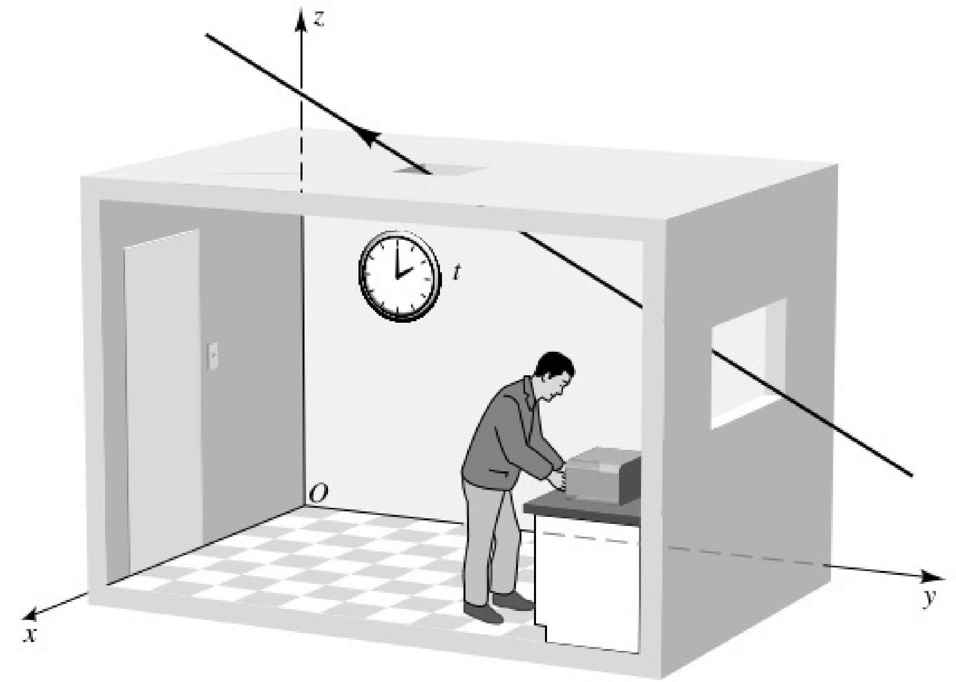
Special Relativity:

The geometry of flat spacetime

* Maxwell's equations for electromagnetism and Galilean transformations are incompatible:
particles moving at c have same speed for all observers

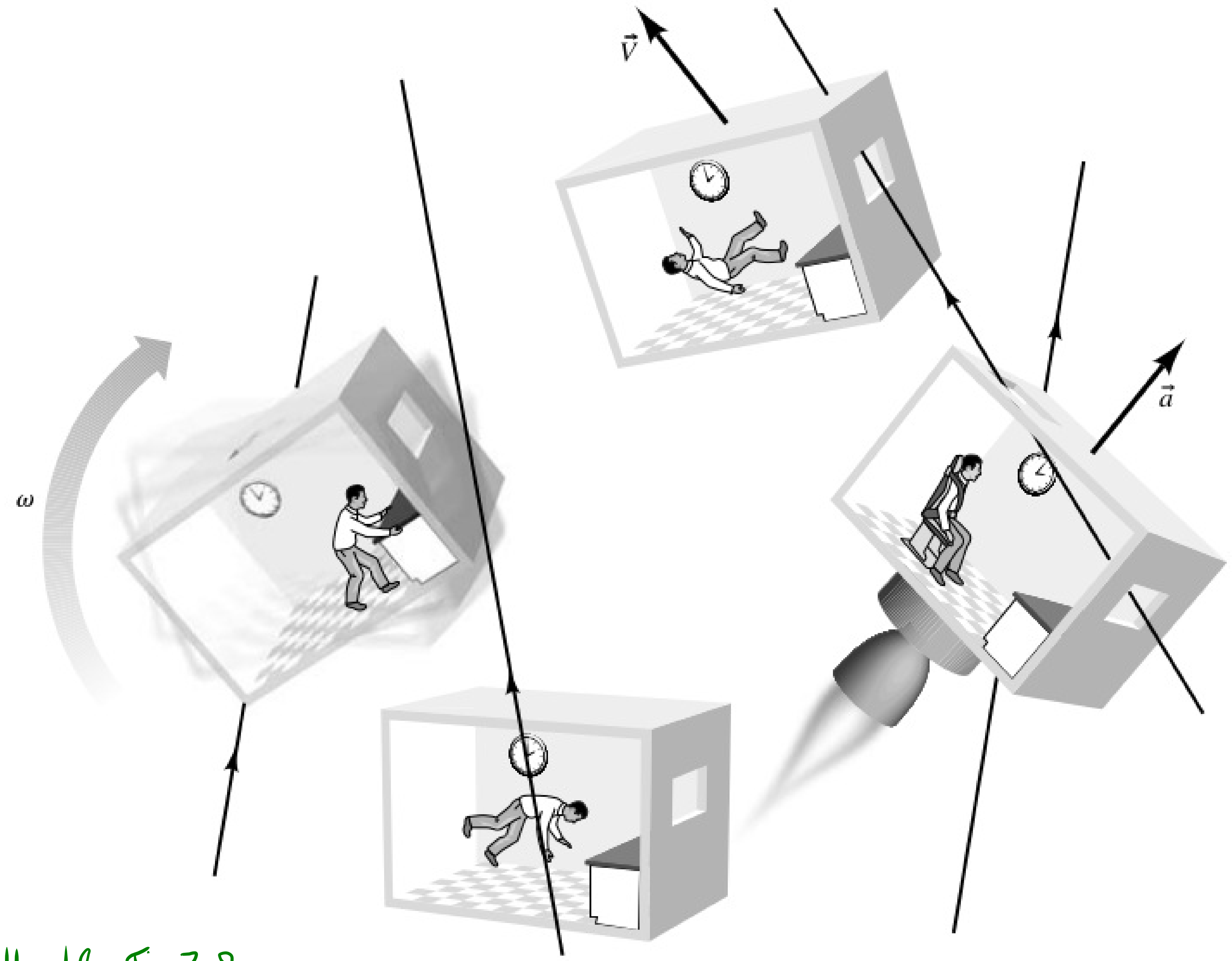
$$c = \frac{\Delta x}{\Delta t} \quad \Rightarrow \quad (\text{absolute}) = \frac{(\text{relative})}{(\text{relative})}$$

* Inertial frames: Labs where free particles move on straight lines @ constant speed



Hartle, Fig 3.1

* Inertial observers have relative constant velocities
* Not all observers are inertial:

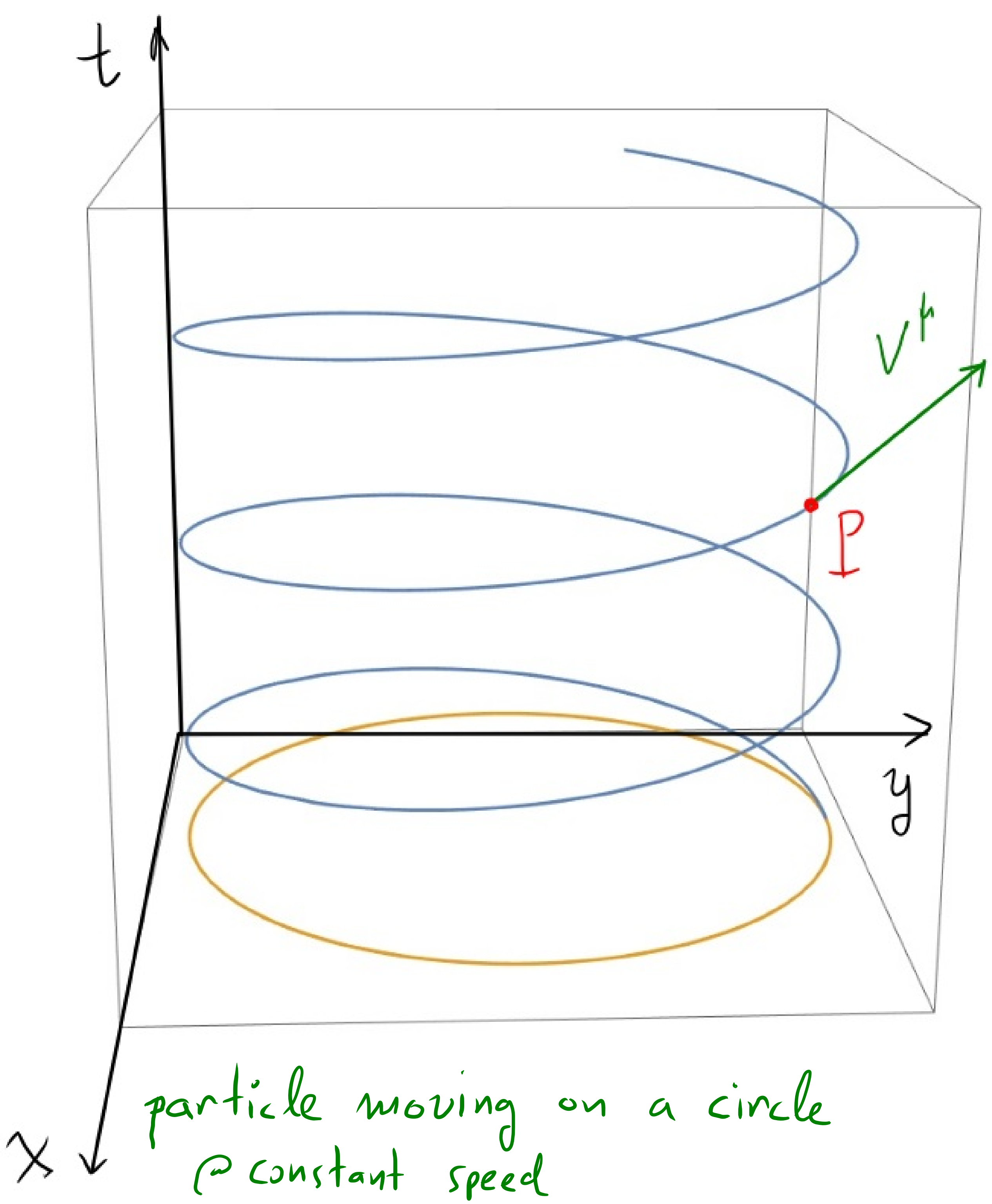


Hartle Fig 3.2

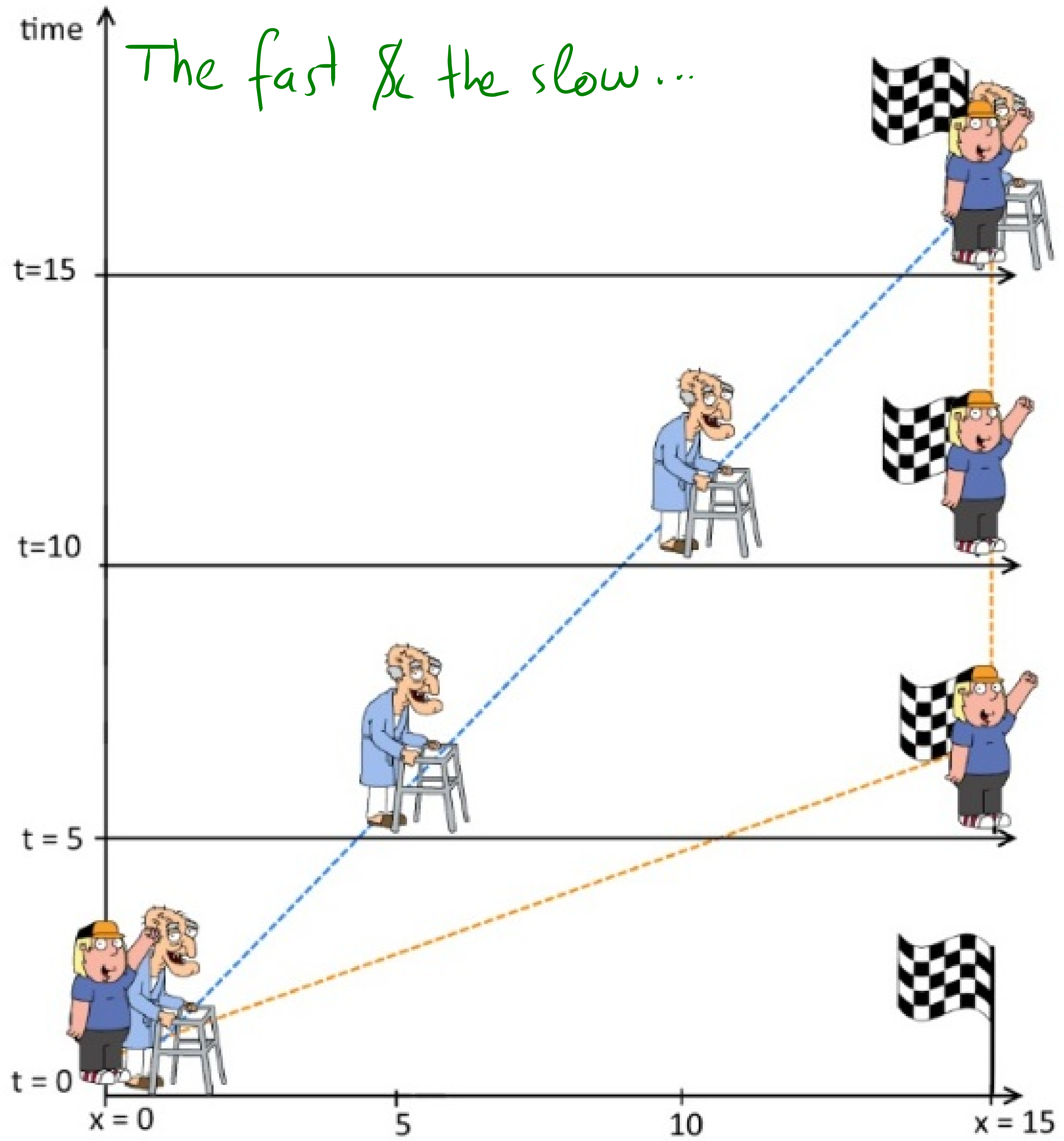
* Spacetime: geometry of events: $P(t,x,y,z)$

something that happens
sometime,
somewhere

* Spacetime: geometry of events: $P(t,x,y,z)$



particle moving on a circle
@ constant speed



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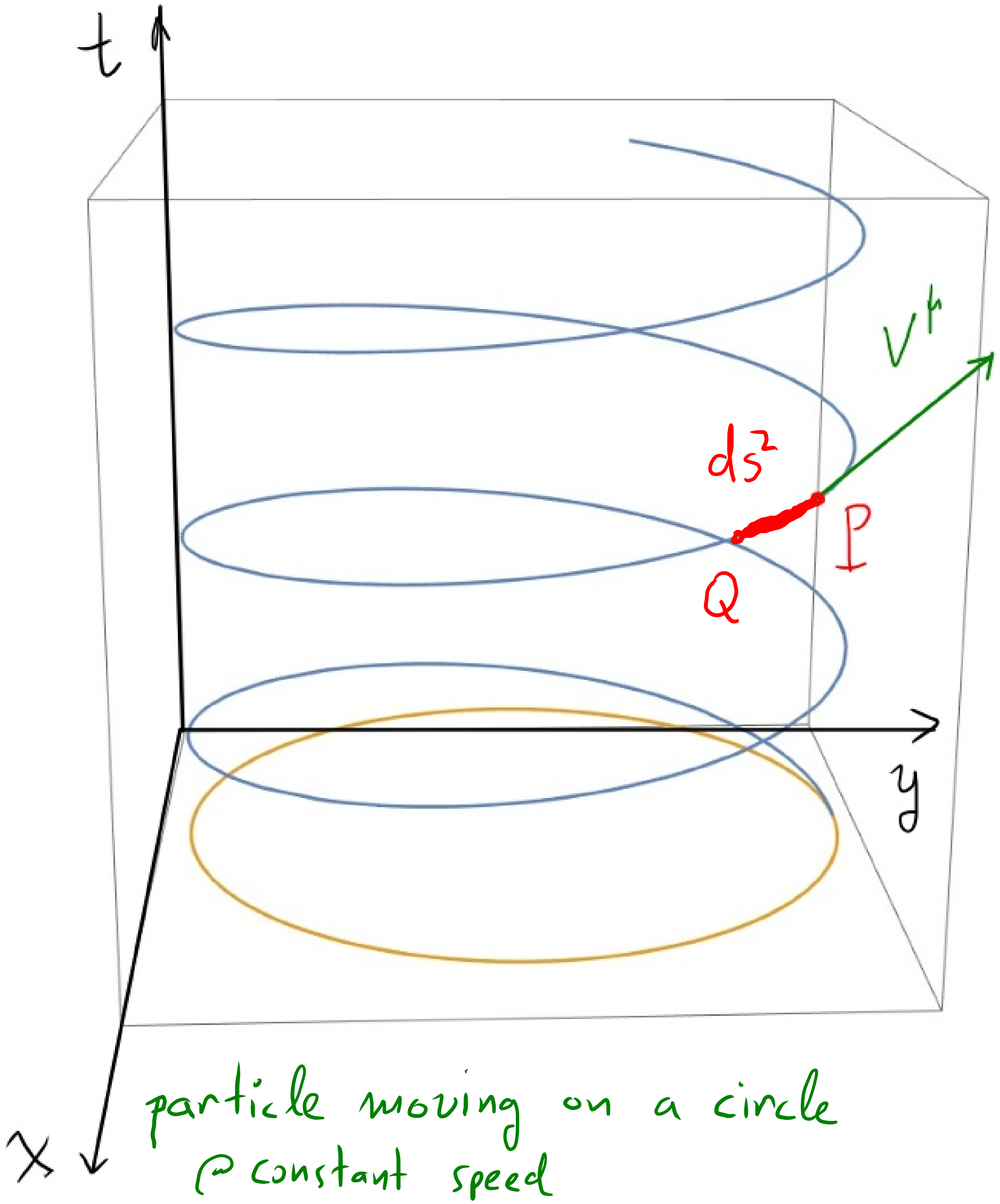


particle moving on a circle
@ constant speed

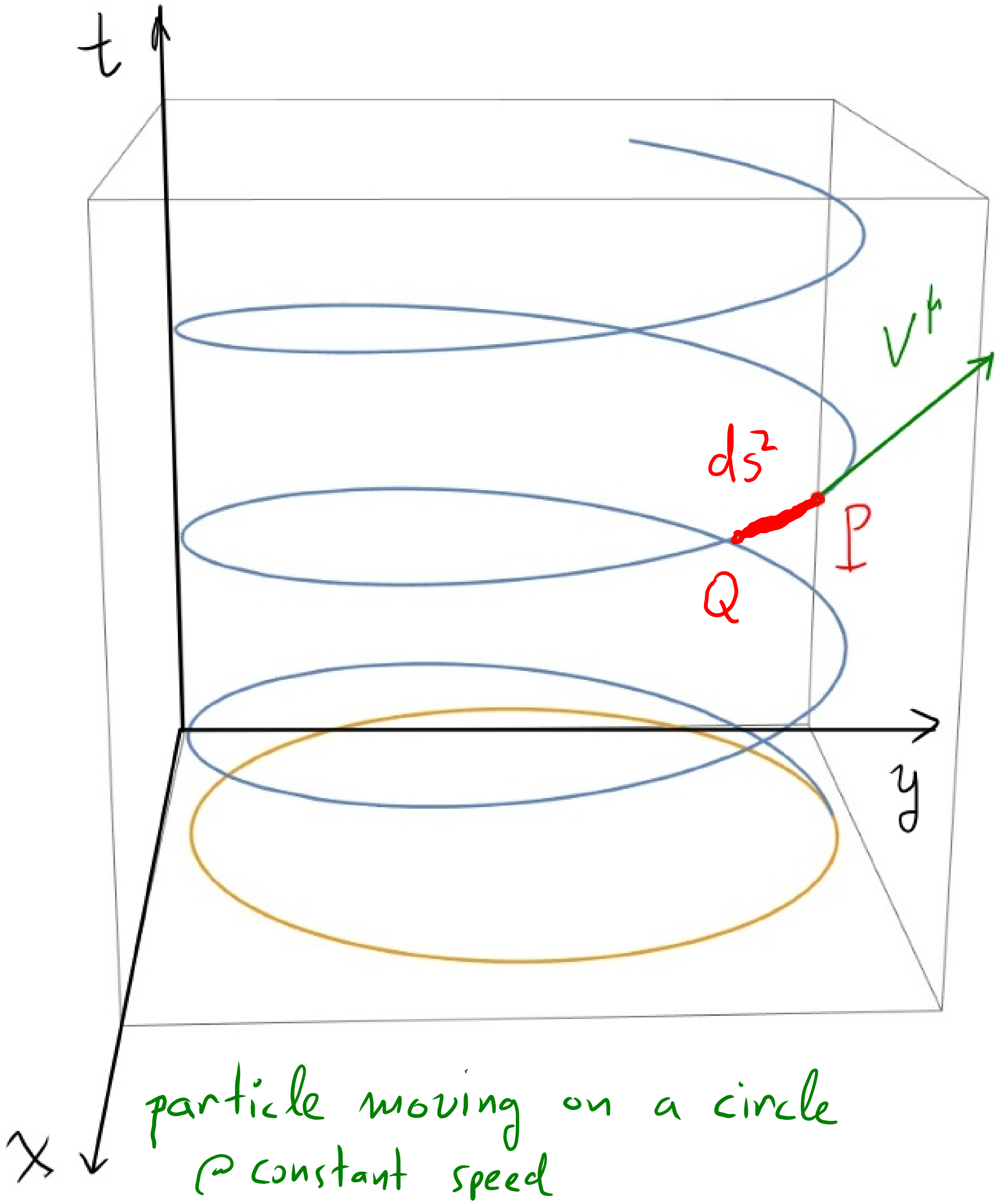
metric: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$= \eta_{\mu\nu}\, dx^\mu dx^\nu$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ds: spacetime distance
observer invariant

* Spacetime: geometry of events: $P(t,x,y,z)$



metric: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$= \eta_{\mu\nu}\, dx^\mu dx^\nu$$

