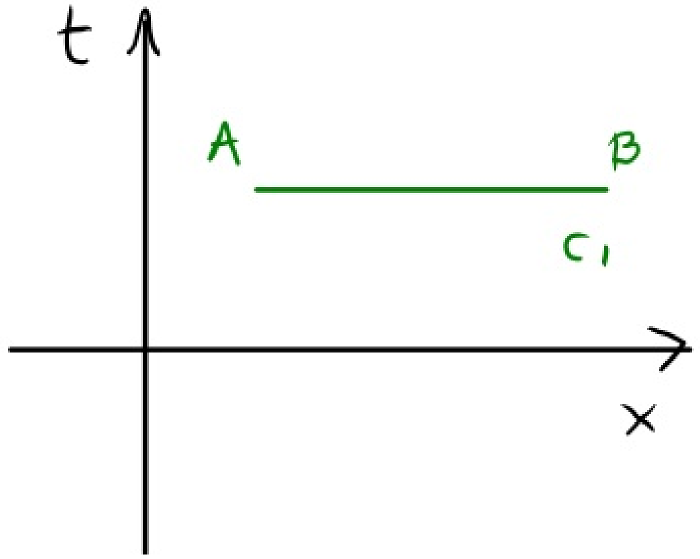

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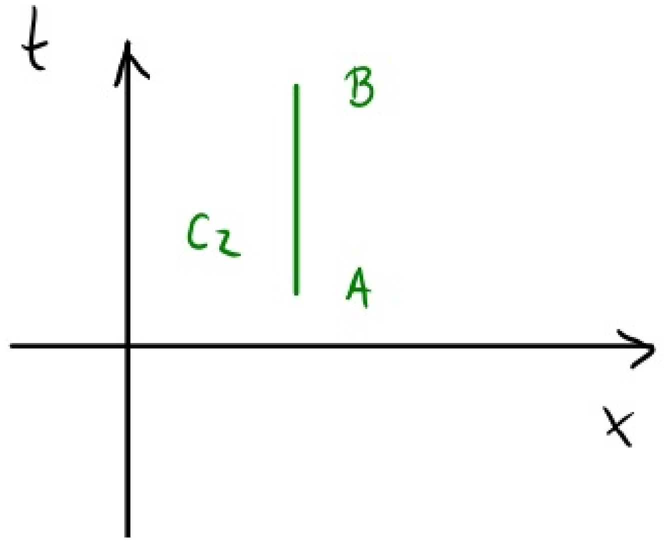
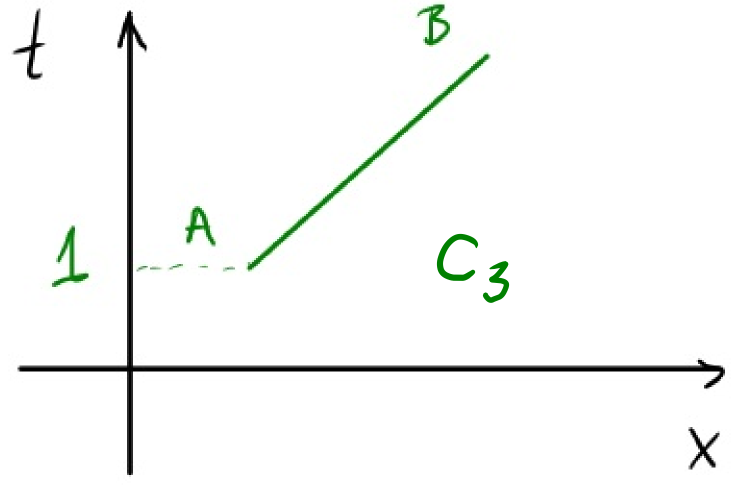


C_3: $x = vt \Rightarrow dx = v \, dt$

$$ds^2 = -(vt)^2 dt^2 + v^2 dt^2 = -v^2 (t^2 - 1) dt^2$$

changes type at $t = 1$



Examples Hartle 7.3

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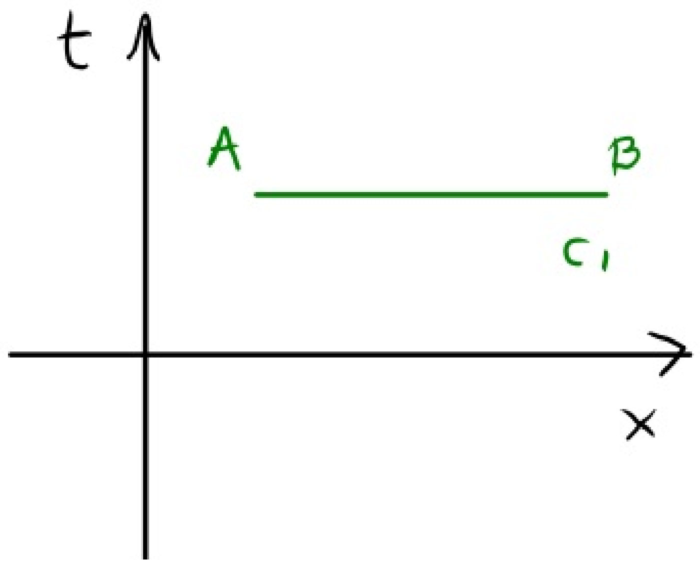

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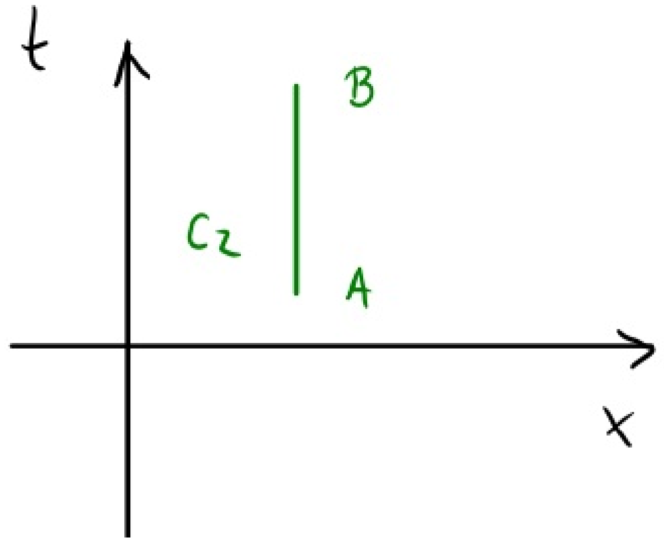

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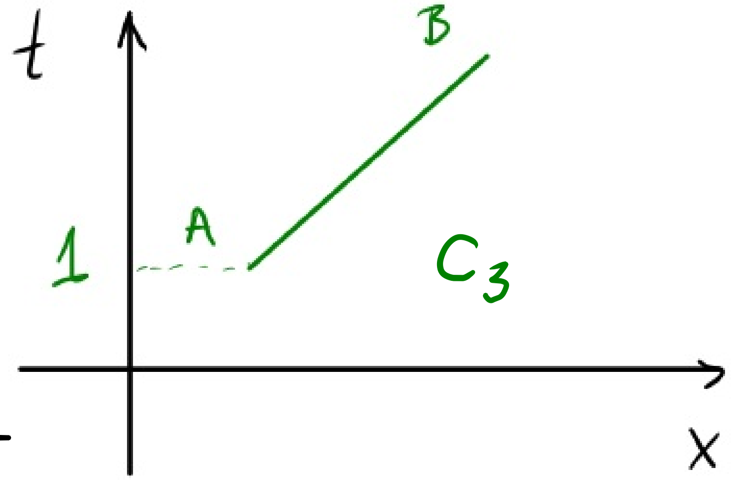
$\overset{t\geq 1}{\Rightarrow}$ $d\tau = v(t^2-1)^{1/2} dt$

$$\Rightarrow \tau_{AB} = \int_A^B d\tau = \int_1^{t_B} v\sqrt{t^2-1}\, dt = \frac{v t_B}{2}(t_B^2 - 1)^{1/2} + v \tanh^{-1} \frac{1 - t_B}{\sqrt{t_B^2 - 1}}$$



Examples   Hartle 7.3

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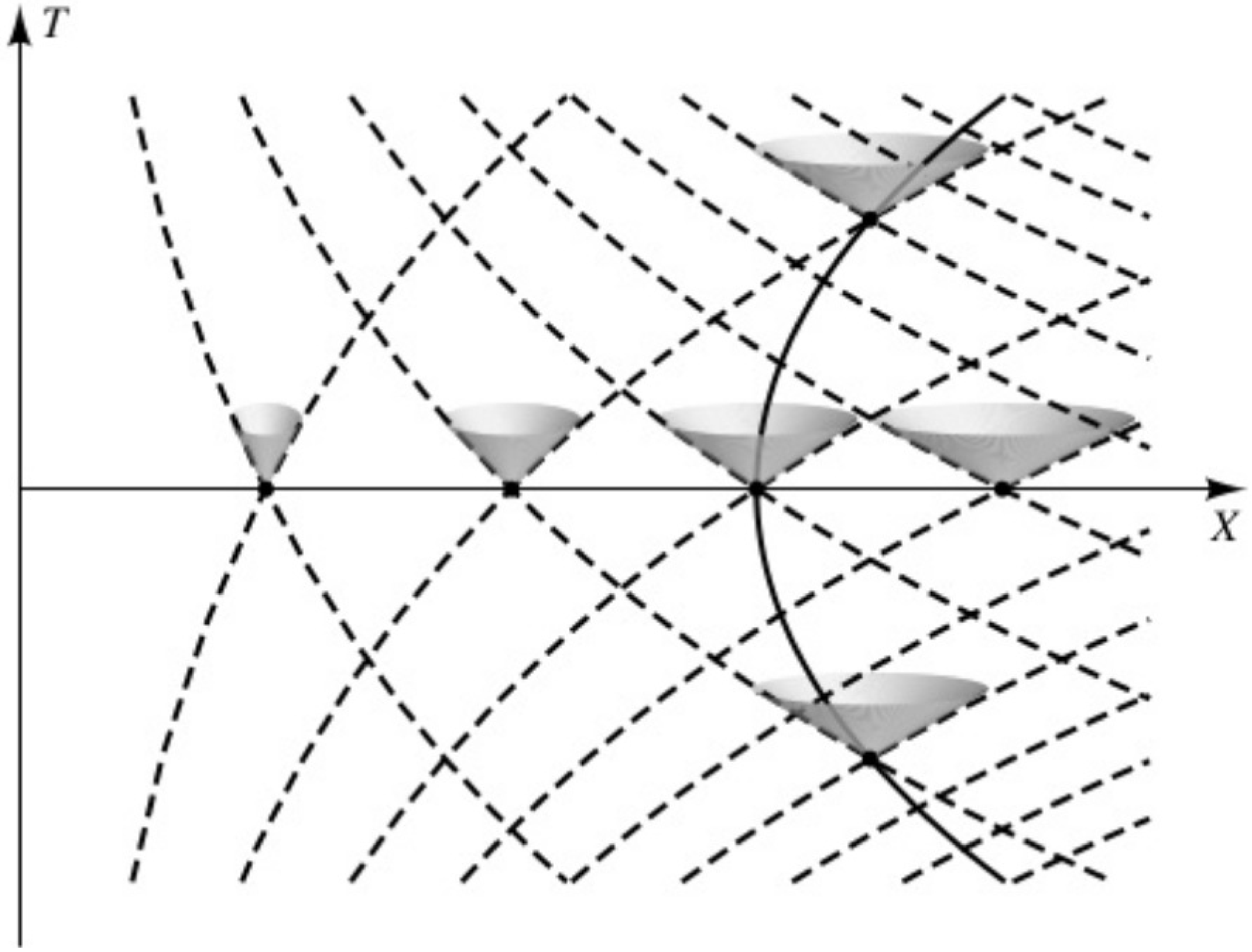
Null curves: $ds^2 = 0 \iff$

$$-x^2\,dt^2 + dx^2 = 0$$



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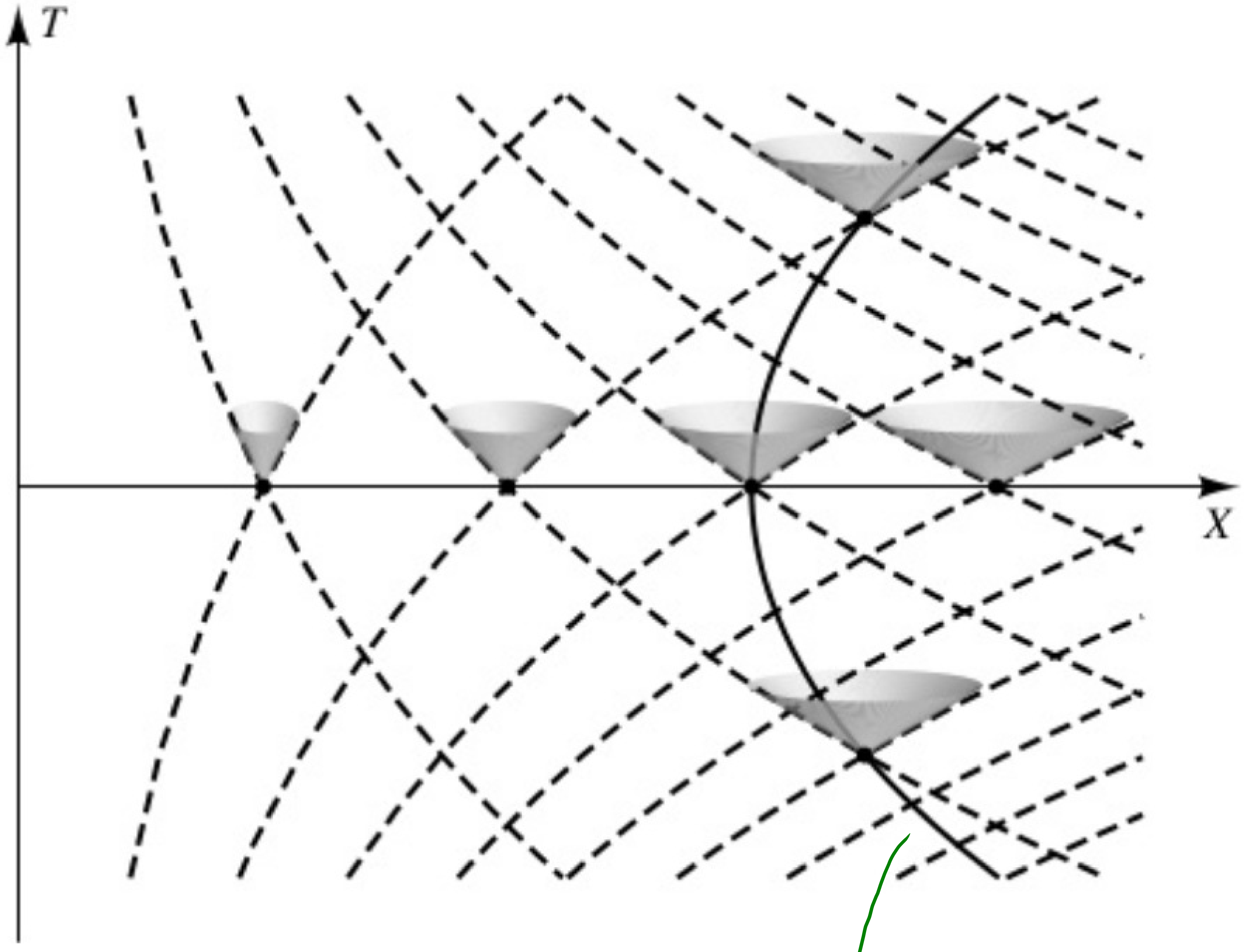
$$\Rightarrow x^2 dt^2 = dx^2 \Rightarrow \frac{dt}{dx} = \pm \frac{1}{x}$$

$$\Rightarrow t = \pm \ln \frac{x}{x_0}$$



a timelike curve: always in local light cone!

Examples Hartle 7.3

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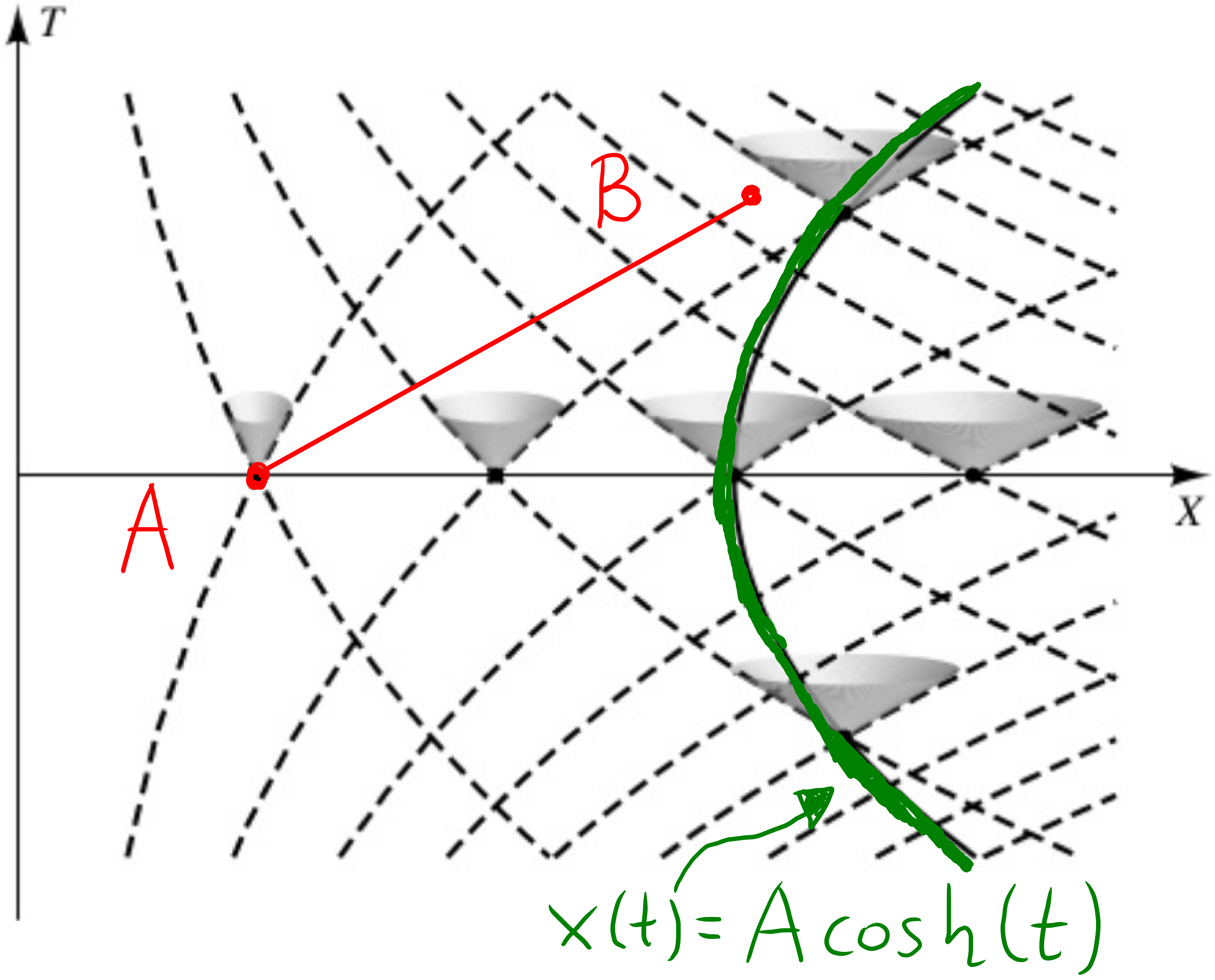

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- we can see why a $x = vt + v_0$ curve spacelike $\to$ timelike

- $x(t) = A\cosh(t)$ timelike : $\frac{dx}{dt} = A\sinh(t) < A\cosh(t) = x \Rightarrow \frac{dt}{dx} > \frac{1}{x}$

Examples Hartle 7.3

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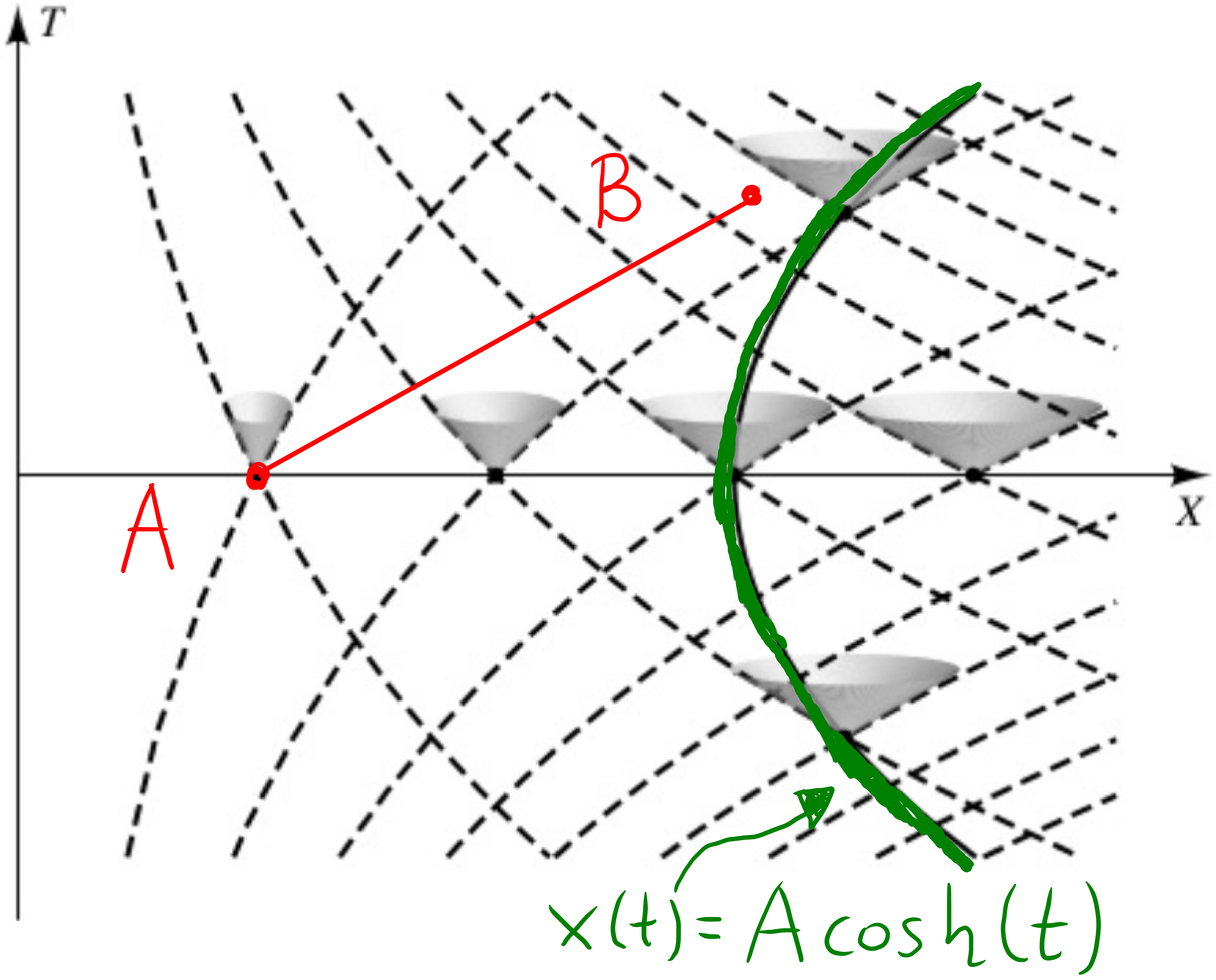

– we can see why a $x = vt + v_0$ curve spacelike $\rightarrow$ timelike

– $x(t) = A\cosh(t)$ timelike : $\frac{dx}{dt} = A\sinh(t) < A\cosh(t) = x \Rightarrow \frac{dt}{dx} > \frac{1}{x}$

$$\Rightarrow dx = A\sinh(t)dt \Rightarrow ds^2 = -A^2\cosh^2 t\, dt^2 + A^2 \sinh^2 t\, dt^2 = -A^2 dt^2 < 0 \Rightarrow d\tau = A\,dt$$

Example: A Wormhole (Hartle 7.7)

$$ds^2 = -dt^2 + dr^2 + (r^2+b^2)(d\theta^2 + \sin^2\theta \, d\varphi^2) \qquad b>0$$

Example: A Wormhole (Hartle 7.7)

$$ds^2 = -dt^2 + dr^2 + (r^2+b^2)(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad b>0$$

$b=0 \Rightarrow$ flat spacetime

$$ds^2 = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$= -dt^2 + dx^2 + dy^2 + dz^2$$

for $x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

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$t = \text{const} \Rightarrow ds^2 = dr^2 + (r^2+b^2)(d\theta^2 + \sin^2\theta \, d\varphi^2)$

$\hookrightarrow$ spacelike surface

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stay at $\theta = \frac{\pi}{2}$, $d\theta = 0$

$\rightsquigarrow d\Sigma^2 = dr^2 + (r^2+b^2)\, d\varphi^2$

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Embed in Euclidean (flat) 3d space:
Find a 2-d surface w/ induced metric $d\Sigma^2$

Consider cylindrical coordinates in 3d:

$$d\sigma^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$ Euclidean flat metric in $\mathbb{R}^3$

$$\rightsquigarrow d\Sigma^2 = dr^2 + (r^2 + b^2)\, d\varphi^2$$

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Find a 2-d surface w/ induced metric $d\Sigma^2$

Consider cylindrical coordinates in 3d:

$$d\sigma^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$ Euclidean flat metric in $\mathbb{R}^3$

The surface $\rho(z,\varphi) = b\cosh\left(\frac{z}{b}\right)$ will have $d\Sigma^2$ if we choose

$\rho^2 = r^2 + b^2 \qquad z(r) = b\sinh^{-1}\left(\frac{r}{b}\right)$, φ the same

---

$$\rightsquigarrow d\Sigma^2 = dr^2 + (r^2+b^2)\,d\varphi^2$$

Embed in Euclidean (flat) 3d space:

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Indeed

$$d\sigma^2 = \underbrace{\left[\frac{\partial\rho}{\partial r}\, dr\right]^2}_{d\rho^2} + \underbrace{\left[r^2+b^2\right]}_{\rho^2} d\varphi^2 + \underbrace{\left[\frac{\partial z}{\partial r}\, dr\right]^2}_{dz^2}$$

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$$\left(\frac{\partial z}{\partial r}\right)^2 = \frac{b^2}{r^2+b^2} \qquad \left(\frac{\partial\rho}{\partial r}\right)^2 = \frac{r^2}{r^2+b^2} \qquad \Rightarrow \qquad \left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial\rho}{\partial r}\right)^2 = 1$$

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Notice that at $r=0$ $d\Sigma^2 = b^2 d\varphi^2$, a circle, not a point!

(in fact in 3d it corresponds to a sphere $ds^2 = b^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$)

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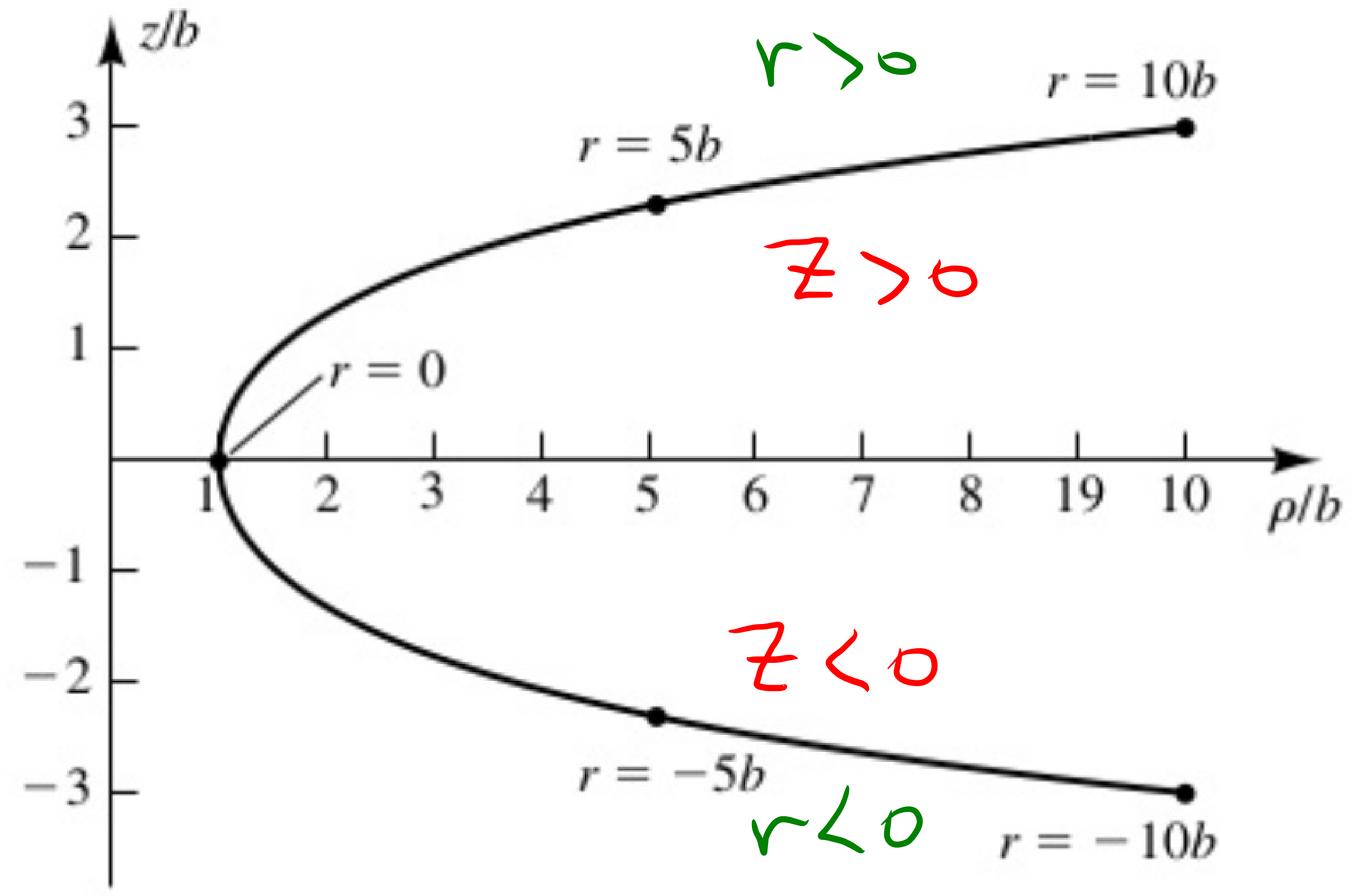

Hartle, Fig 7.4

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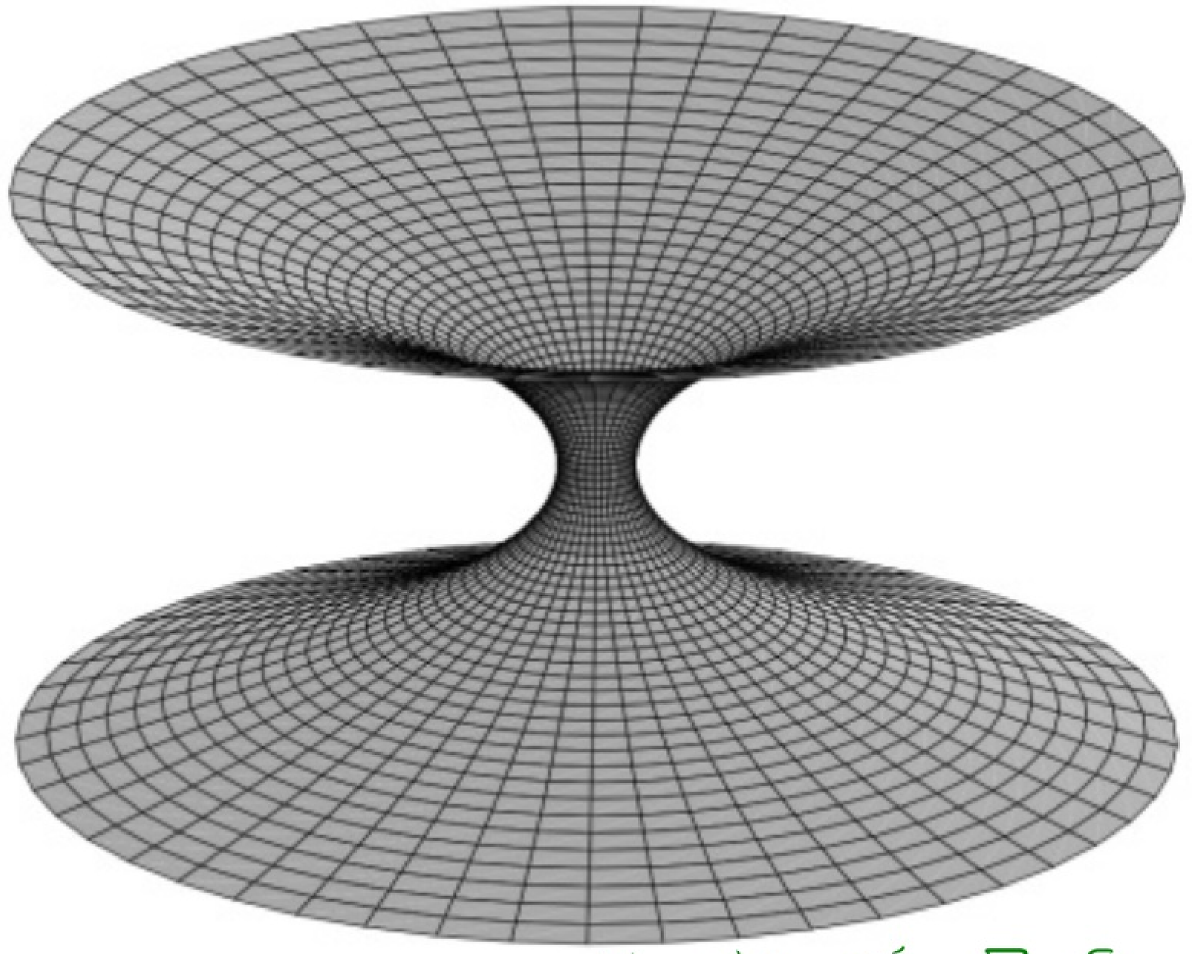
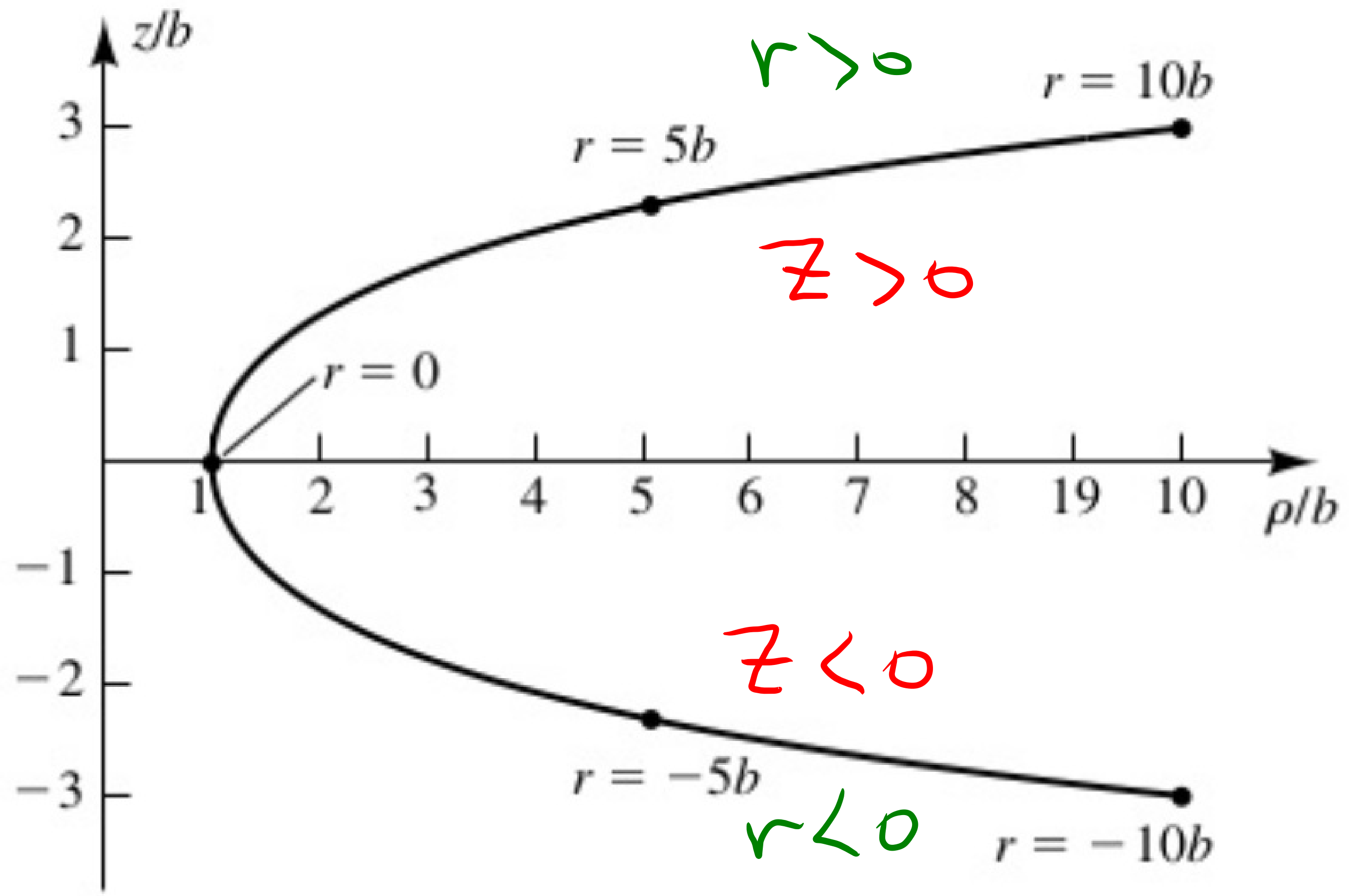


Hartle, Fig 7.5



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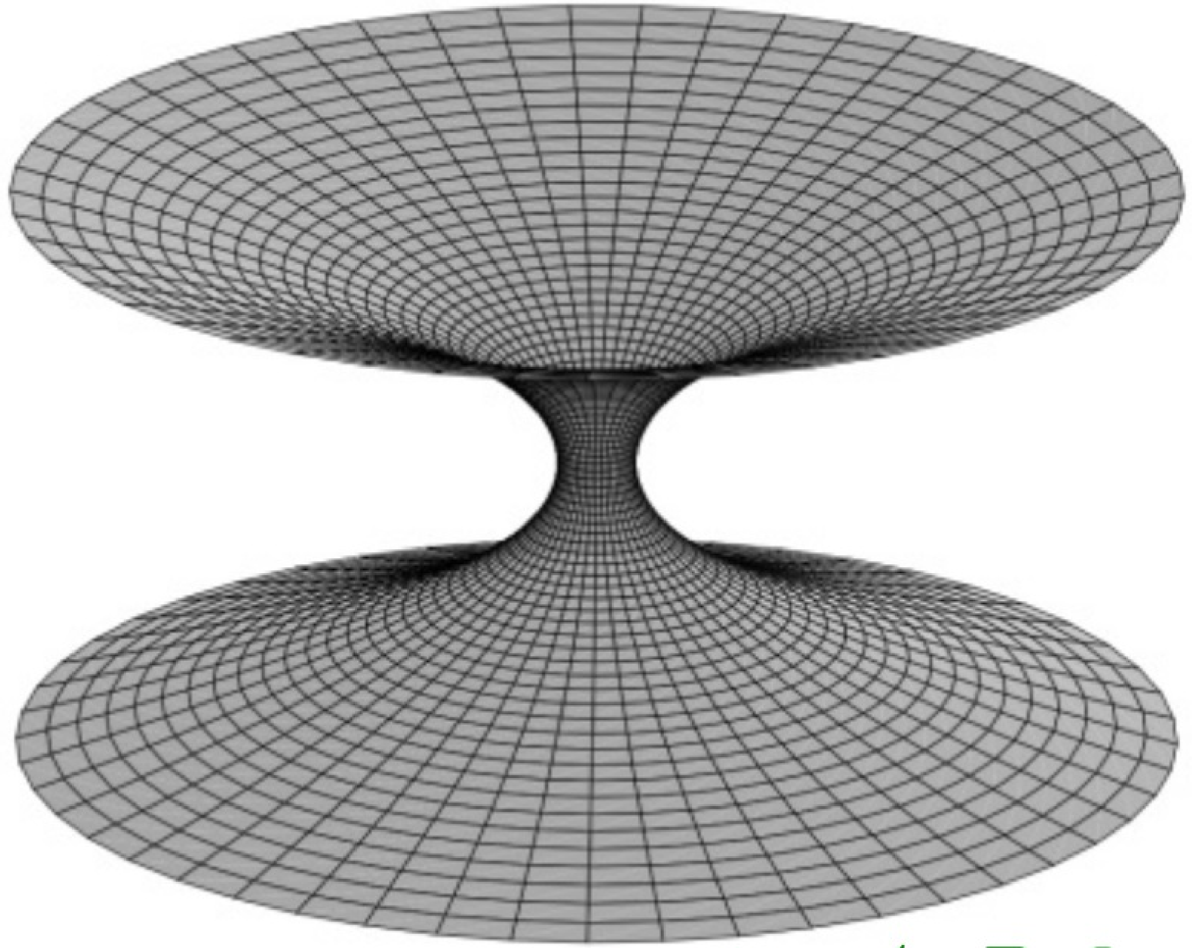

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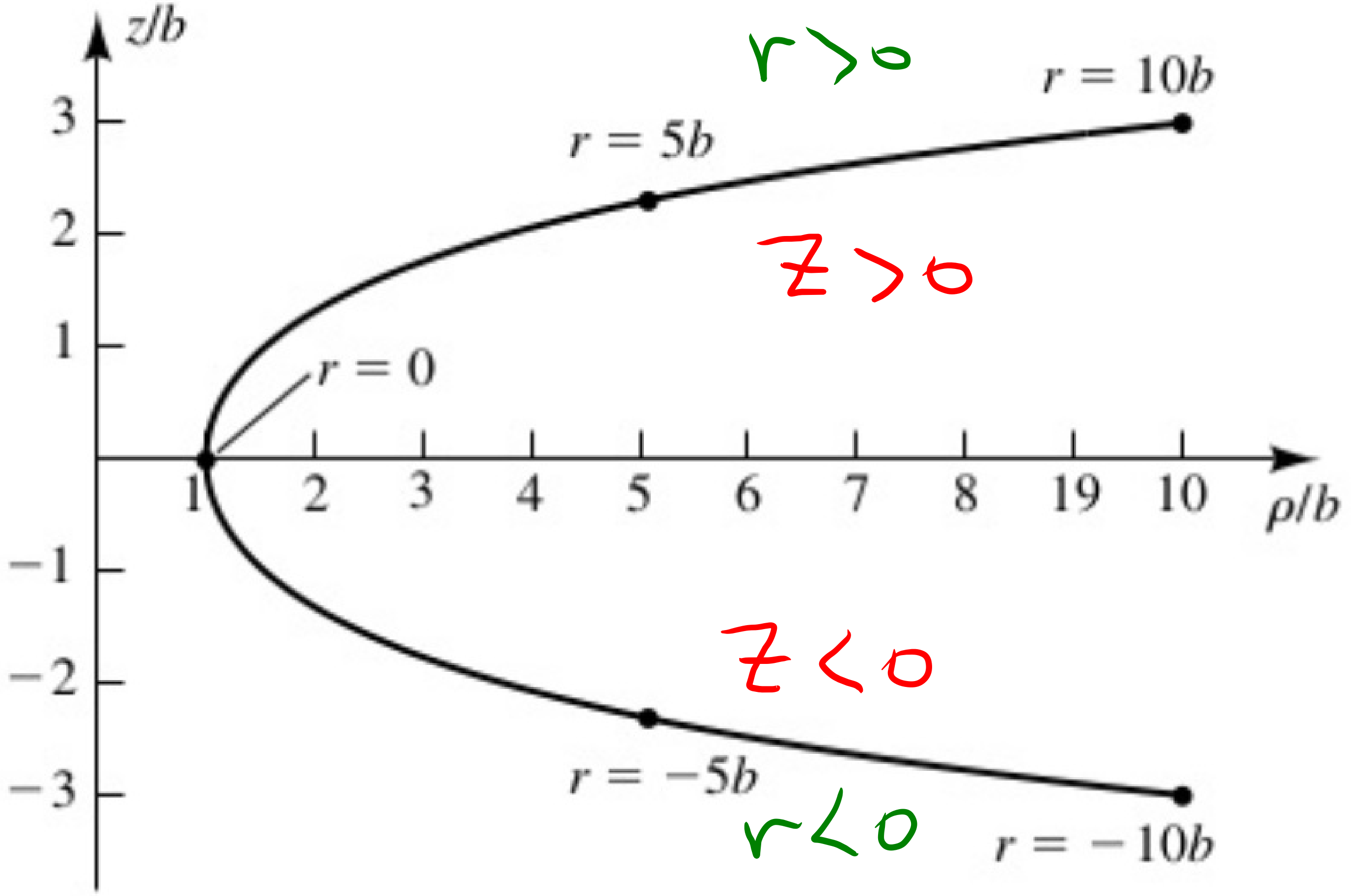

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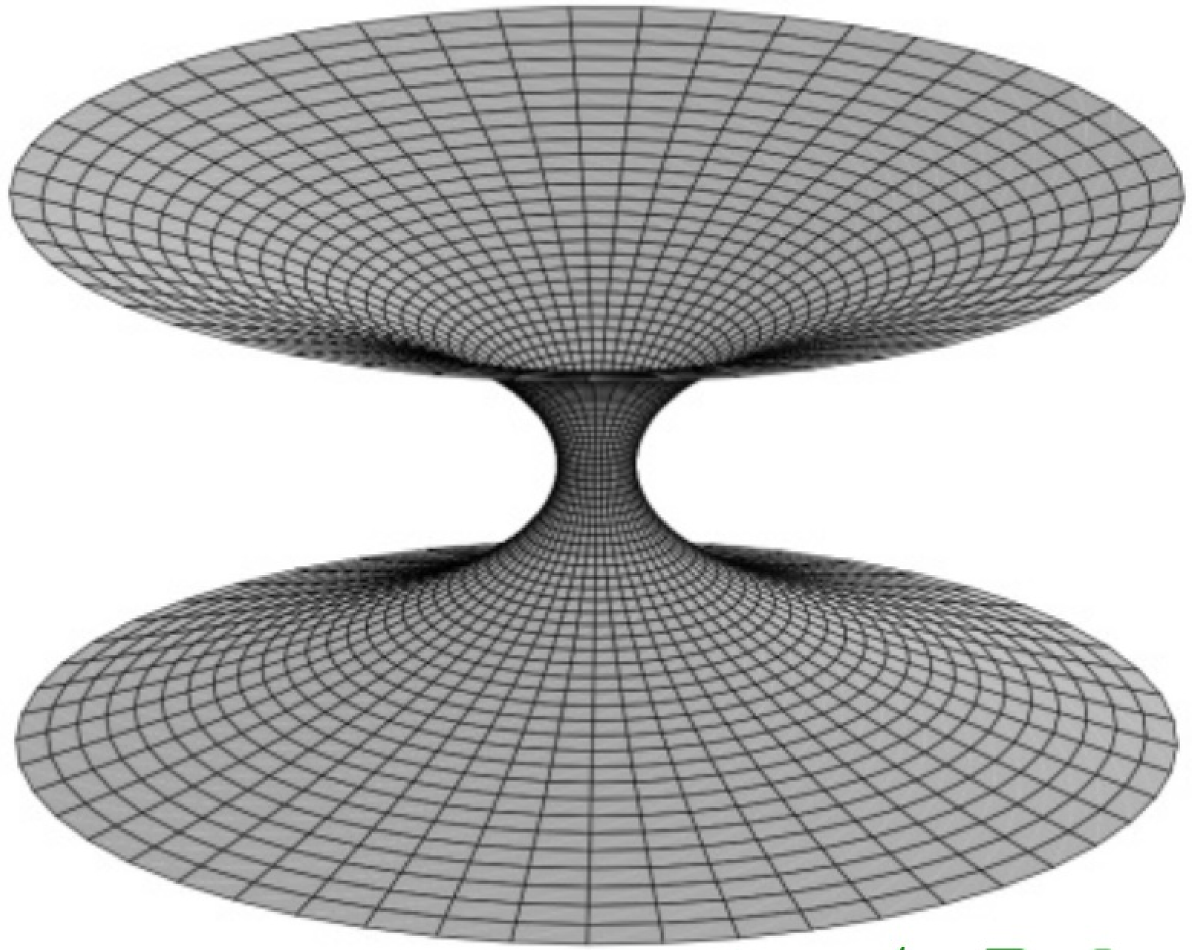

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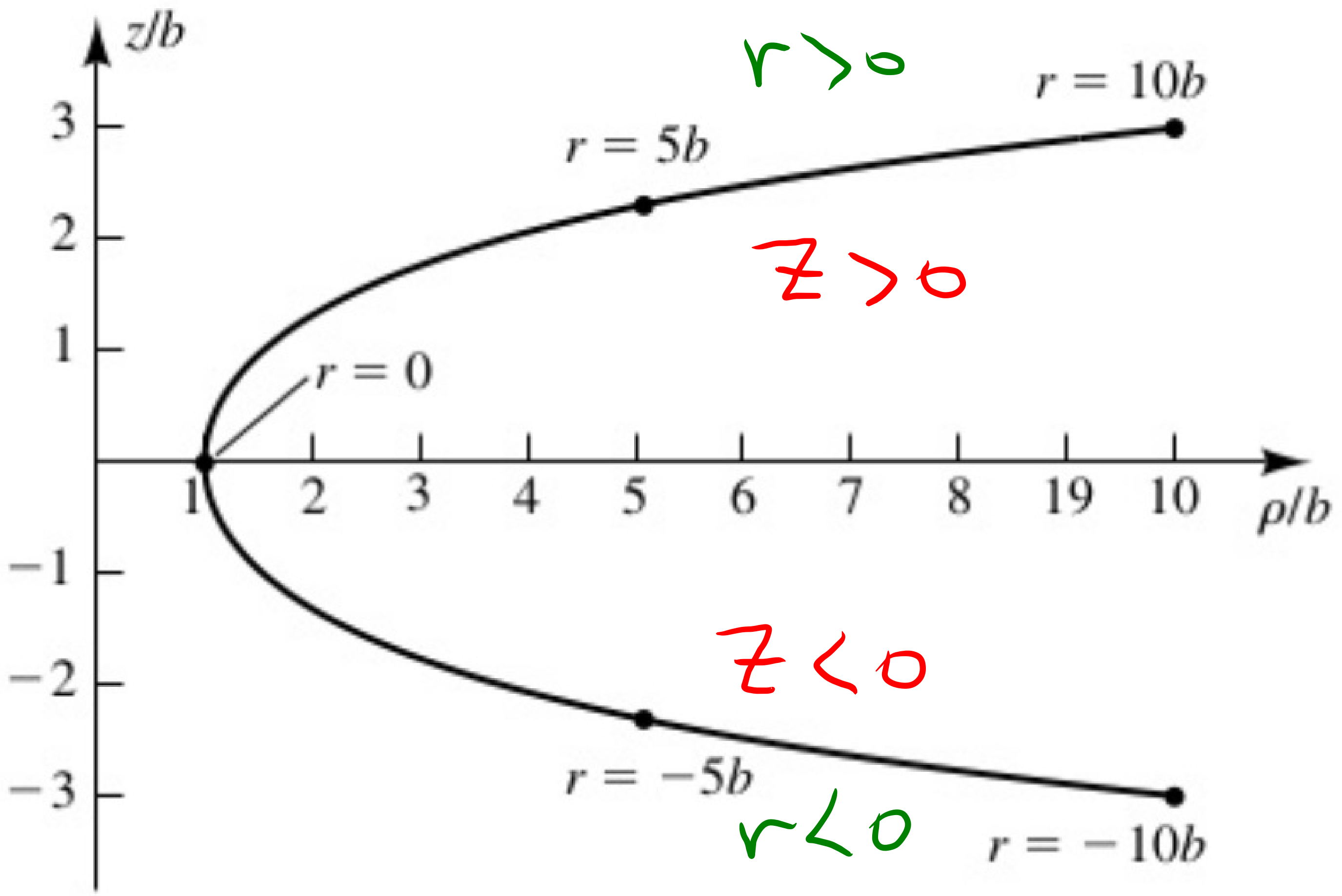

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constant for fixed t

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- fixed point in space: $dx = dy = dz = 0$ $\Rightarrow$ $ds^2 = -dt^2$ comoving frame/observer

t: their proper time

a "galaxy" is approximately sitting at fixed (x,y,z) at all times

$\rightsquigarrow$ a comoving frame

$\rightsquigarrow$ t measures the "local time", same for all such "galaxies"

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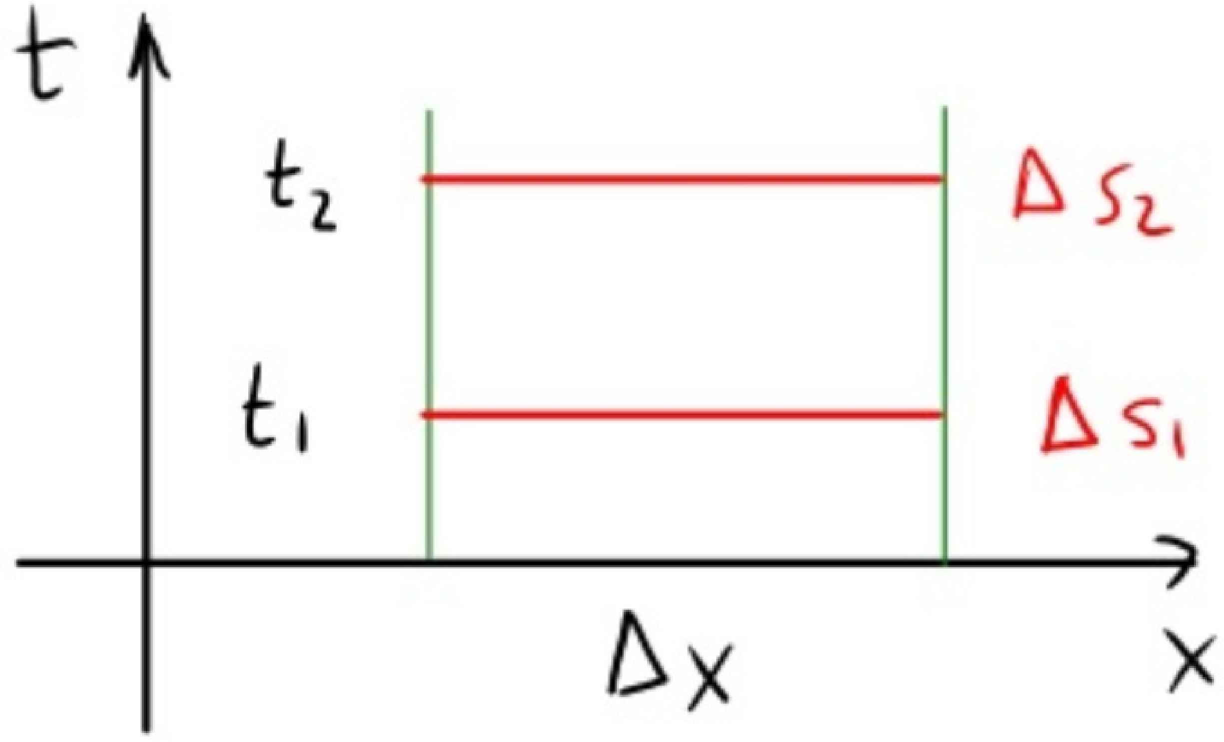
$$\left.\begin{aligned}\Delta s_1 &= a_1\,\Delta x\\ \Delta s_2 &= a_2\,\Delta x\end{aligned}\right\}\Rightarrow \frac{\Delta s_1}{\Delta s_2} = \frac{a_1}{a_2}$$



* typically $a(t) = t^q \quad 0<q<1$

$\Rightarrow \lim_{t\to 0} a(t) = 0 \quad \Rightarrow$ singularity at $t=0$

spacetime ends @ finite time in the past

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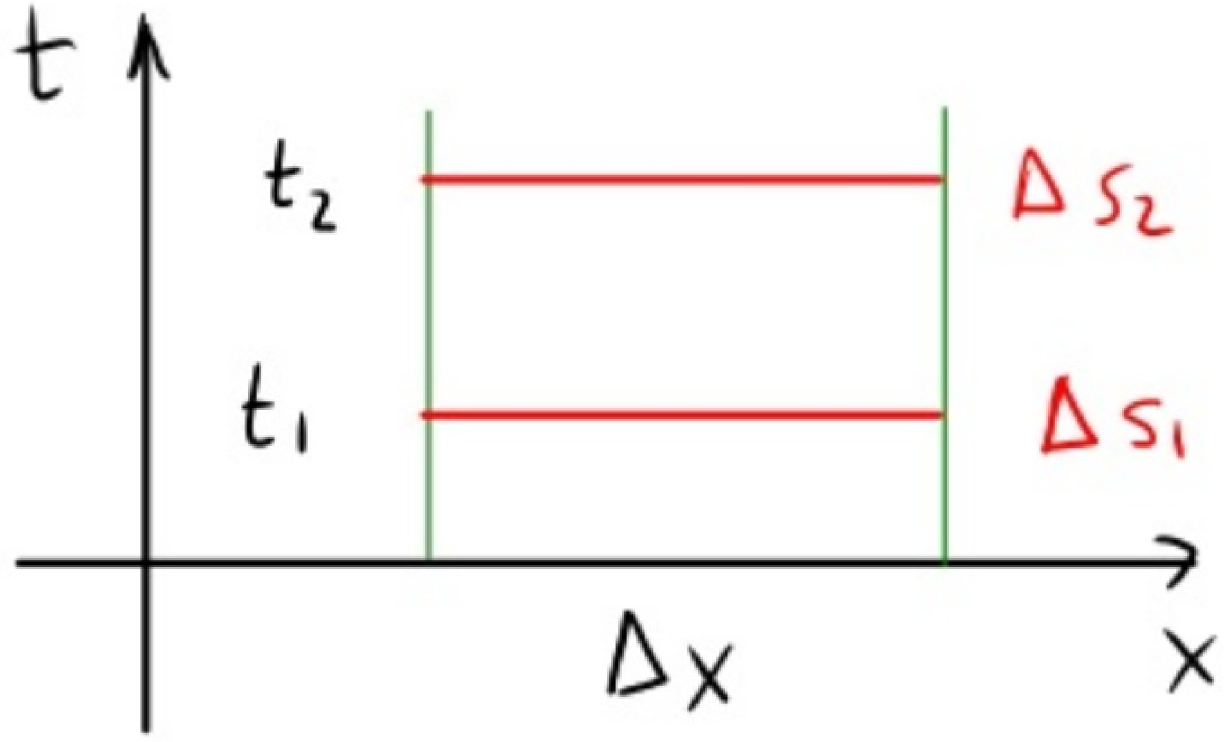

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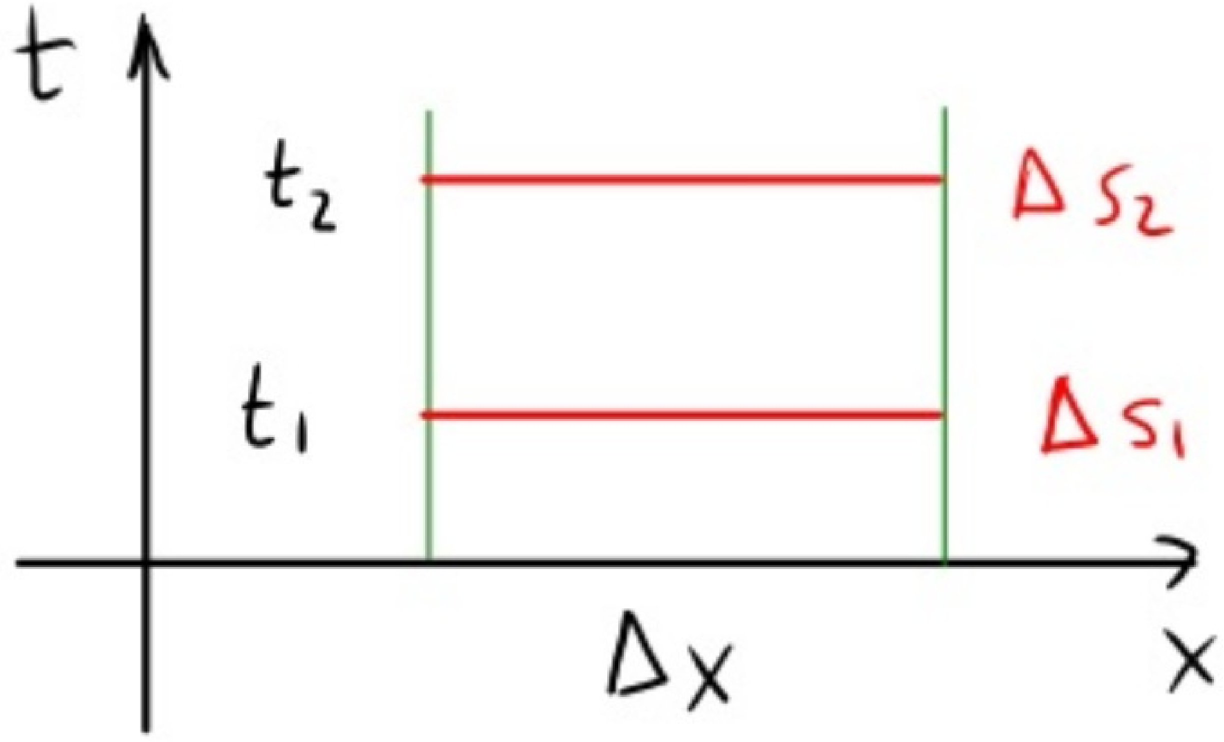