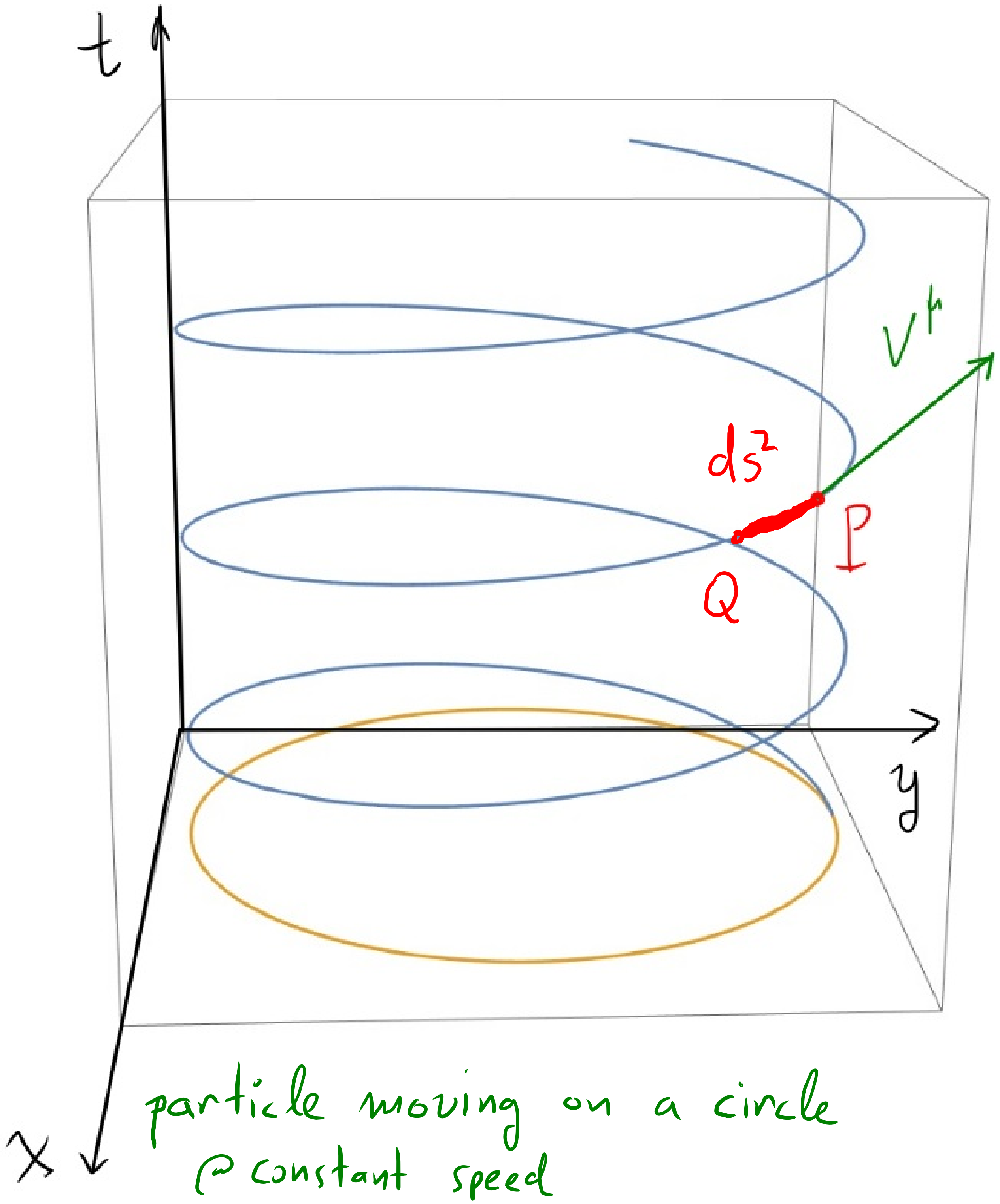
(α) $dx = dy = dz = 0$

$\rightsquigarrow ds^2 = -dt^2$

define $d\tau^2 = -ds^2 = -dt^2$

$d\tau$: proper time

* Spacetime: geometry of events: $P(t,x,y,z)$



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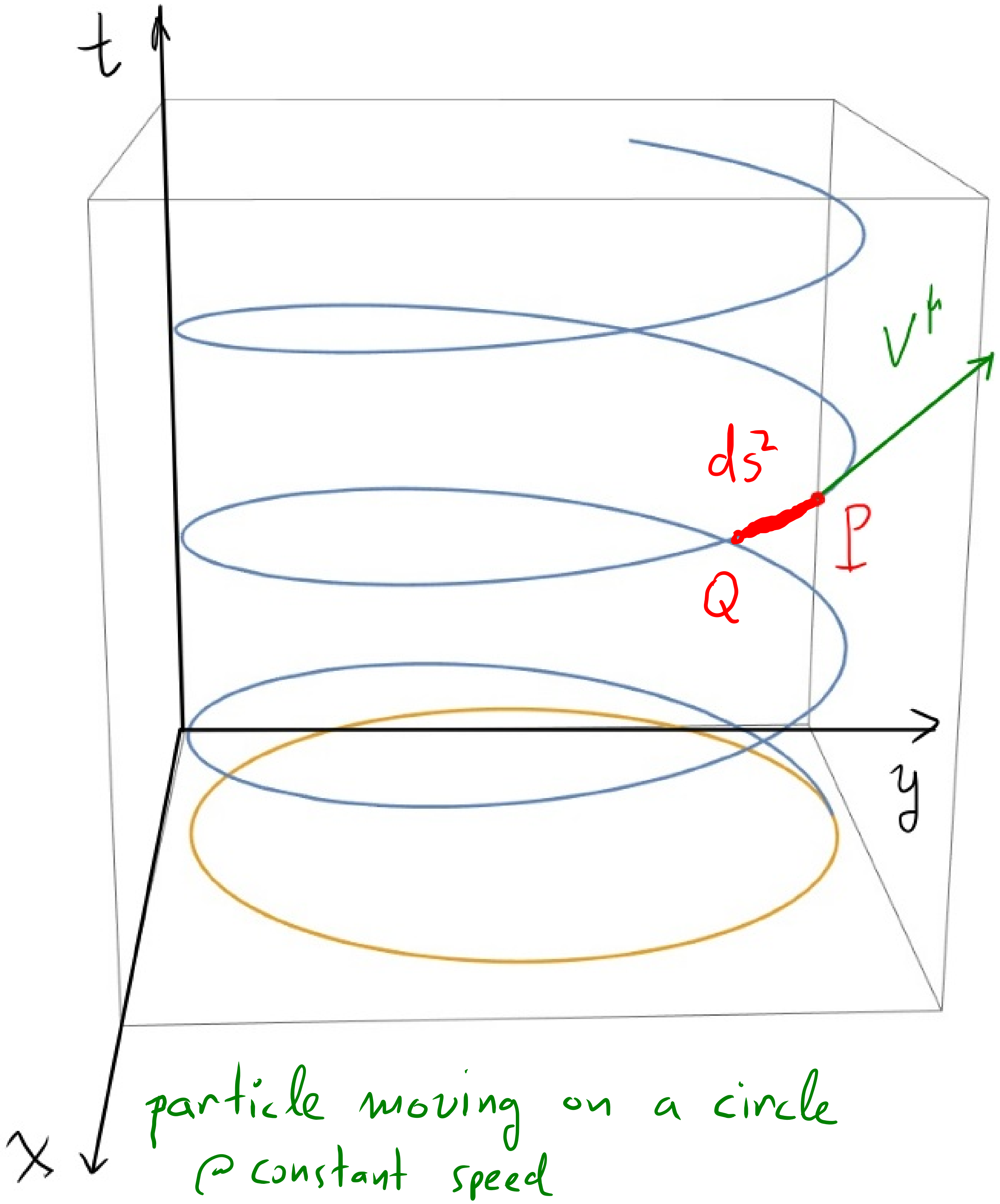
$d\tau$: proper time

(β) $dt = 0 \rightsquigarrow ds^2 = dx^2 + dy^2 + dz^2$

ds: distance / length

* Spacetime: geometry of events: $P(t,x,y,z)$



(γ) $dy = dz = 0$

$ds^2 = -dt^2 + dx^2$

if dx changes $\Rightarrow$ dt changes to keep ds fixed

(α) $dx = dy = dz = 0$

$\rightsquigarrow ds^2 = -dt^2$

define $d\tau^2 = -ds^2 = -dt^2$

$d\tau$: proper time

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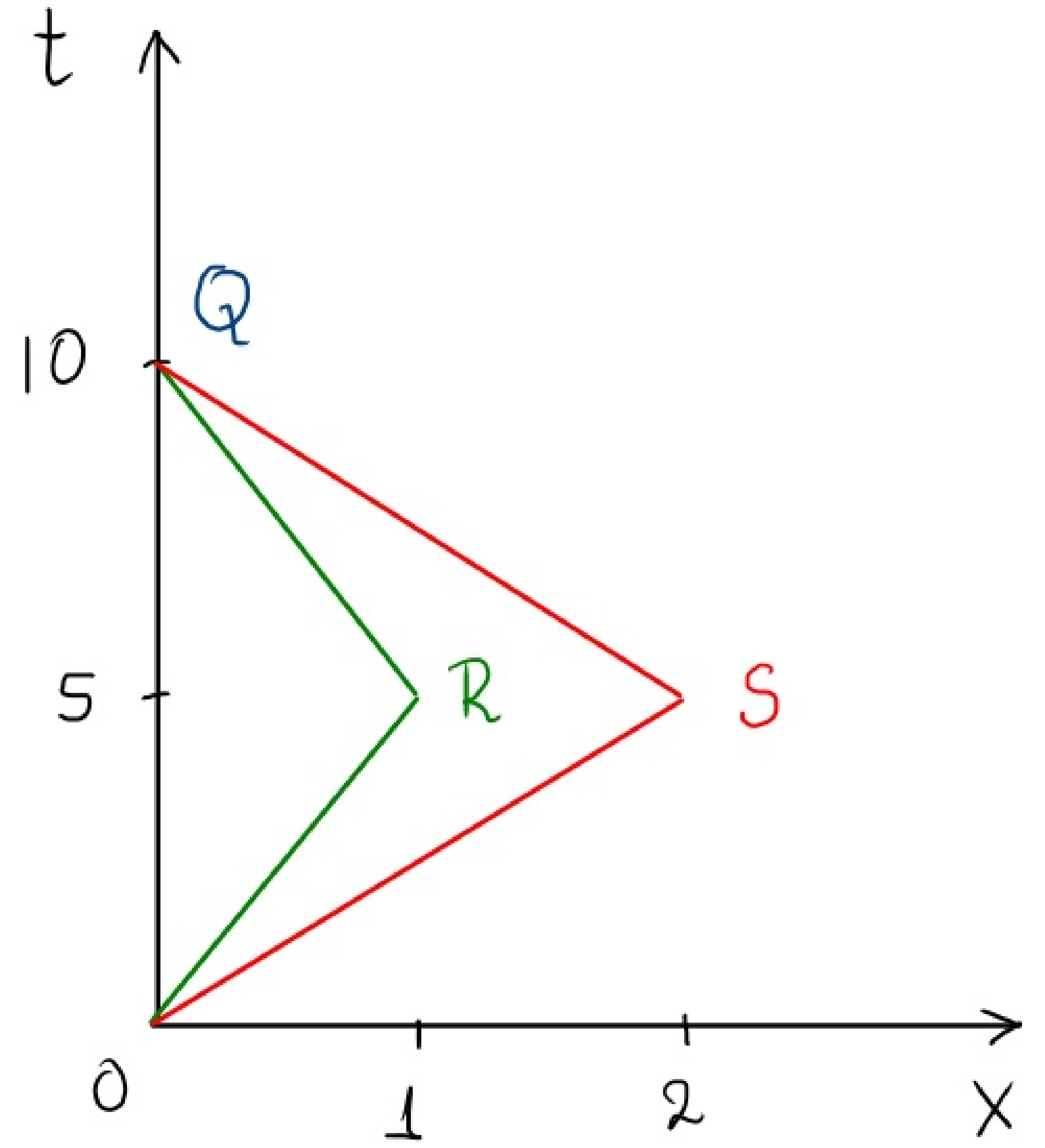
ds: distance / length

* Minkowski geometry: not to be confused by Euclidean!

e.g the spacetime length

$$|S_{OQ}| > |S_{ORQ}| > |S_{OSQ}|$$

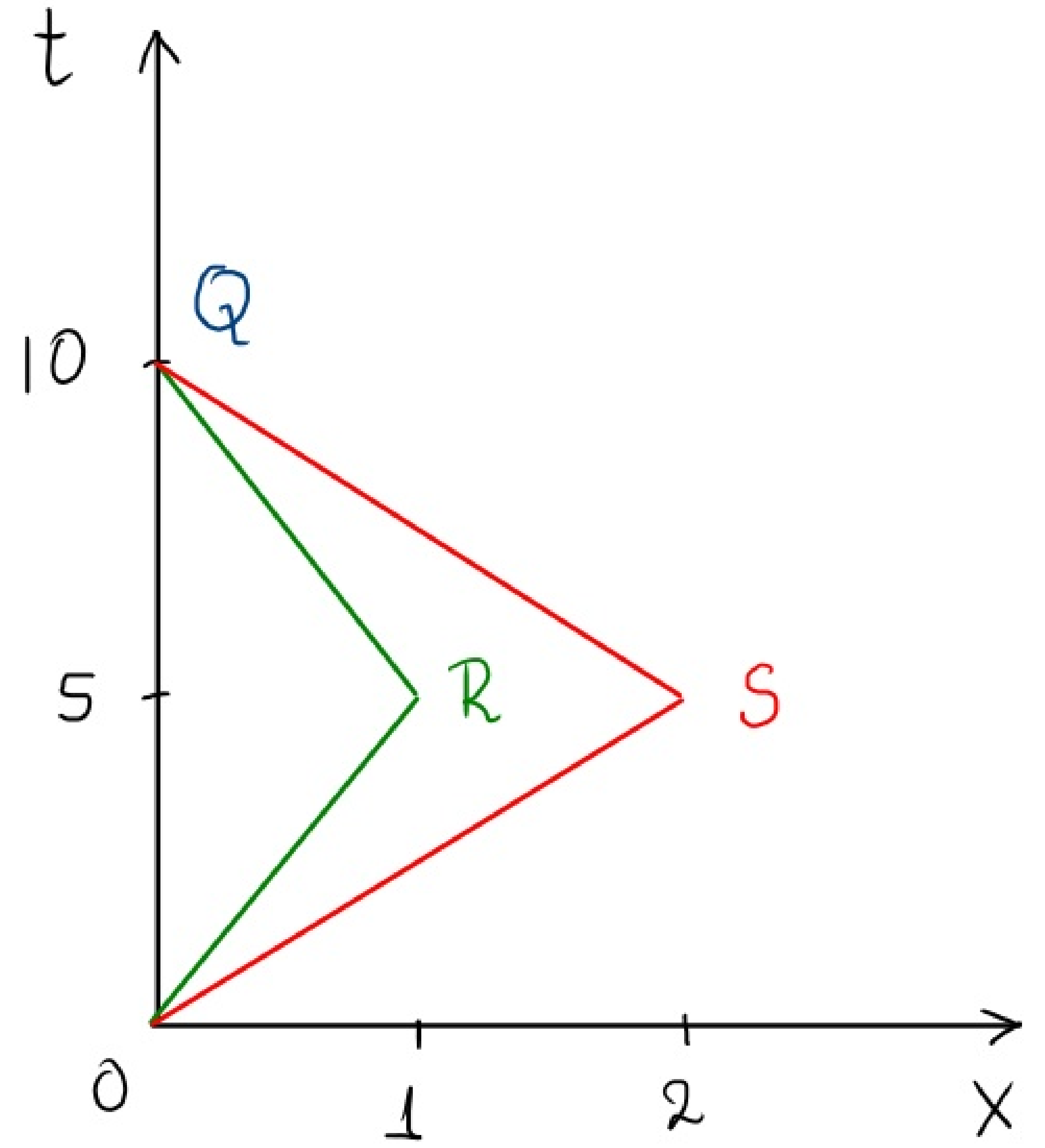


* Minkowski geometry: not to be confused by Euclidean!

e.g the spacetime length

$$|S_{OQ}| > |S_{ORQ}| > |S_{OSQ}|$$

$$S_{OQ}^2 = -t_{OQ}^2 + 0 = -10^2 \Rightarrow |S_{OQ}| = 10$$



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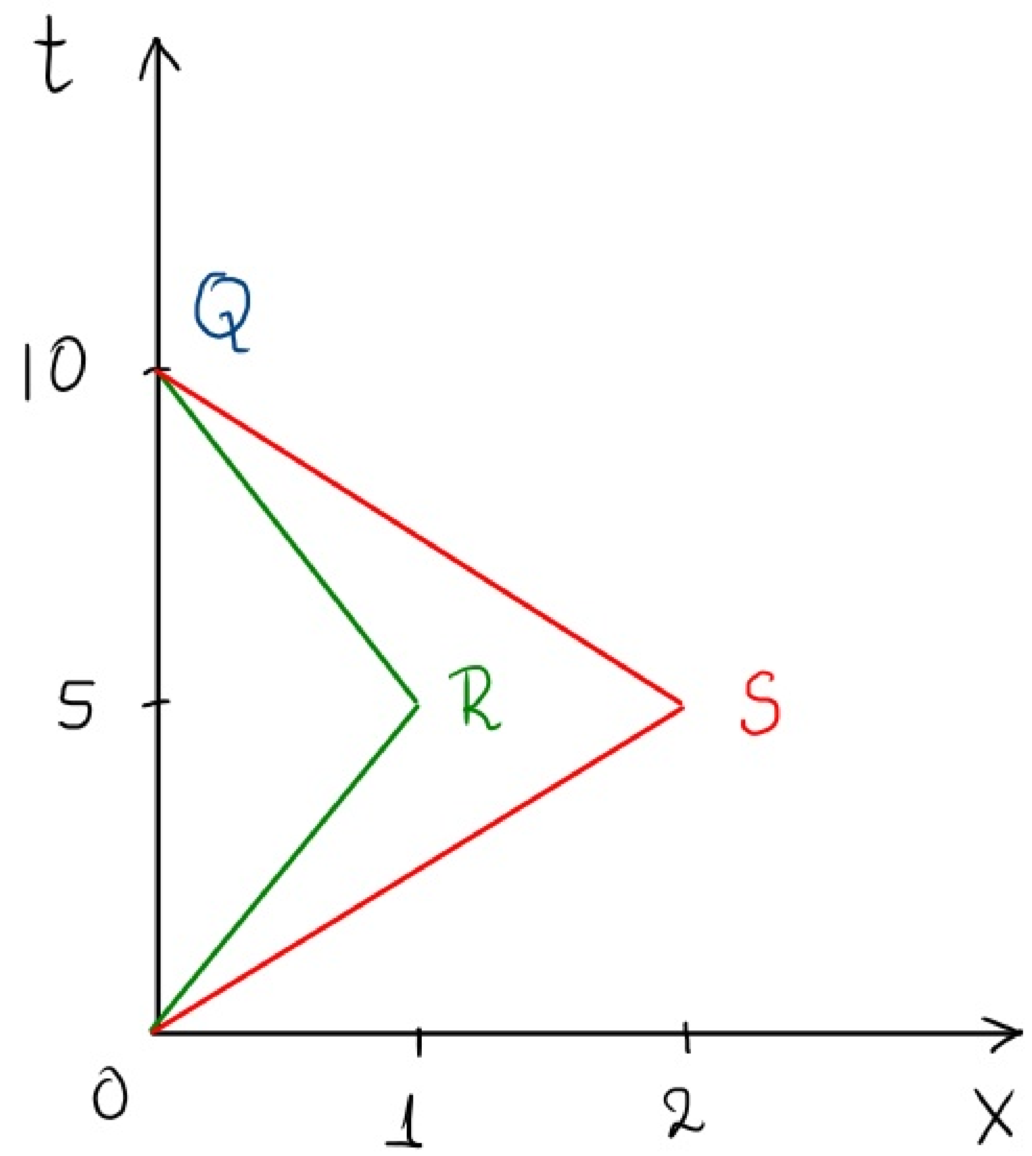


$$S_{OQ}^2 = -t_{OQ}^2 + 0 = -10^2 \Rightarrow |S_{OQ}| = 10$$

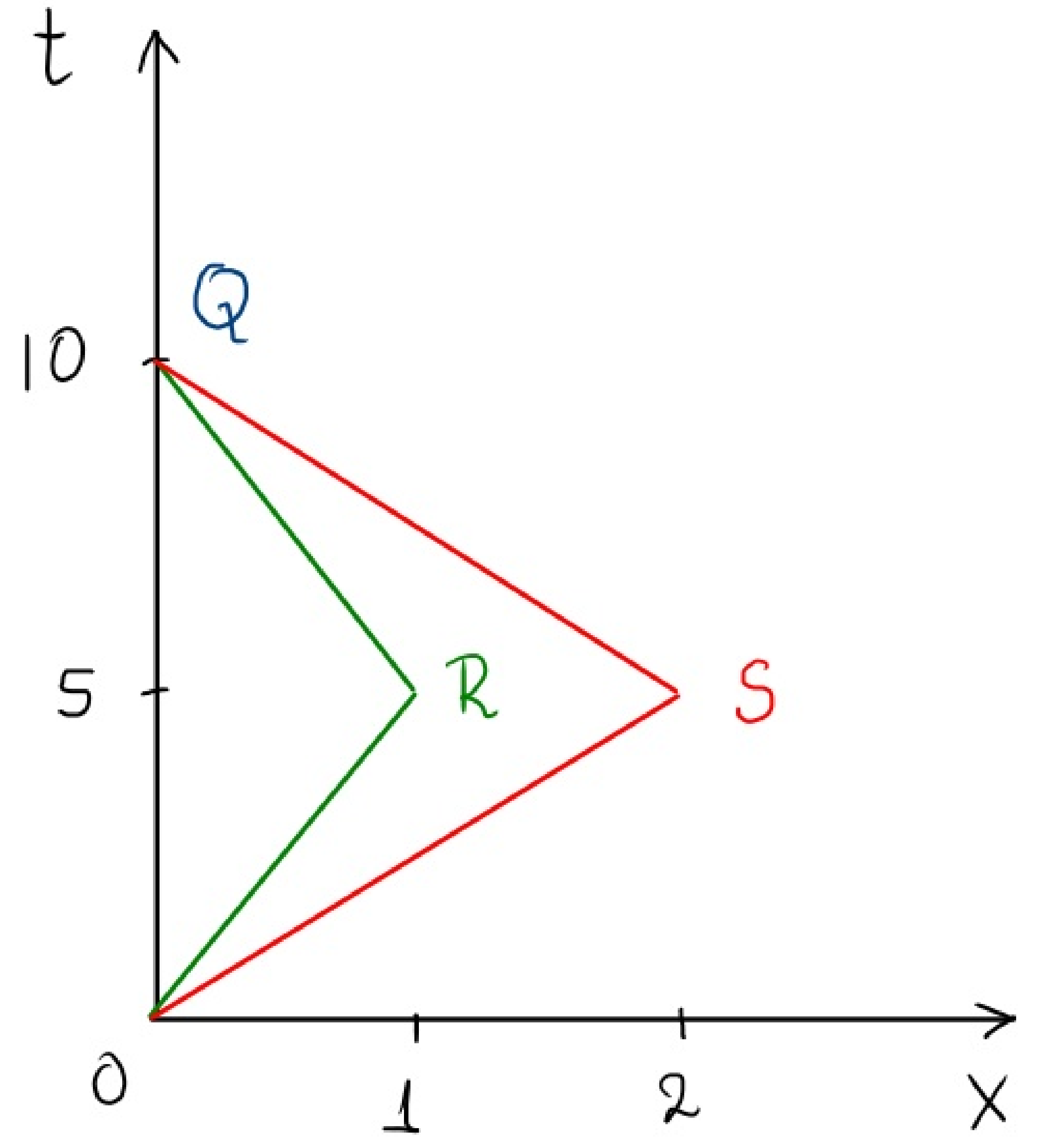
$$S_{ORQ}^2 = 2\, S_{OR}^2 = 2(-t_{OR}^2 + X_{OR}^2) = 2(-5^2 + 1^2)$$

$$= 2(-24) = -48 \qquad \Rightarrow |S_{ORQ}| = \sqrt{48}$$

* Minkowski geometry: not to be confused by Euclidean!