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$$\Rightarrow t = A(\pm x - x_0)^r \qquad r = \frac{1}{1-q}$$

$r=3$ radiation dominated era

$r=2$ matter " "

* typically $a(t) = t^q \quad 0<q<1$

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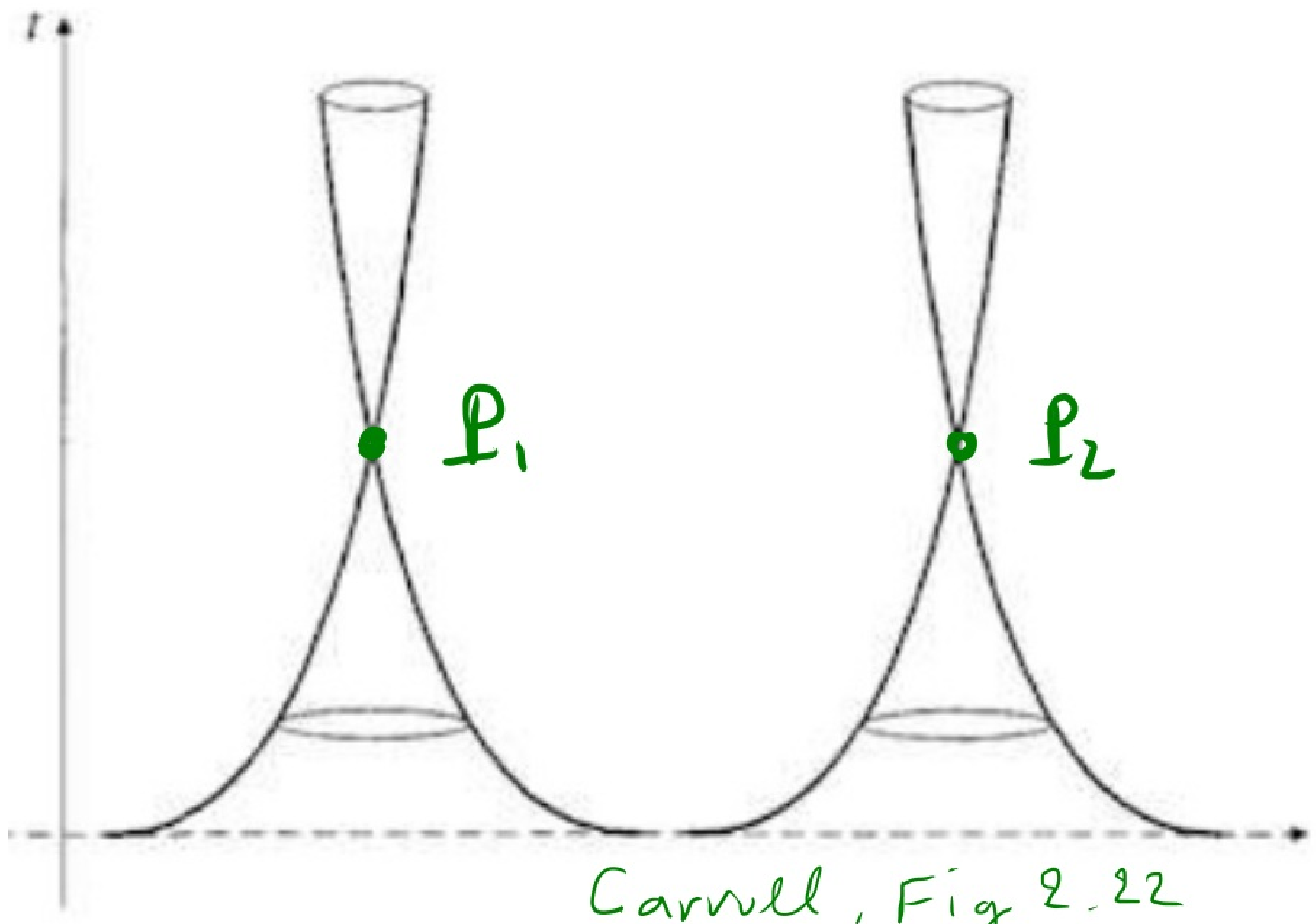
* The past of P_1, P_2 non overlapping, events define horizons. Events outside horizons have no causal contact

* Light cones tangent to $t=0$ (singularity)



Carroll, Fig 2.22

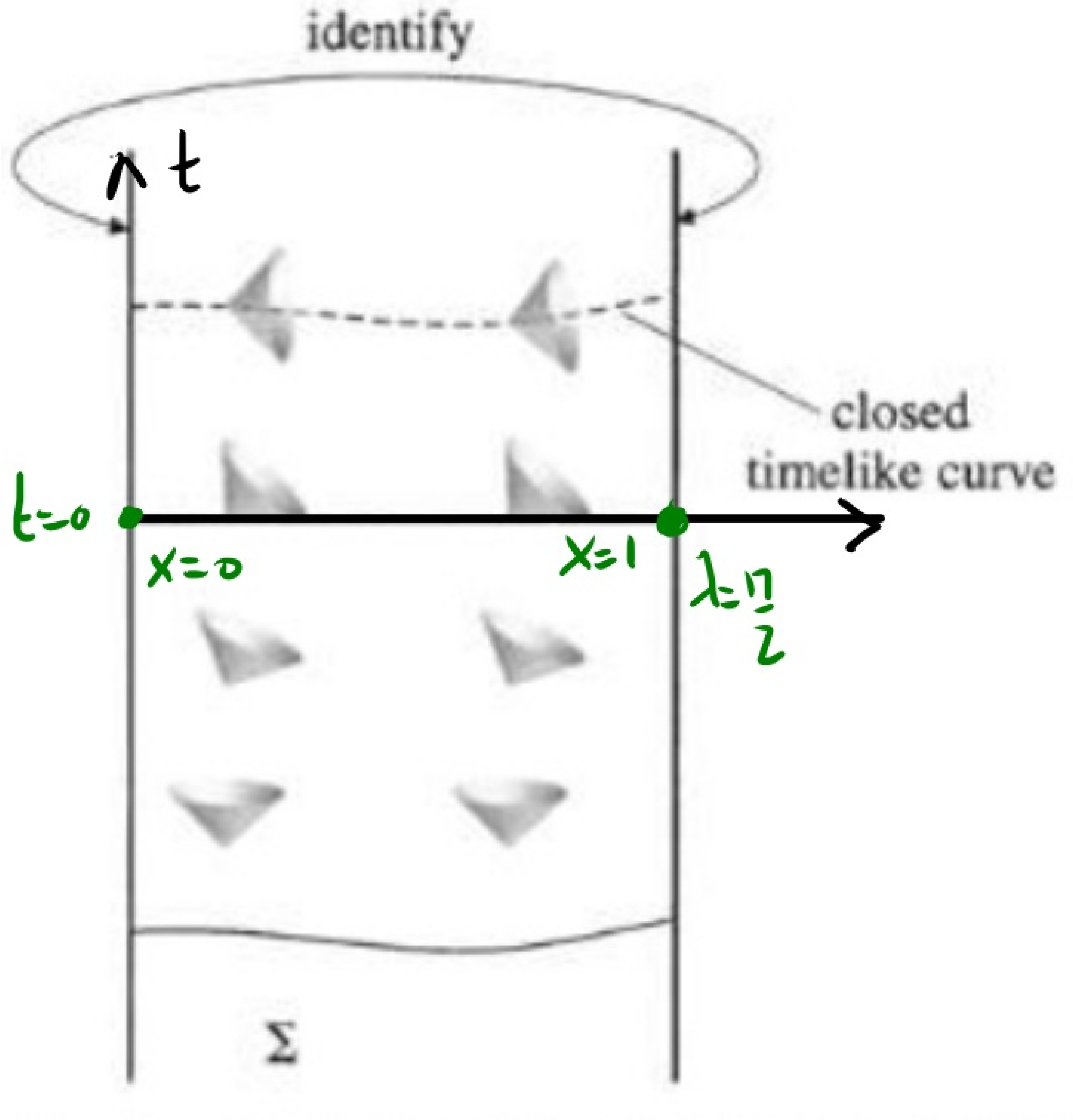
Example: Misner space (Carroll §2.7)

$$ds^2 = -\cos\lambda\, dt^2 - \sin\lambda\,[dt\,dx + dx\,dt] + \cos\lambda\, dx^2$$

$$t = \cot\lambda$$

$$-\infty < t < +\infty \qquad 0 < \lambda < \pi$$

topology: $\mathbb{R} \times S^1$: $(t, x) \sim (t, x+1)$



Carrol, Fig 2.25

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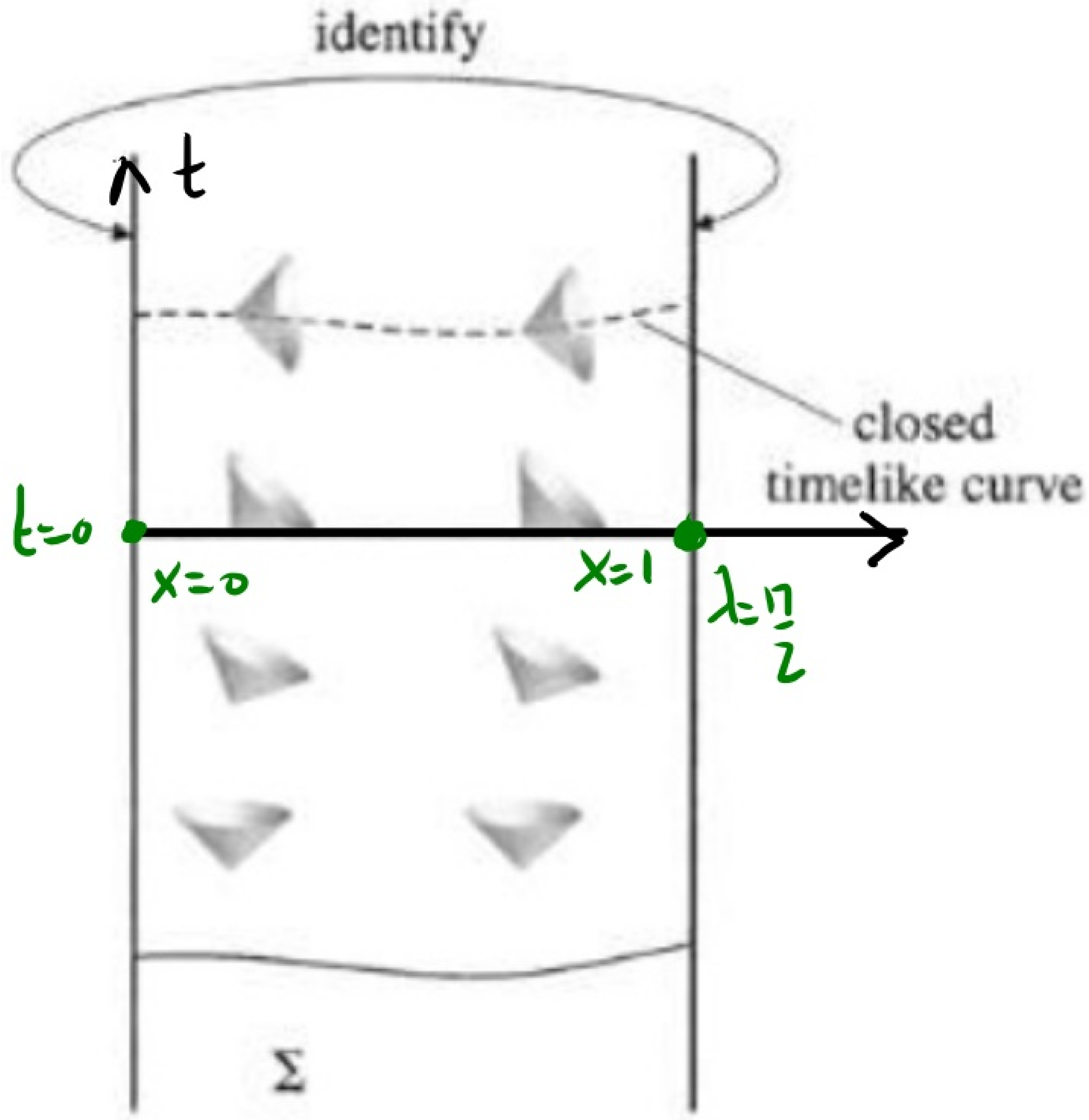
identify



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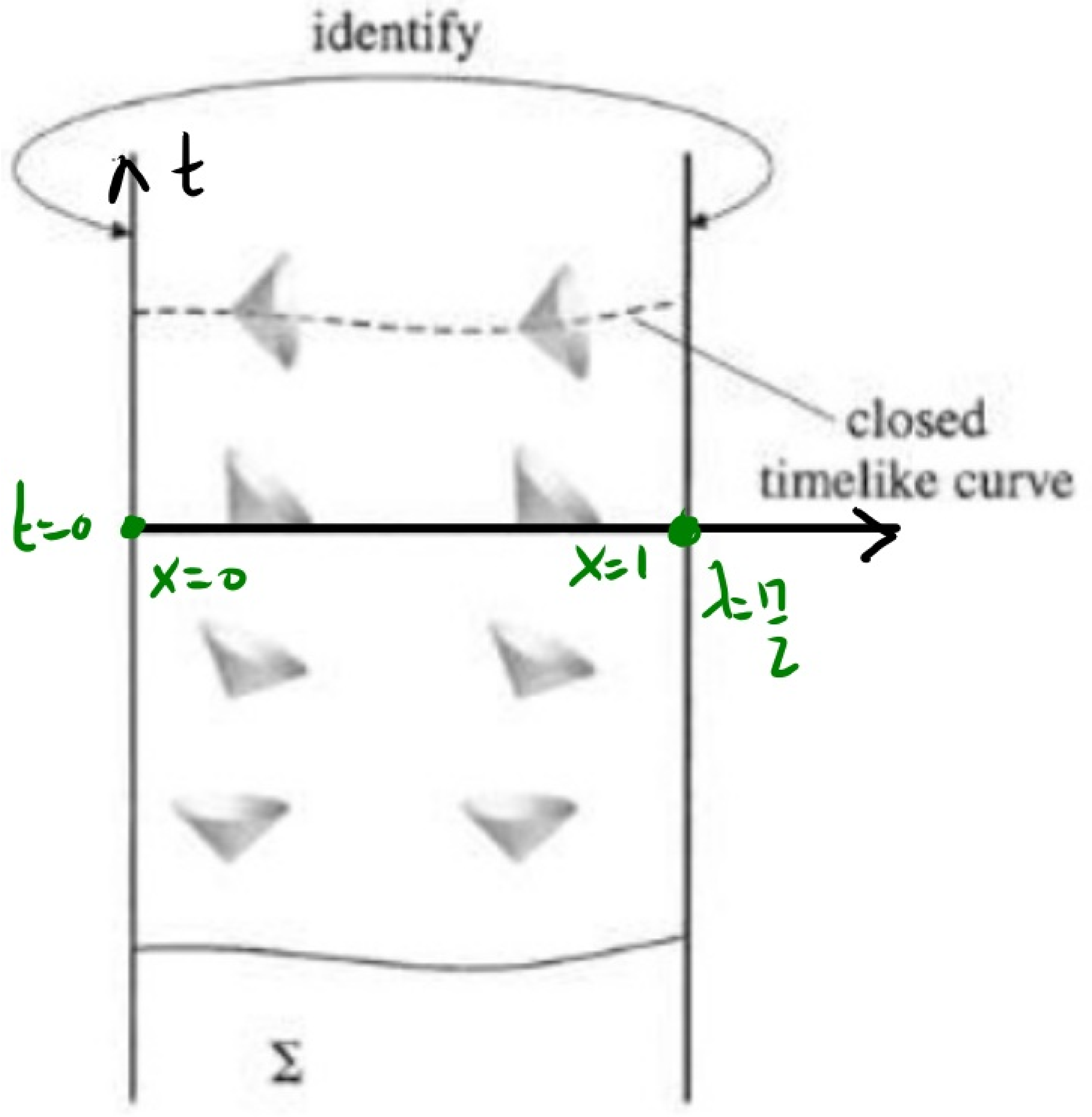

Carrol, Fig 2.25

non-degenerate:

$$g = \begin{vmatrix} -\cos\lambda & -\sin\lambda \\ -\sin\lambda & \cos\lambda \end{vmatrix} = -\cos^2\lambda - \sin^2\lambda = -1 \neq 0$$

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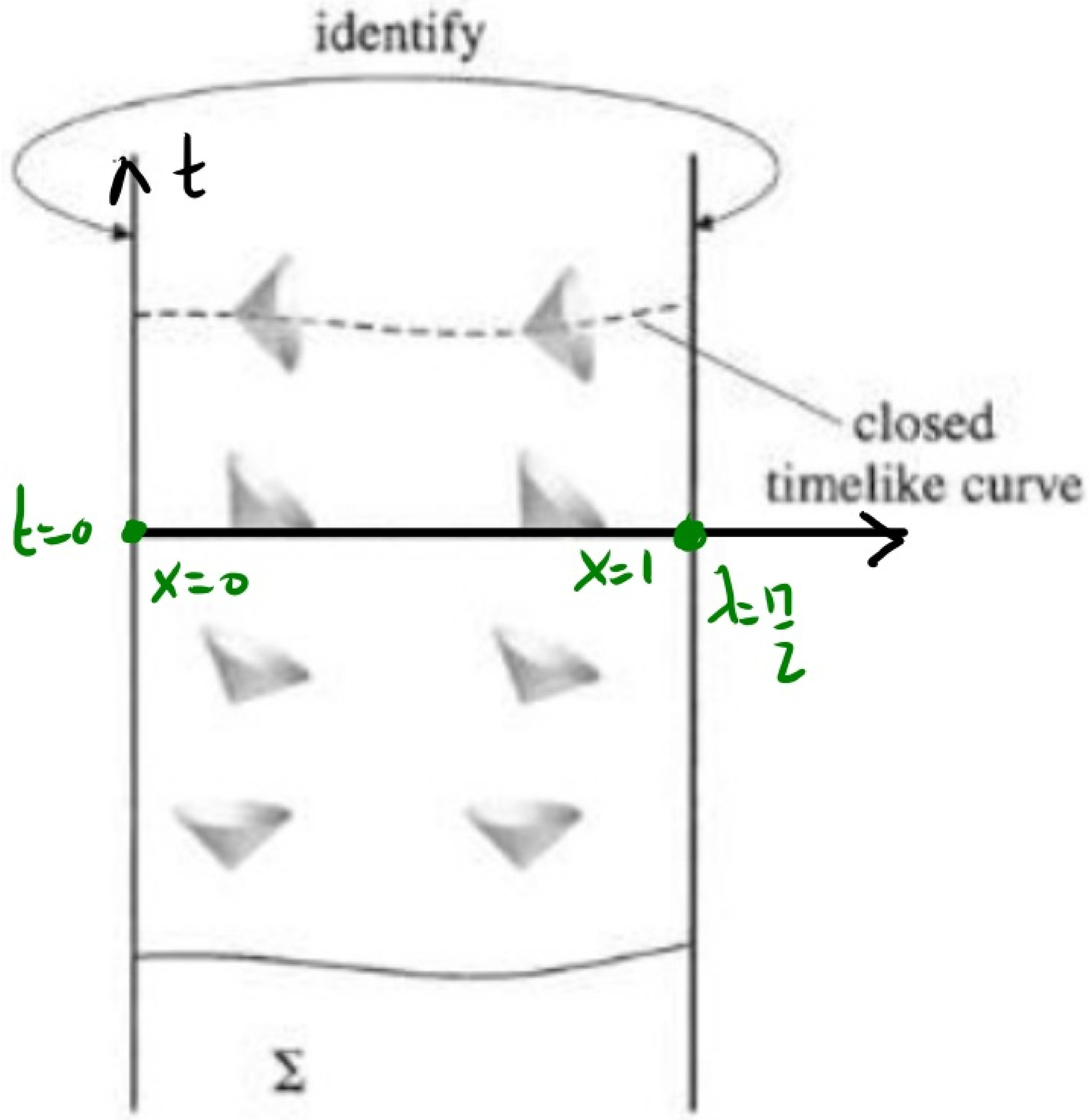

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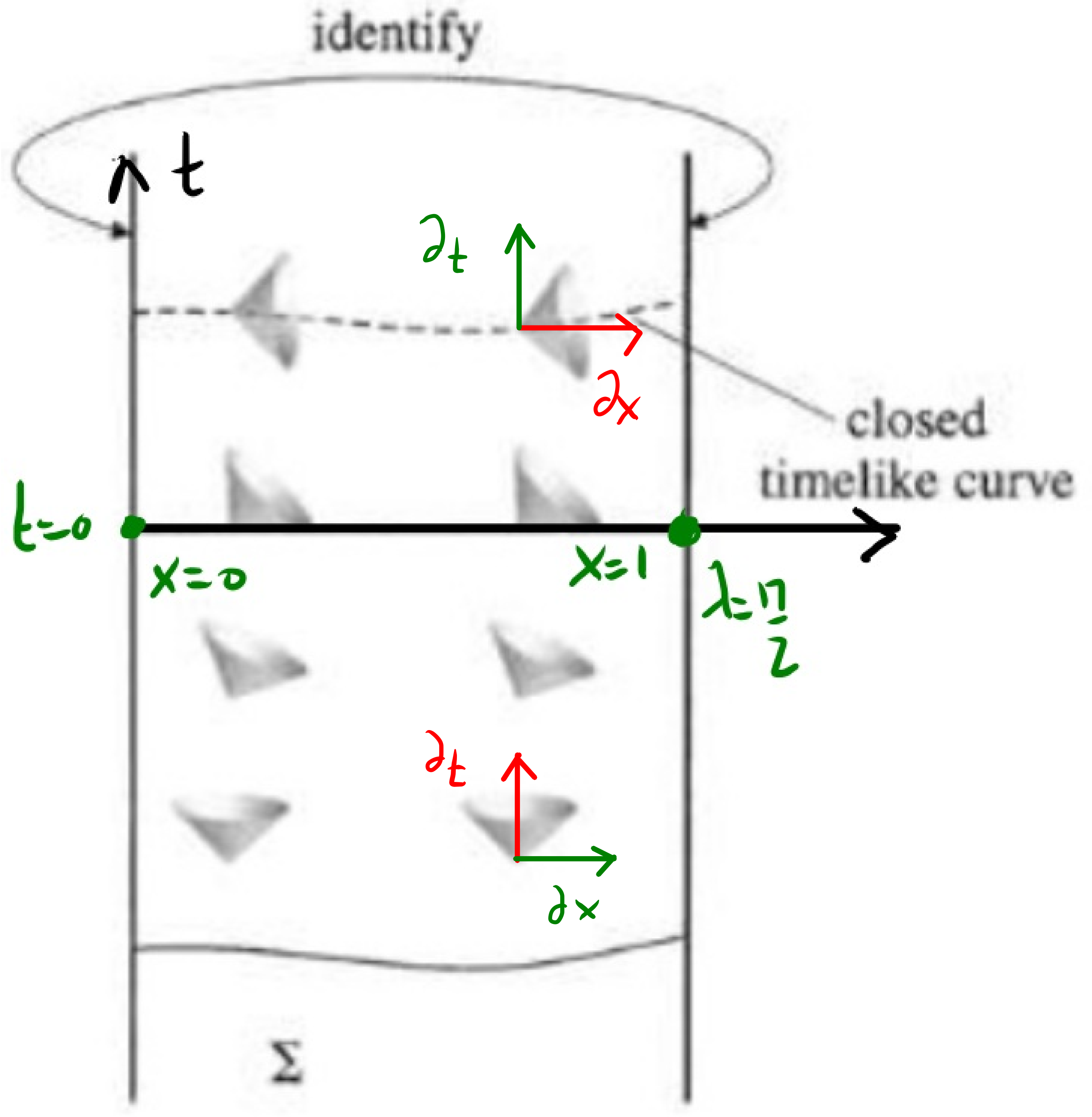

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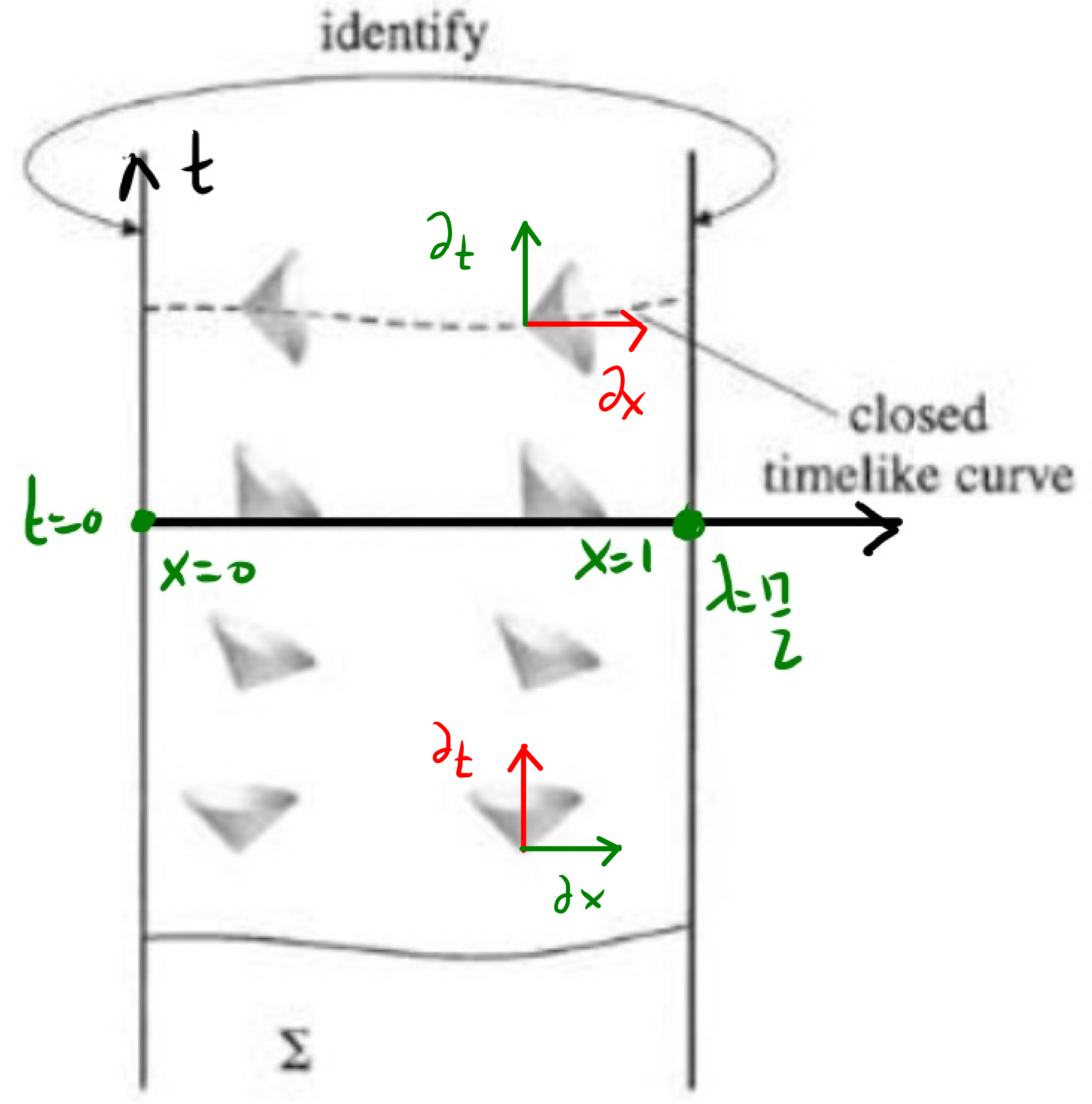

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* For $\lambda > \frac{\pi}{2}$ $(t>0)$, ∂_x is timelike: direction of time in x-direction!

$\Rightarrow$ $\exists$ timelike closed curves!

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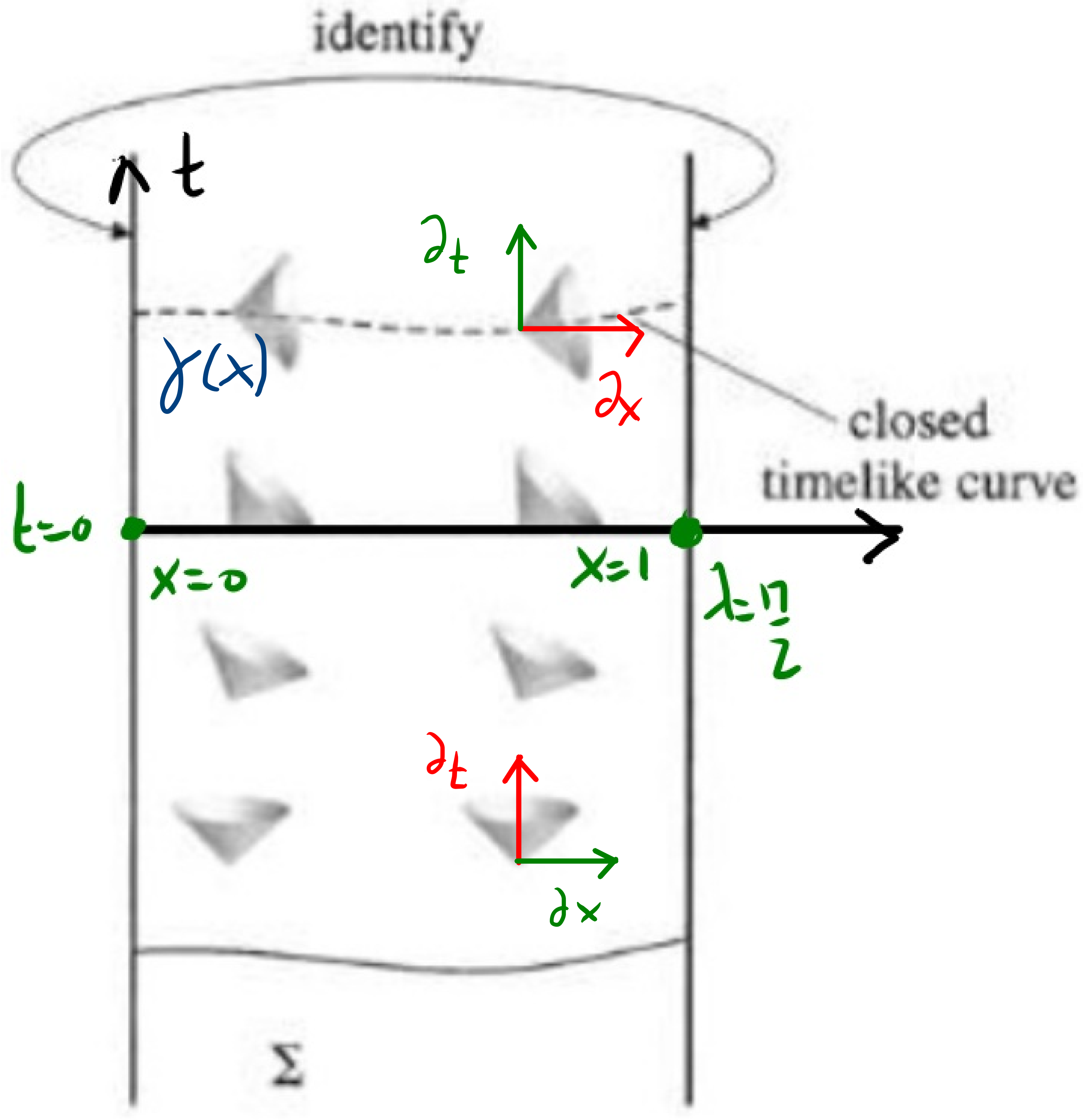

Carrol, Fig 2.25

* the events on the closed timelike curve $\gamma(x)$ not causally connected to the events on Σ

or

initial data on Σ cannot determine events on $\gamma(x)$

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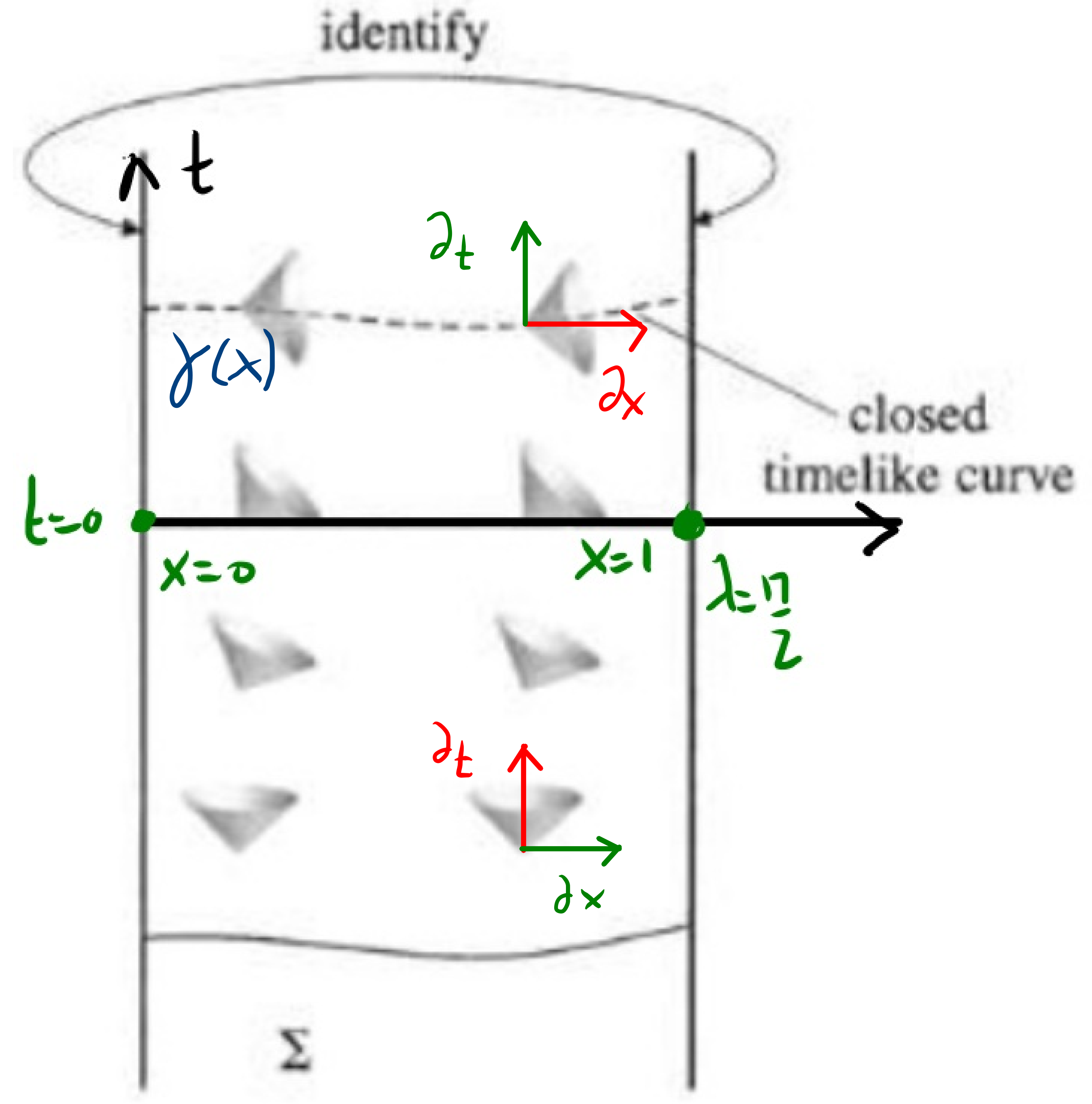

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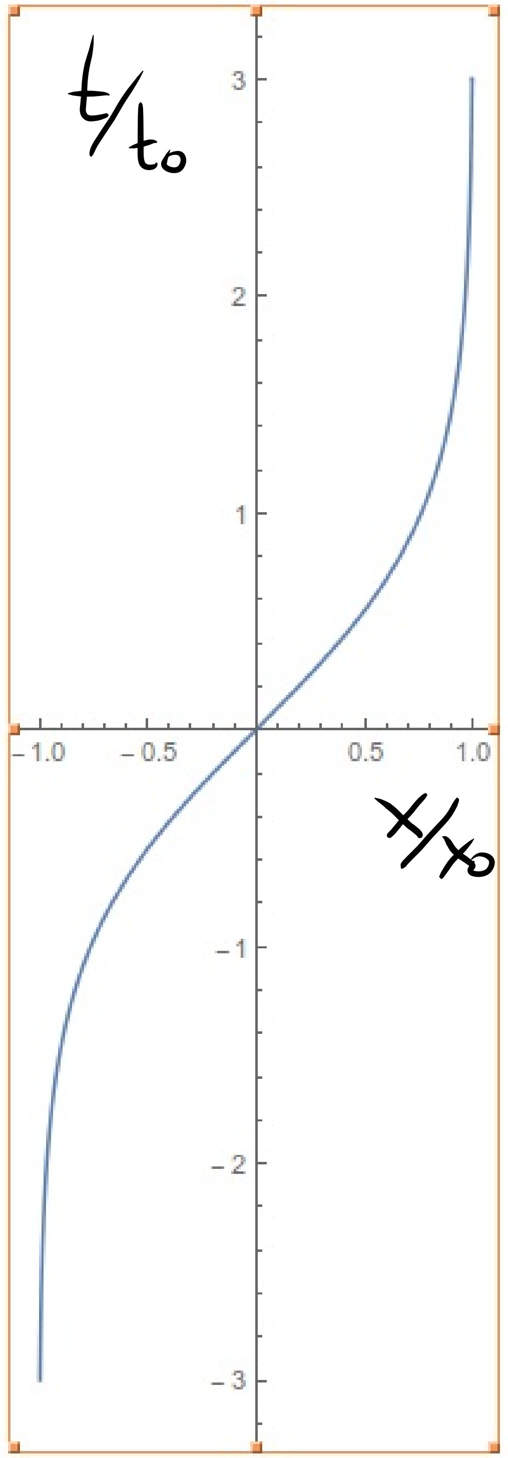
$\rightsquigarrow$ a "Cauchy horizon" forms at $t=0$, makes initial value problem ill defined

(don't worry, not a solution to Einstein equations ...)

Example: Travelling faster than c (but not faster than light) (Hartle 7.4)

Let $x_s = x_0 \tanh \frac{t}{t_0}$ the trajectory of a spaceship



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Alcubierre, Class Quant. Grav 11, L73, 1994

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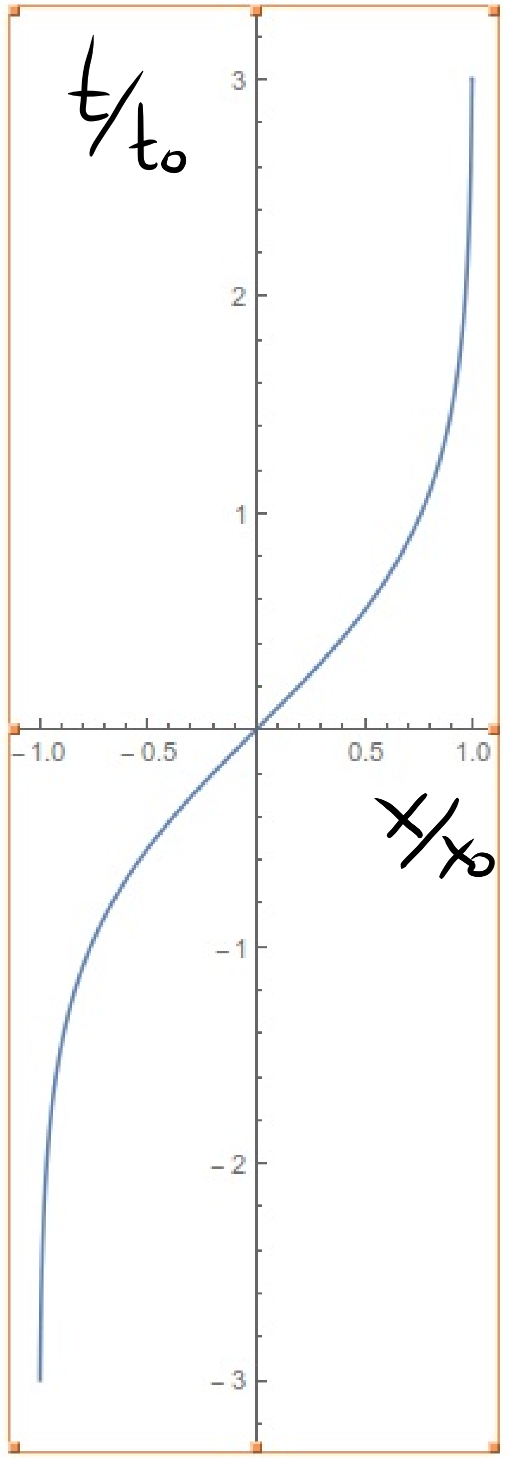
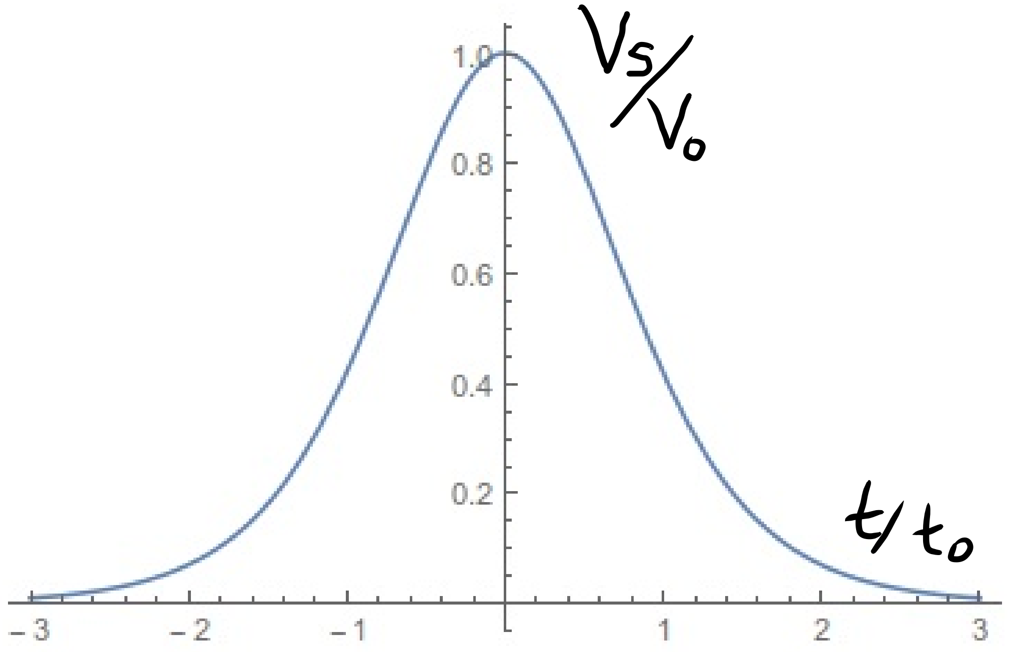


$$V_s(t) = \frac{dx_s}{dt} = \frac{x_0}{t_0} \frac{1}{\cosh^2\left(\frac{t}{t_0}\right)} = V_0 \frac{1}{\cosh^2\left(\frac{t}{t_0}\right)} \qquad V_0 \equiv \frac{x_0}{t_0}$$



V_0 can be chosen to be $V_0 > 1$

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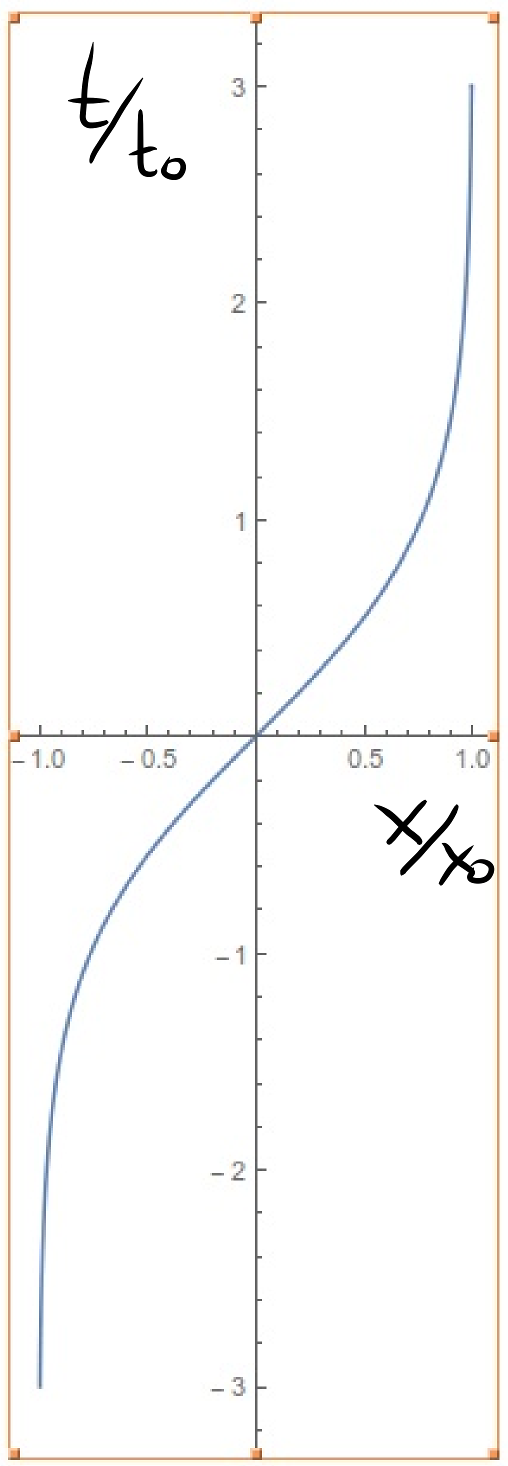

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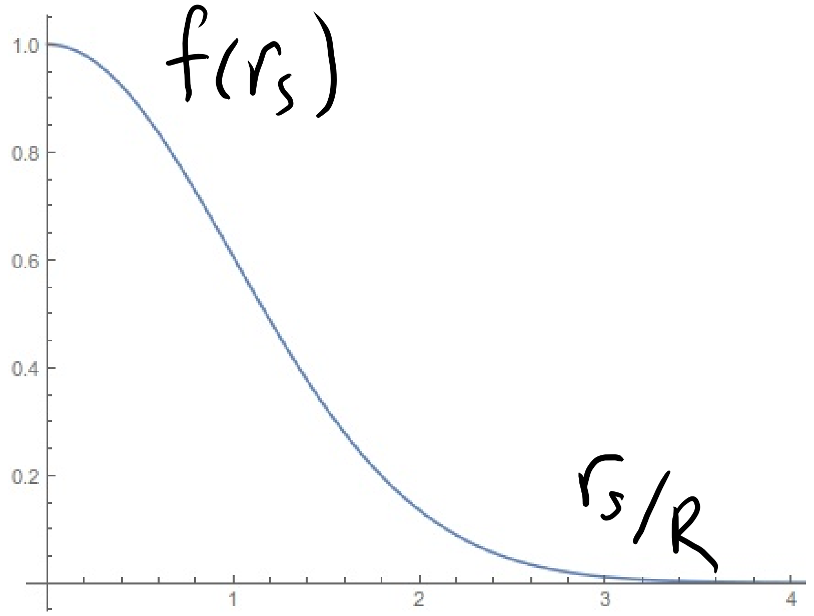
$$r_s^2 = (x - x_s(t))^2 + y^2 + z^2 = \left(\begin{array}{l} \text{distance of event } (t, x, y, z) \\ \text{from } (t, x_s(t), 0, 0) \end{array} \right)$$

$$f(r_s) = e^{-\frac{r_s^2}{2R^2}} \qquad R \ll x_0$$



$$\rightsquigarrow f(0) = 1 \,, \quad f(r_s \gg R) \approx 0$$

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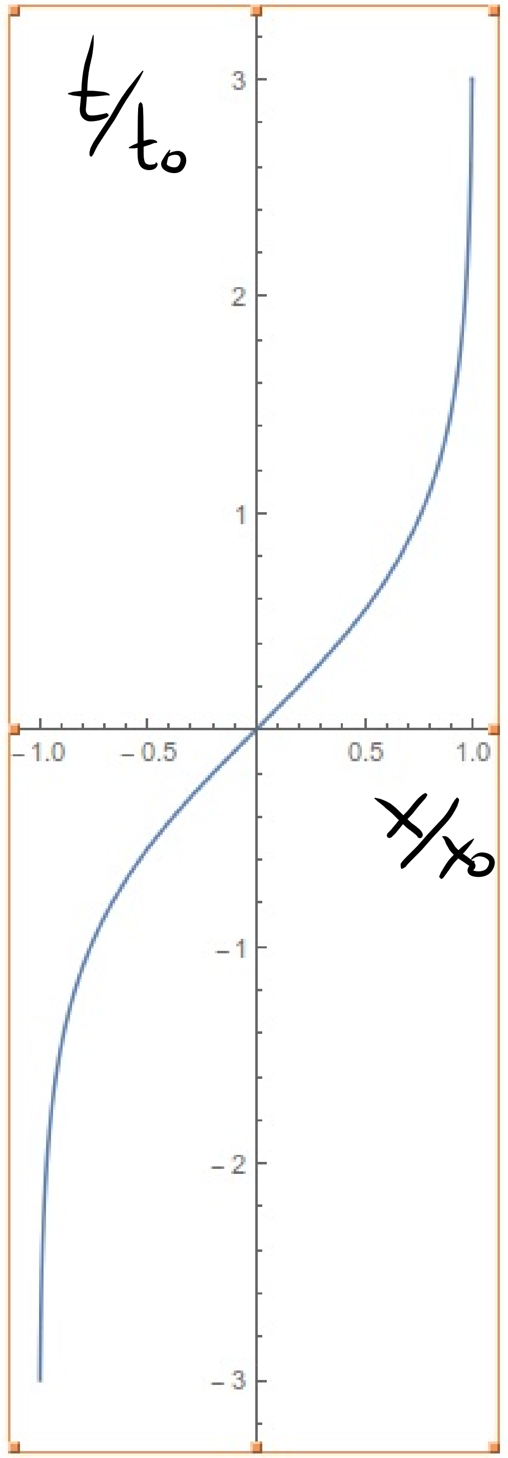

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Take events for $dy = dz = 0$

$$ds^2 = (-1 + V_s^2 f^2)dt^2 - V_s f\,(dt\,dx + dx\,dt) + dx^2$$

$$(g_{\mu\nu}) = \begin{pmatrix} -1 + V_s^2 f^2 & -V_s f \\ -V_s f & 1 \end{pmatrix} \Rightarrow \det g = -1 + V_s^2 f^2 - V_s^2 f^2 = -1$$

eigenvalues: $\frac{1}{2}\left(V_s^2 f^2 - \sqrt{4 + V_s^4 f^4}\right) < 0, \quad \frac{1}{2}\left(V_s^2 f^2 + \sqrt{4 + V_s^4 f^4}\right) > 0$

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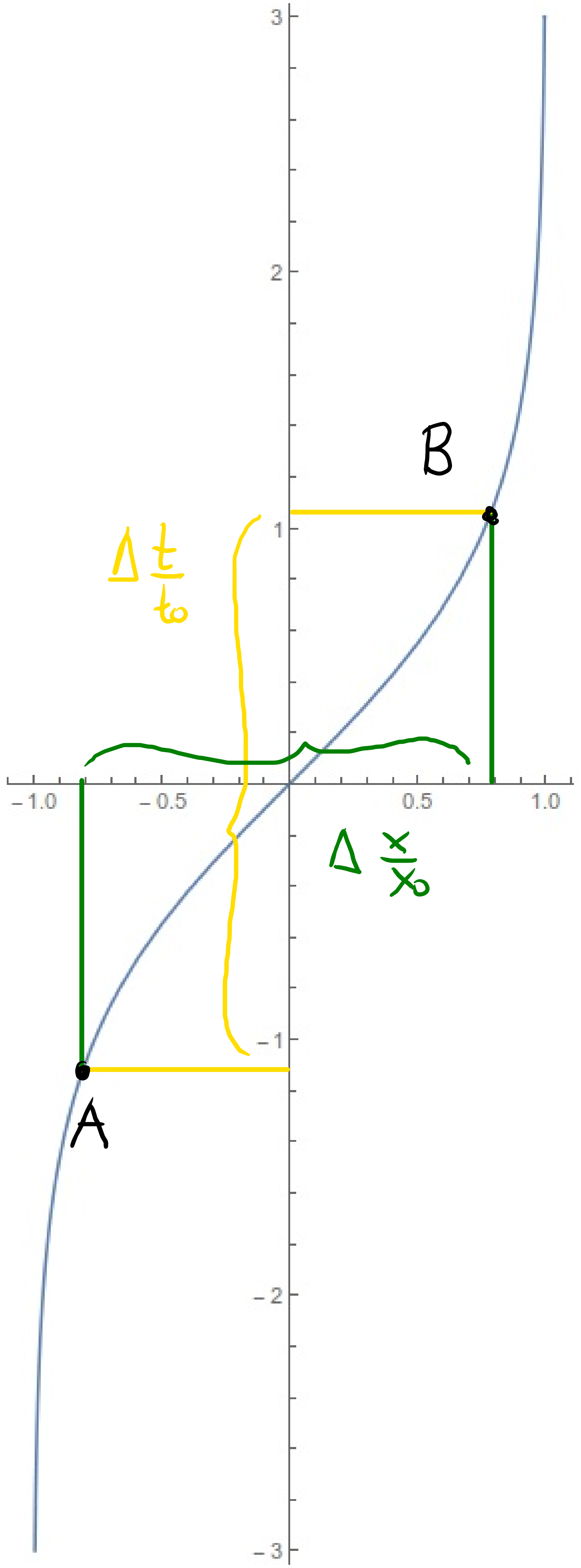
$$\frac{x_B}{x_0} = \tanh \frac{t_B}{t_0}$$

green curves are world lines of "space stations" at fixed positions



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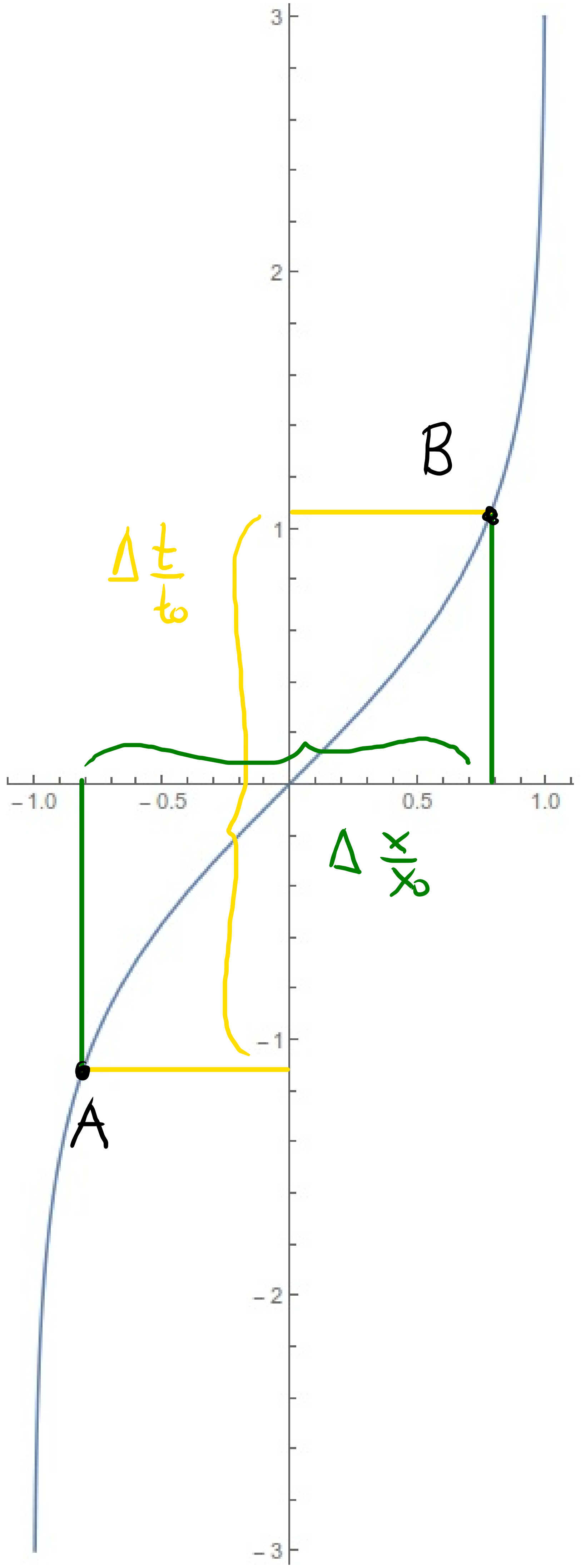

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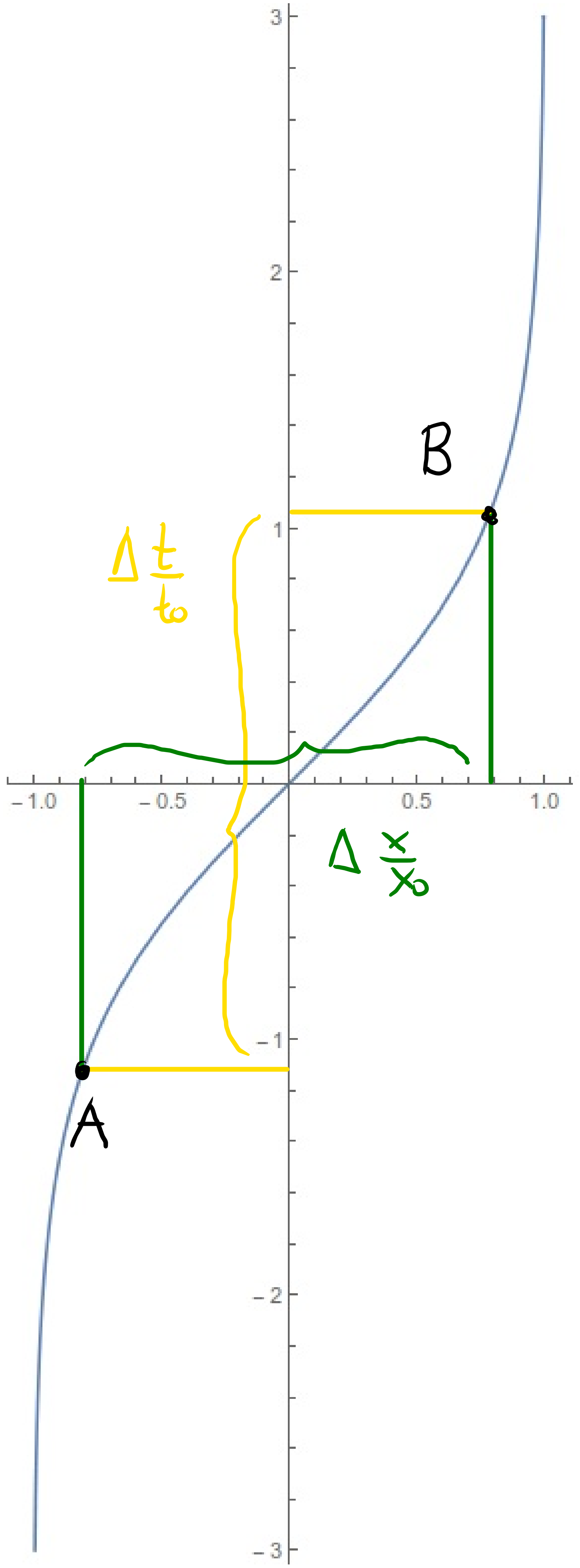