e.g the spacetime length

$$|S_{OQ}| > |S_{ORQ}| > |S_{OSQ}|$$



$$S_{OQ}^2 = -t_{OQ}^2 + 0 = -10^2 \Rightarrow |S_{OQ}| = \sqrt{100}$$

$$S_{ORQ}^2 = 2\,S_{OR}^2 = 2(-t_{OR}^2 + X_{OR}^2) = 2(-5^2 + 1^2)$$

$$= 2(-24) = -48 \qquad \Rightarrow |S_{ORQ}| = \sqrt{48}$$

$$S_{OSQ}^2 = 2\,S_{OS}^2 = 2(-t_{OS}^2 + X_{OS}^2) = 2(-5^2 + 2^2)$$

$$= 2(-21) = -42 \qquad \Rightarrow |S_{OSQ}| = \sqrt{42}$$

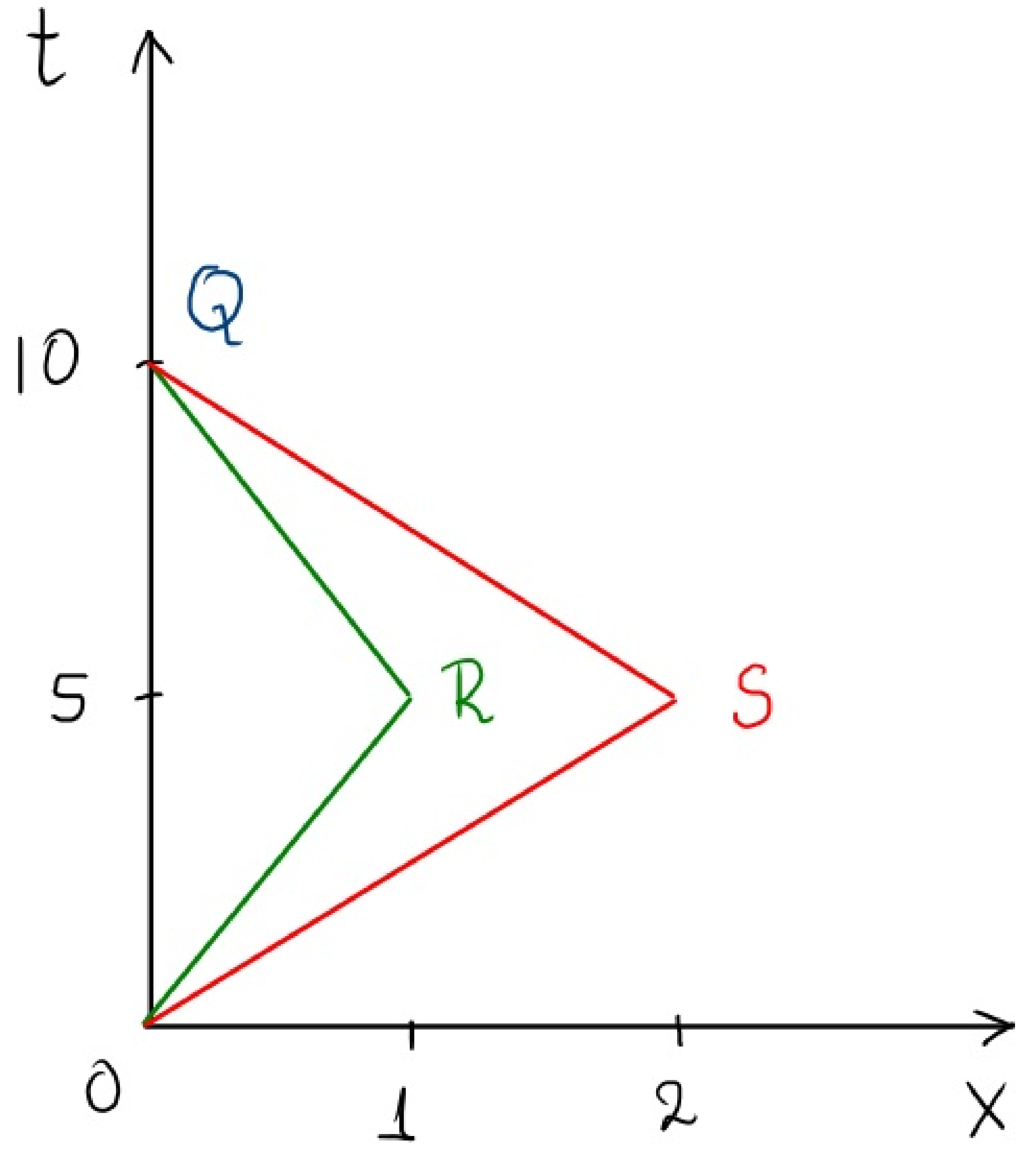
* Minkowski geometry: not to be confused by Euclidean!

e.g the spacetime length

$$|S_{OQ}| > |S_{ORQ}| > |S_{OSQ}|$$

$$\tau_{OQ} = \sqrt{100} > \tau_{ORQ} = \sqrt{48} > \tau_{OSQ} = \sqrt{42}$$

- the twin paradox: straight line connecting two timelike separated events is of longest proper time



* Minkowski geometry: not to be confused by Euclidean!

e.g: a "circle": locus of points at constant distance from a point.

Euclidean: $s^2 = x^2 + y^2 = R^2$ $\quad z=0$

Minkowski: $s^2 = -t^2 + x^2 = -R^2$ $\quad y=z=0$

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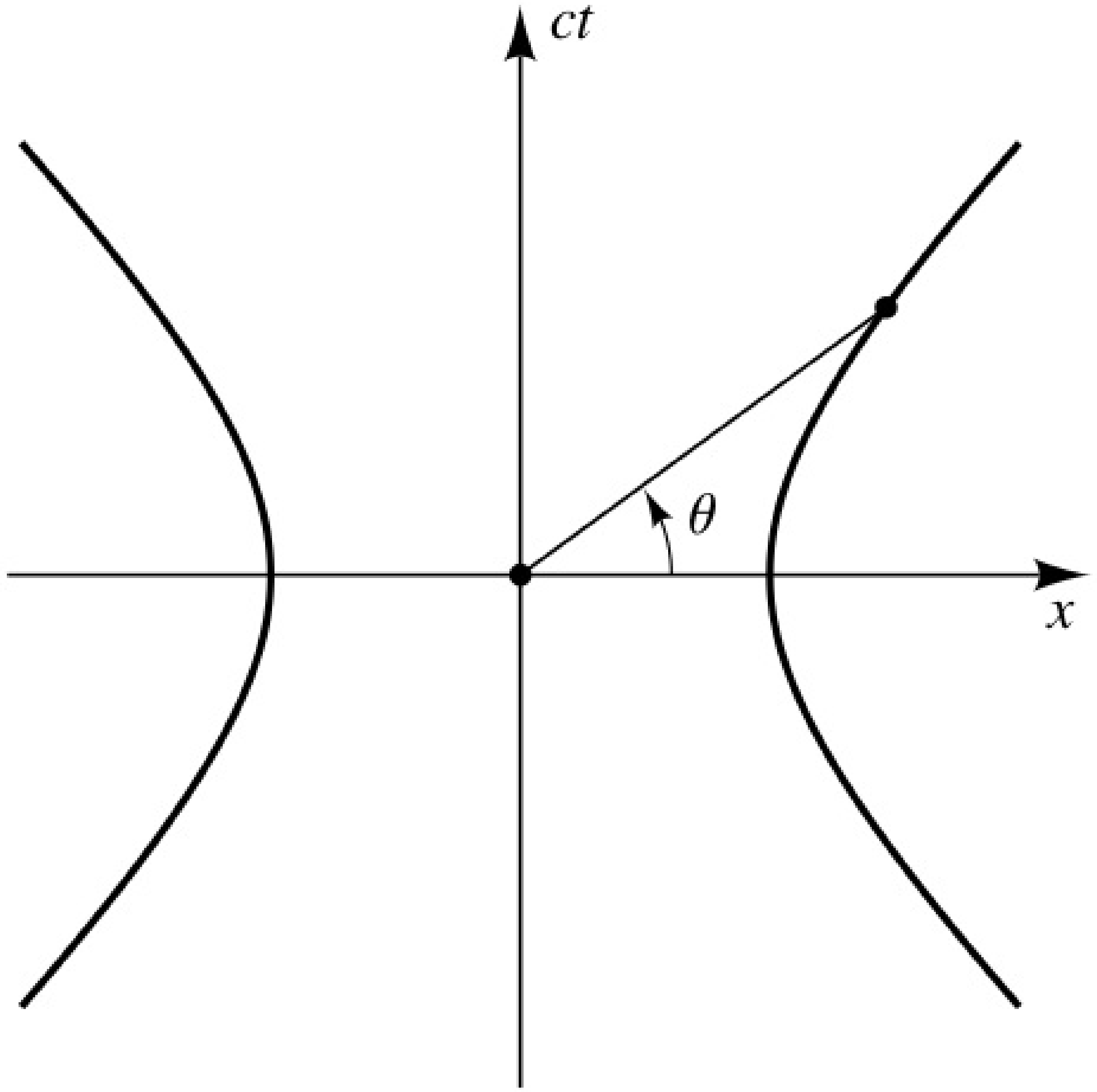

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set $y=z=0$, then

$x^2 - t^2 = R^2$ hyperbola

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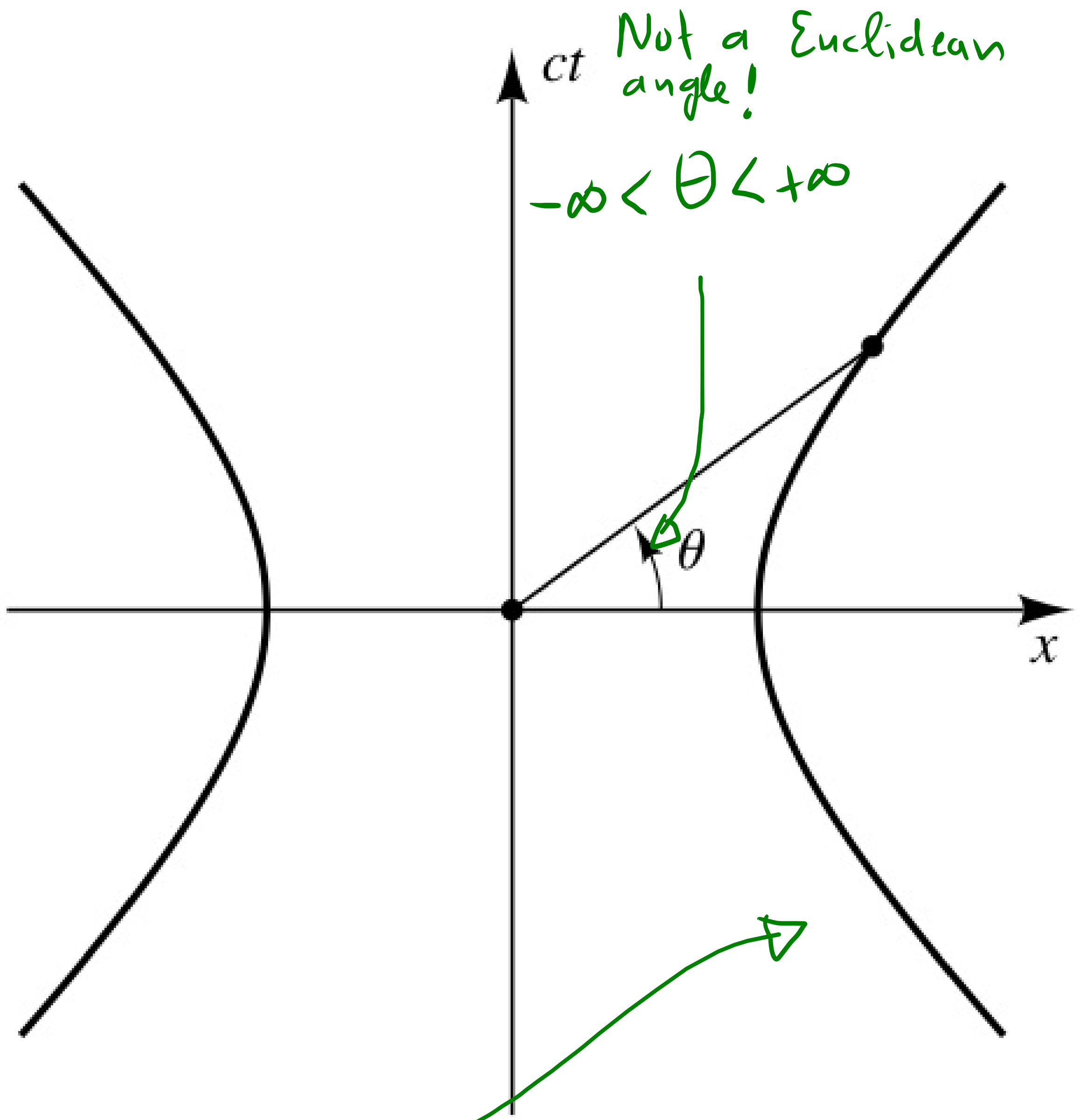


Parametric Equations: (only for right branch)

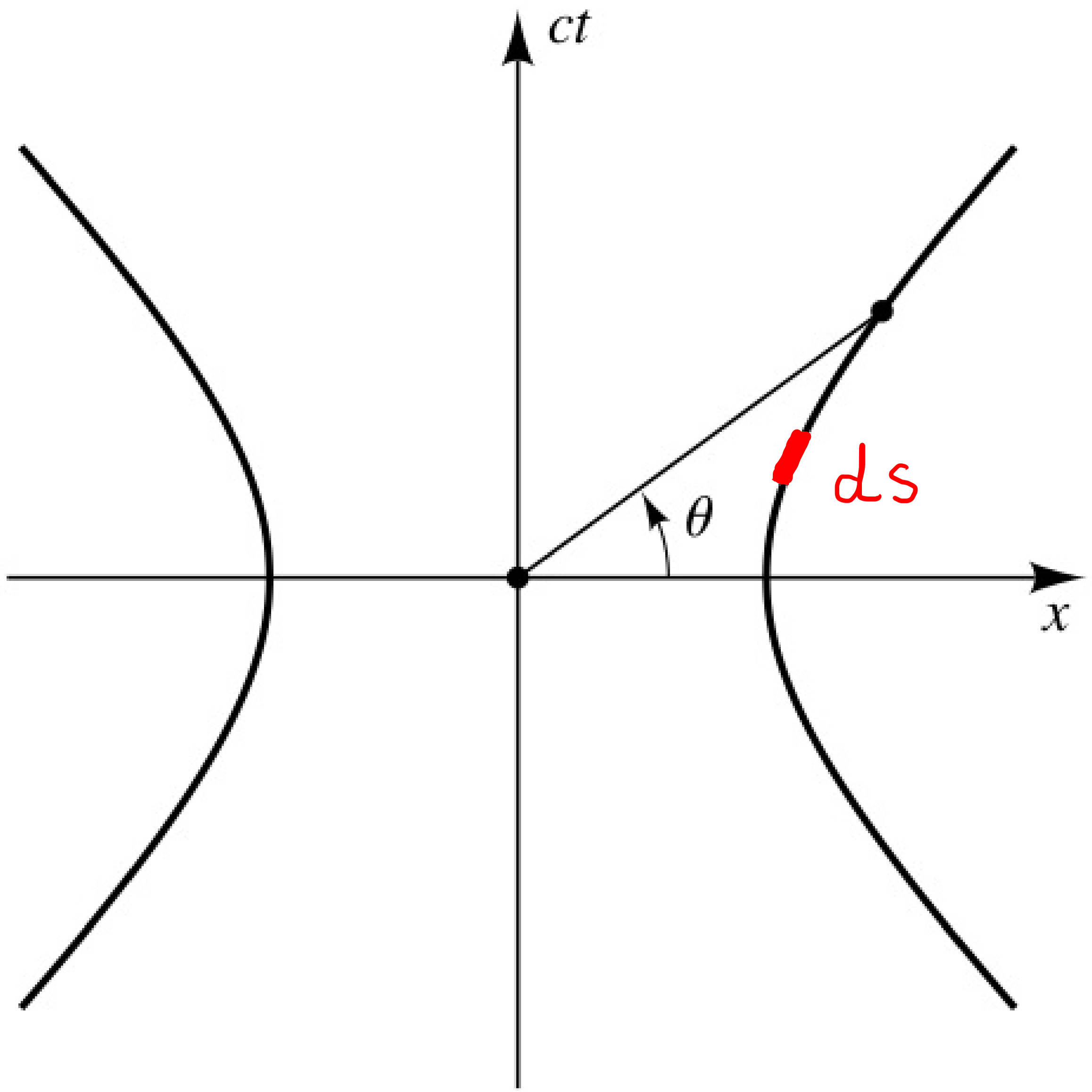
$t = R\sinh\theta$

$x = R\cosh\theta$

$\rightsquigarrow x^2 - t^2 = R^2\cosh^2\theta - R^2\sinh^2\theta = R^2$

* Minkowski geometry: not to be confused by Euclidean!

e.g.: a "circle": locus of points at constant distance from a point.



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Parametric Equations:

$t = R\sinh\theta$

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$$s = \int ds = \int \{|-dt^2 + dx^2|\}^{1/2}$$

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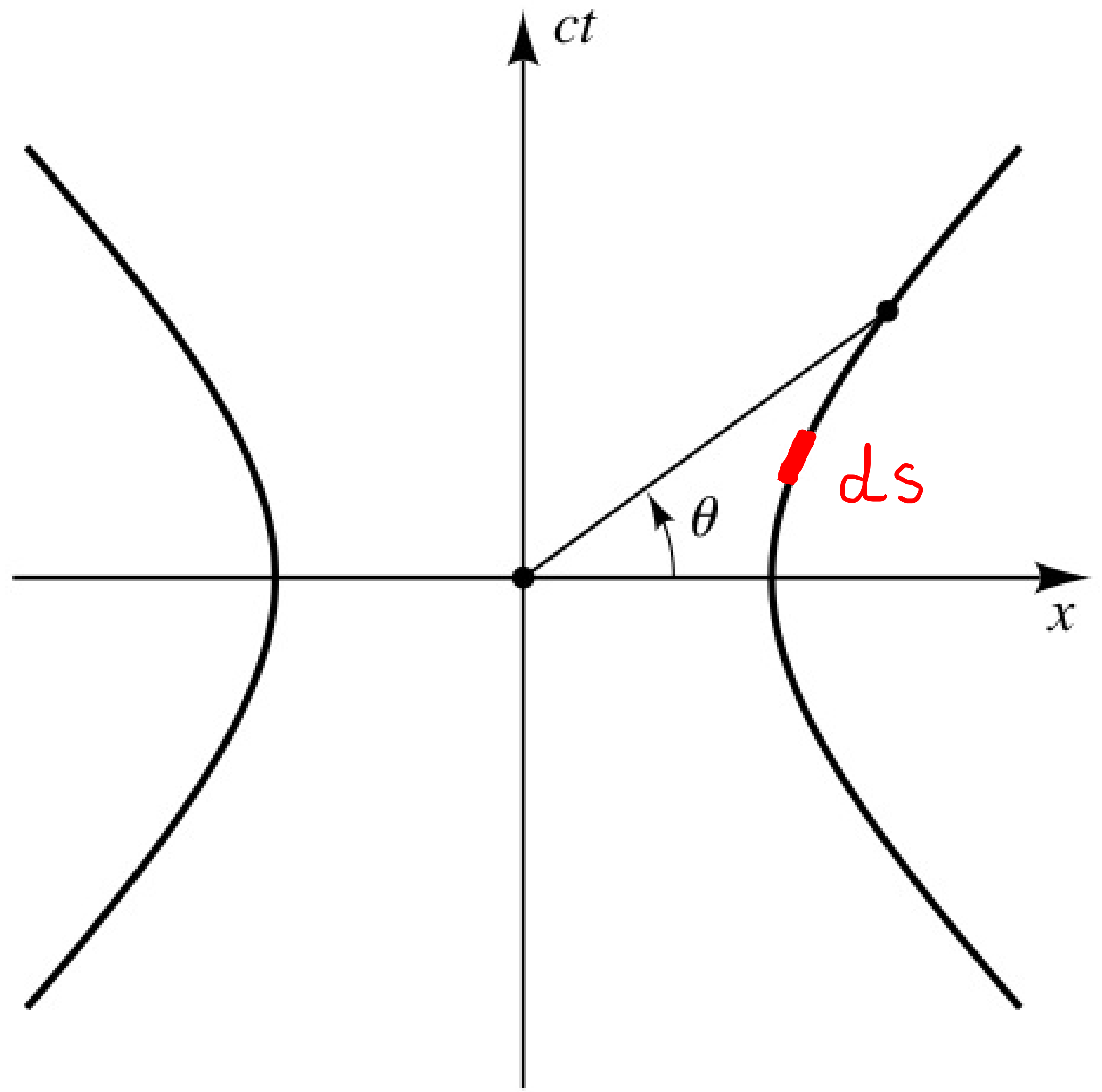


$$\frac{dt}{d\theta} = R\cosh\theta$$

$$\frac{dx}{d\theta} = R\sinh\theta$$

Parametric Equations:

$$t = R\sinh\theta$$

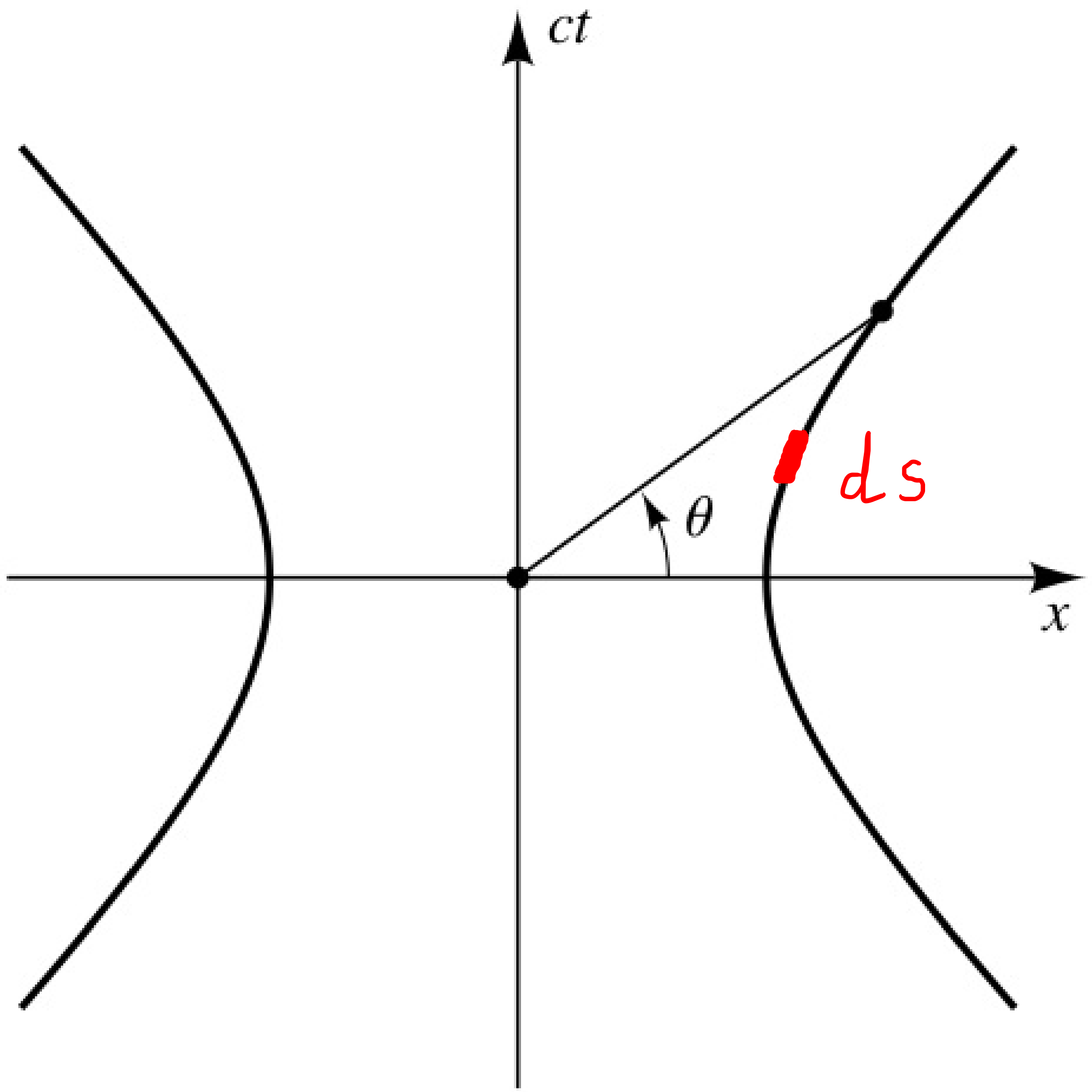
$$x = R\cosh\theta$$

$$s = \int ds = \int \{|-dt^2 + dx^2|\}^{1/2}$$

$$= \int d\theta \left\{\left|-\left(\frac{dt}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2\right|\right\}^{1/2}$$

* Minkowski geometry: not to be confused by Euclidean!

e.g.: a "circle": locus of points at constant distance from a point:



$$= \int d\theta \left\{ (R\cosh\theta)^2 - (R\sinh\theta)^2 \right\}^{1/2}$$

$$= \int d\theta \cdot R = R \cdot \theta$$

$$s = \int ds = \int \left\{ \left| -dt^2 + dx^2 \right| \right\}^{1/2}$$

$$= \int d\theta \left\{ \left| -\left(\frac{dt}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2 \right| \right\}^{1/2}$$

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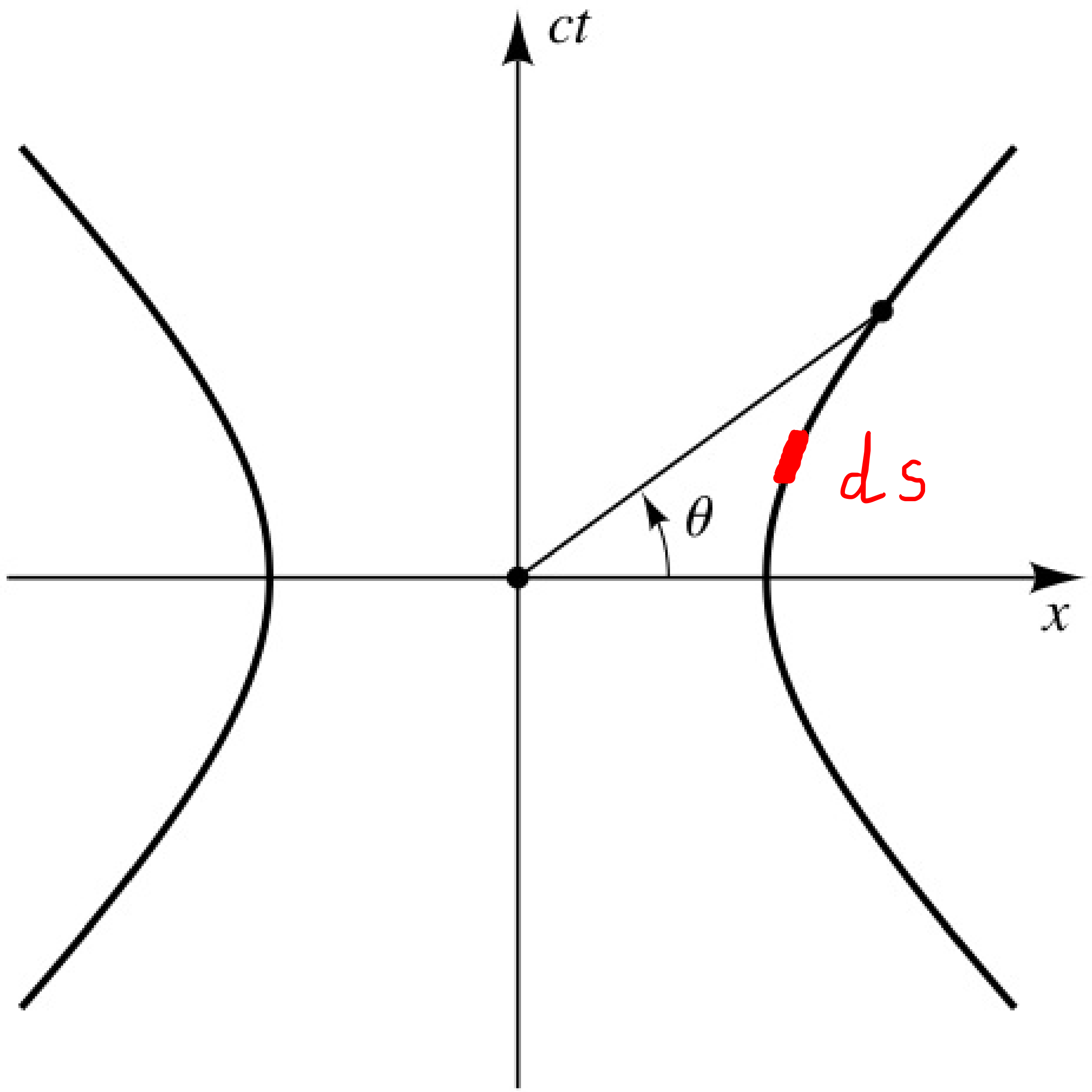
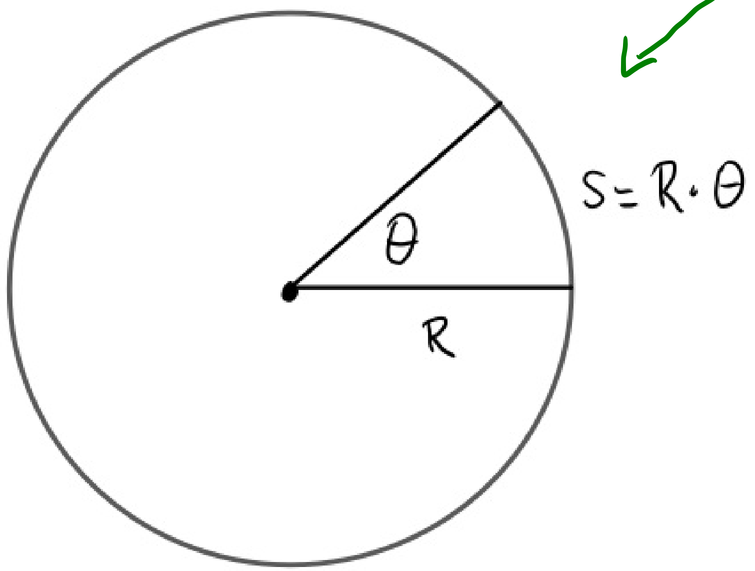


not compact

compact



similar to the Euclidean circle!

$$S = \int ds = \int \left\{ |-dt^2 + dx^2| \right\}^{1/2}$$

$$= \int d\theta \left\{ \left| -\left(\frac{dt}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2 \right| \right\}^{1/2}$$

* Causal Structure

$\Delta s^2 < 0$ timelike separated events

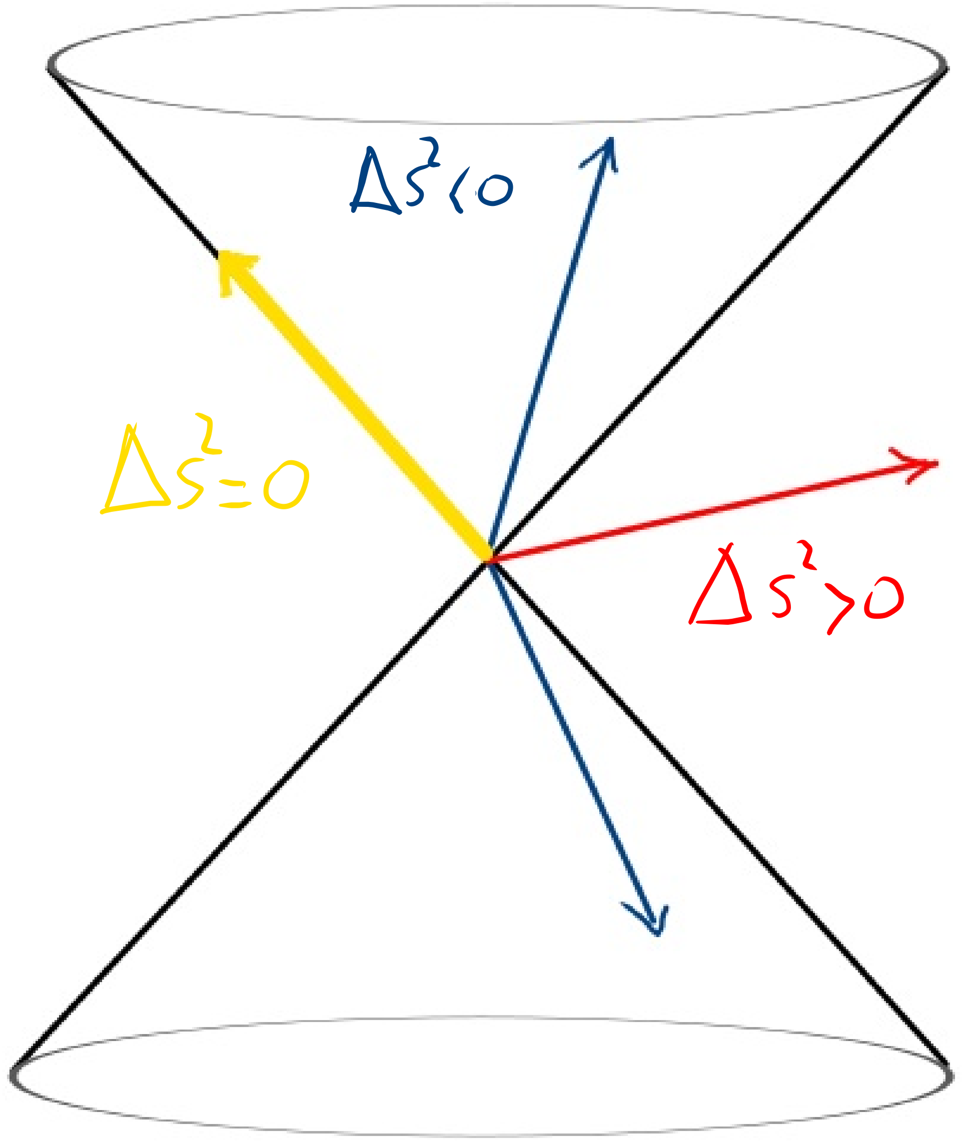
$\Delta s^2 = 0$ null/lightlike " "

$\Delta s^2 > 0$ spacelike separated events

$\Delta s^2 = 0$ define the lightcone of an event

future light cone

past light cone



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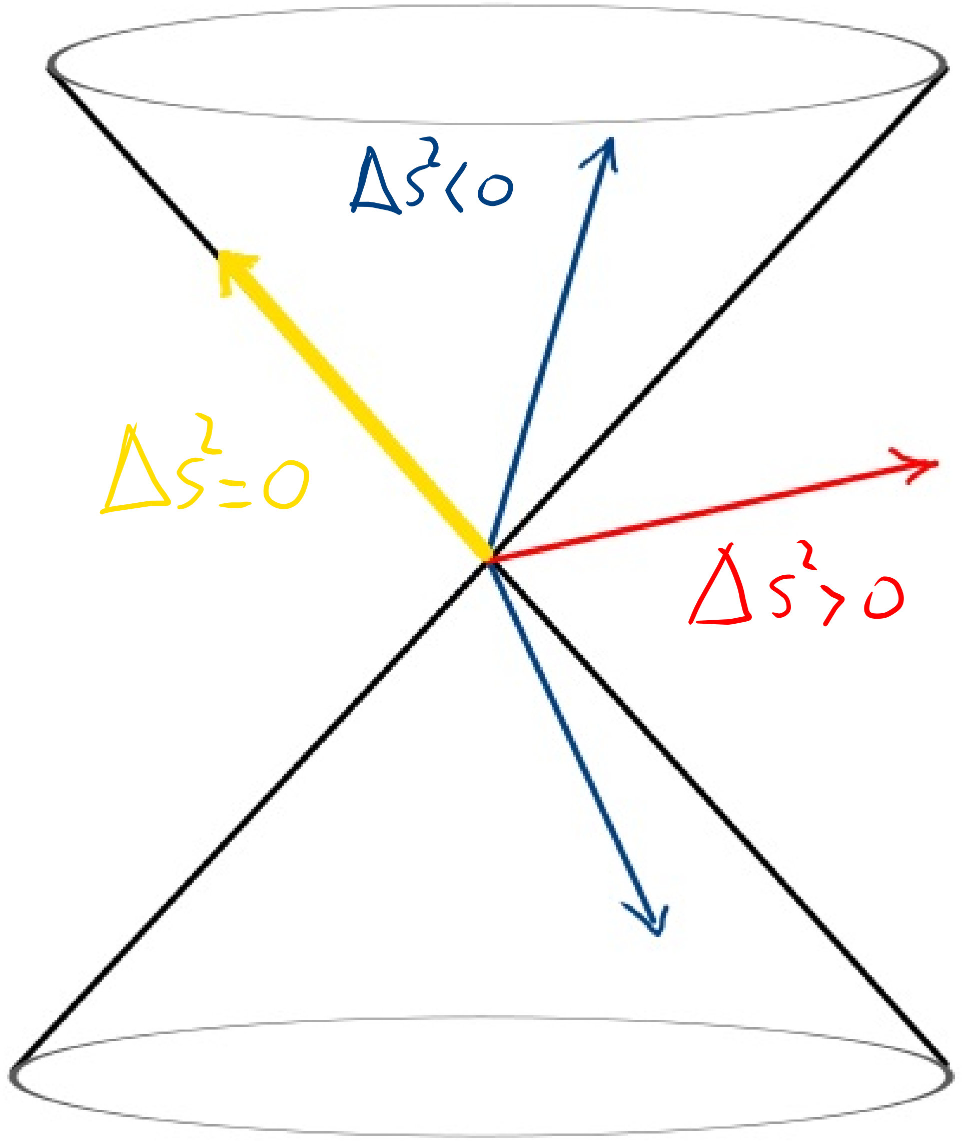


$\Delta s^2 = 0$ define the lightcone of an event

future light cone

past light cone

An event can influence/be influenced by events on+within its future/past light cone

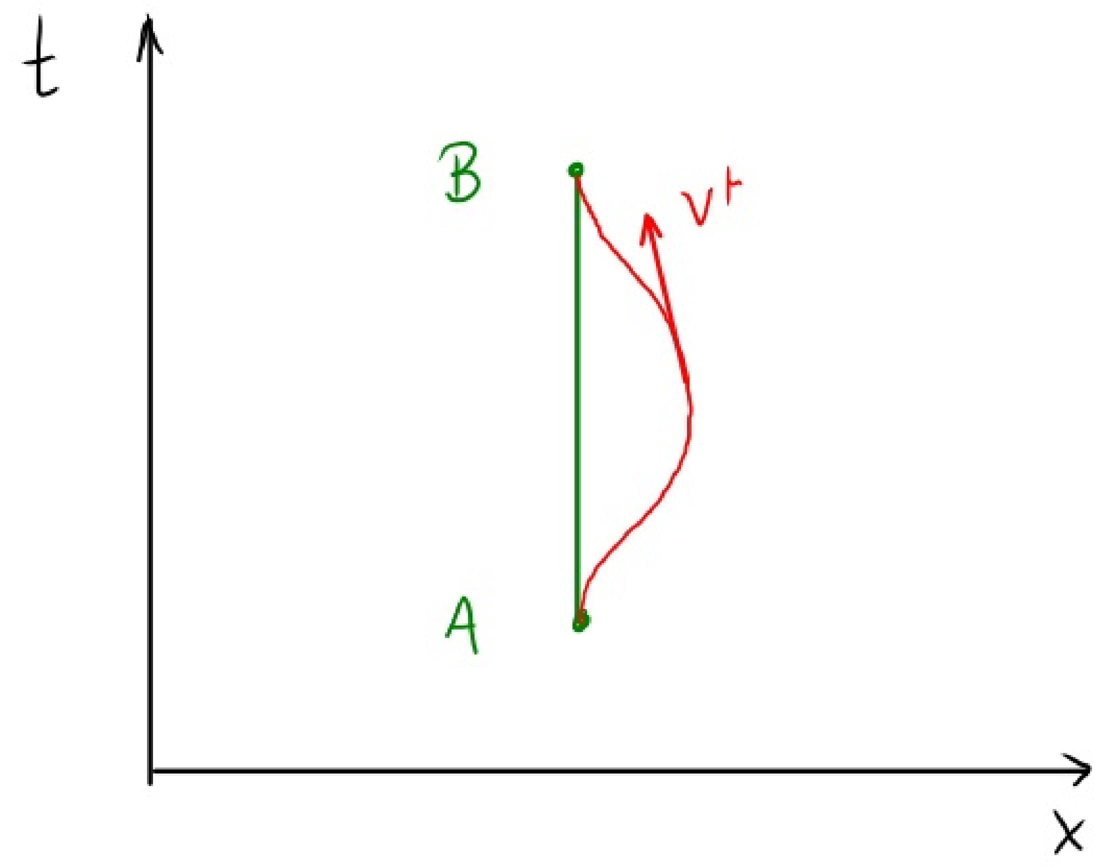
* Proper time: on timelike curve

$$d\tau^2 = -ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) > 0$$

* Proper time: on timelike curve

$$d\tau^2 = -ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) > 0$$

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left\{ dt^2 - (dx^2 + dy^2 + dz^2) \right\}^{1/2}$$

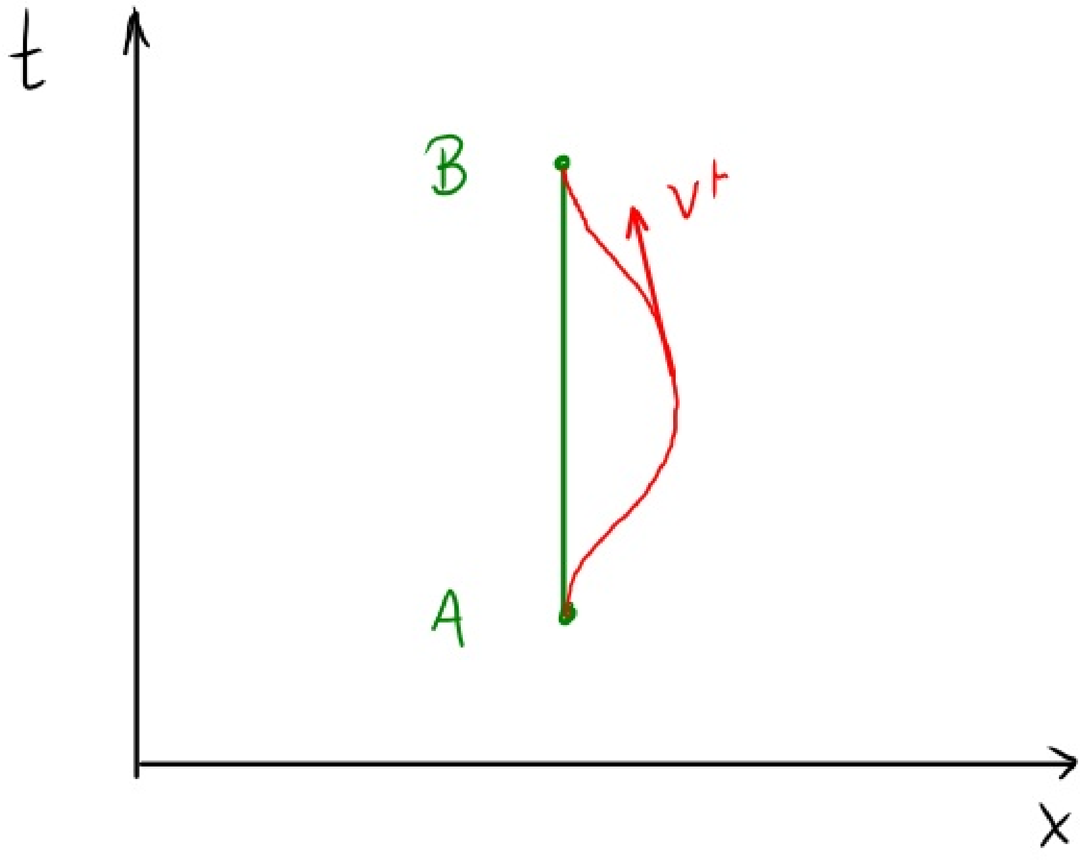


* Proper time: on timelike curve

$$d\tau^2 = -ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) > 0$$

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left\{ dt^2 - (dx^2 + dy^2 + dz^2) \right\}^{1/2}$$

$$= \int_{t_A}^{t_B} dt \left\{ 1 - \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right) \right\}^{1/2}$$