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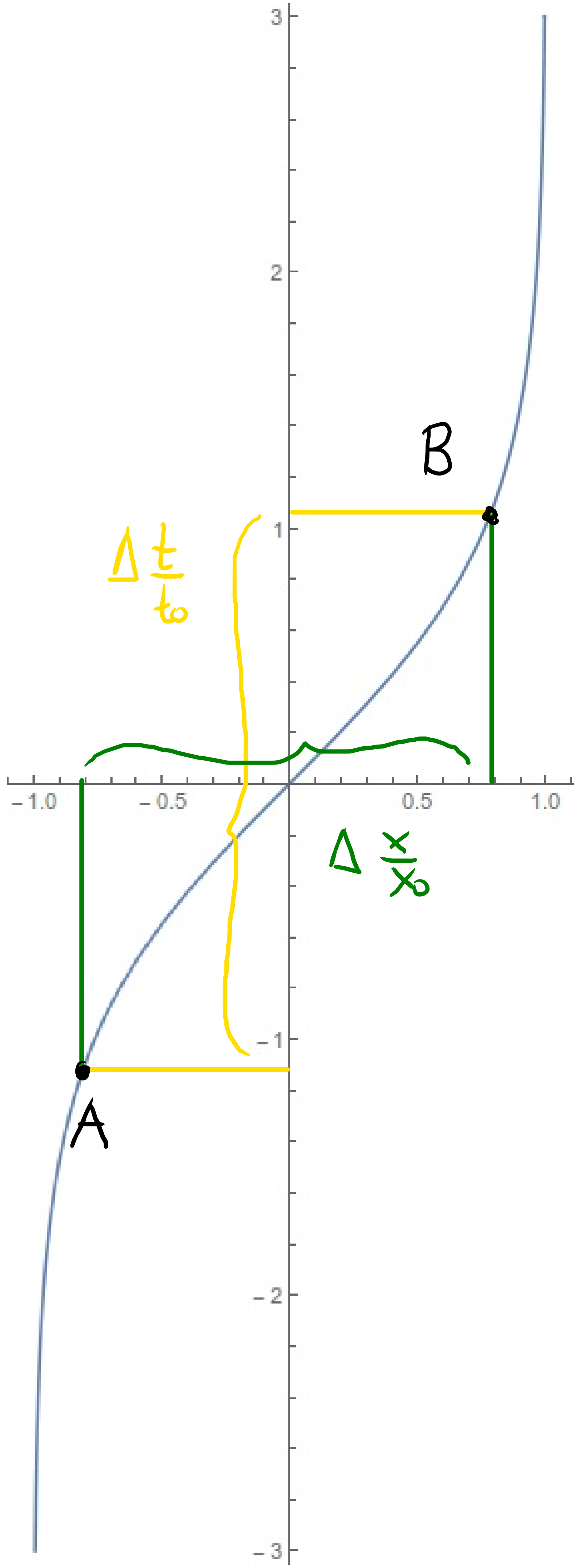

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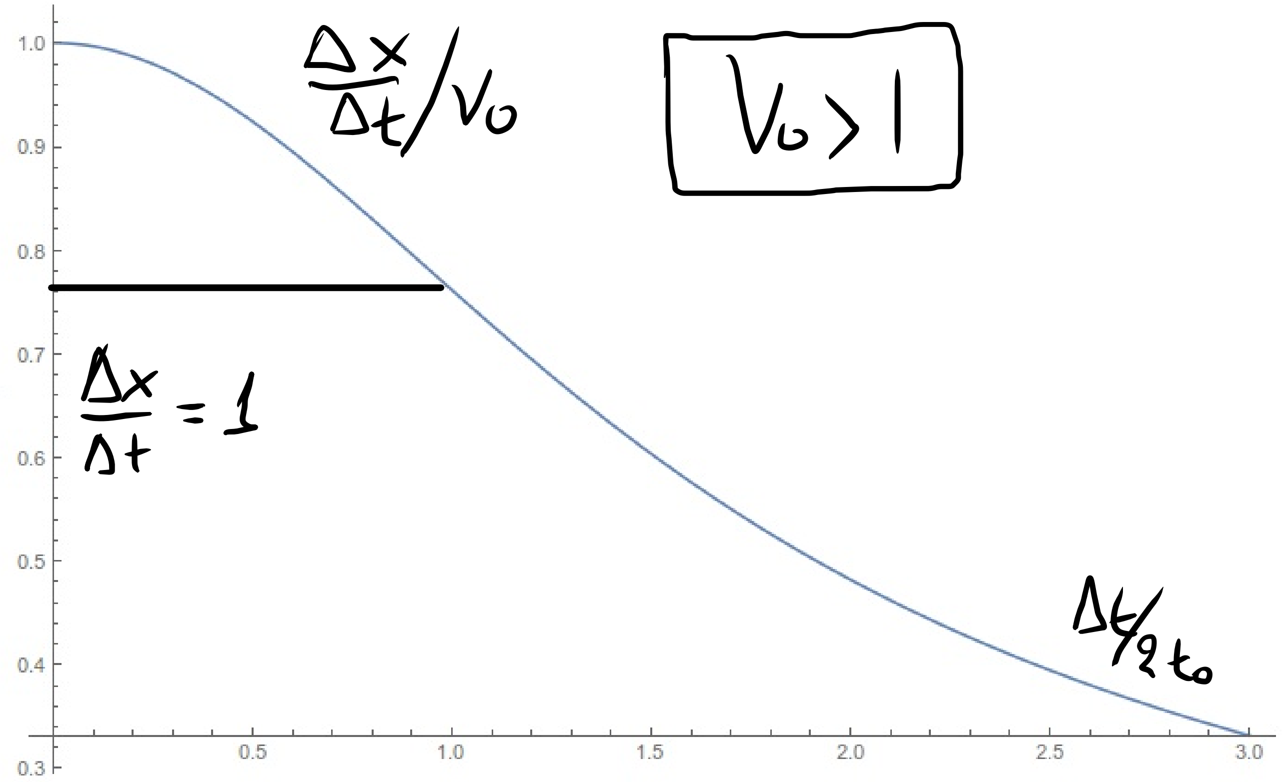
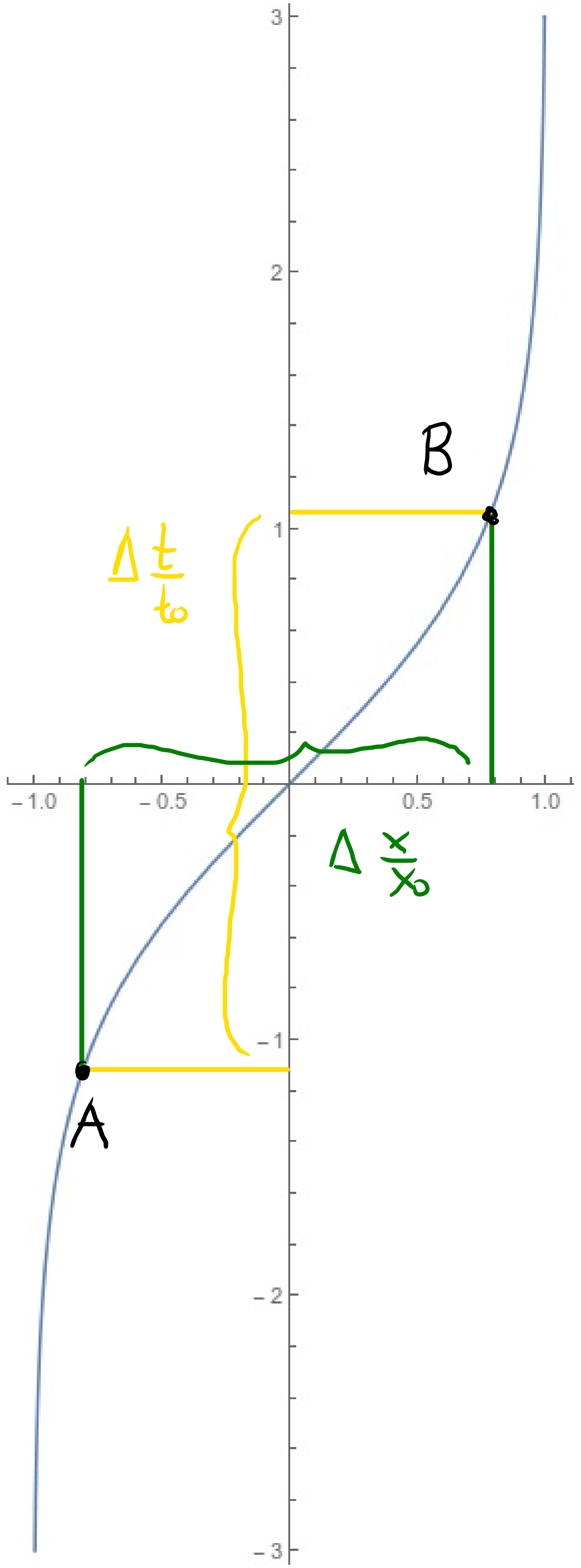


$\boxed{V_0 > 1}$



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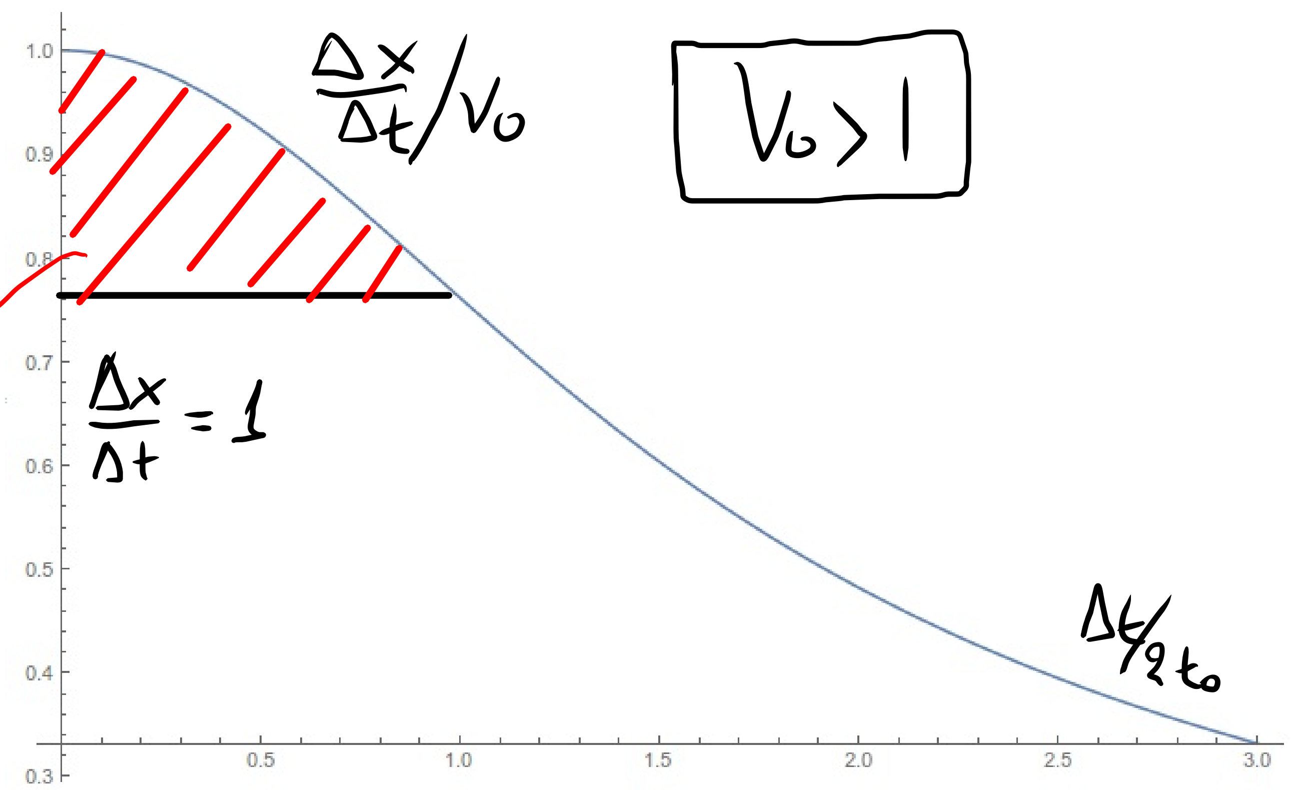

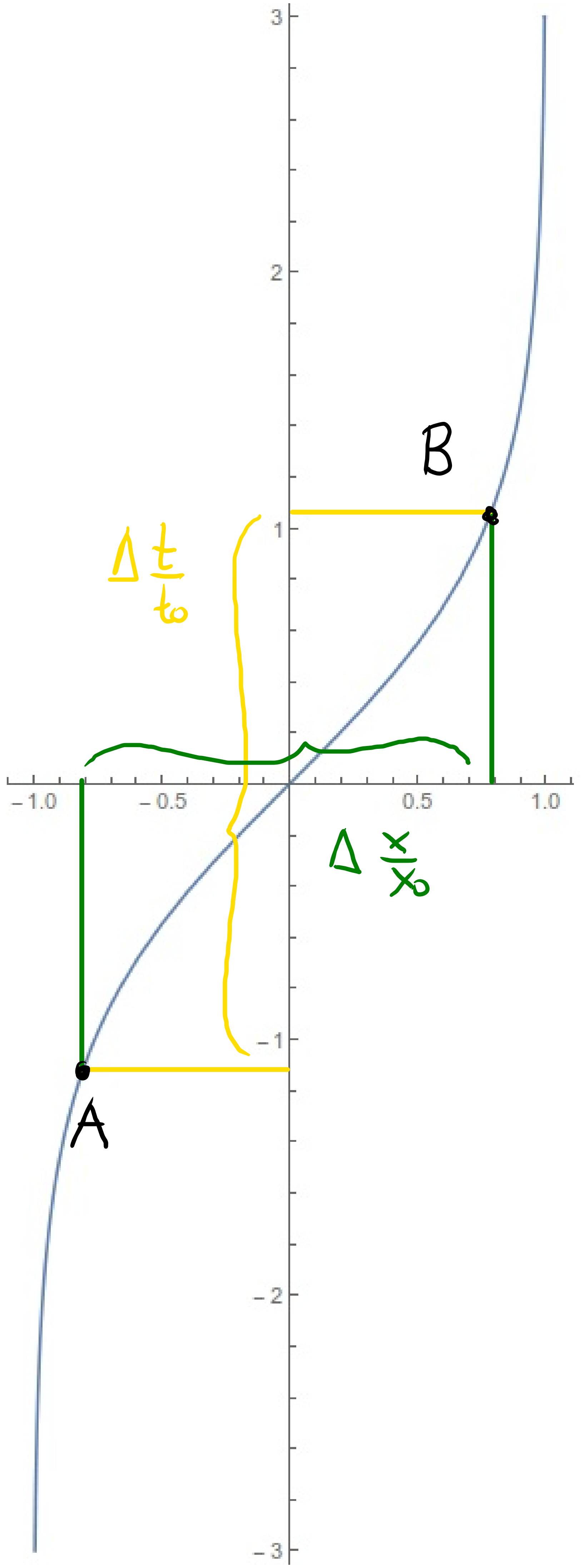

travels from $A \rightarrow B$ with

$$\frac{\Delta x}{\Delta t} > 1 \equiv c$$

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How is this possible? Well, light does not travel on straight lines with a (± 1) slope close to the ship...

$$ds^2 = 0 \Rightarrow -dt^2 + \left[dx - V_s f\, dt\right]^2 = 0$$

$$\Rightarrow dt = \pm \left[dx - V_s f\, dt\right] \Rightarrow$$

$$\Rightarrow \frac{dx}{dt} = \pm 1 + V_s f$$

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light cones are tilted

The 4-velocity is $u^\mu = (u^0, u^1) = \left(\frac{dt}{dt}, \frac{dx_s}{dt}\right) = (1, V_s)$

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$$u_\mu u^\mu = g_{\mu\nu} u^\mu u^\nu = g_{00} u^0 u^0 + 2 g_{01} u^0 u^1 + g_{11} u^1 u^1$$

$$= (-1+V_s^2)\cdot 1 \cdot 1 + 2(-V_s)\cdot 1 \cdot V_s + 1 \cdot V_s \cdot V_s$$

$$= -1 + V_s^2 - 2V_s^2 + V_s^2 = -1$$ $\rightsquigarrow$ timelike always!

$$ds^2 = 0 \Rightarrow -dt^2 + [dx - V_s f\, dt]^2 = 0$$

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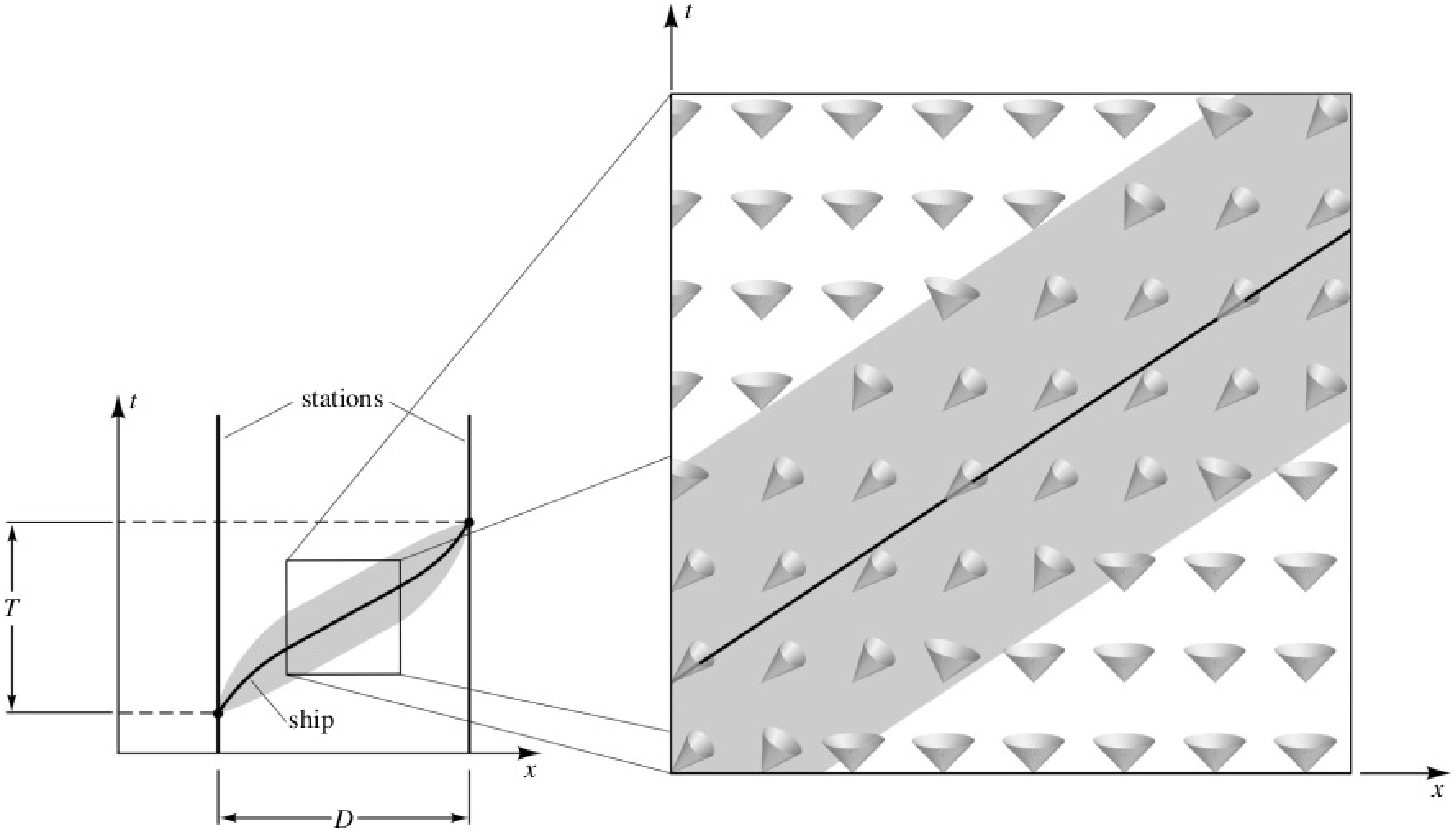
so that!



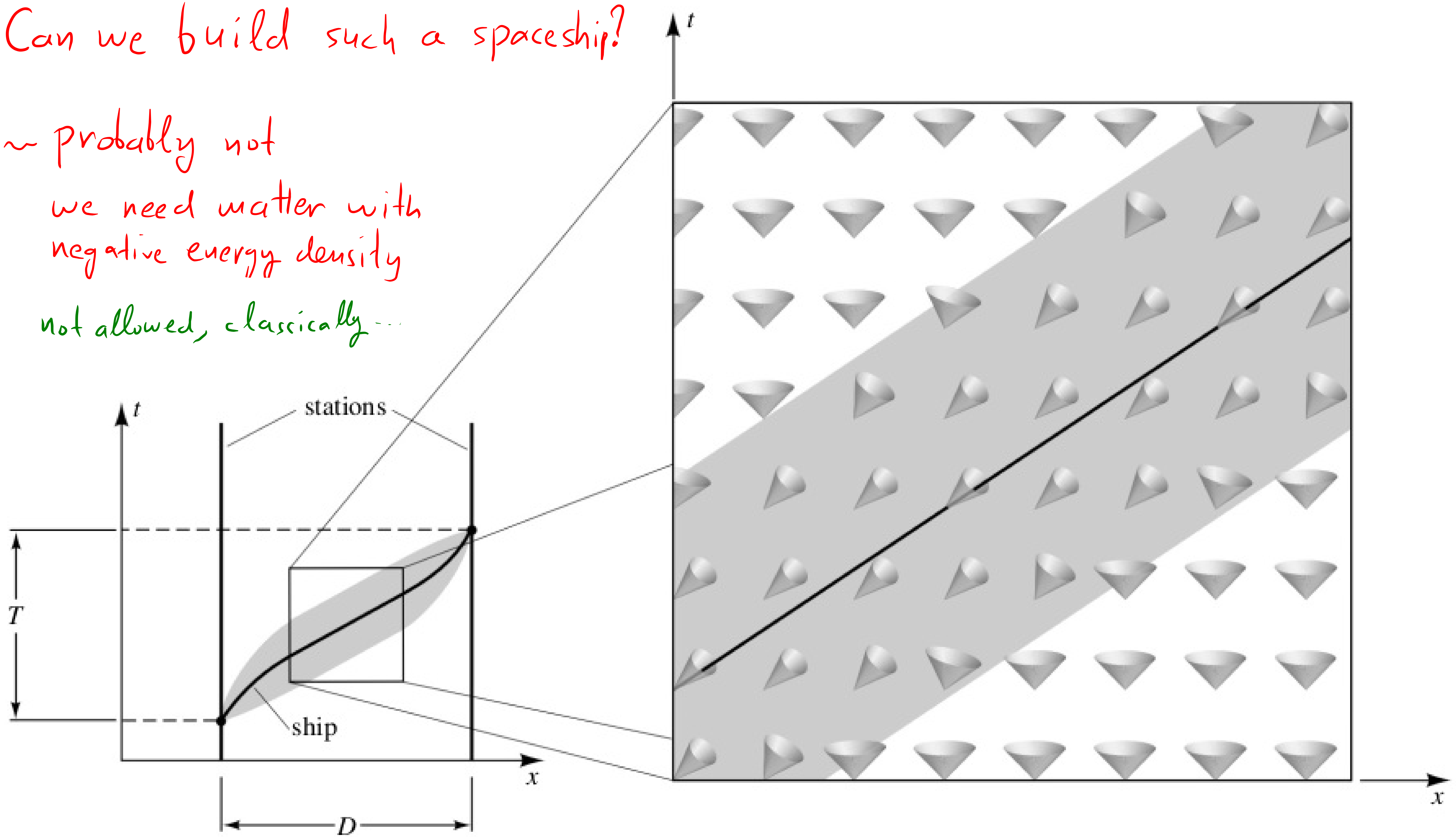
Hartle, Fig 7.2

Can we build such a spaceship?

~ probably not

we need matter with negative energy density

not allowed, classically ...



Hartle, Fig 7.2

* Penrose Diagram of Minkowski Spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \equiv -dt^2 + dr^2 + r^2 \, d\Omega^2$$

$$-\infty < t < +\infty \qquad 0 \leq r < +\infty$$

Reminder:

$$d\Sigma^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$$= dx^2 + dy^2 + dz^2$$

$$x = r \sin\theta \cos\varphi$$

$$y = r \sin\theta \sin\varphi$$

$$z = r \cos\theta$$

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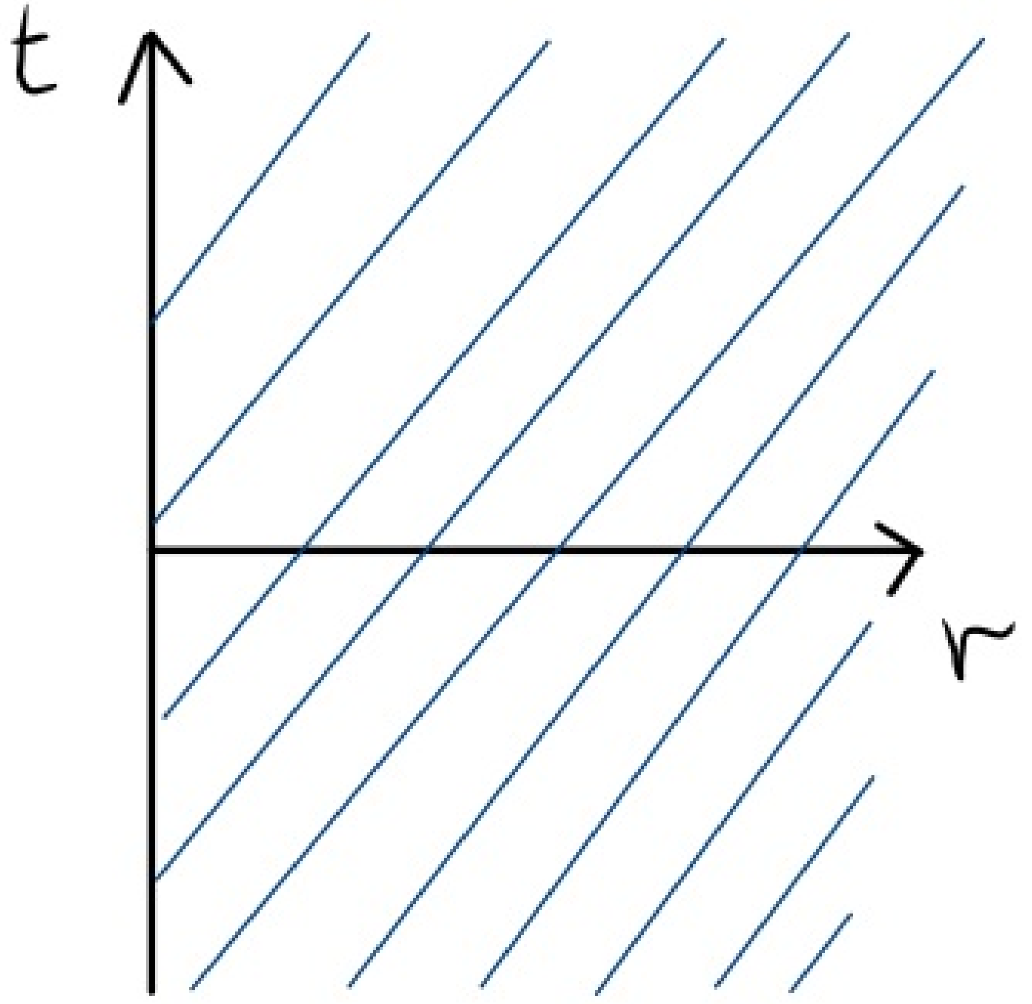

suppress the θ-φ coordinates,

draw events + worldlines on

t-r half plane

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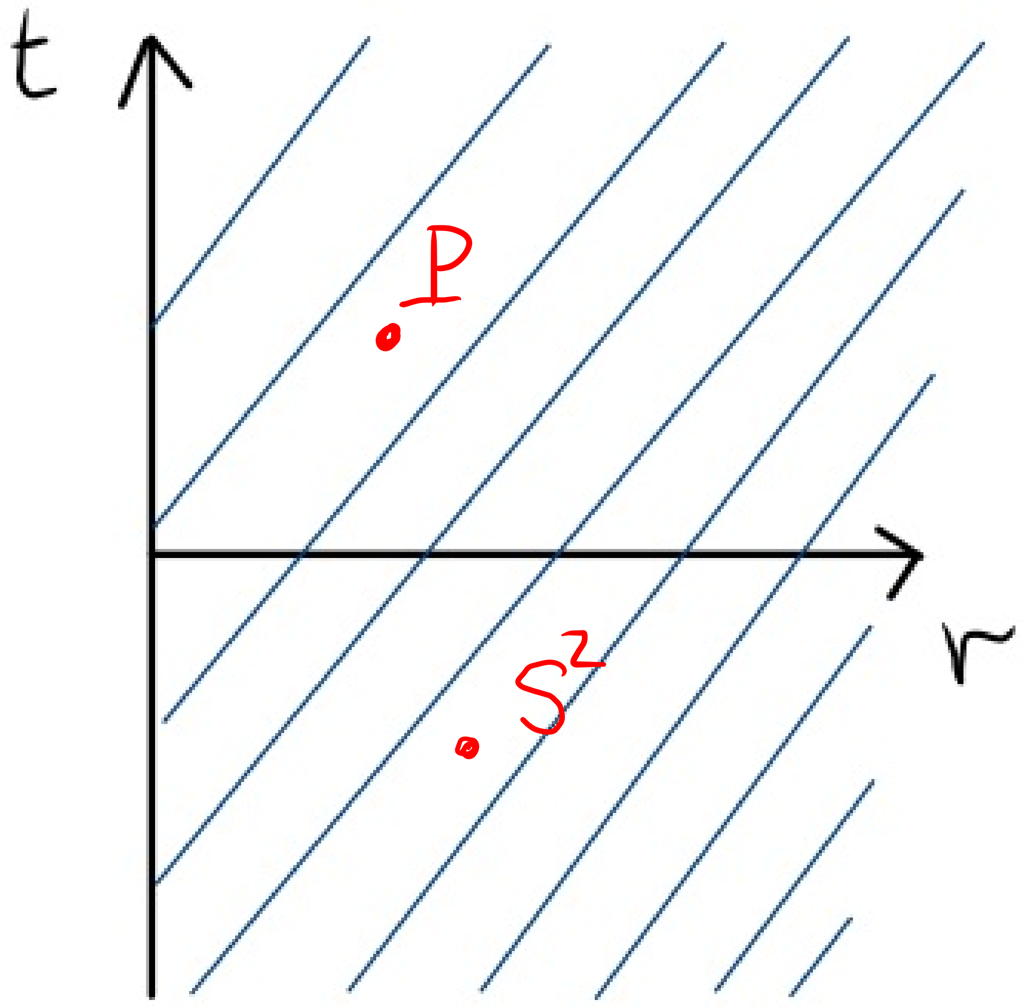

→ In fact, a point on this diagram may represent an event P at $(\theta, \varphi) = (\pi/2, 0)$, or all the events on a sphere S^2

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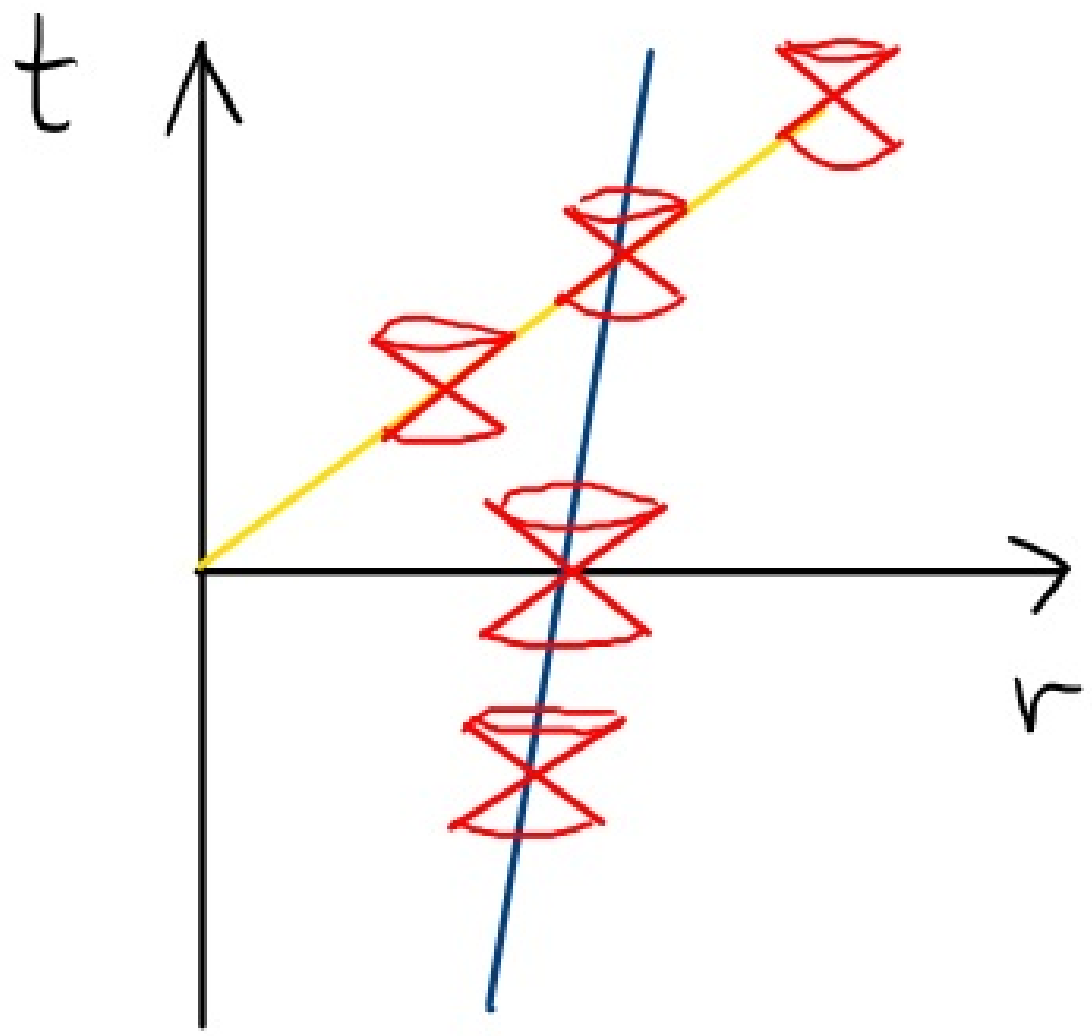

- Pictorial representation of causal structure of spacetime helps us understand key properties of spacetime geometry

  ⇝ device a means for a compact view that captures global properties + causal structure

  "compactification"

  → preserves lightcones

  → represents spacetime in finite region w/boundary the infinities

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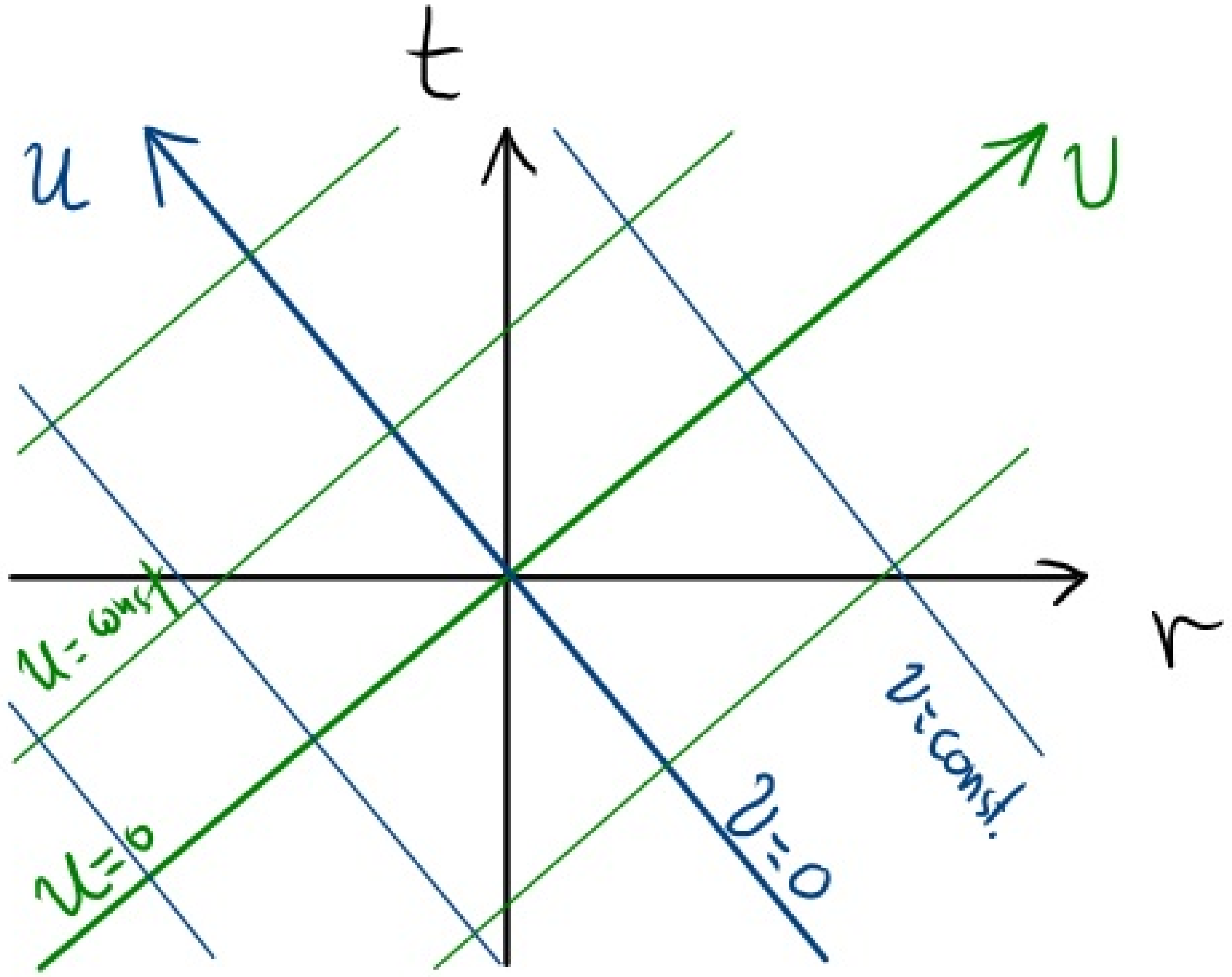

* New coordinates (light cone coordinates)

$$u = t - r \qquad \Longleftrightarrow \qquad t = \frac{v+u}{2}$$

$$v = t + r \qquad\qquad r = \frac{v-u}{2}$$

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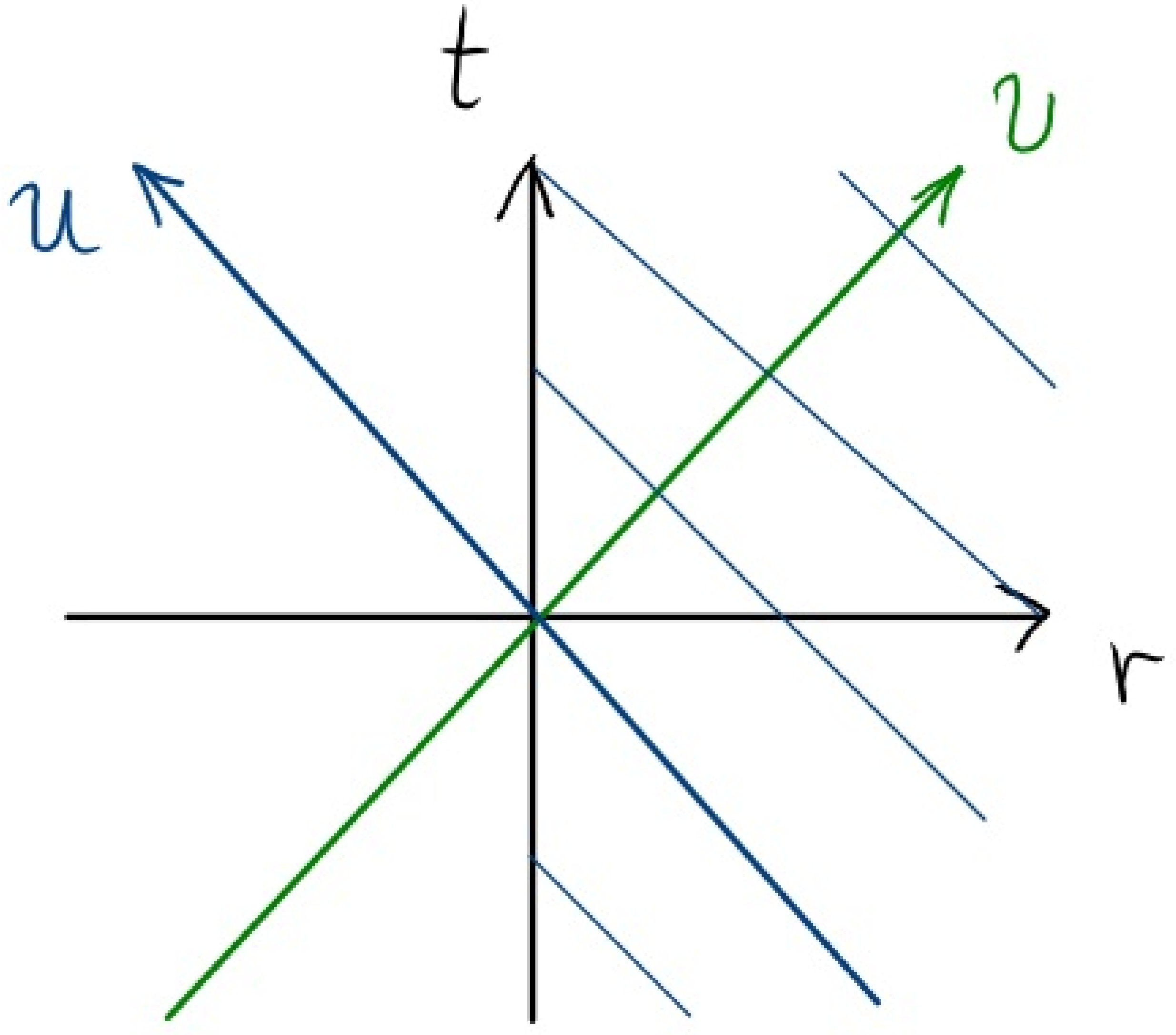

* New coordinates (light cone coordinates)

$$u = t - r \qquad v = t + r \iff t = \frac{v+u}{2} \qquad r = \frac{v-u}{2}$$

$$ds^2 = -du\,dv + \frac{(u-v)^2}{4}\, d\Omega^2$$

$\rightsquigarrow$ still, the diagram is infinite ...

$$r \geq 0 \Rightarrow v \geq u$$

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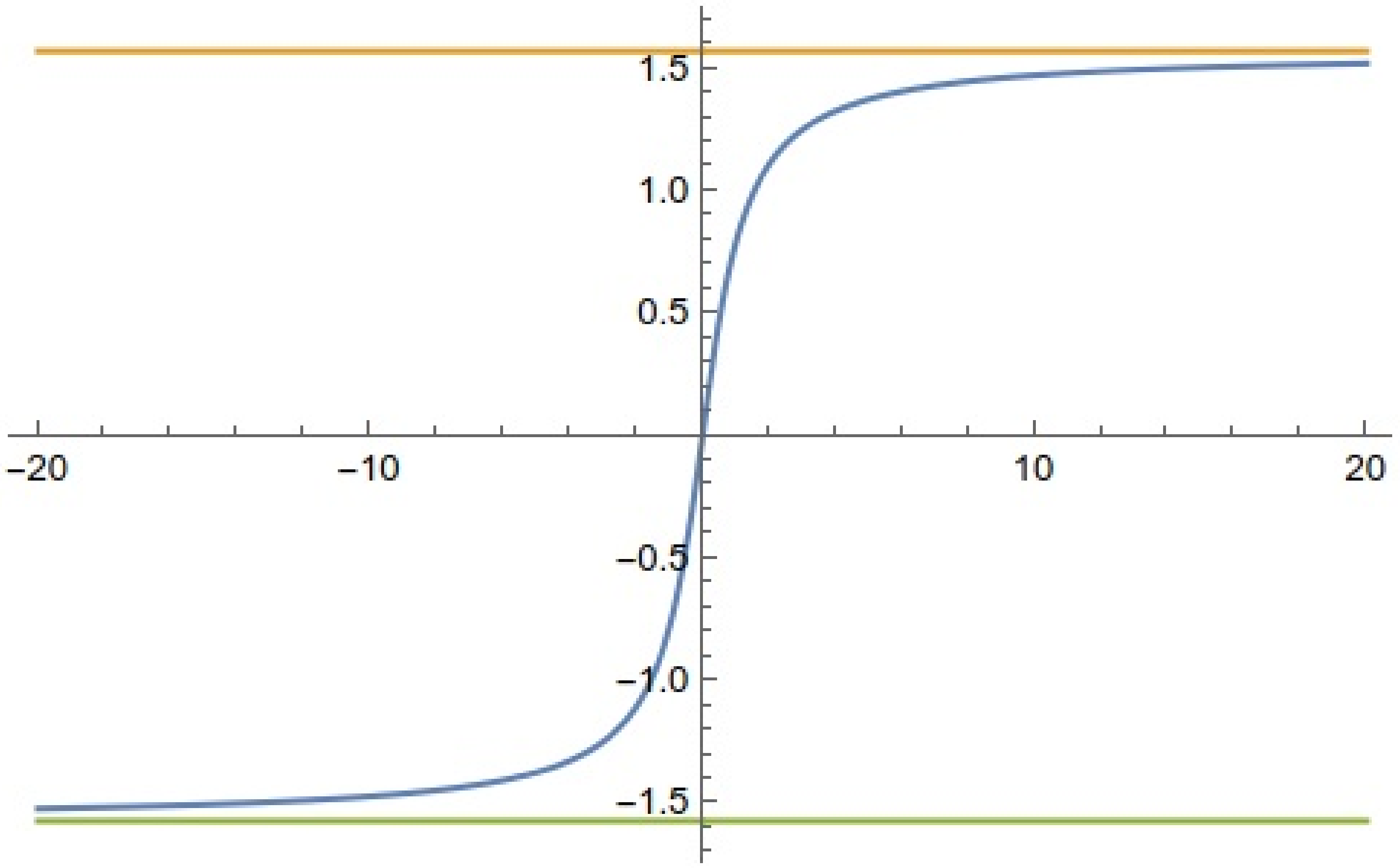
*



$+\infty$

coordinates)

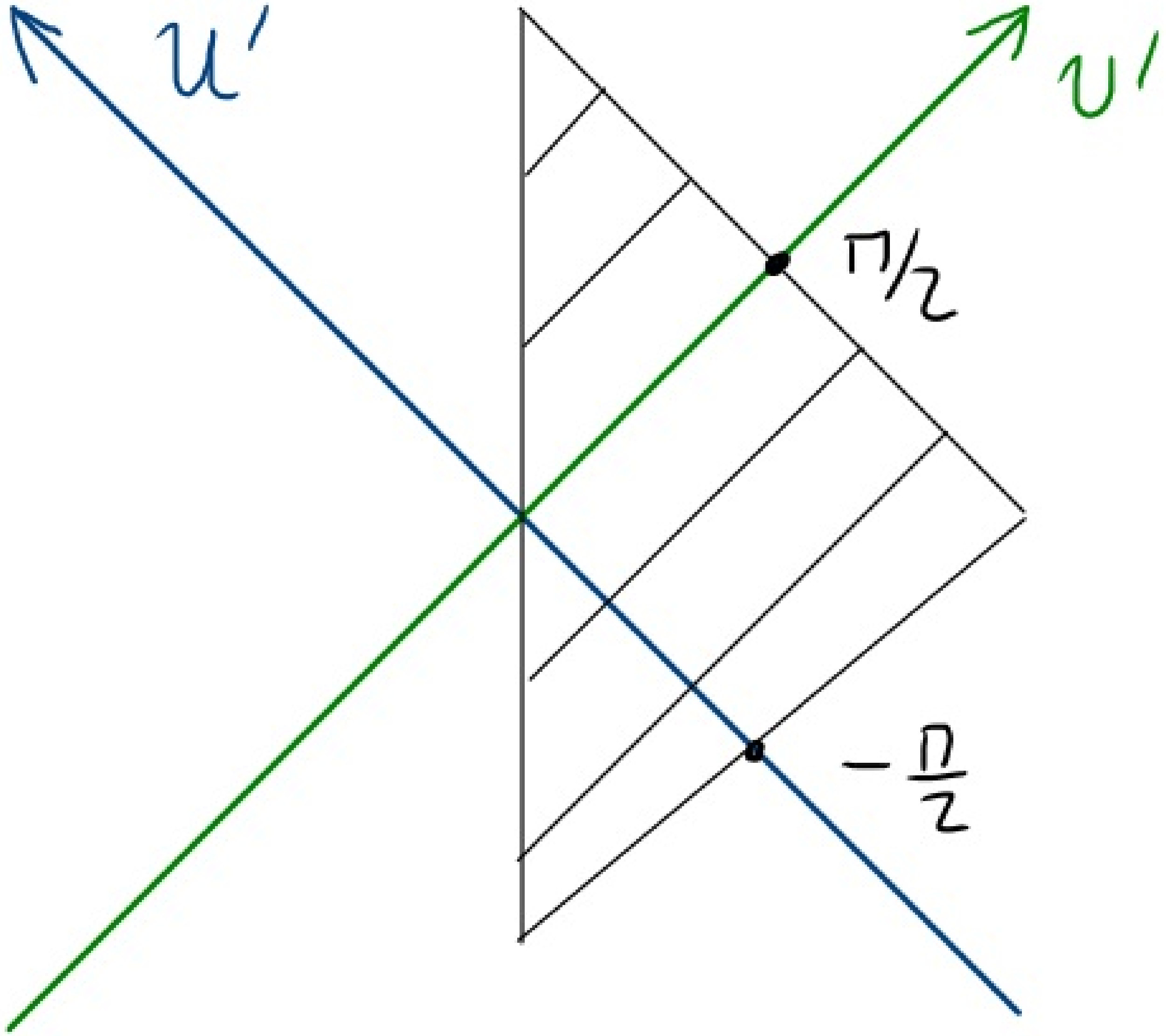
$\frac{+u}{2}$

$\frac{-u}{2}$

* Transform again: Bring infinities @ finite distance:

$$u' = \tan^{-1} u \qquad v' = \tan^{-1} v$$

$$-\frac{\pi}{2} < u' < \frac{\pi}{2} \qquad -\frac{\pi}{2} < v' < \frac{\pi}{2}$$



$$v > u \Rightarrow v' > u'$$

($\tan^{-1}$ increasing function)

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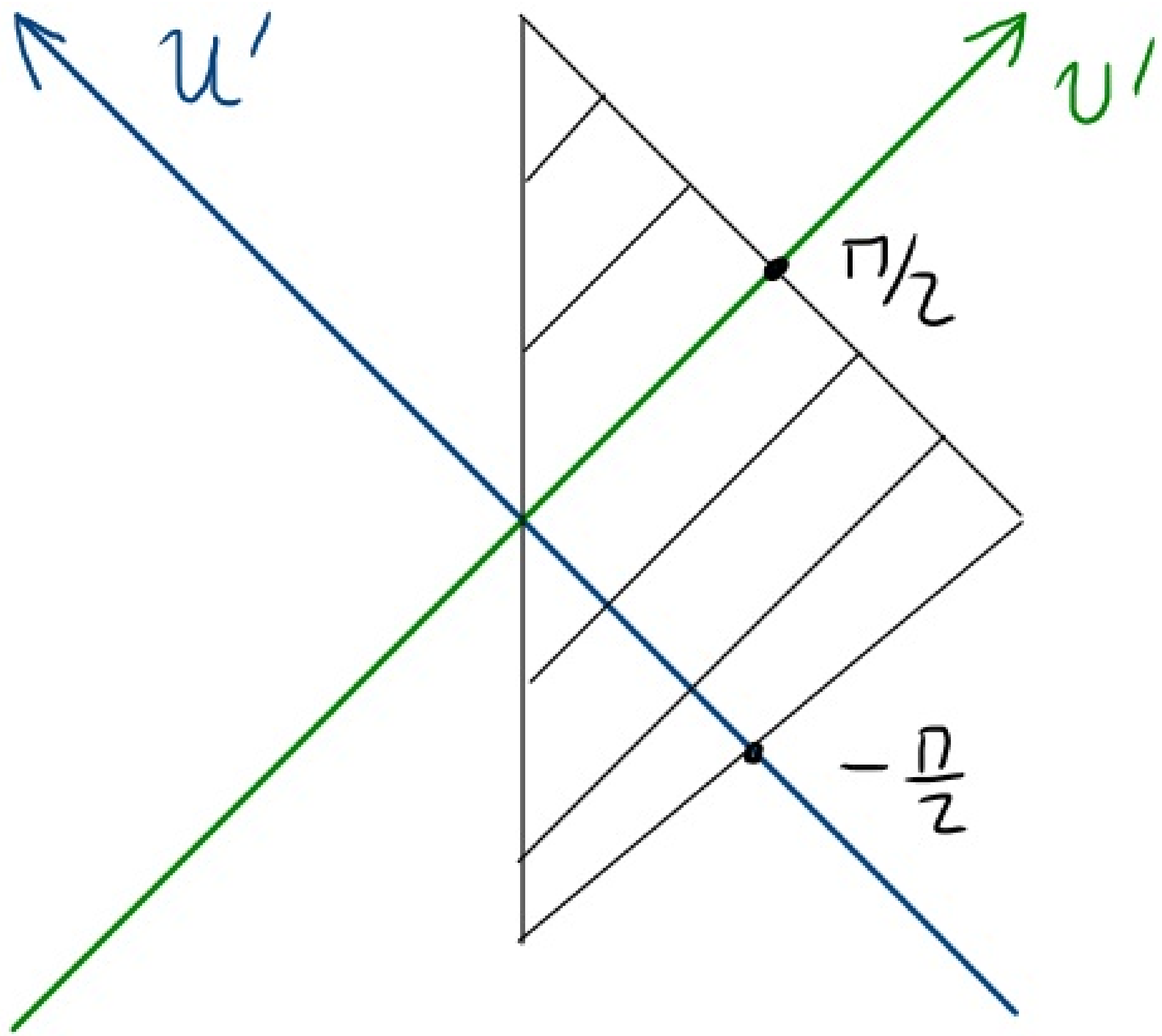

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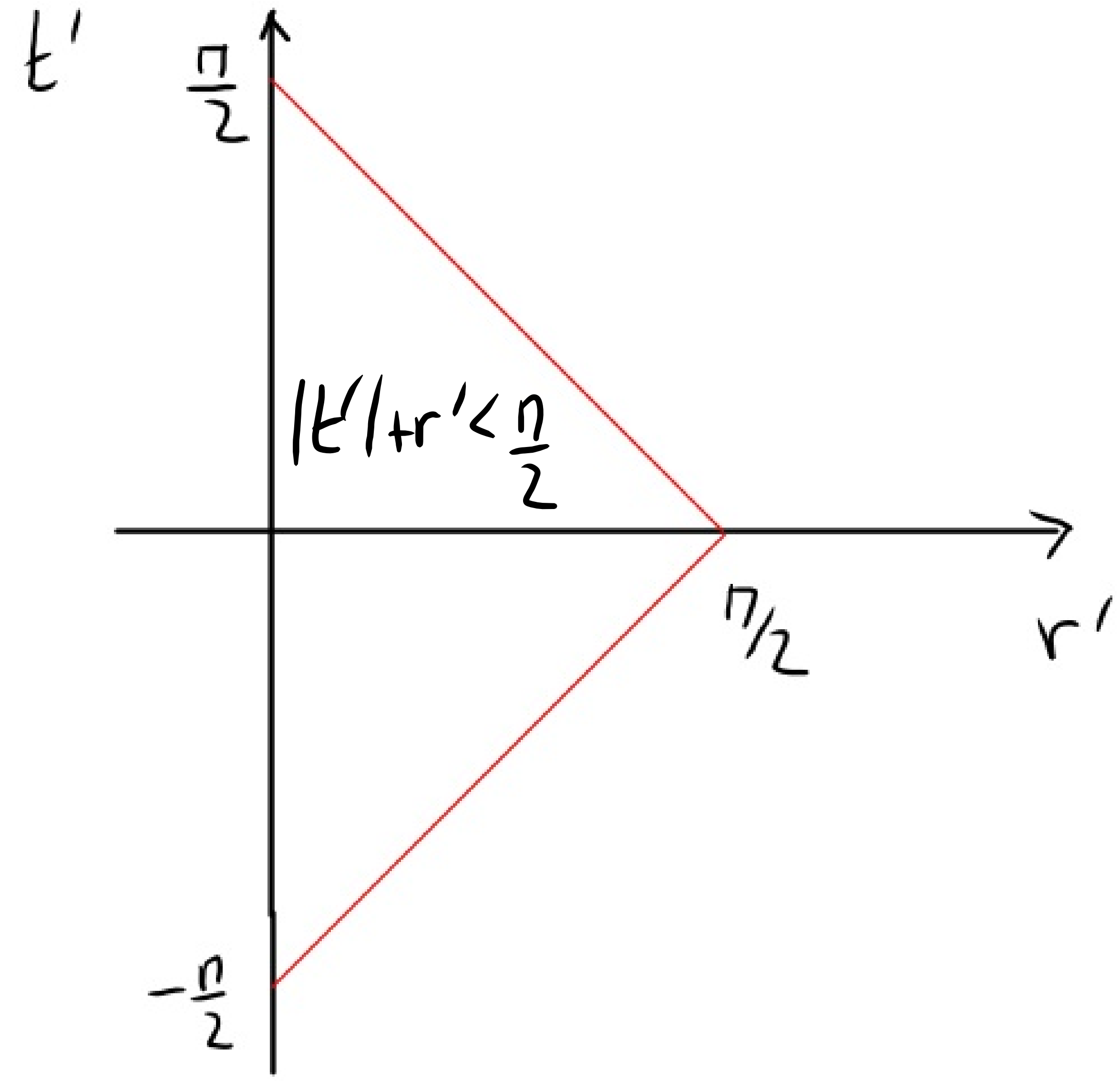

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($\tan^{-1}$ increasing function)