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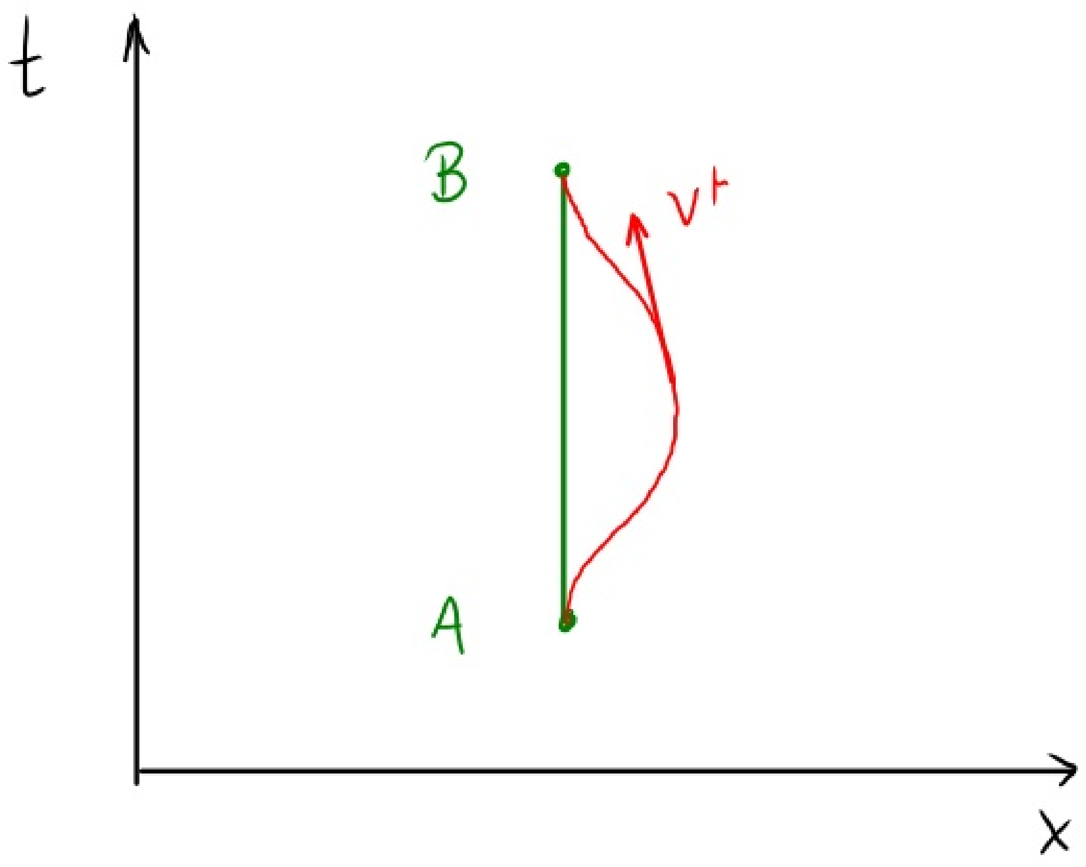

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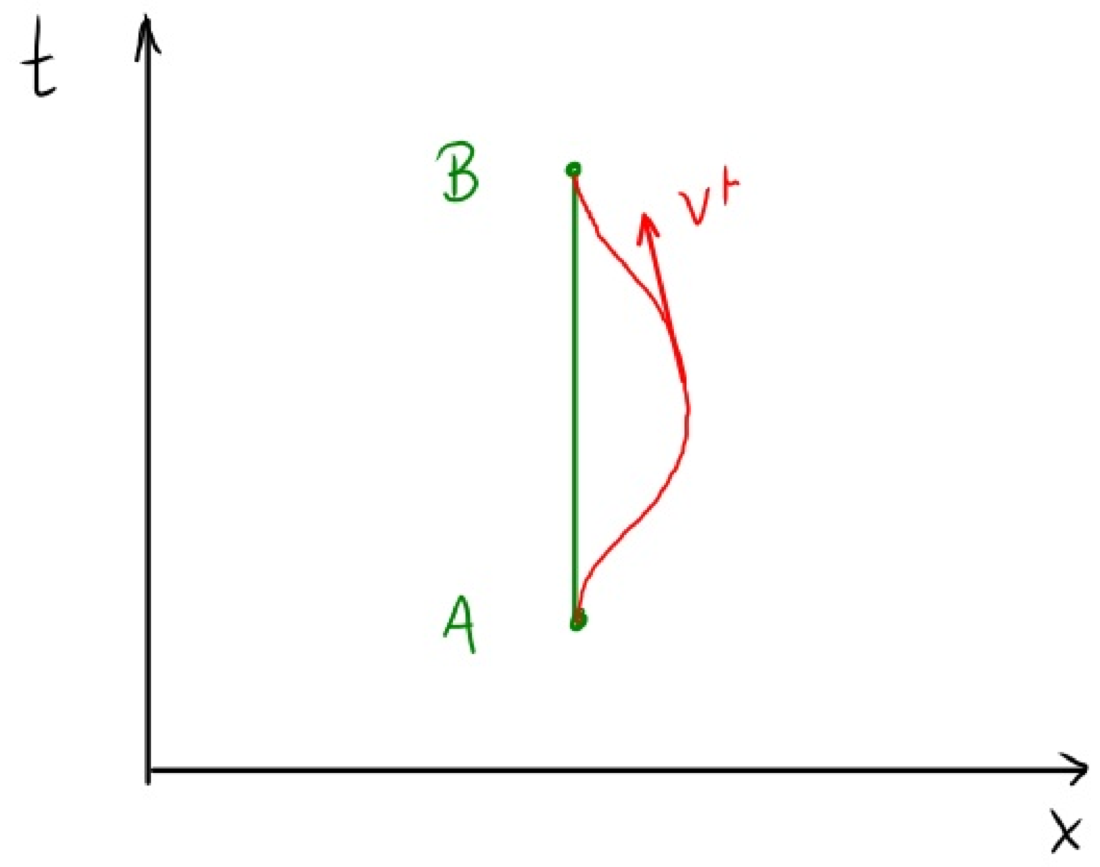
$$= \int_{t_A}^{t_B} dt \left\{ 1 - \left(V^{x\,2} + V^{y\,2} + V^{z\,2} \right) \right\}^{1/2}$$

* Proper time : on timelike curve

$$d\tau^2 = -ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) > 0$$



$$\tau_{AB} = \int_A^B d\tau = \int_A^B \{ dt^2 - (dx^2 + dy^2 + dz^2) \}^{1/2}$$

$$= \int_{t_A}^{t_B} dt \left\{ 1 - \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right) \right\}^{1/2}$$

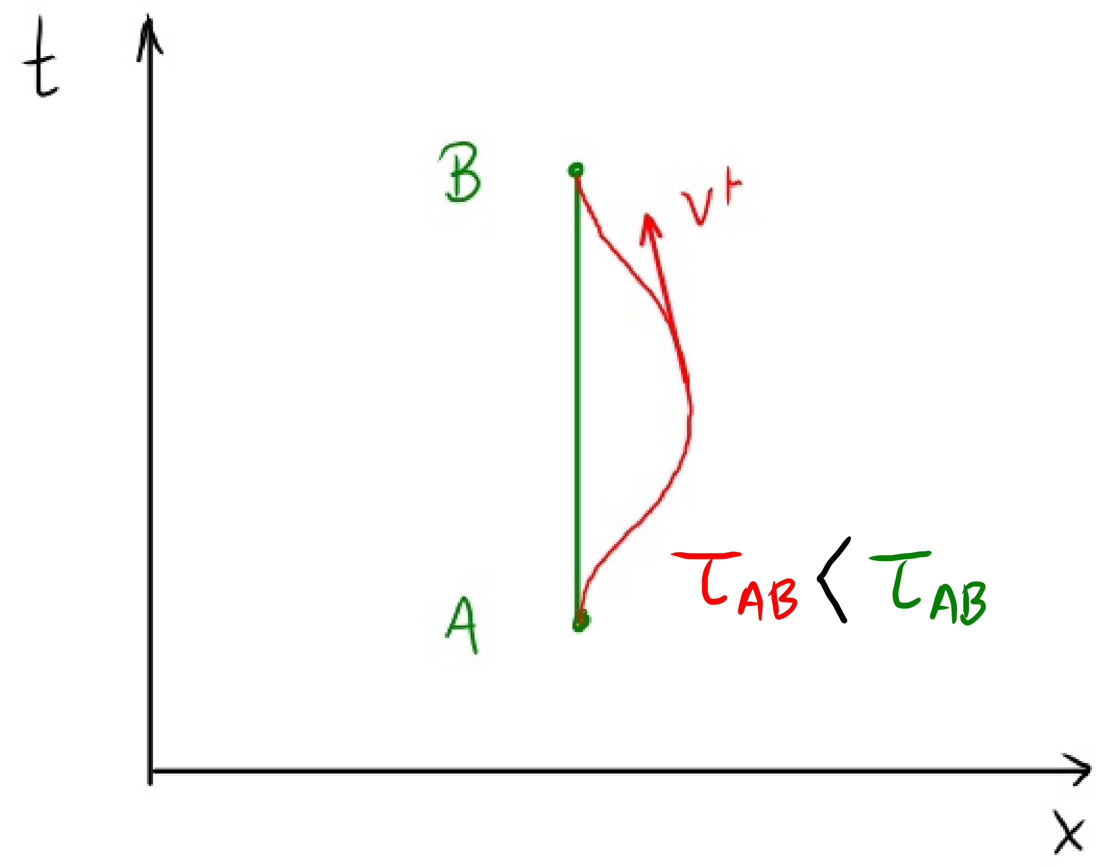
$$= \int_{t_A}^{t_B} dt \{ 1 - (V^{x\,2} + V^{y\,2} + V^{z\,2}) \}^{1/2}$$

$$= \int_{t_A}^{t_B} dt \{ 1 - V^2(t) \}^{1/2} < \int_{t_A}^{t_B} dt \cdot 1 = t_B - t_A$$

$\hookrightarrow \frac{1}{\gamma} = \sqrt{1-V^2} \Rightarrow \gamma = 1/\sqrt{1-V^2}$

* Proper time : on timelike curve

$$d\tau^2 = -ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) > 0$$



$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left\{ dt^2 - (dx^2 + dy^2 + dz^2) \right\}^{1/2}$$

$$= \int_{t_A}^{t_B} dt \left\{ 1 - \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right) \right\}^{1/2}$$

$$= \int_{t_A}^{t_B} dt \left\{ 1 - \left(V^{x\,2} + V^{y\,2} + V^{z\,2} \right) \right\}^{1/2}$$

$$d\tau = \sqrt{1 - v^2}\, dt$$

$$\Updownarrow$$

$$dt = \gamma\, d\tau$$

$$= \int_{t_A}^{t_B} dt \left\{ 1 - V^2(t) \right\}^{1/2} < \int_{t_A}^{t_B} dt \cdot 1 = t_B - t_A$$

But $\tau_{AB}^{(0)} = t_B - t_A > \tau_{AB}$ (twin paradox)

* Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

$$= -dt'^2 + dx'^2 + dy'^2 + dz'^2 = \eta_{\mu'\nu'}\, dx^{\mu'} dx^{\nu'}$$

* Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$
$$= -dt'^2 + dx'^2 + dy'^2 + dz'^2 = \eta_{\mu'\nu'}\, dx^{\mu'} dx^{\nu'}$$

$$t' = \cosh\theta \; t - \sinh\theta \; x$$

$$x' = -\sinh\theta \; t + \cosh\theta \; x$$

$$y' = y \qquad z' = z$$

* Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$
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$$\left.\begin{aligned} t' &= \cosh\theta\; t - \sinh\theta\; x \\ x' &= -\sinh\theta\; t + \cosh\theta\; x \\ y' &= y \qquad z' = z \end{aligned}\right\} \Rightarrow \begin{aligned} dt' &= \cosh\theta\; dt - \sinh\theta\; dx \\ dx' &= -\sinh\theta\; dt + \cosh\theta\; dx \\ dy' &= dy \qquad dz' = dz \end{aligned}$$

* ## Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$
$$= -dt'^2 + dx'^2 + dy'^2 + dz'^2 = \eta_{\mu'\nu'}\, dx^{\mu'} dx^{\nu'}$$

$$\left.\begin{aligned} t' &= \cosh\theta\ t - \sinh\theta\ x \\ x' &= -\sinh\theta\ t + \cosh\theta\ x \\ y' &= y \qquad z' = z \end{aligned}\right\} \Rightarrow \left.\begin{aligned} dt' &= \cosh\theta\ dt - \sinh\theta\ dx \\ dx' &= -\sinh\theta\ dt + \cosh\theta\ dx \\ dy' &= dy \qquad dz' = dz \end{aligned}\right\} \Rightarrow$$

$$-dt'^2 + dx'^2 = -(\cosh\theta\, dt - \sinh\theta\, dx)^2 + (-\sinh\theta\ dt + \cosh\theta\ dx)^2$$

* Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$
$$= -dt'^2 + dx'^2 + dy'^2 + dz'^2 = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'}$$

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$$-dt'^2 + dx'^2 = -(\cosh\theta\, dt - \sinh\theta\, dx)^2 + (-\sinh\theta\ dt + \cosh\theta\ dx)^2$$
$$= -\cosh^2\theta\, dt^2 + 2\cosh\theta \sinh\theta\, dt\, dx - \sinh^2\theta\ dx^2$$
$$+ \sinh^2\theta\, dt^2 - 2\sinh\theta \cosh\theta\ dt\, dx + \cosh^2\theta\ dx^2$$

* Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$
$$= -dt'^2 + dx'^2 + dy'^2 + dz'^2 = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'}$$

$$\left.\begin{aligned} t' &= \cosh\theta\, t - \sinh\theta\, x \\ x' &= -\sinh\theta\, t + \cosh\theta\, x \\ y' &= y \qquad z' = z \end{aligned}\right\} \Rightarrow \left.\begin{aligned} dt' &= \cosh\theta\, dt - \sinh\theta\, dx \\ dx' &= -\sinh\theta\, dt + \cosh\theta\, dx \\ dy' &= dy \qquad dz' = dz \end{aligned}\right\} \Rightarrow$$

$$-dt'^2 + dx'^2 = -(\cosh\theta\, dt - \sinh\theta\, dx)^2 + (-\sinh\theta\, dt + \cosh\theta\, dx)^2$$
$$= -\cosh^2\theta\, dt^2 + \cancel{2\cosh\theta\sinh\theta\, dt\, dx} - \sinh^2\theta\, dx^2$$
$$+ \sinh^2\theta\, dt^2 - \cancel{2\sinh\theta\cosh\theta\, dt\, dx} + \cosh^2\theta\, dx^2$$
$$= -dt^2 + dx^2$$

* Lorentz Transformations

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Boost in x-direction

* Lorentz Transformations

$$\begin{pmatrix} \cosh\theta_x & -\sinh\theta_x & 0 & 0 \\ -\sinh\theta_x & \cosh\theta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \cosh\theta_y & 0 & -\sinh\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\theta_y & 0 & \cosh\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Boost in x-direction   Boost in y-direction

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Boost in x-direction — Boost in y-direction — Boost in z-direction

* Lorentz Transformations

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Boost in x-direction   Boost in y-direction   Boost in z-direction

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi_x & \sin\varphi_x \\ 0 & 0 & -\sin\varphi_x & \cos\varphi_x \end{pmatrix}$$

Rotation around x-axis

* Lorentz Transformations

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Boost in x-direction

$$\begin{pmatrix} \cosh\theta_y & 0 & -\sinh\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\theta_y & 0 & \cosh\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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$$\begin{pmatrix} \cosh\theta_z & 0 & 0 & -\sinh\theta_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\theta_z & 0 & 0 & \cosh\theta_z \end{pmatrix}$$

Boost in z-direction

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi_x & \sin\varphi_x \\ 0 & 0 & -\sin\varphi_x & \cos\varphi_x \end{pmatrix}$$

Rotation around x-axis

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi_y & 0 & -\sin\varphi_y \\ 0 & 0 & 1 & 0 \\ 0 & \sin\varphi_y & 0 & \cos\varphi_y \end{pmatrix}$$

Rotation around y-axis

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi_z & \sin\varphi_z & 0 \\ 0 & -\sin\varphi_z & \cos\varphi_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation around z-axis

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$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Boost in x-direction

$$dt' = \cosh\theta \, dt - \sinh\theta \, dx$$

$$dx' = -\sinh\theta \, dt + \cosh\theta \, dx$$

$$dy' = dy$$

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Observer sits at $x' = y' = z' = 0 \rightsquigarrow dx' = 0$

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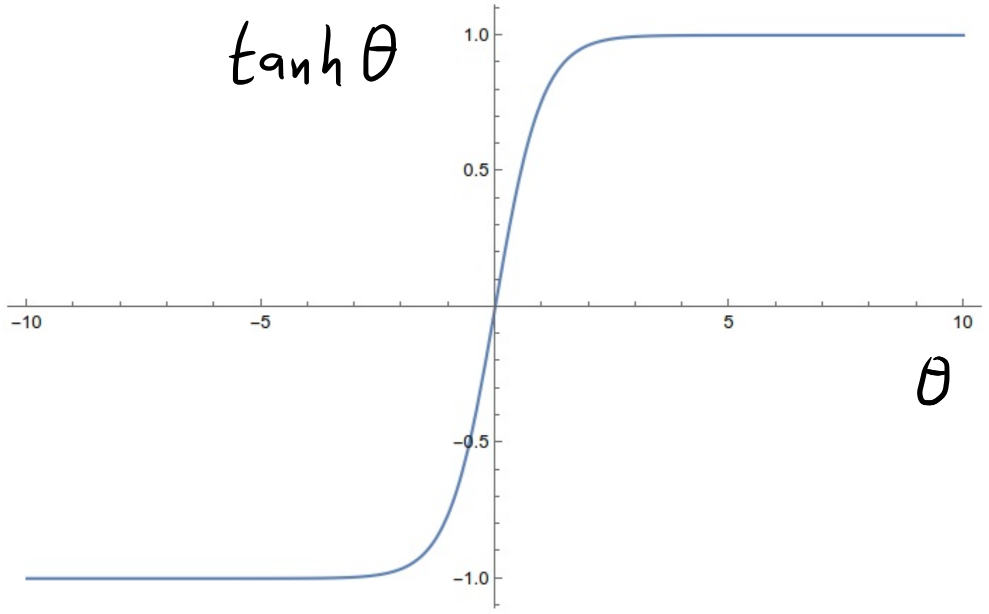
Boost in x-direction

$-\infty < \theta_x, \theta_y, \theta_z < +\infty$ rapidities

$\rightsquigarrow \quad -1 < \tanh\theta_i < +1$



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* ## Lorentz Transformations

General proper Lorentz xfm is a product of boosts + rotations

→ a linear xfm

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} \, x^{\mu}$$

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* Lorentz Transformations

$\Rightarrow \quad \eta = \Lambda^T \eta \Lambda \qquad$ since $\eta_{\mu\nu} = \eta_{\mu'\nu'}$

$\Rightarrow \quad \Lambda \in O(3,1) \qquad$ Lorentz group

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Any other tensor:

$$V^{\mu'} = \Lambda^{\mu'}{}_{\mu} V^{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}$$

$$\omega_{\mu'} = \Lambda^{\mu}{}_{\mu'} \omega_{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \omega_{\mu}$$

$$F_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} F_{\mu\nu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} F_{\mu\nu} \quad , \text{e.t.c.}$$

* **Lorentz Transformations**

Index raising and lowering:

$v_\mu = \eta_{\mu\nu} v^\nu$ $\qquad$ v^ν a vector $\qquad$ v_μ a 1-form

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$U_\mu = \eta_{\mu\nu} U^\nu$ $\qquad$ U^ν a vector $\qquad$ U_μ a 1-form

$U_0 = \eta_{00} U^0 + \eta_{01} U^1 + \eta_{02} U^2 + \eta_{03} U^3$

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$$U_1 = \underset{0}{\cancel{\eta_{10}}} U^0 + \underset{1}{\eta_{11}} U^1 + \underset{0}{\cancel{\eta_{12}}} U^2 + \underset{0}{\cancel{\eta_{13}}} U^3 = U^1$$

$$U_2 = \underset{0}{\cancel{\eta_{20}}} U^0 + \underset{0}{\cancel{\eta_{21}}} U^1 + \underset{1}{\eta_{22}} U^2 + \underset{0}{\cancel{\eta_{23}}} U^3 = U^2$$

$$U_3 = \underset{0}{\cancel{\eta_{30}}} U^0 + \underset{0}{\cancel{\eta_{31}}} U^1 + \underset{0}{\cancel{\eta_{32}}} U^2 + \underset{1}{\eta_{33}} U^3 = U^3$$

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$F_\mu{}^\nu = \eta_{\mu\rho} F^{\rho\nu}$ $\quad$ $F_{\mu\nu} = \eta_{\nu\rho} F_\mu{}^\rho = \eta_{\nu\rho} \eta_{\mu\sigma} F^{\rho\sigma}$ $\quad$ $\mu, \nu = 0, 1, 2, 3$
$i, j = 1, 2, 3$