* And again: $u' = t' - r'$ $\quad \Leftrightarrow \quad$ $t' = \frac{v'+u'}{2}$

$v' = t' + r'$ $\qquad\qquad r' = \frac{v'-u'}{2}$

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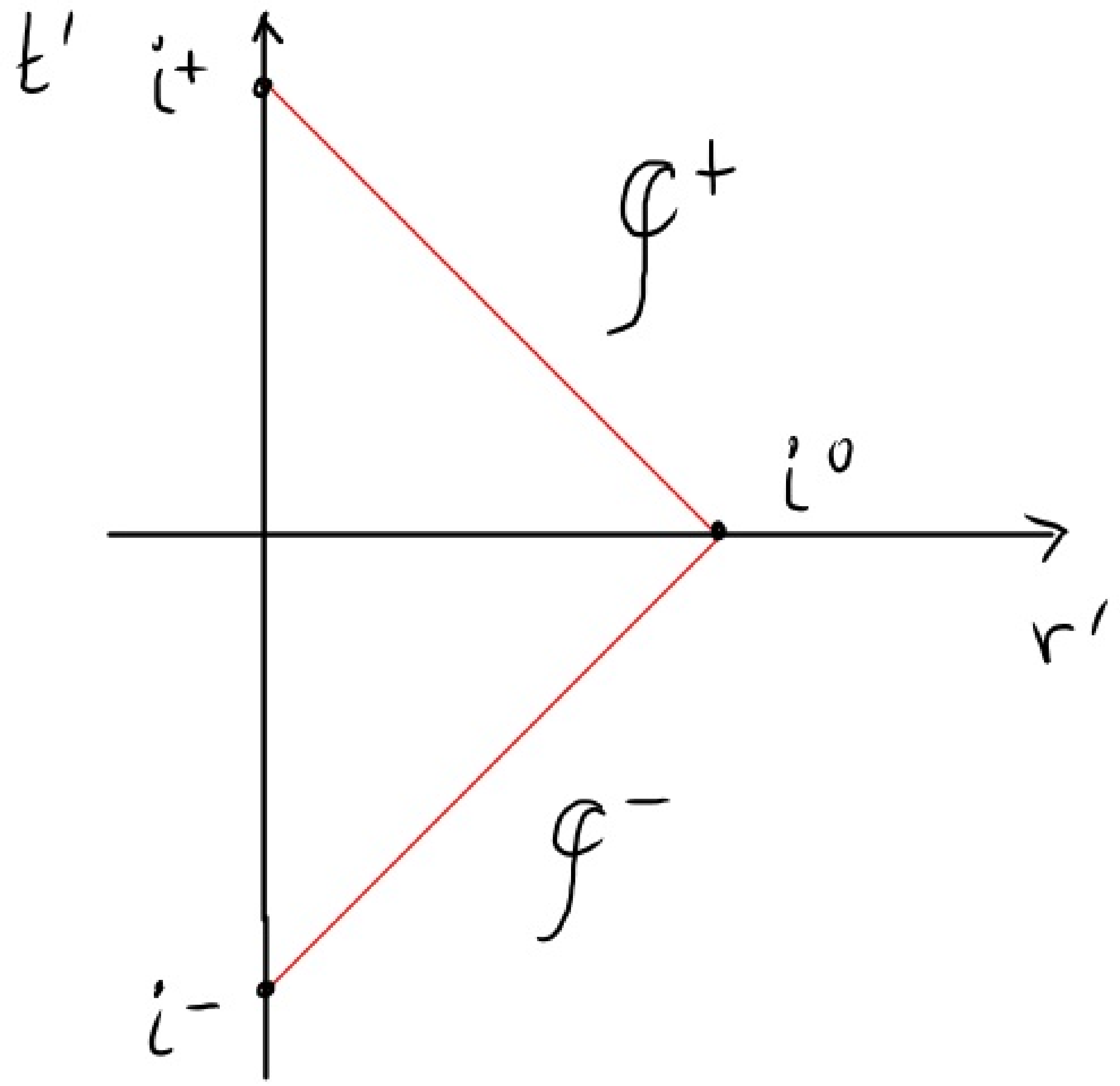
$$t' = \frac{1}{2}\left\{\tan^{-1}(t+r) + \tan^{-1}(t-r)\right\}$$

$$r' = \frac{1}{2}\left\{\tan^{-1}(t+r) - \tan^{-1}(t-r)\right\}$$

$$|t'| + r < \frac{\pi}{2}$$



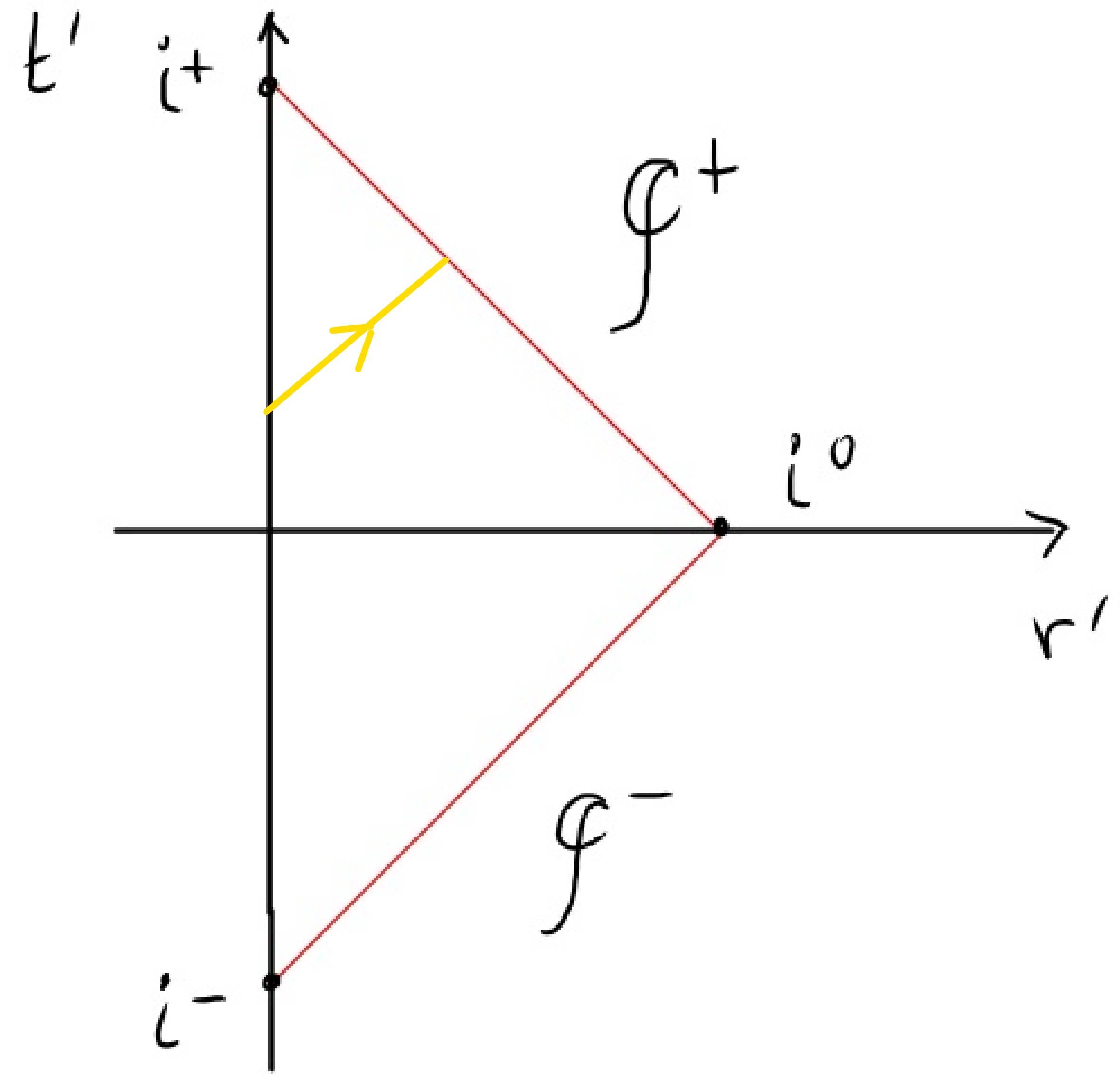
* Penrose Diagram of Minkowski Spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \equiv -dt^2 + dr^2 + r^2\, d\Omega^2$$

$$t' = \frac{1}{2}\left\{ \tan^{-1}(t+r) + \tan^{-1}(t-r) \right\}$$

$$r' = \frac{1}{2}\left\{ \tan^{-1}(t+r) - \tan^{-1}(t-r) \right\}$$

$$|t'| + r < \frac{\pi}{2}$$



- Outgoing light rays:

$$t = r + r_0 \Rightarrow u = \text{const} \Rightarrow u' = \text{const} \Rightarrow t' - r' = \text{const} \Rightarrow$$

$$t' = r' + r_0'$$

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$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \equiv -dt^2 + dr^2 + r^2\, d\Omega^2$$

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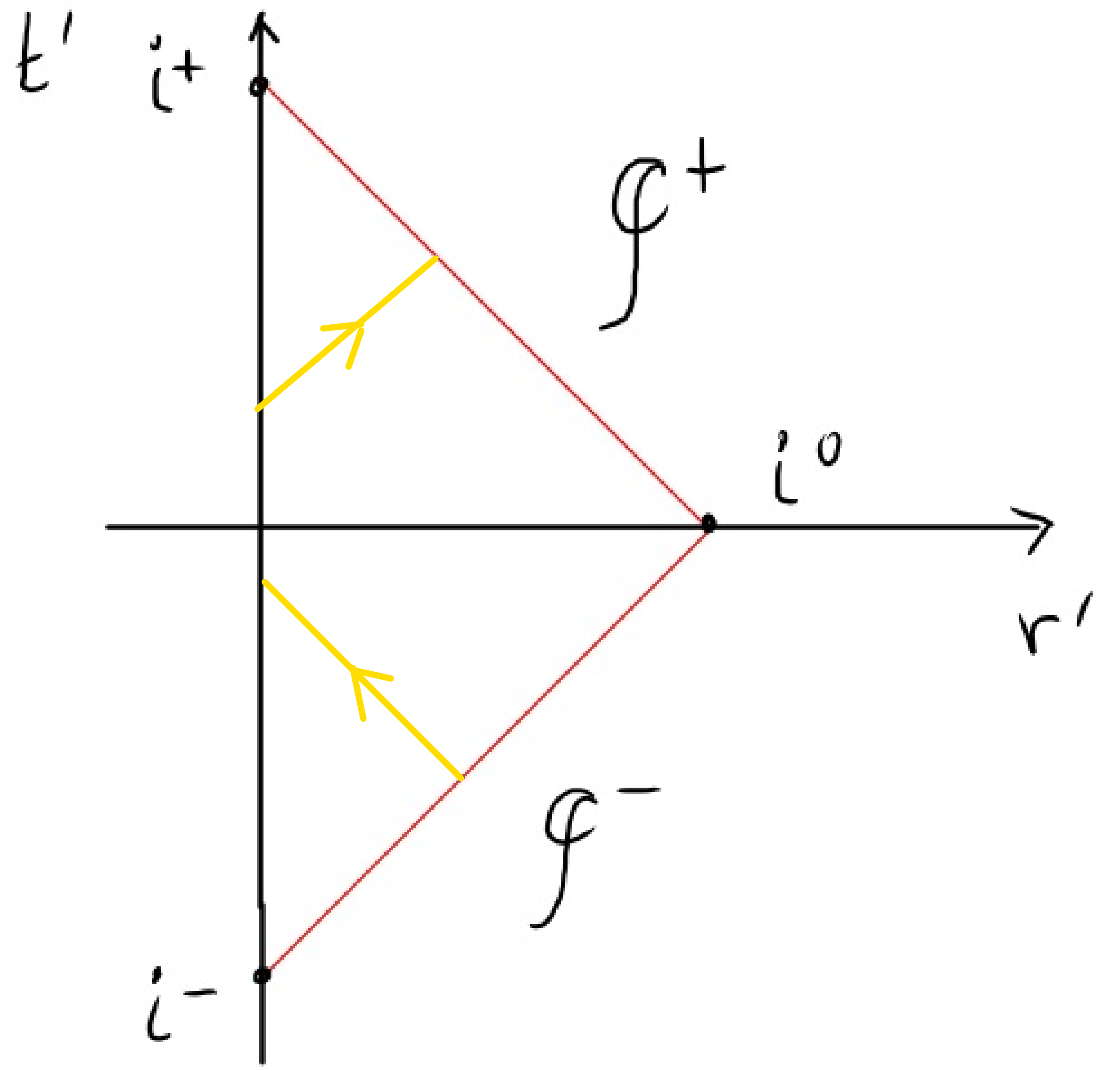
$$|t'| + r < \frac{\pi}{2}$$



- Outgoing light rays:

$$t = r + r_0 \Rightarrow u = \text{const} \Rightarrow u' = \text{const} \Rightarrow t' - r' = \text{const} \Rightarrow$$

$$t' = r' + r_0'$$

- Incoming light rays:

$$t = -r + r_0 \Rightarrow v = \text{const} \Rightarrow v' = \text{const} \Rightarrow t' + r' = \text{const} \Rightarrow$$

$$t' = -r' + r_0'$$

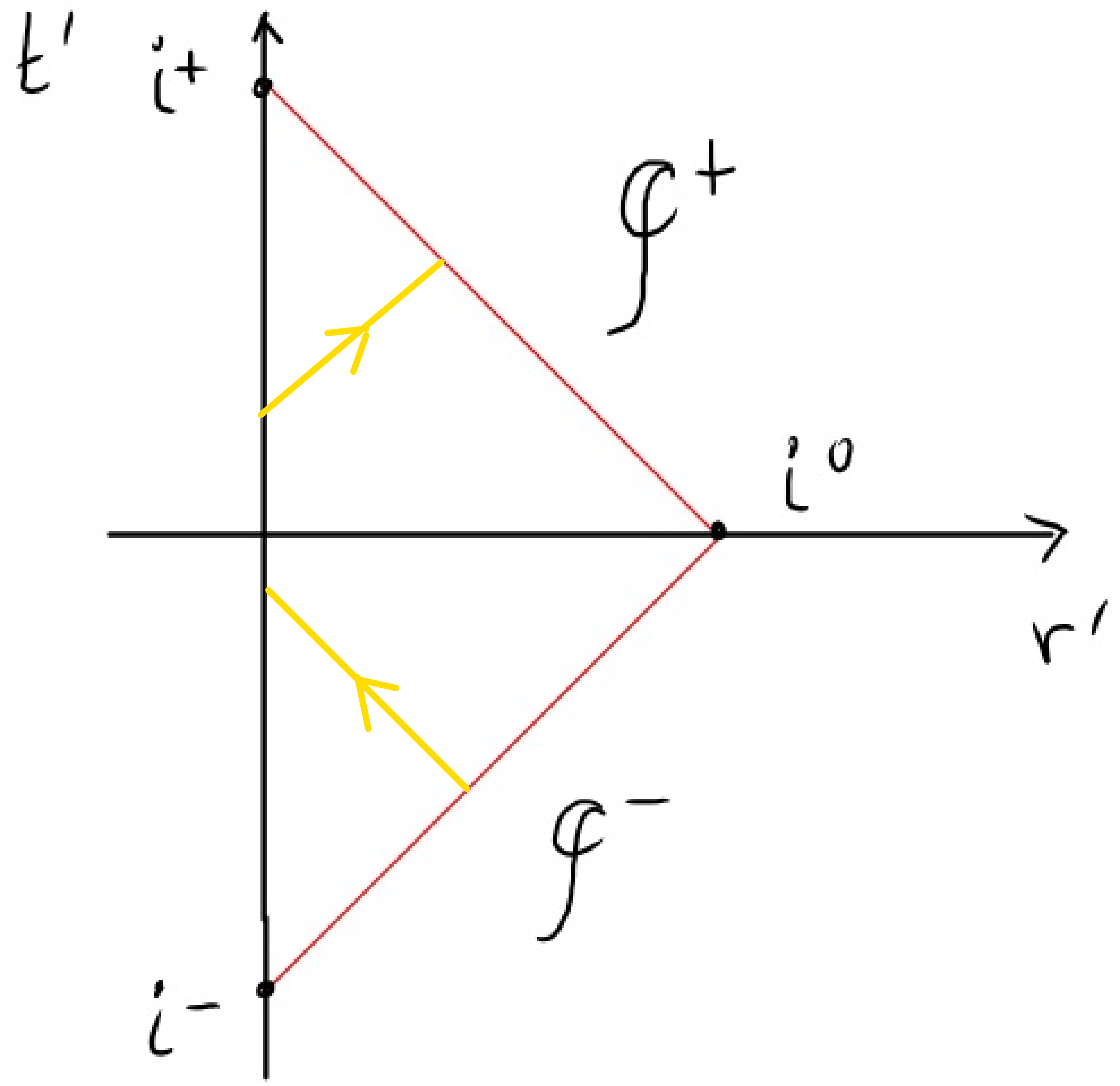
* Penrose Diagram of Minkowski Spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \equiv -dt^2 + dr^2 + r^2 d\Omega^2$$

$$t' = \frac{1}{2}\left\{ \tan^{-1}(t+r) + \tan^{-1}(t-r) \right\}$$

$$r' = \frac{1}{2}\left\{ \tan^{-1}(t+r) - \tan^{-1}(t-r) \right\}$$

$$|t'| + r < \frac{\pi}{2}$$



* light cones at 45°

* light rays (i) originate "on" $\mathscr{I}^-$

(ii) end up "on" $\mathscr{I}^+$

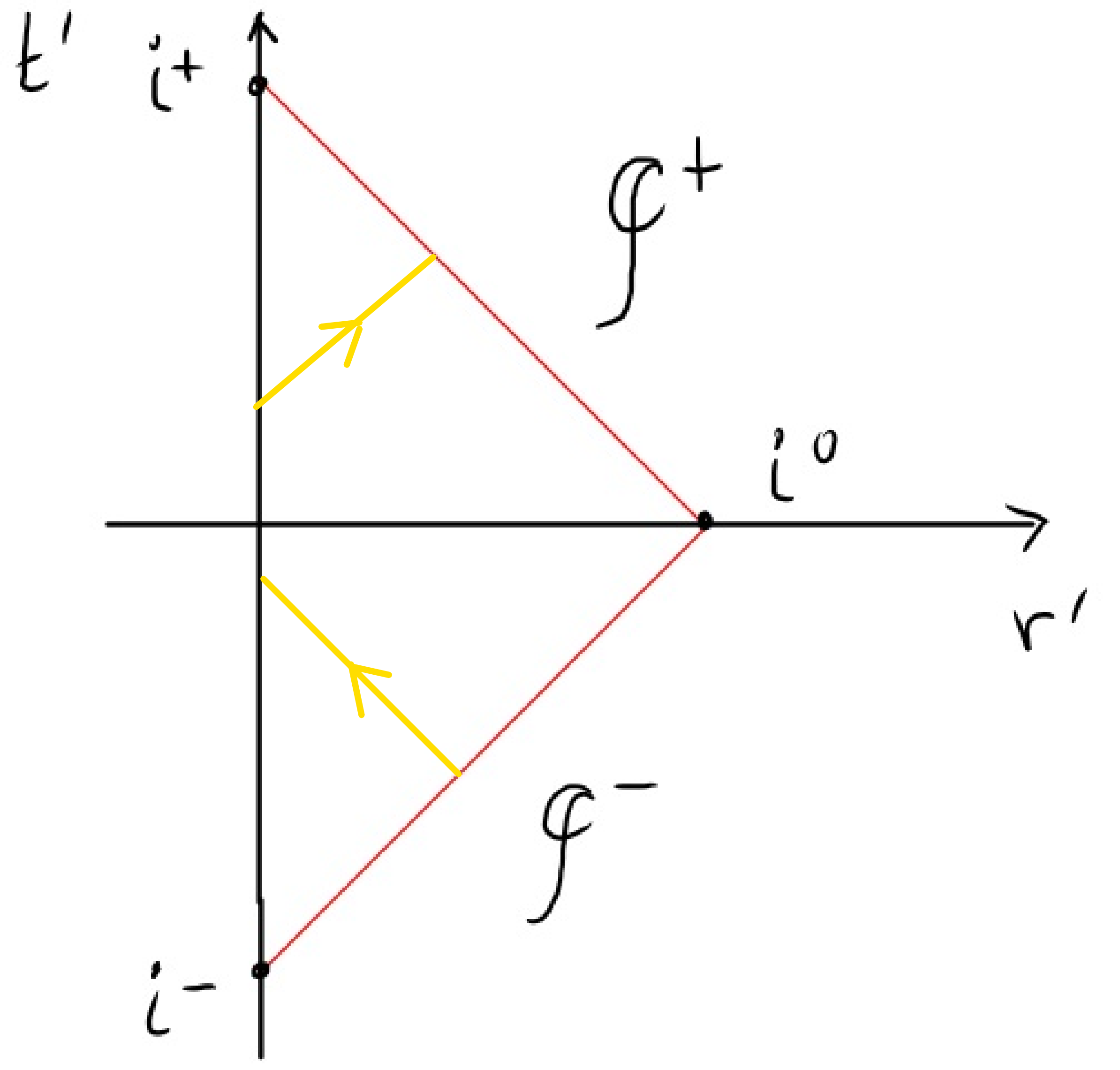
* Penrose Diagram of Minkowski Spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \equiv -dt^2 + dr^2 + r^2 d\Omega^2$$

$$t' = \frac{1}{2}\left\{ \tan^{-1}(t+r) + \tan^{-1}(t-r) \right\}$$

$$|t'| + r < \frac{\pi}{2}$$

$$r' = \frac{1}{2}\left\{ \tan^{-1}(t+r) - \tan^{-1}(t-r) \right\}$$



* light cones at 45°

* light rays (i) originate "on" $\mathcal{J}^-$ past null infinity

(ii) end up "on" $\mathcal{J}^+$ future null infinity

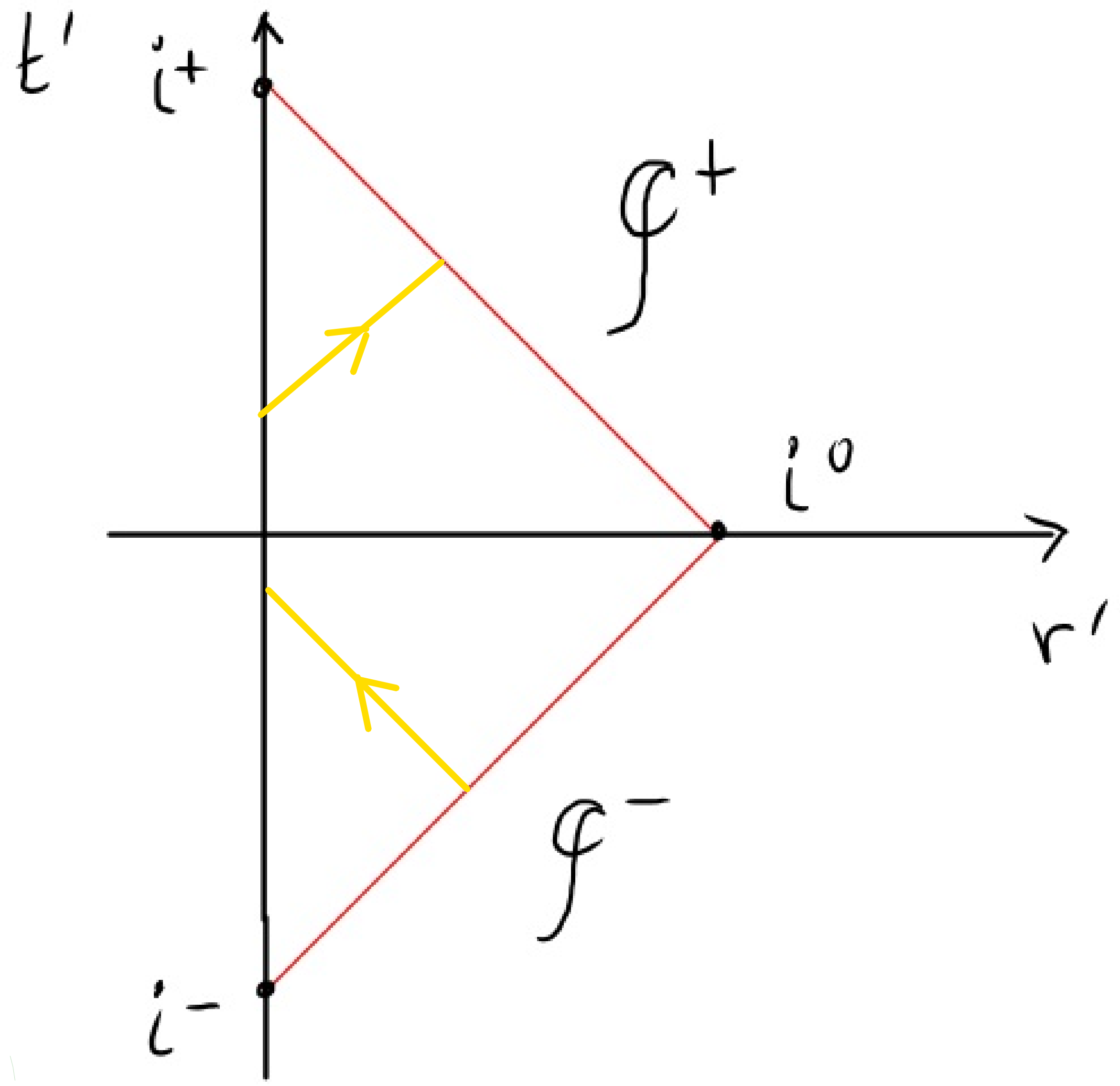
* Penrose Diagram of Minkowski Spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \equiv -dt^2 + dr^2 + r^2 \, d\Omega^2$$

$$t' = \frac{1}{2}\left\{ \tan^{-1}(t+r) + \tan^{-1}(t-r) \right\}$$

$$r' = \frac{1}{2}\left\{ \tan^{-1}(t+r) - \tan^{-1}(t-r) \right\}$$

$$|t'| + r < \frac{\pi}{2}$$



* Massive particle moving in radial direction

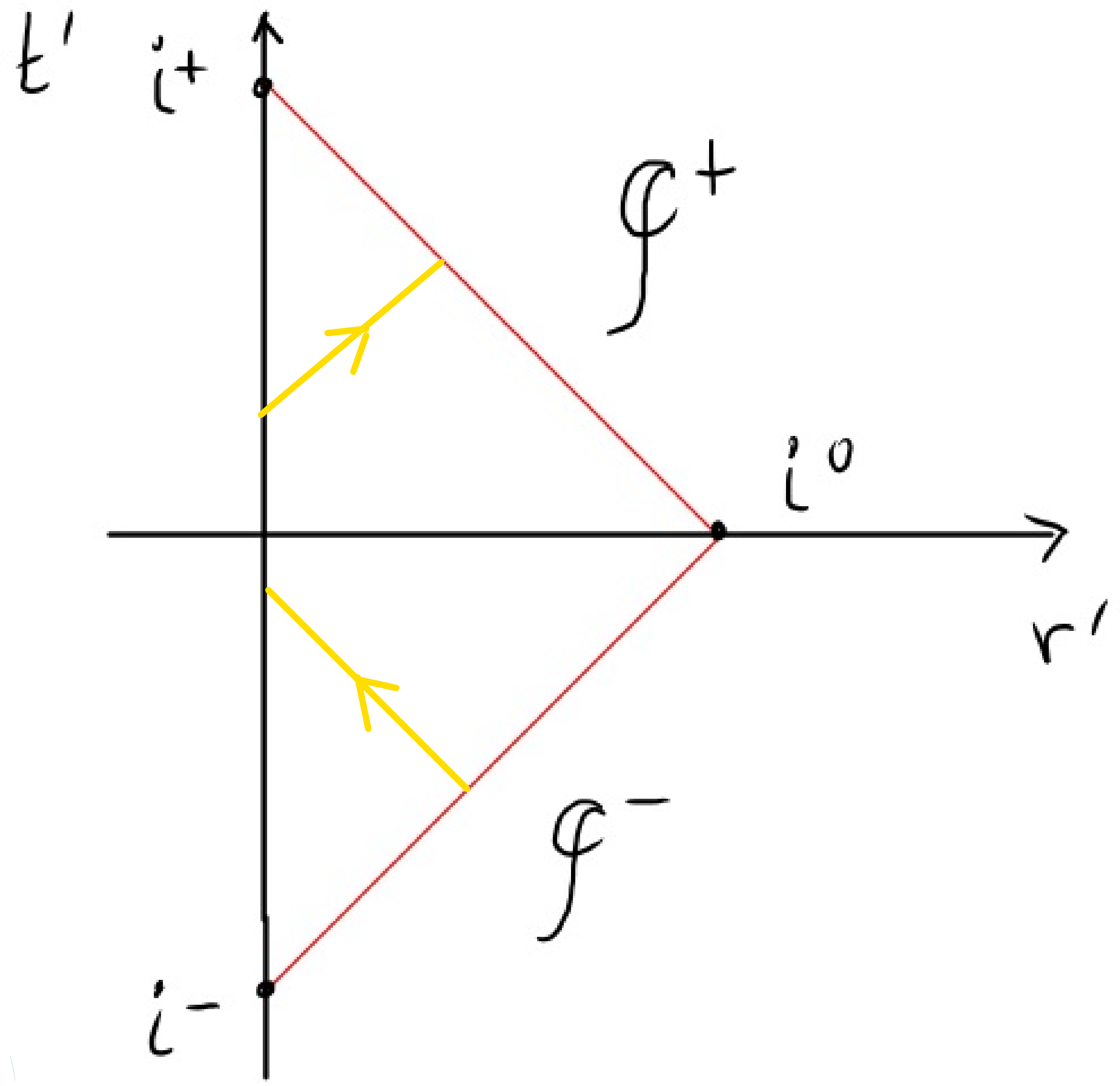
$$r = Vt \qquad 0 < V < 1$$

* Penrose Diagram of Minkowski Spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \equiv -dt^2 + dr^2 + r^2 d\Omega^2$$

$$t' = \frac{1}{2}\{\tan^{-1}(1+v)t + \tan^{-1}(1-v)t\}$$

$$r' = \frac{1}{2}\{\tan^{-1}(1+v)t - \tan^{-1}(1-v)t\}$$



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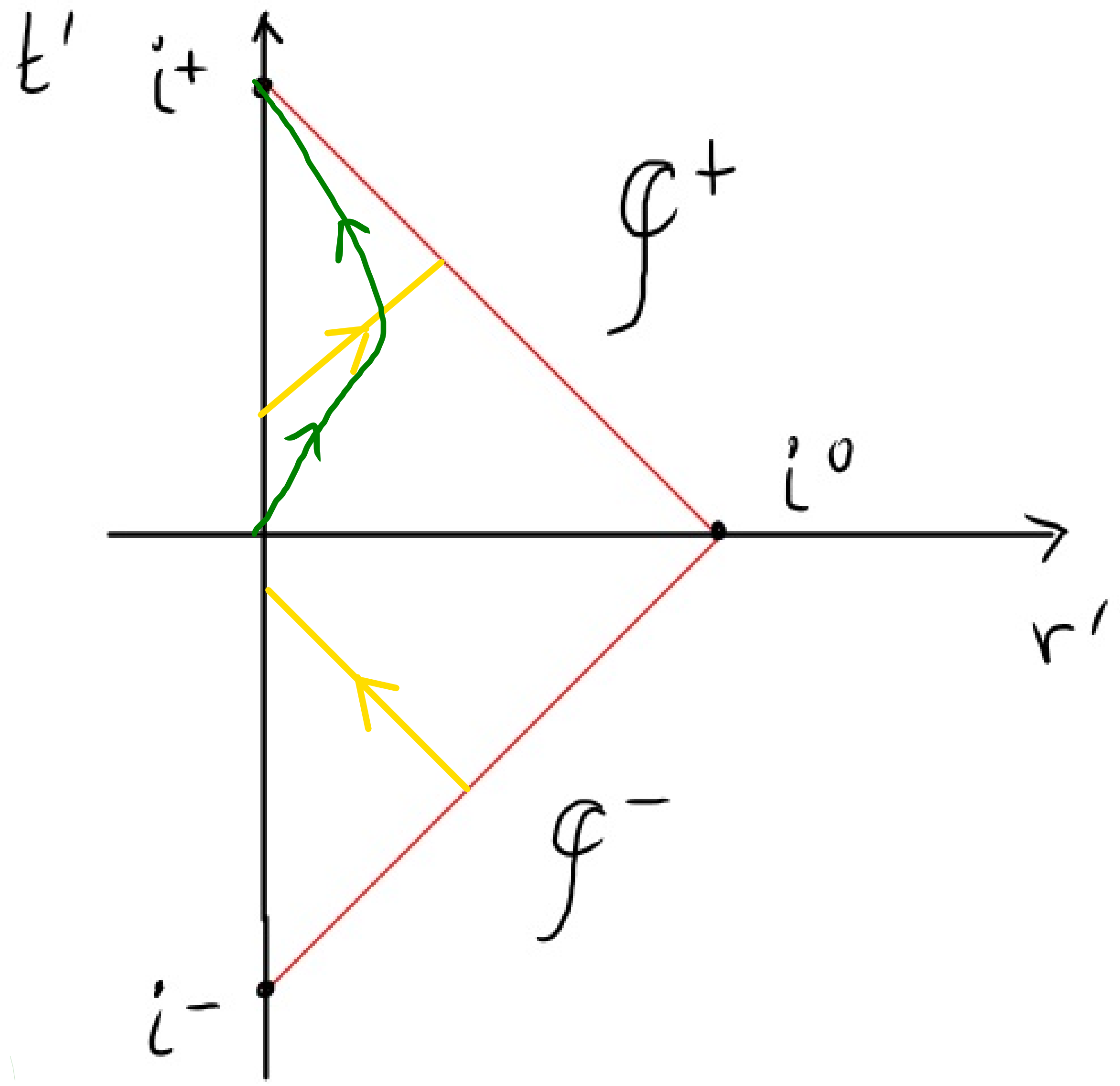
$$r' = \frac{1}{2}\left\{ \tan^{-1}(1+V)t - \tan^{-1}(1-V)t \right\}$$