$F_0{}^\nu = -F^{0\nu}$, $F_i{}^\nu = F^{i\nu}$; $F_{00} = F^{00}$, $F_{0i} = -F^{0i}$, $F_{i0} = -F^{i0}$, $F_{ij} = F^{ij}$

every 0 raised/lowered gives a (−) factor

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Inner product:

$$v \cdot w = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3$$

$$= \eta_{\mu\nu} v^\mu w^\nu$$

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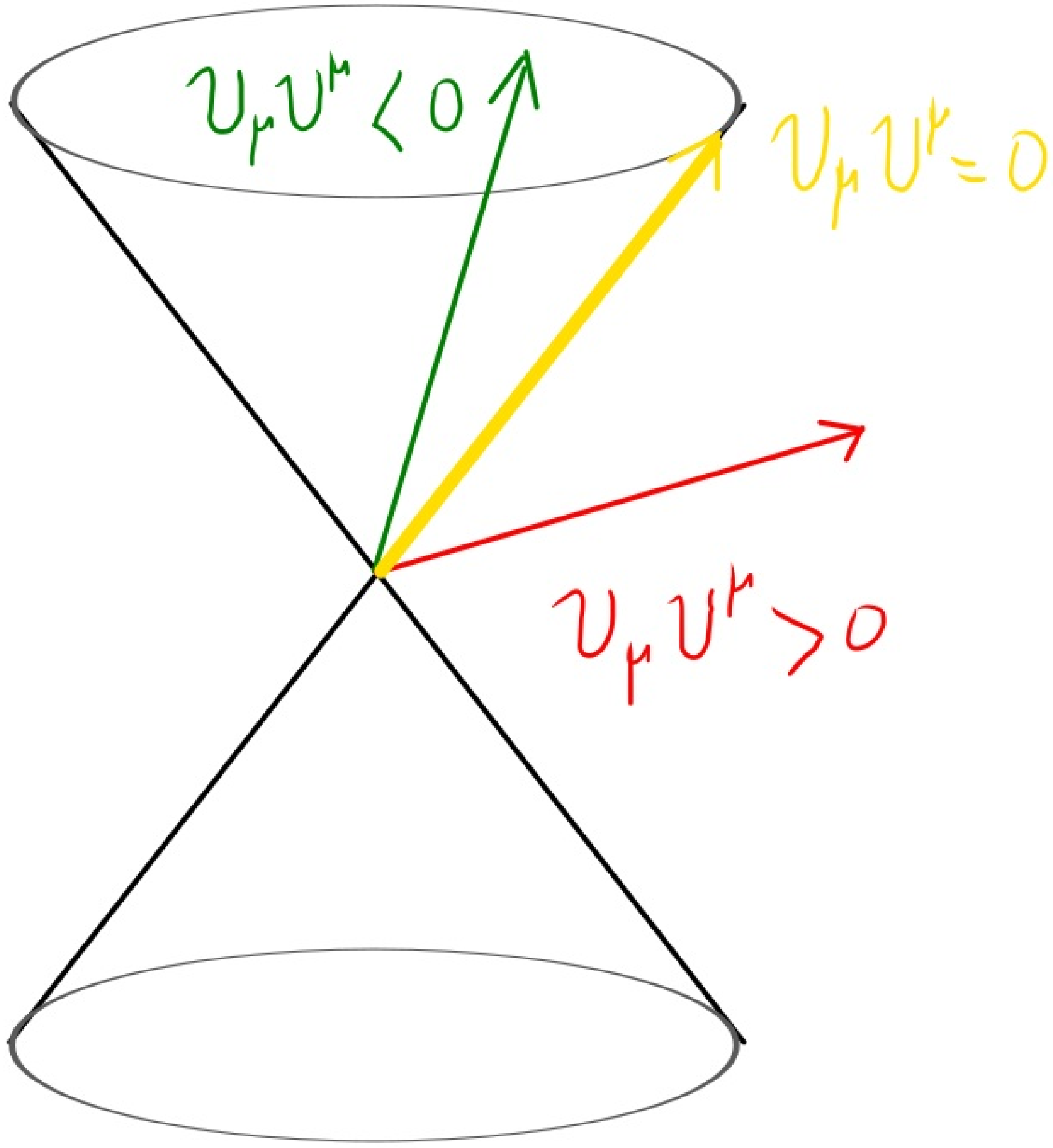

(Spacetime) length of vector:

$$v \cdot v = v_\mu v^\mu = \eta_{\mu\nu} v^\mu v^\nu = -(v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2$$

timelike $\quad v_\mu v^\mu < 0$

null / lightlike $\quad v_\mu v^\mu = 0$

spacelike $\quad v_\mu v^\mu > 0$

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We have described the proper Lorentz group : continuously connected to $\mathbb{1}$

$\hookrightarrow$ boosts + rotations

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We have described the proper Lorentz group : continuously connected to 1
This is $O^+(3,1)$. But we also have

parity xfm $P: (t, x, y, z) \to (t, -x, -y, -z)$

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$$x^\mu \to x^\mu + a^\mu$$

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Lorentz xfms + translations = Poincaré group.

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- 3 rotations
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* Poincaré Group is the symmetry group of Minkowski spacetime

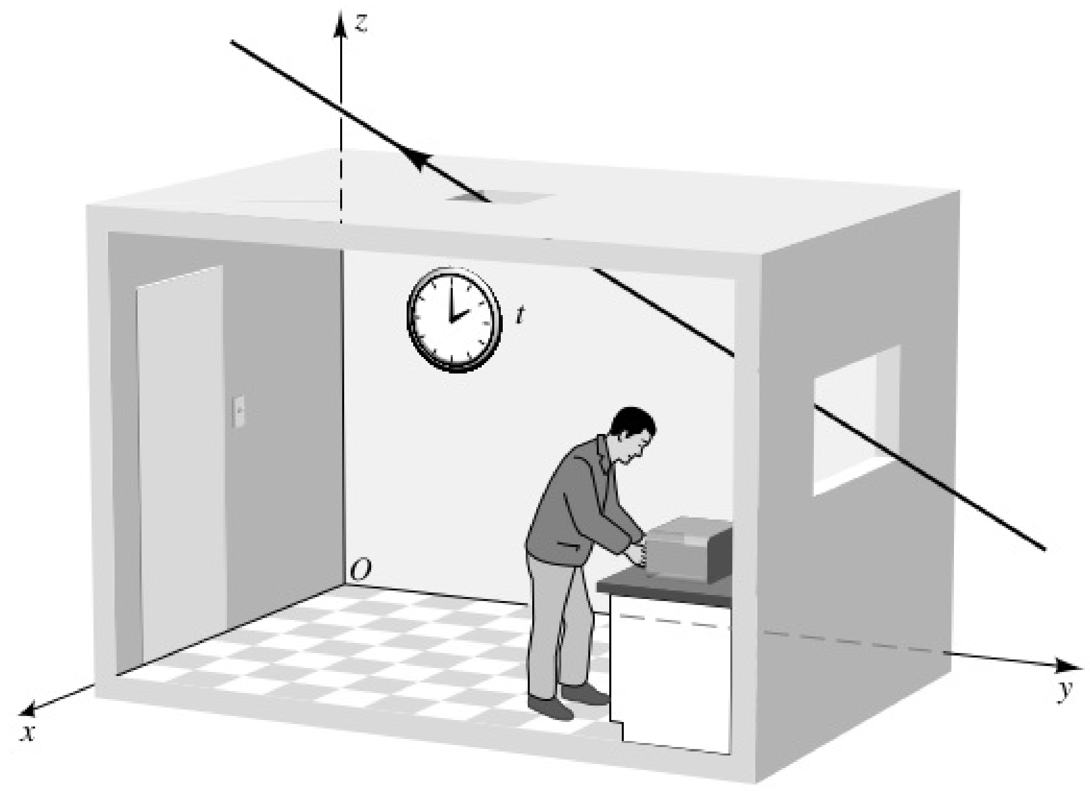
In GR, we only have a local Lorentz symmetry:

- It acts on the tangent space at each point
- Approximate symmetry in local inertial frames

* Dynamics of particles

Free massive particle: $\frac{dU^\mu}{d\tau} = 0$, where

$$U^\mu = \frac{dx^\mu}{d\tau}$$



Hartle, Fig 3.1

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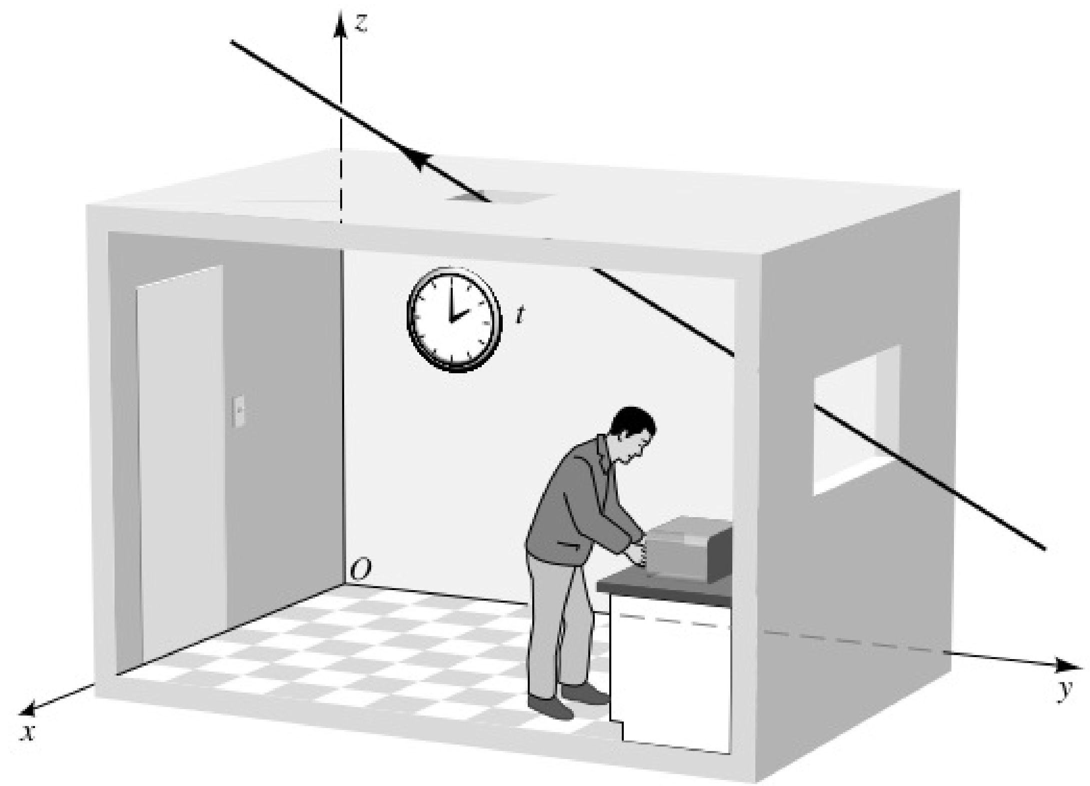
$$\boxed{\begin{gathered} \gamma = \frac{1}{\sqrt{1-v^2}} \\ dt = \gamma\, d\tau \end{gathered}}$$



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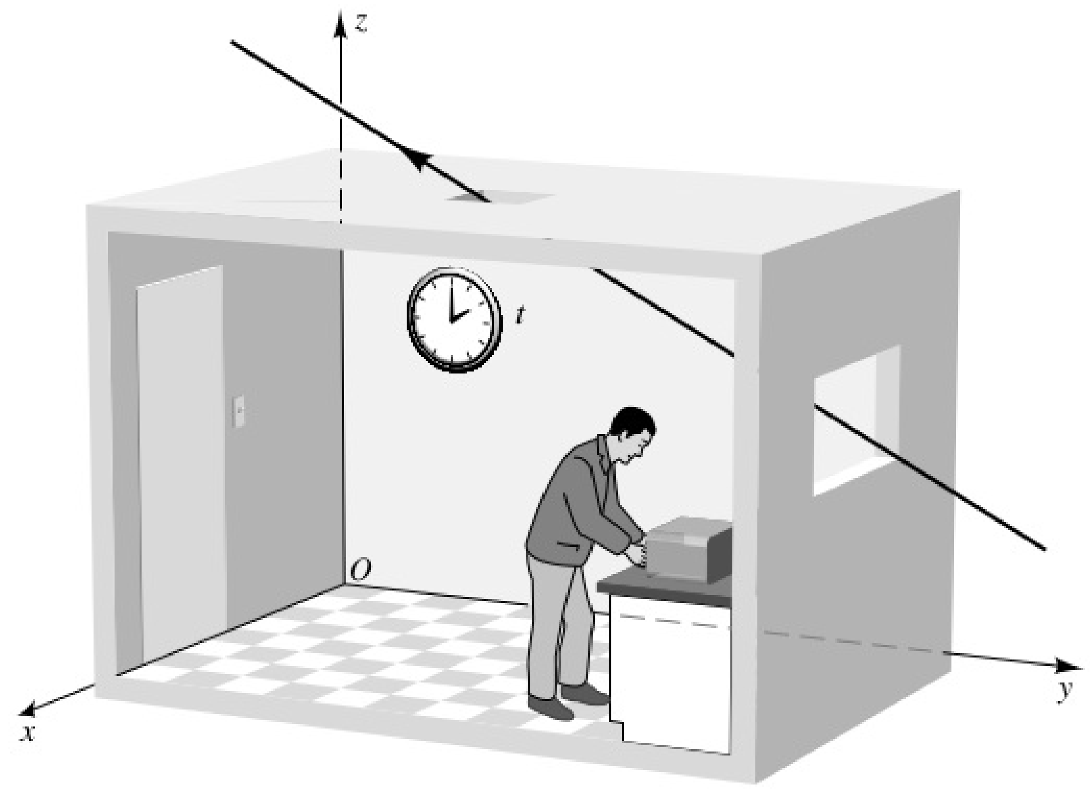

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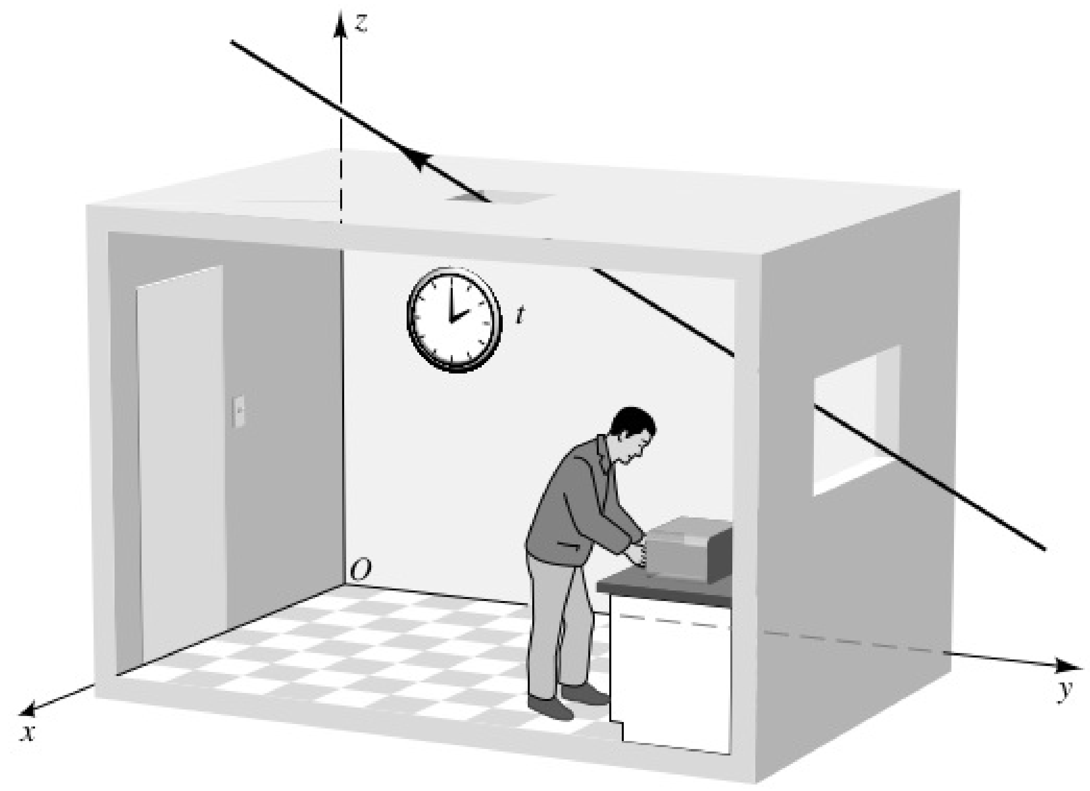

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Free massive particle: $\frac{dU^\mu}{d\tau} = 0$, where

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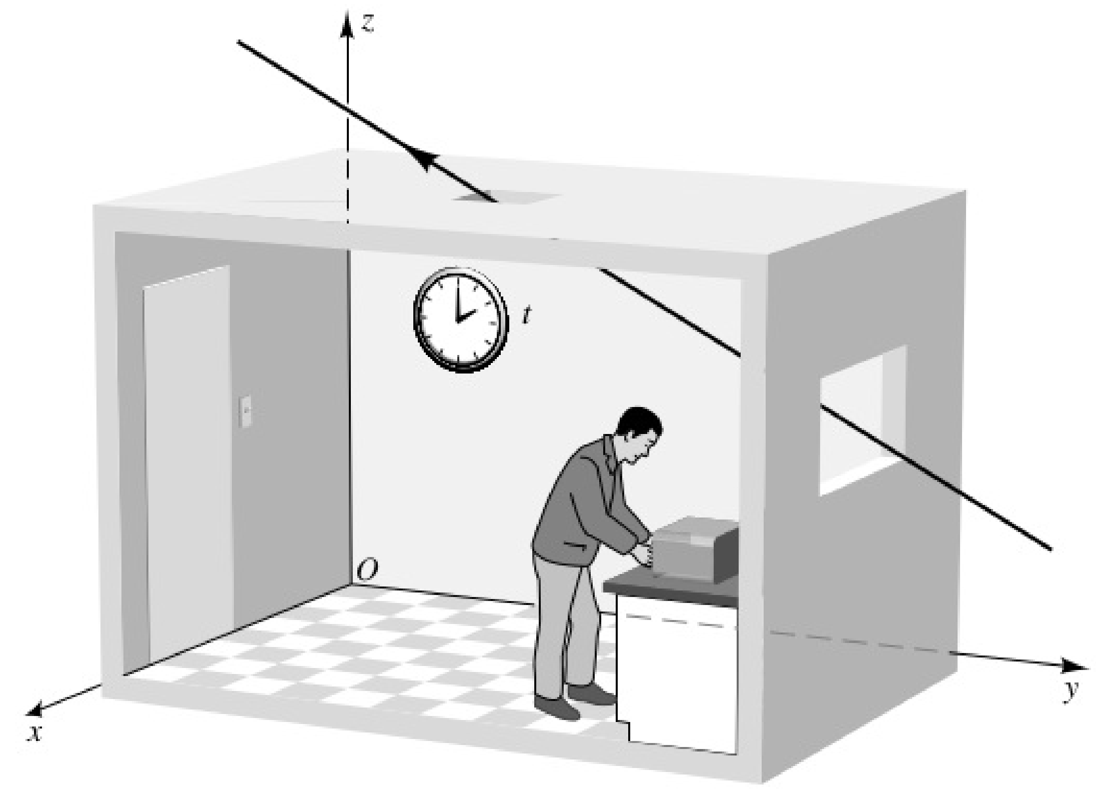

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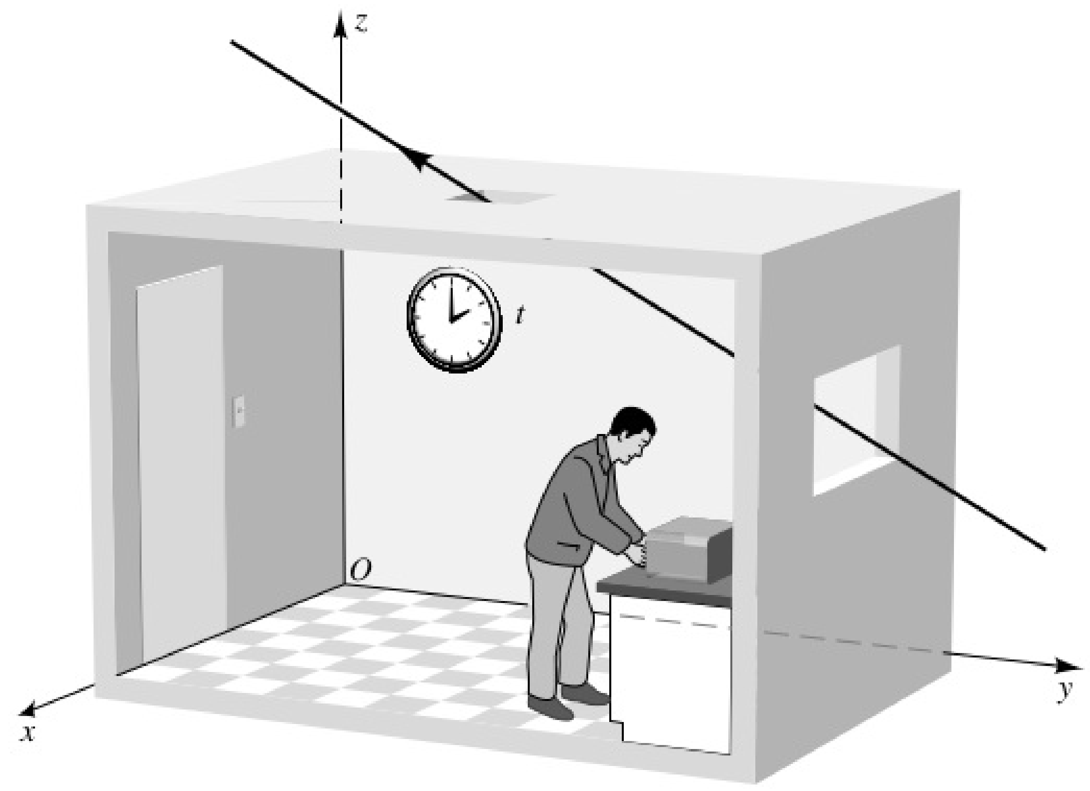

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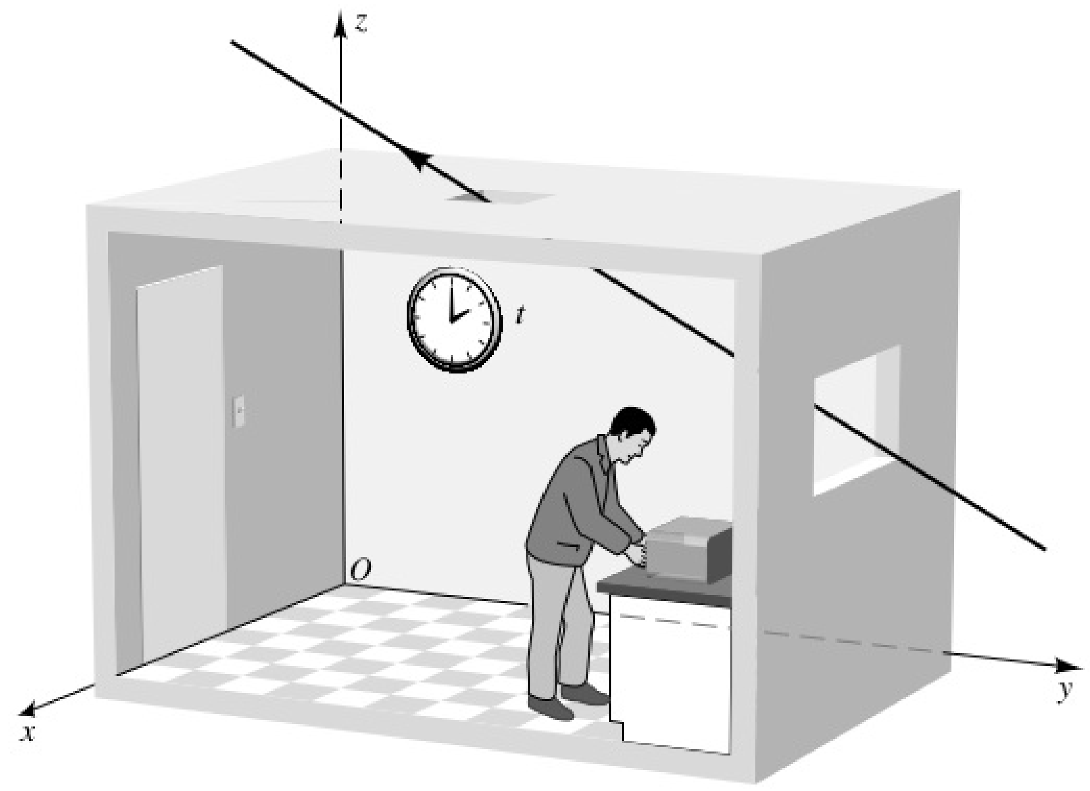

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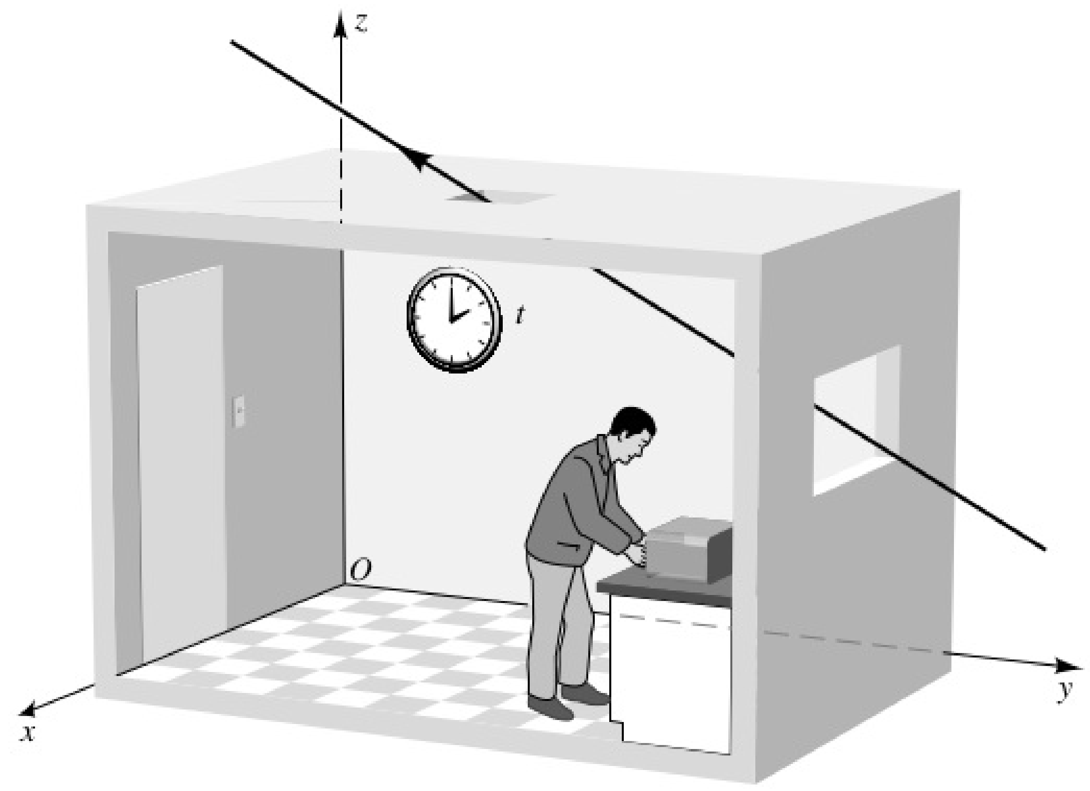

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check it also from: $U_\mu U^\mu = -(U^0)^2 + U^i U^i = -\gamma^2 + \gamma^2 {V^i}^2 = -1$

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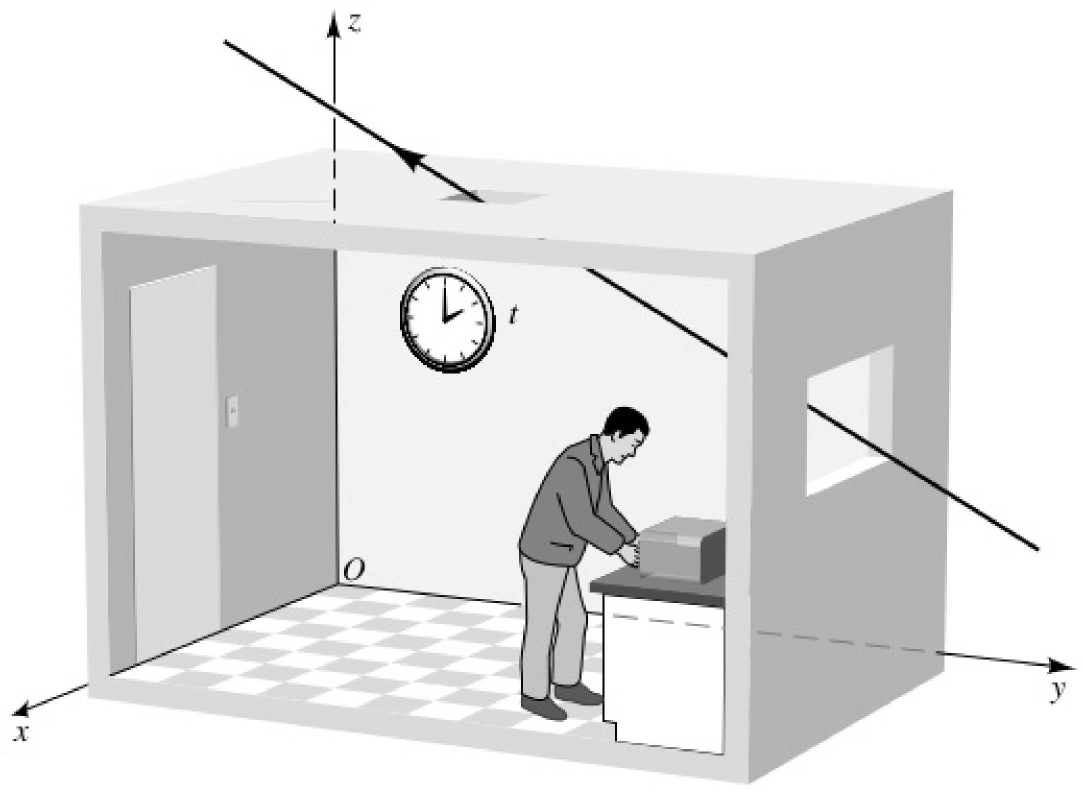

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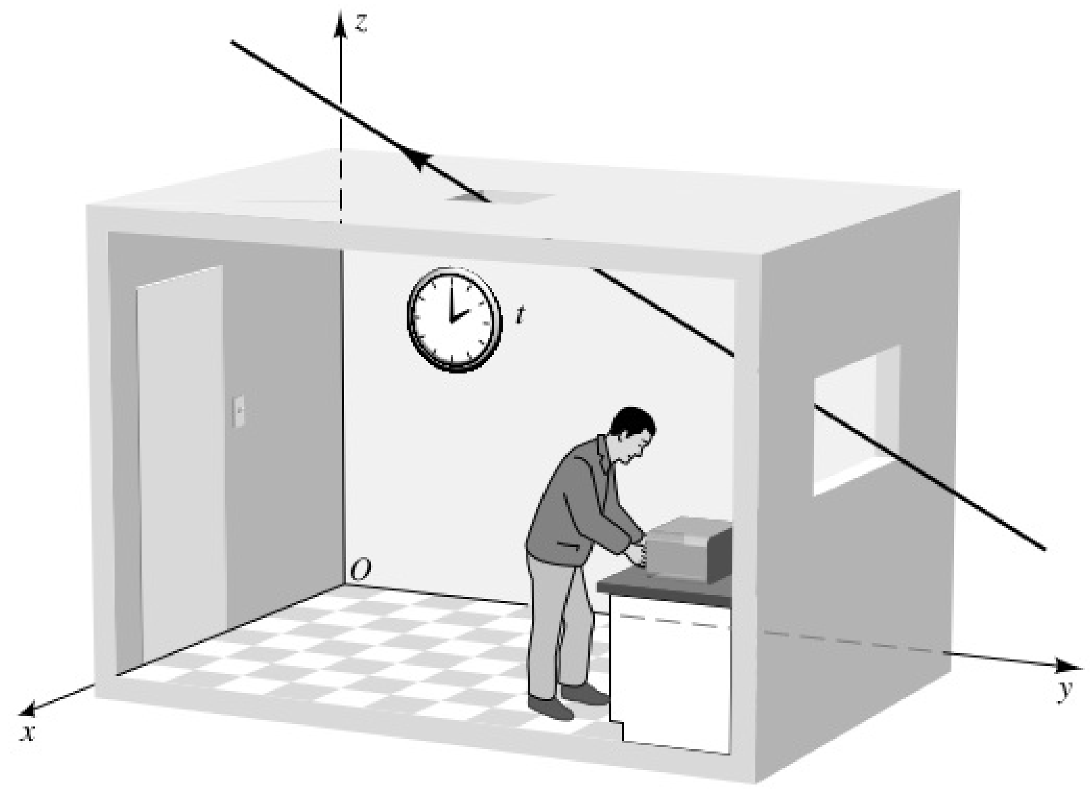

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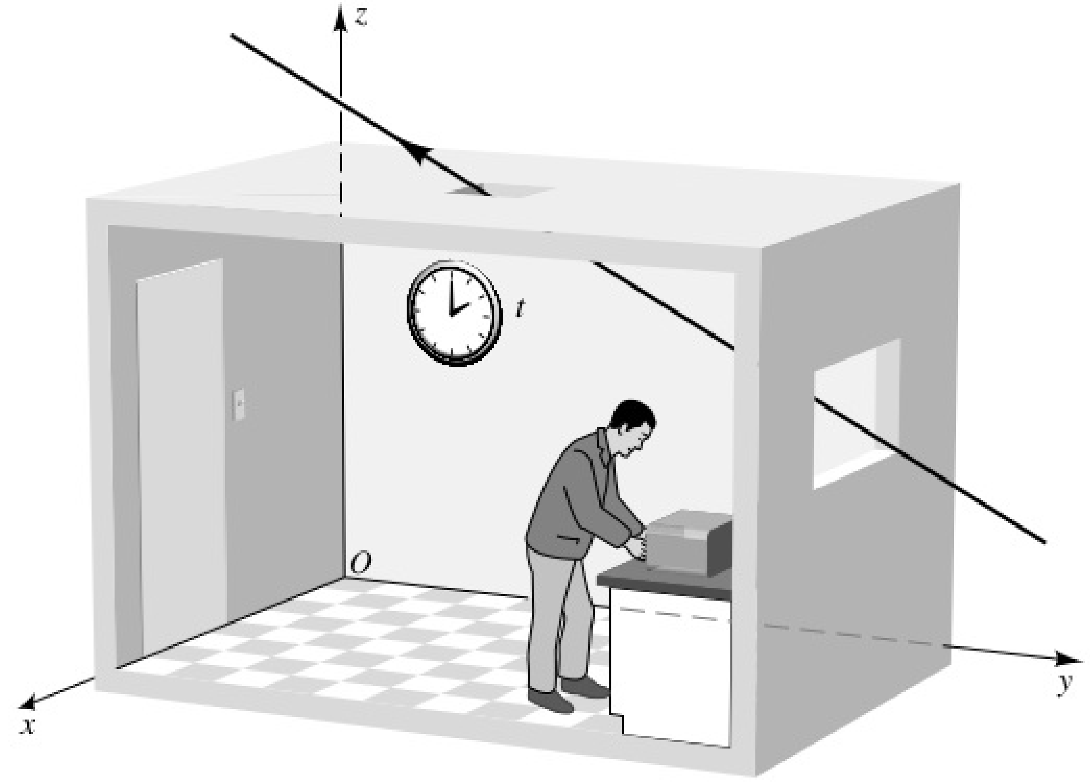

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But $E = m\gamma = p^0$ $\qquad \vec{p} = m\gamma \vec{V}$, so $E^2 = p^2 + m^2$

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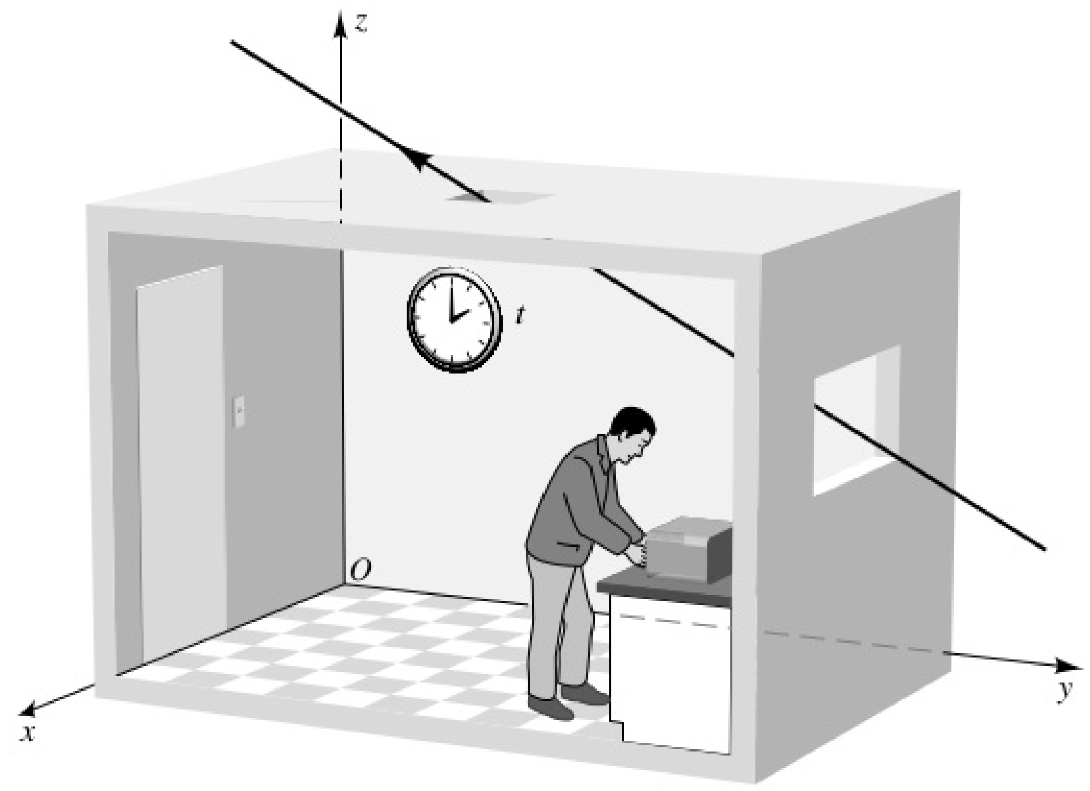

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$$f^\mu = \frac{dp^\mu}{d\tau} \quad \text{4-force}$$

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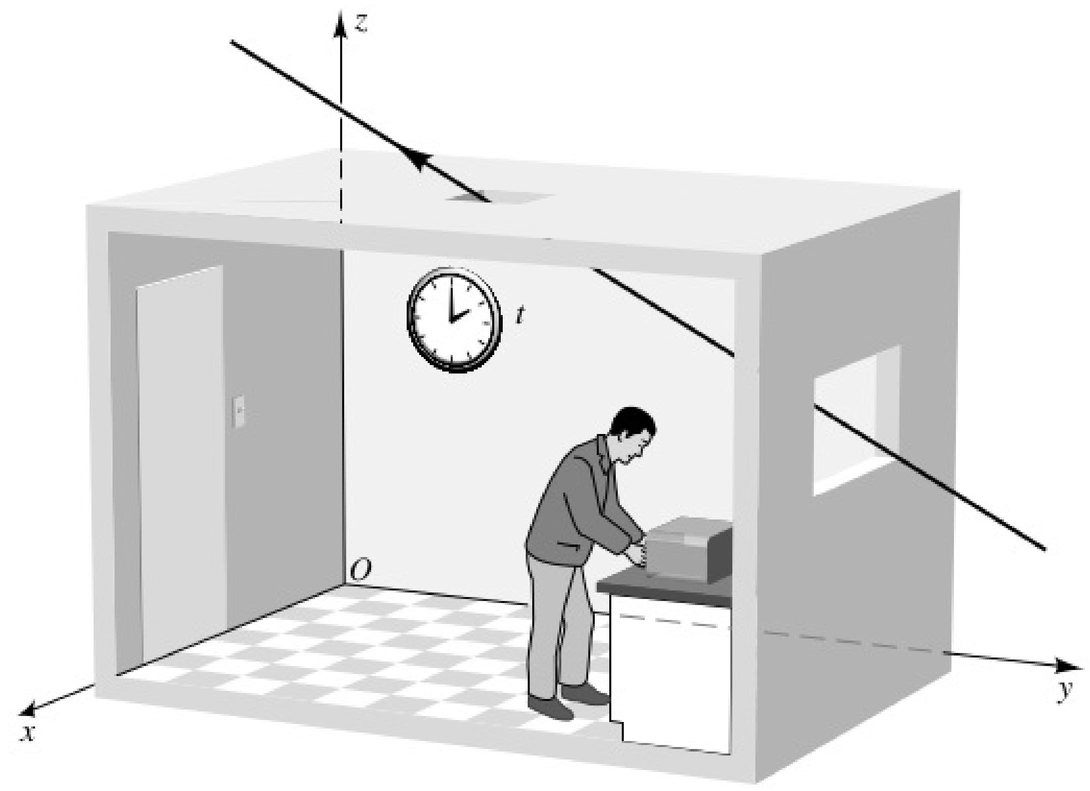

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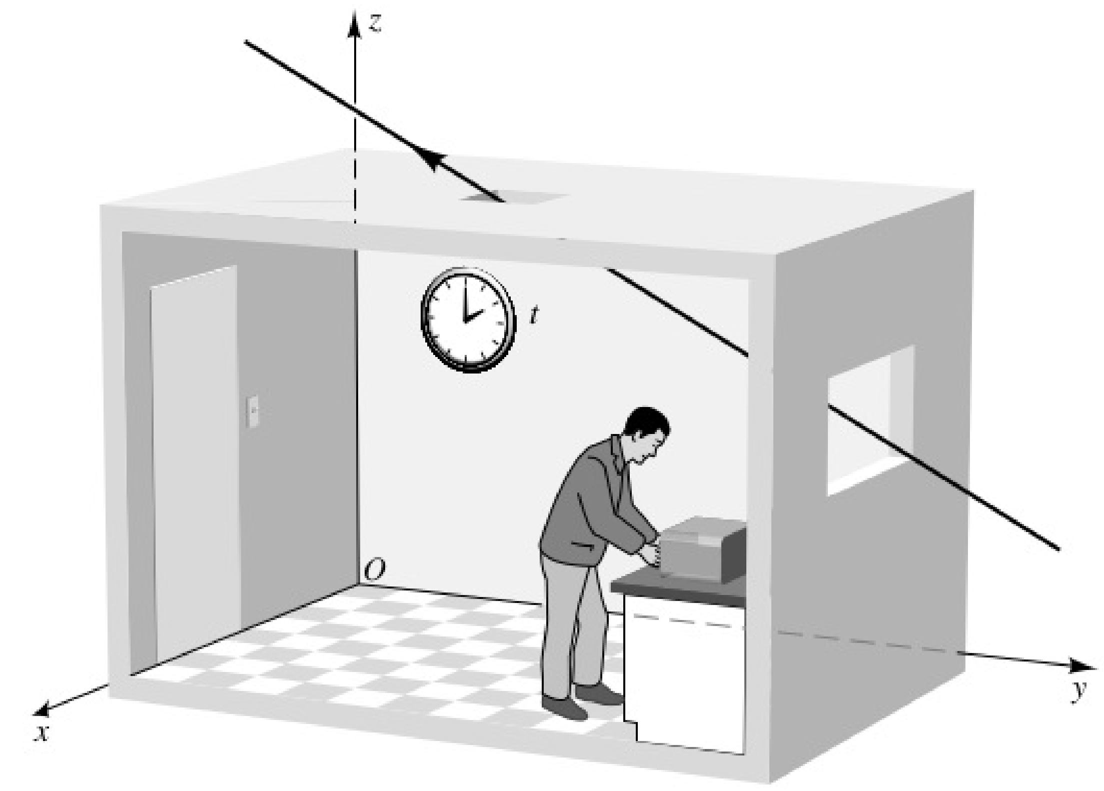