* Massive particle moving in radial direction

$$r = Vt \qquad 0 < V < 1 \quad \Rightarrow \quad 1 - V > 0$$

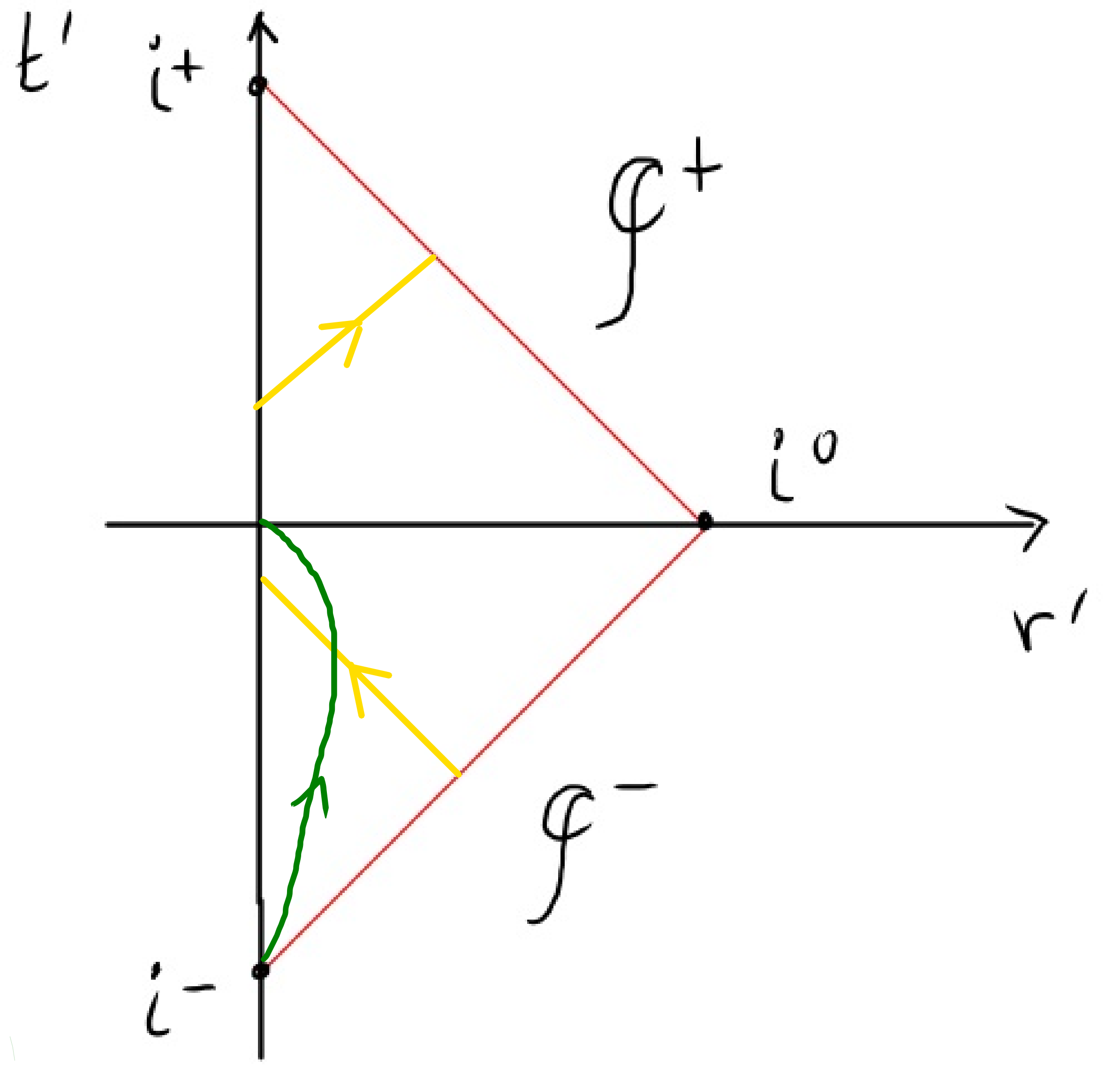
$$\left.\begin{aligned} \lim_{t\to+\infty} t' &= \frac{1}{2}\left\{ \frac{\pi}{2} + \frac{\pi}{2} \right\} = \frac{\pi}{2} \\ \lim_{t\to+\infty} r' &= \frac{1}{2}\left\{ \frac{\pi}{2} - \frac{\pi}{2} \right\} = 0 \end{aligned}\right\} \longrightarrow i^+$$

* Penrose Diagram of Minkowski Spacetime

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \equiv -dt^2 + dr^2 + r^2\, d\Omega^2$$

$$t' = \frac{1}{2}\left\{ \tan^{-1}\left((1+v)t\right) + \tan^{-1}\left((1-v)t\right) \right\}$$

$$r' = \frac{1}{2}\left\{ \tan^{-1}\left((1+v)t\right) - \tan^{-1}\left((1-v)t\right) \right\}$$



* Massive particle moving in radial direction

$$r = Vt \qquad 0 < V < 1 \quad \Rightarrow \quad 1 - V > 0$$

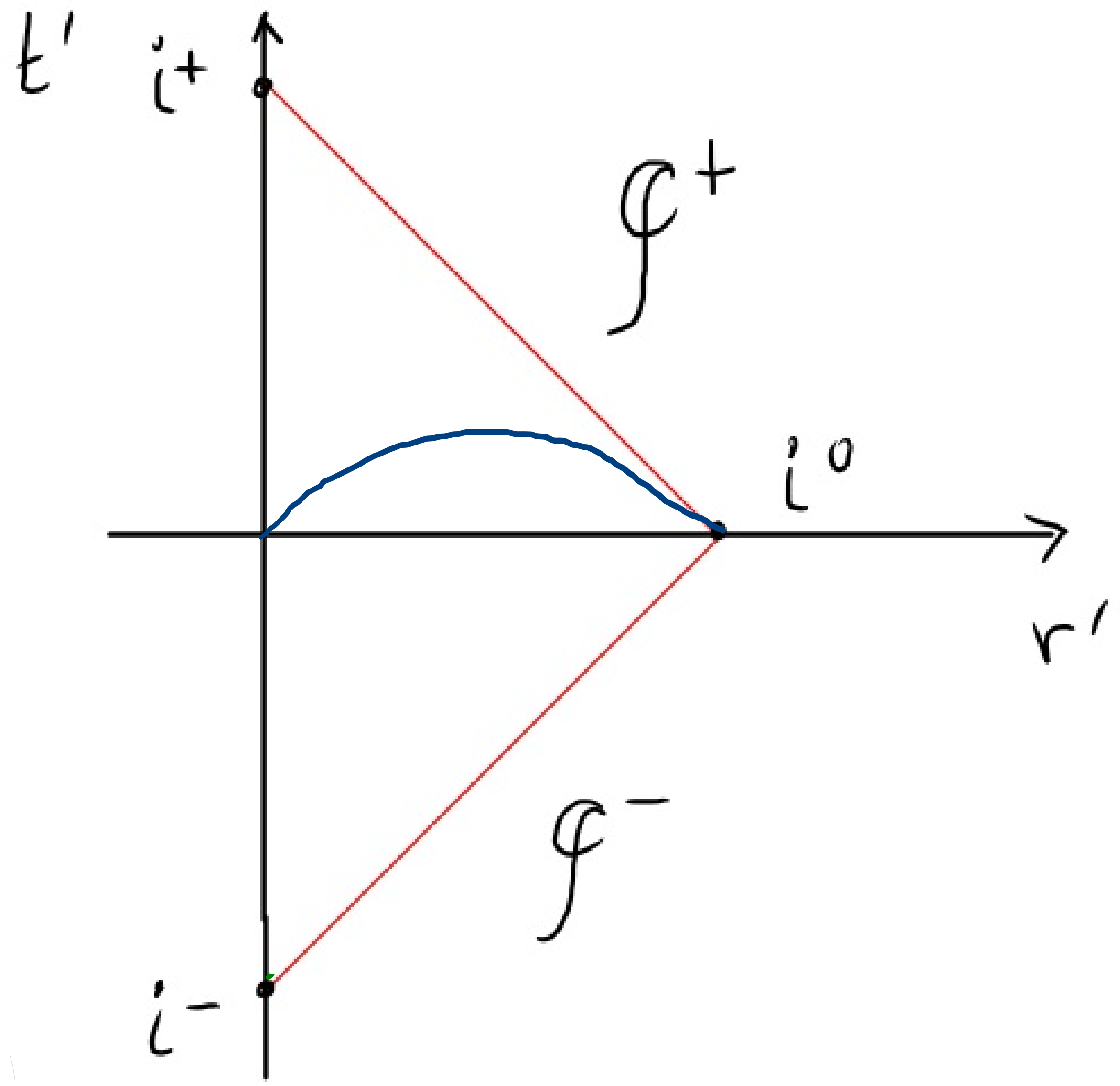
$$\left.\begin{aligned} \lim_{t\to-\infty} t' &= \frac{1}{2}\left\{\left(-\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}\right)\right\} = -\frac{\pi}{2} \\ \lim_{t\to-\infty} r' &= \frac{1}{2}\left\{\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)\right\} = 0 \end{aligned}\right\} \longrightarrow i^-$$

* Penrose Diagram of Minkowski Spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \equiv -dt^2 + dr^2 + r^2 d\Omega^2$$

$$t' = \frac{1}{2}\left\{ \tan^{-1}(1+V)t + \tan^{-1}(1-V)t \right\}$$

$$r' = \frac{1}{2}\left\{ \tan^{-1}(1+V)t - \tan^{-1}(1-V)t \right\}$$



* Spatial Slice

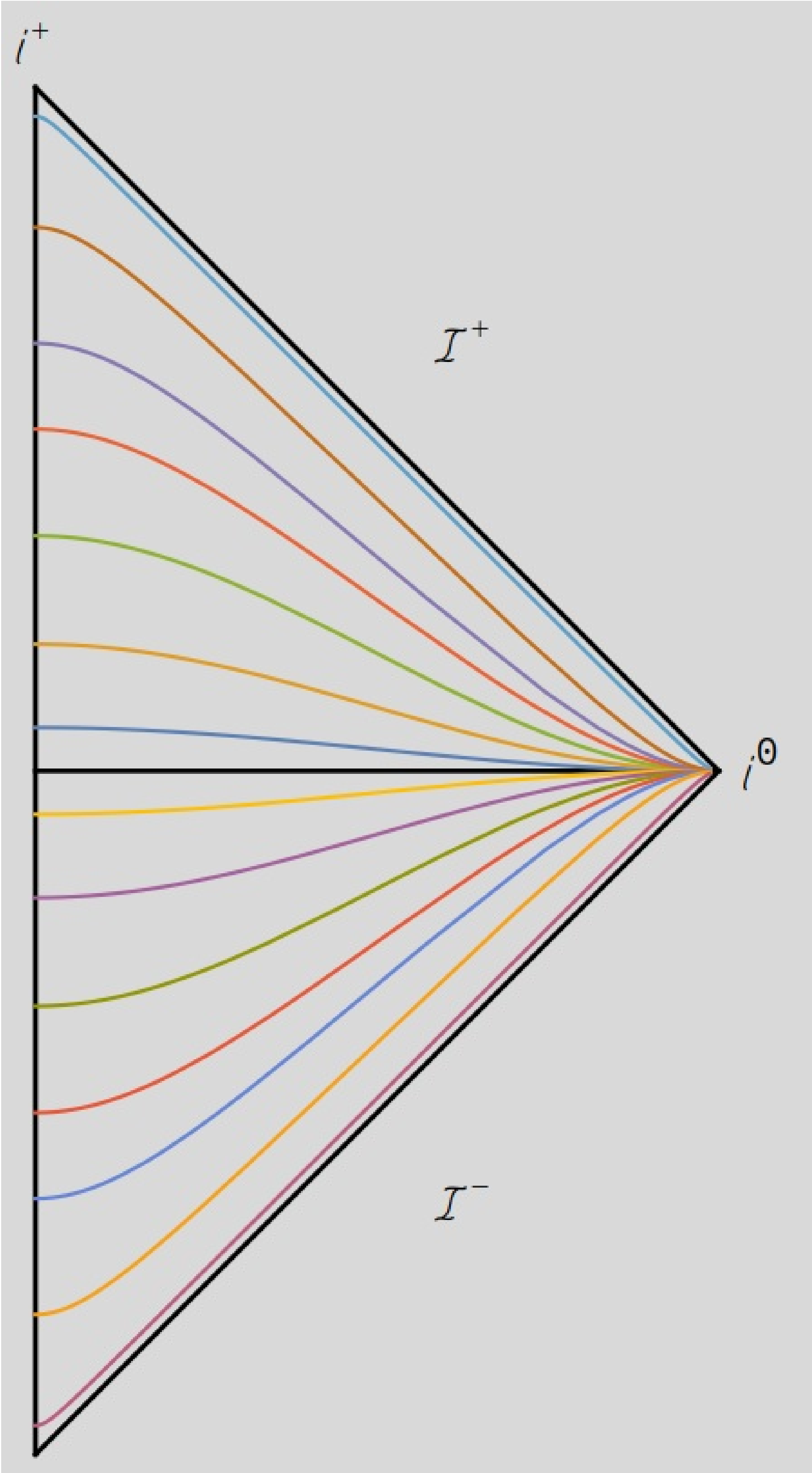
$r = Vt \qquad 1 < V \qquad \Rightarrow 1 - V < 0$

$$\lim_{t \to +\infty} t' = \frac{1}{2}\left\{\left(+\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}\right)\right\} = 0$$

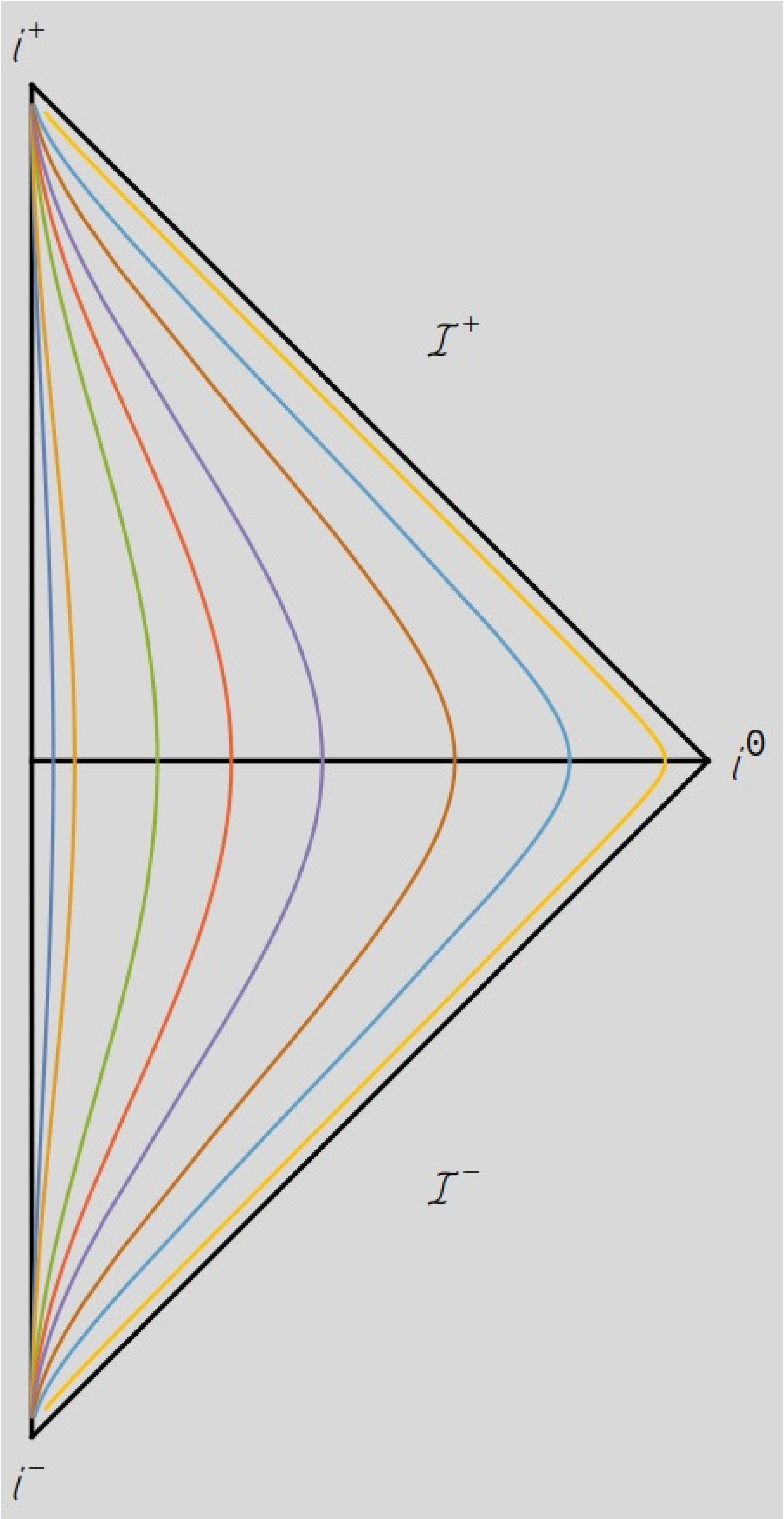
$$\lim_{t \to +\infty} r' = \frac{1}{2}\left\{\left(+\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)\right\} = \frac{\pi}{2}$$

$\longrightarrow i^0$

spatial slices of constant t



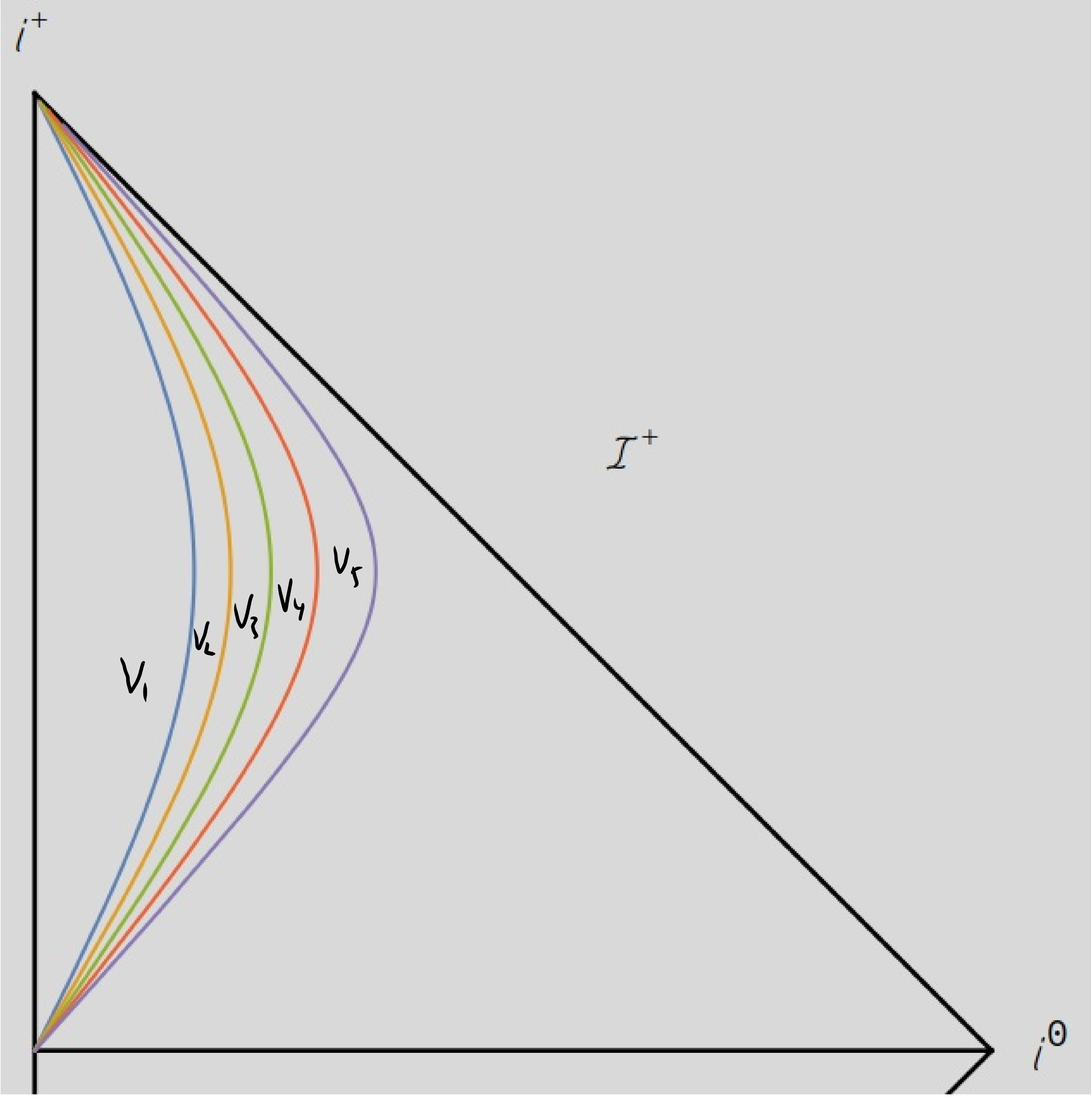
worldlines of constant r



worldlines of massive particles moving outwards with radial velocity $r = Vt$

$$V_1 < V_2 < \cdots < V_5$$

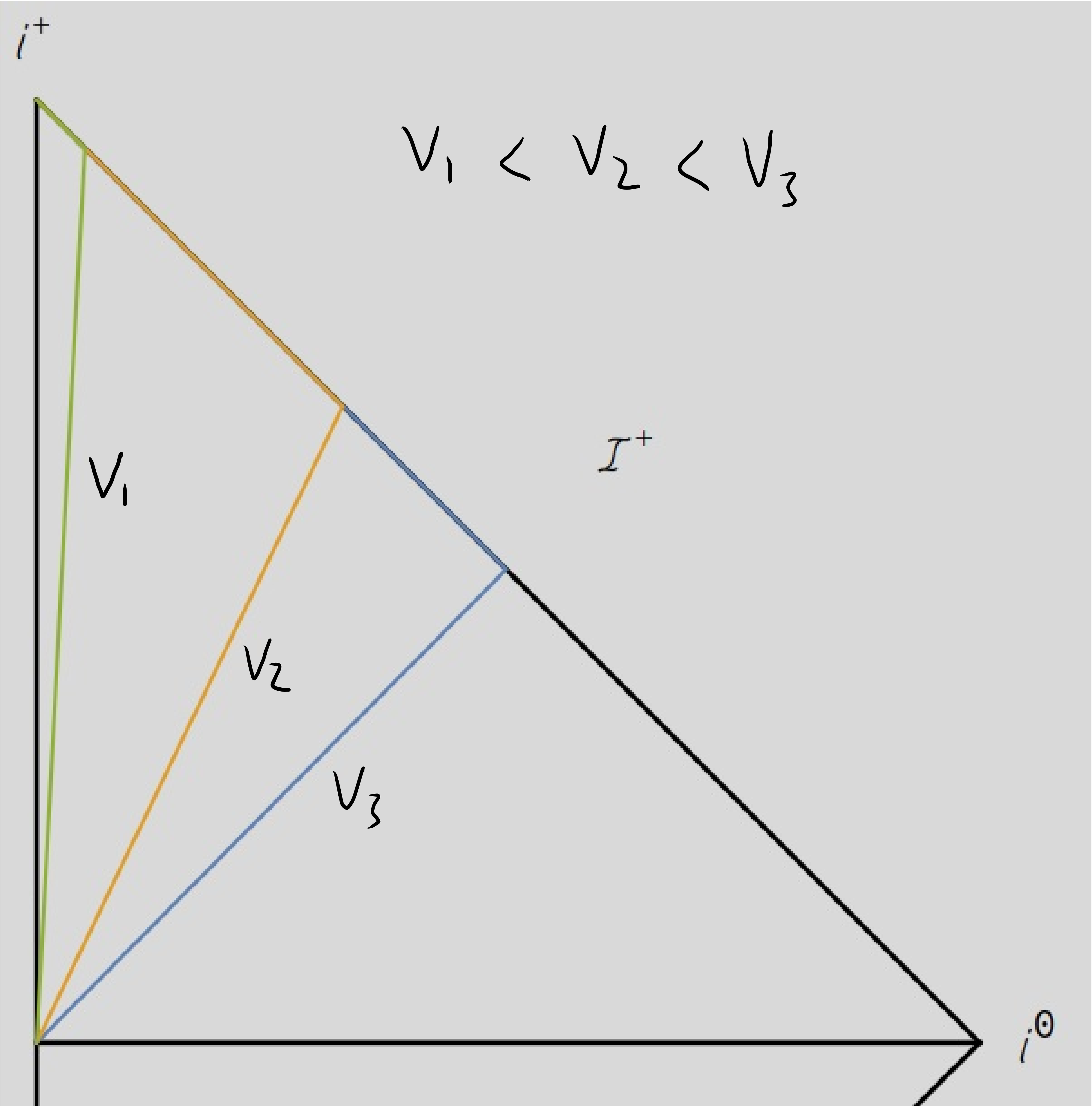
As $V \to 1$, the worldlines approach $\mathscr{I}^+$ and then move asymptotically close to it



worldlines of massive particles moving outwards with radial velocity $r = V t$

$V_1 < V_2 < \dots < V_5$

As $V \to 1$, the worldlines approach $\mathscr{I}^+$ and then move asymptotically close to it



Make the plots using Mathematica:

Define the boundaries:

```
gboundary = Graphics[{
   Thick, Black, Line[{{π/2, 0  }, {0  ,  π/2}}],
   Thick, Black, Line[{{π/2, 0  }, {0  , -π/2}}],
   Thick, Black, Line[{{0  , π/2}, {0  , -π/2}}],
   Thick, Black, Line[{{0  , 0  }, {π/2, 0   }}],
   Black, Text[Style["𝒾+", Large], {0        ,  π/2 + 0.1}, FormatType → StandardForm],
   Black, Text[Style["𝒾-", Large], {0        , -π/2 - 0.1}, FormatType → StandardForm],
   Black, Text[Style["𝒾0", Large], {π/2 + 0.1,  0         }, FormatType → StandardForm],
   Black, Text[Style["ℐ+", Large], {π/4 + 0.2,  π/4 + 0.2}, FormatType → StandardForm],
   Black, Text[Style["ℐ-", Large], {π/4 + 0.2, -π/4 - 0.2}, FormatType → StandardForm]
  }];
```

Plot $r = vt$ curves

```
r [t_] = v t;
tp[t_] = 0.5 (ArcTan[t + r[t]] + ArcTan[t - r[t]]);
rp[t_] = 0.5 (ArcTan[t + r[t]] - ArcTan[t - r[t]]);
gplot = ParametricPlot[
   {
    {rp[t], tp[t]} /. v → 0.5,
    {rp[t], tp[t]} /. v → 0.6,
    {rp[t], tp[t]} /. v → 0.7,
    {rp[t], tp[t]} /. v → 0.8,
    {rp[t], tp[t]} /. v → 0.9
   }, {t, 0, 100}, PlotRange → All];
Show[gboundary, gplot]
```

Plot $r = \text{const.}$ curves

```
rr[τ_] = a;
tt[τ_] = τ;
tp[τ_] = 0.5 (ArcTan[tt[τ] + rr[τ]] + ArcTan[tt[τ] - rr[τ]]);
rp[τ_] = 0.5 (ArcTan[tt[τ] + rr[τ]] - ArcTan[tt[τ] - rr[τ]]);
gplot = ParametricPlot[
   {
    {rp[τ], tp[τ]} /. a → .05,
    {rp[τ], tp[τ]} /. a → .1,
    {rp[τ], tp[τ]} /. a → .3,
    {rp[τ], tp[τ]} /. a → .5,
    {rp[τ], tp[τ]} /. a → .8,
    {rp[τ], tp[τ]} /. a → 1.5,
    {rp[τ], tp[τ]} /. a → 3,
    {rp[τ], tp[τ]} /. a → 10
   }, {τ, -20, 20}, PlotRange → All];
Show[gboundary, gplot]
```

Make the plots using Mathematica:

Define the boundaries:

```
gboundary = Graphics[{
    Thick, Black, Line[{{π/2, 0   }, {0   , π/2}}],
    Thick, Black, Line[{{π/2, 0   }, {0   , -π/2}}],
    Thick, Black, Line[{{0   , π/2}, {0   , -π/2}}],
    Thick, Black, Line[{{0   , 0   }, {π/2, 0    }}],
    Black, Text[Style["i+", Large], {0         , π/2 + 0.1}, FormatType → StandardForm],
    Black, Text[Style["i-", Large], {0         , -π/2 - 0.1}, FormatType → StandardForm],
    Black, Text[Style["i0", Large], {π/2 + 0.1, 0          }, FormatType → StandardForm],
    Black, Text[Style["ℐ+", Large], {π/4 + 0.2, π/4 + 0.2}, FormatType → StandardForm],
    Black, Text[Style["ℐ-", Large], {π/4 + 0.2, -π/4 - 0.2}, FormatType → StandardForm]
  }];
```

Plot r = vt curves

```
r [t_] = v t ;
tp[t_] = 0.5 (ArcTan[t + r[t]] + ArcTan[t - r[t]]);
rp[t_] = 0.5 (ArcTan[t + r[t]] - ArcTan[t - r[t]]);
gplot = ParametricPlot[
   {
    {rp[t], tp[t]} /. v → 0.5,
    {rp[t], tp[t]} /. v → 0.6,
    {rp[t], tp[t]} /. v → 0.7,
    {rp[t], tp[t]} /. v → 0.8,
    {rp[t], tp[t]} /. v → 0.9
   }, {t, 0, 100}, PlotRange → All];
Show[gboundary, gplot]
```

Plot t = const. curves

```
rr[τ_] = τ;
tt[τ_] = a;
tp[τ_] = 0.5 (ArcTan[tt[τ] + rr[τ]] + ArcTan[tt[τ] - rr[τ]]);
rp[τ_] = 0.5 (ArcTan[tt[τ] + rr[τ]] - ArcTan[tt[τ] - rr[τ]]);
gplot = ParametricPlot[
   {
    {rp[τ], tp[τ]} /. a → .1,
    {rp[τ], tp[τ]} /. a → .3,
    {rp[τ], tp[τ]} /. a → .6,
    {rp[τ], tp[τ]} /. a → 1,
    {rp[τ], tp[τ]} /. a → 1.5,
    {rp[τ], tp[τ]} /. a → 3,
    {rp[τ], tp[τ]} /. a → 15,
    {rp[τ], tp[τ]} /. a → -.1,
    {rp[τ], tp[τ]} /. a → -.3,
    {rp[τ], tp[τ]} /. a → -.6,
    {rp[τ], tp[τ]} /. a → -1,
    {rp[τ], tp[τ]} /. a → -1.5,
    {rp[τ], tp[τ]} /. a → - 3,
    {rp[τ], tp[τ]} /. a → - 15
   }, {τ, 0, 50}, PlotRange → All];
Show[gboundary, gplot]
```