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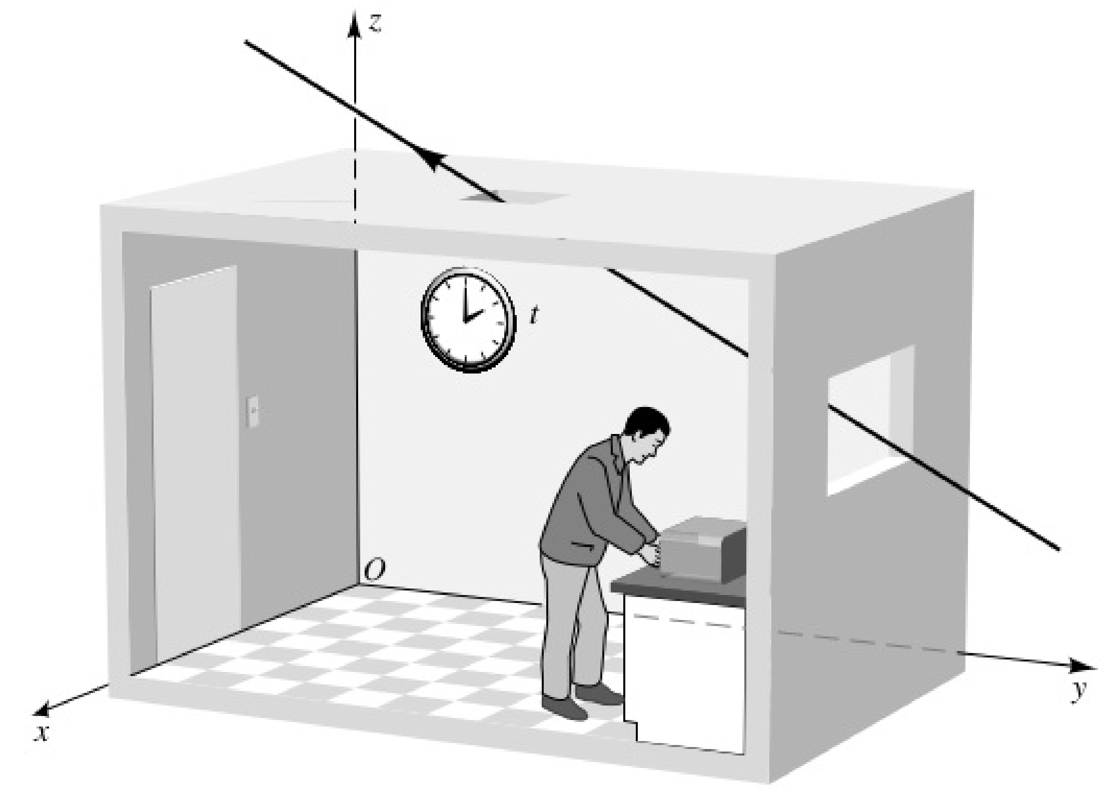

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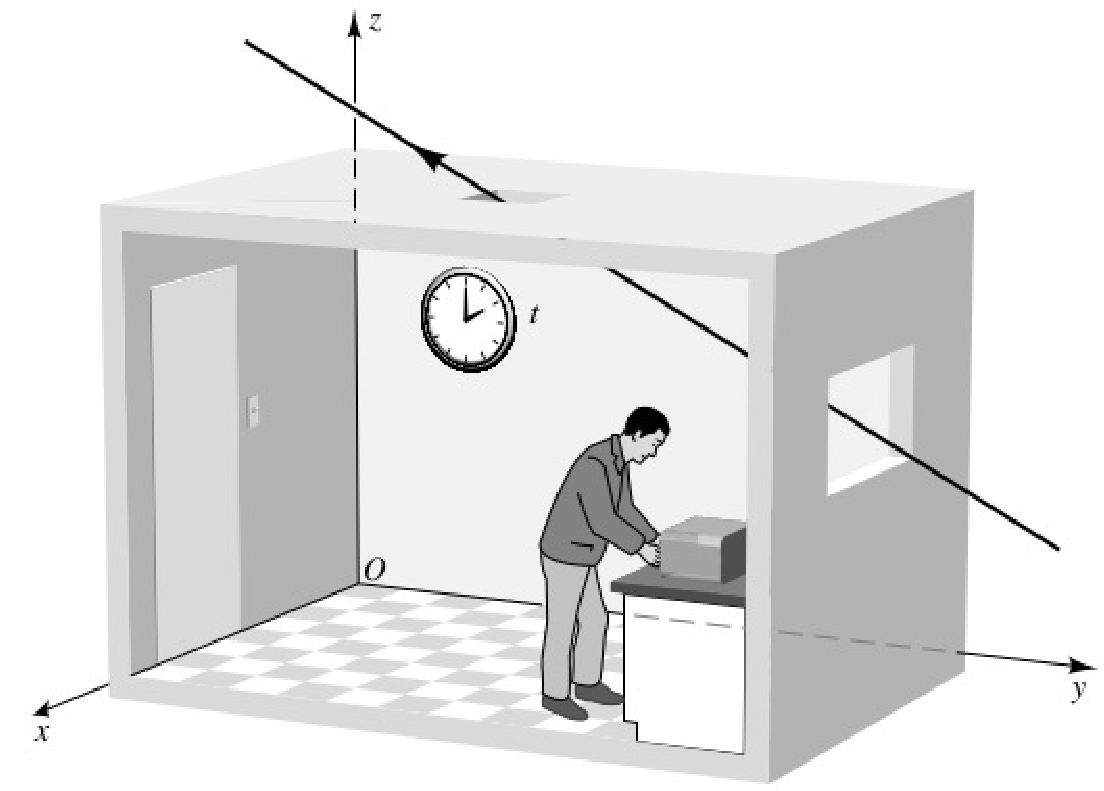

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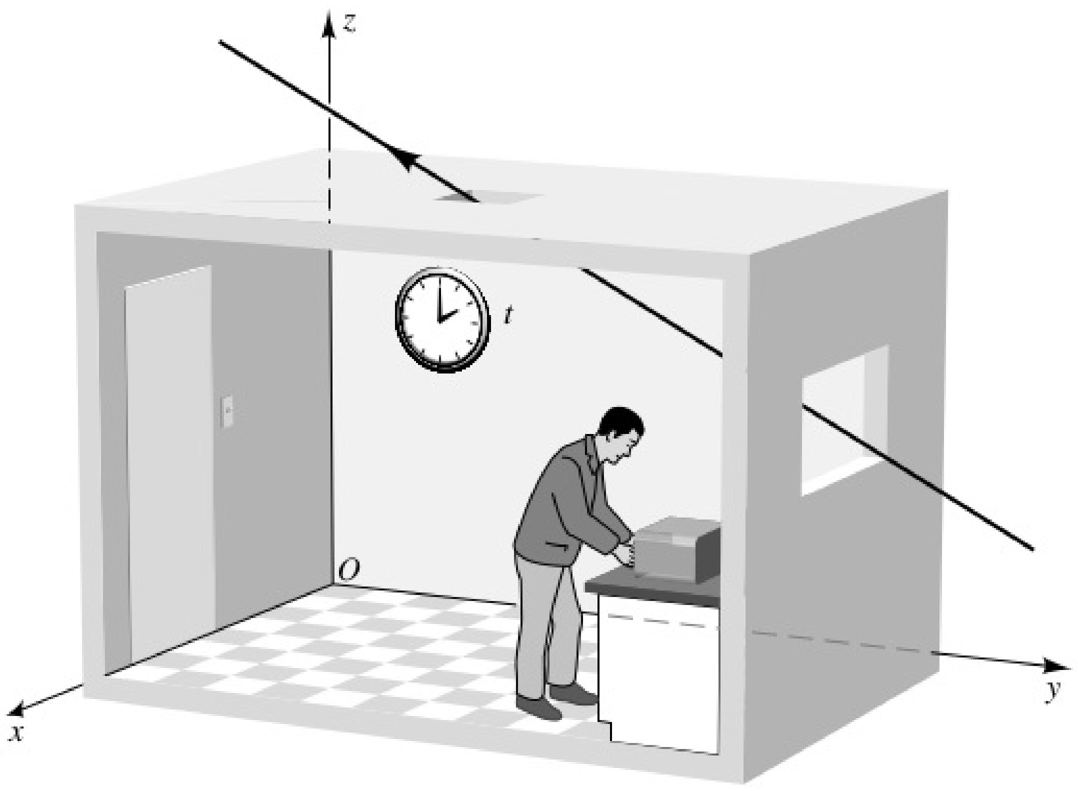

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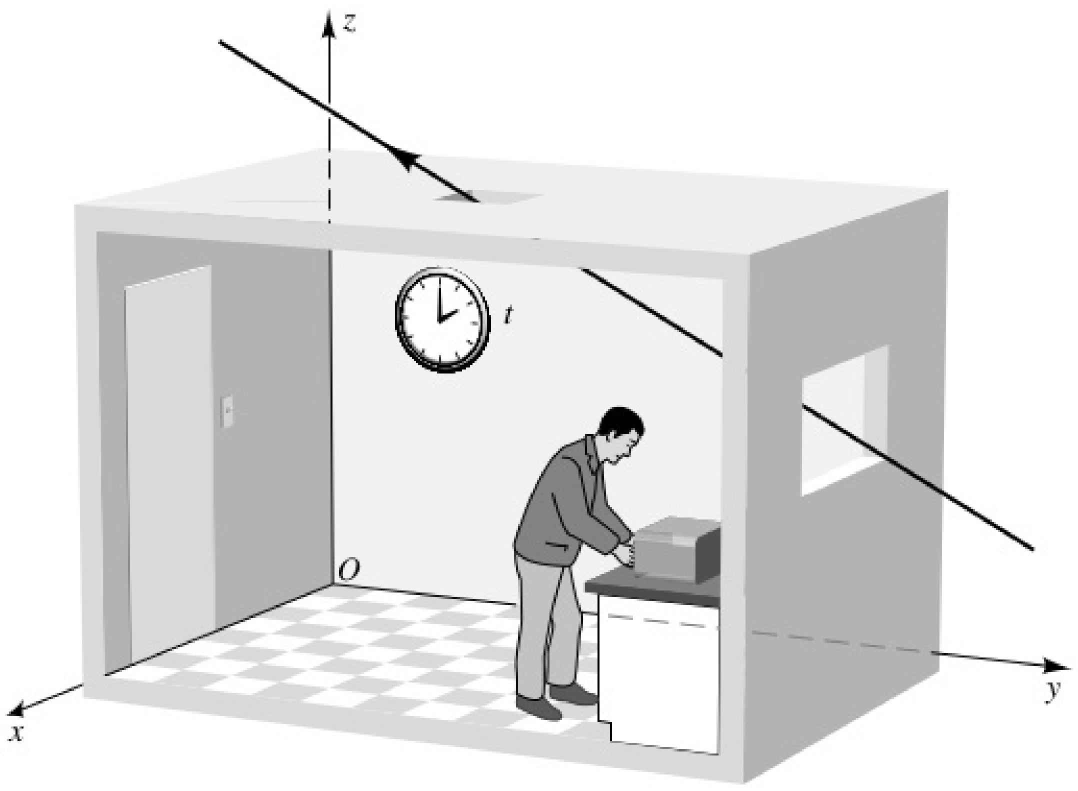

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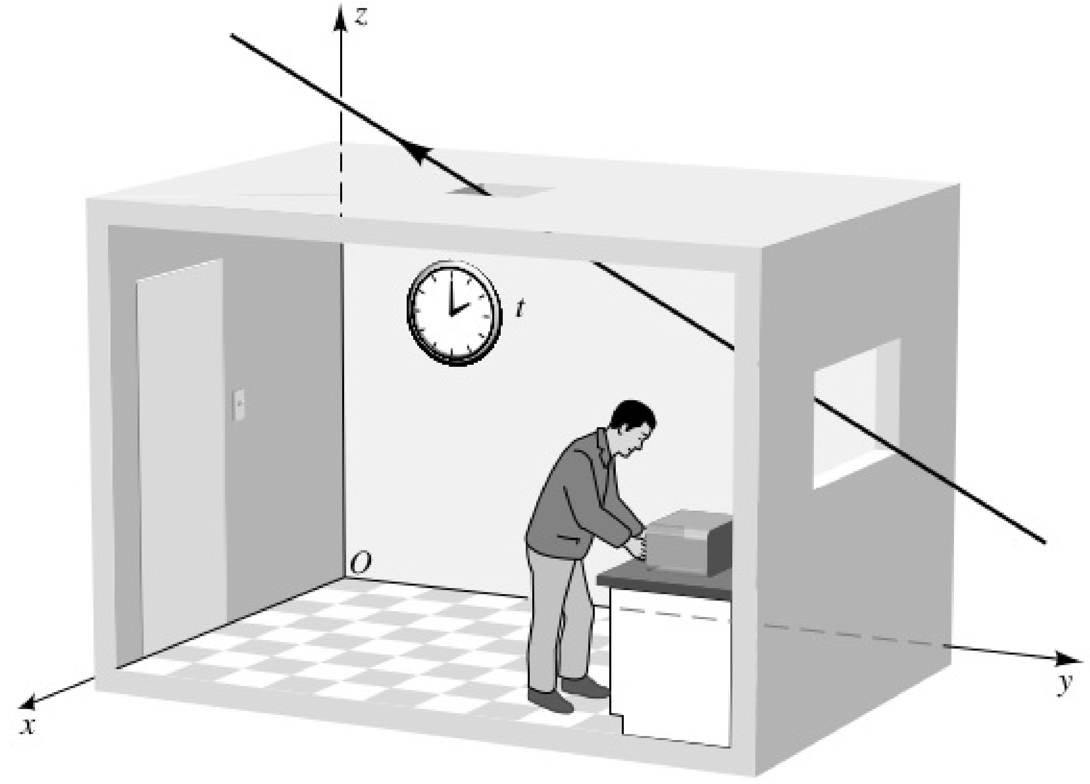

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Therefore $\frac{dp^\mu}{d\tau} = f^\mu$ has 3-independent equations to solve (due to $p_\mu p^\mu = -m^2$)

* Photons

massless particles on null lines, e.g.

$\rightsquigarrow$ $x^\mu = u^\mu \cdot \lambda$, $u^\mu = (1,1,0,0)$, $u_\mu u^\mu = 0$ (null)

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λ: affine parameter, any $\lambda' = \alpha\lambda + \beta$ is also an affine parameter

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$p_\mu p^\mu = 0 \qquad k_\mu k^\mu = 0$ null vectors

λ affine $\Leftrightarrow \dfrac{dp^\mu}{d\lambda} = 0$. We usually choose λ, so that $p^\mu = \dfrac{dx^\mu}{d\lambda}$

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$$-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

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use antisymmetry

$\epsilon_{ijk} = -\epsilon_{ikj}$

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* # Electromagnetism (EM)

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$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = \mathcal{E}_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

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* <u>Electromagnetism</u> (EM)

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -\mathcal{E}_1 & -\mathcal{E}_2 & -\mathcal{E}_3 \\ \mathcal{E}_1 & 0 & B_3 & -B_2 \\ \mathcal{E}_2 & -B_3 & 0 & B_1 \\ \mathcal{E}_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad (F^{\mu\nu}) = \begin{pmatrix} 0 & \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 \\ -\mathcal{E}_1 & 0 & B_3 & -B_2 \\ -\mathcal{E}_2 & -B_3 & 0 & B_1 \\ -\mathcal{E}_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = \mathcal{E}_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -\mathcal{E}_i \qquad B_k = \frac{1}{2}\epsilon_{kij} F_{ij}$$

Maxwell's Equations: dynamics

$$(\nabla \times B)^i - \partial_t \mathcal{E}^i = J^i$$

$$\nabla \cdot \mathcal{E} = \rho$$

$$(\nabla \times \mathcal{E})^i + \partial_t B^i = 0$$

$$\nabla \cdot B = 0$$

* Electromagnetism (EM)

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

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$$F_{i0} = \mathcal{E}_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

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Maxwell's Equations: dynamics

$$(\nabla\times B)^i - \partial_t \mathcal{E}^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 \mathcal{E}_i = J_i \qquad (1)$$

$$\nabla\cdot\mathcal{E} = \rho \qquad \partial_i \mathcal{E}_i = \rho \equiv J^0 \qquad (2)$$

$$\Longleftrightarrow$$

$$(\nabla\times\mathcal{E})^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\partial_j \mathcal{E}_k + \partial_0 B_i = 0 \qquad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \qquad (4)$$

$$\epsilon_{ijk}\partial_j B_k = \epsilon_{ijk}\partial_j\left(\tfrac{1}{2}\epsilon_{klm}F_{lm}\right)$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$$

$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij}F_{ij}$$

$$(\nabla\times B)^i - \partial_t \mathcal{E}^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 \mathcal{E}_i = J_i \quad (1)$$

$$\nabla\cdot\mathcal{E} = \rho \qquad \partial_i \mathcal{E}_i = \rho \equiv J^0 \quad (2)$$

$$\Leftrightarrow$$

$$(\nabla\times\mathcal{E})^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\partial_j \mathcal{E}_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\epsilon_{ijk}\partial_j B_k = \epsilon_{ijk}\partial_j\left(\tfrac{1}{2}\epsilon_{klm}F_{lm}\right)$$

$$= \tfrac{1}{2}\epsilon_{kij}\epsilon_{klm}\partial_j F_{lm}$$

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$$

$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij}F_{ij}$$

$$(\nabla\times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 E_i = J_i \quad (1)$$

$$\nabla\cdot E = \rho \qquad \partial_i E_i = \rho \equiv J^0 \quad (2)$$

$$\Longleftrightarrow$$

$$(\nabla\times E)^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\epsilon_{ijk}\partial_j B_k = \epsilon_{ijk}\partial_j\left(\tfrac{1}{2}\epsilon_{klm}F_{lm}\right)$$
$$= \tfrac{1}{2}\epsilon_{kij}\,\epsilon_{klm}\,\partial_j F_{lm}$$
$$= \tfrac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_j F_{lm}$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$

$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij}F_{ij}$

$$(\nabla\times B)^i - \partial_t \mathcal{E}^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 \mathcal{E}_i = J_i \quad (1)$$
$$\nabla\cdot\mathcal{E} = \rho \qquad \partial_i \mathcal{E}_i = \rho \equiv J^0 \quad (2)$$
$$\Longleftrightarrow$$
$$(\nabla\times\mathcal{E})^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\partial_j \mathcal{E}_k + \partial_0 B_i = 0 \quad (3)$$
$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\epsilon_{ijk}\partial_j B_k = \epsilon_{ijk}\partial_j\left(\tfrac{1}{2}\epsilon_{klm}F_{lm}\right)$$
$$=\tfrac{1}{2}\epsilon_{kij}\epsilon_{klm}\partial_j F_{lm}$$
$$=\tfrac{1}{2}\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\partial_j F_{lm}$$
$$=\tfrac{1}{2}\left(\partial_j F_{ij}-\partial_j F_{ji}\right)=\partial_j F_{ij}$$

$$\epsilon_{kij}\epsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$$

$F_{i0}=E_i$	$F_{ij}=\epsilon_{ijk}B_k$
$F^{i0}=-E_i$	$B_k=\tfrac{1}{2}\epsilon_{kij}F_{ij}$

$$(\nabla\times B)^i-\partial_t\mathcal{E}^i=J^i \qquad \epsilon_{ijk}\partial_j B_k-\partial_0\mathcal{E}_i=J_i \quad (1)$$
$$\nabla\cdot\mathcal{E}=\rho \qquad \partial_i\mathcal{E}_i=\rho\equiv J^0 \quad (2)$$
$$\Longleftrightarrow$$
$$(\nabla\times\mathcal{E})^i+\partial_t B^i=0 \qquad \epsilon_{ijk}\partial_j\mathcal{E}_k+\partial_0 B_i=0 \quad (3)$$
$$\nabla\cdot B=0 \qquad \partial_i B_i=0 \quad (4)$$

$$\epsilon_{ijk}\partial_j B_k = \epsilon_{ijk}\partial_j\left(\tfrac{1}{2}\epsilon_{klm}F_{lm}\right)$$
$$= \tfrac{1}{2}\epsilon_{kij}\,\epsilon_{klm}\,\partial_j F_{lm}$$
$$= \tfrac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_j F_{lm}$$
$$= \tfrac{1}{2}\left(\partial_j F_{ij} - \partial_j F_{ji}\right) = \partial_j F_{ij}$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$$
$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij}F_{ij}$$

$$(1) \Rightarrow \quad \partial_j F_{ij} + \partial_0 F^{i0} = J^i$$

$$(\nabla\times B)^i - \partial_t \mathcal{E}^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 \mathcal{E}_i = J_i \quad (1)$$
$$\nabla\cdot\mathcal{E} = \rho \qquad \partial_i \mathcal{E}_i = \rho \equiv J^0 \quad (2)$$
$$\Leftrightarrow$$
$$(\nabla\times\mathcal{E})^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\partial_j \mathcal{E}_k + \partial_0 B_i = 0 \quad (3)$$
$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij} F_{ij}$$

$$(2) \Rightarrow \quad \partial_0 F^{00} + \partial_i F^{0i} = J^0$$

$$(1) \Rightarrow \quad \partial_j F_{ij} + \partial_0 F^{i0} = J^i$$

$$(\nabla \times B)^i - \partial_t \mathcal{E}^i = J^i \qquad \epsilon_{ijk}\,\partial_j B_k - \partial_0 \mathcal{E}_i = J_i \qquad (1)$$

$$\nabla \cdot \mathcal{E} = \rho \qquad \partial_i \mathcal{E}_i = \rho \equiv J^0 \qquad (2)$$

$$\Leftrightarrow$$

$$(\nabla \times \mathcal{E})^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\,\partial_j \mathcal{E}_k + \partial_0 B_i = 0 \qquad (3)$$

$$\nabla \cdot B = 0 \qquad \partial_i B_i = 0 \qquad (4)$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij} F_{ij}$$

$$\left.\begin{aligned} (2) &\Rightarrow \partial_0 F^{00} + \partial_i F^{0i} = J^0 \\ (1) &\Rightarrow \partial_j F_{ij} + \partial_0 F^{i0} = J^i \end{aligned}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$(\nabla \times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 E_i = J_i \quad (1)$$

$$\nabla \cdot E = \rho \qquad \partial_i E_i = \rho \equiv J^0 \quad (2)$$

$$\Leftrightarrow$$

$$(\nabla \times E)^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla \cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk}F_{jk}\right) = 0$$

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$$

$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij}F_{ij}$$

$$(\nabla\times B)^i - \partial_t \mathcal{E}^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 \mathcal{E}_i = J_i \quad (1)$$

$$\nabla\cdot\mathcal{E} = \rho \qquad \partial_i\mathcal{E}_i = \rho \equiv J^0 \quad (2)$$

$$\left.\begin{matrix}(1)\\(2)\end{matrix}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\Longleftrightarrow$$

$$(\nabla\times\mathcal{E})^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\partial_j\mathcal{E}_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\, \partial_j F_{k0} + \partial_0 \left(\frac{1}{2} \epsilon_{ijk} F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk} \left[\partial_j F_{k0} + \frac{1}{2} \partial_0 F_{jk} \right] = 0 \qquad (5)$$

$$\epsilon_{kij}\, \epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2} \epsilon_{kij} F_{ij}$$

$$(\nabla \times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\, \partial_j B_k - \partial_0 E_i = J_i \quad (1)$$

$$\nabla \cdot E = \rho \qquad \partial_i E_i = \rho \equiv J^0 \quad (2)$$

$$\Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\Leftrightarrow$$

$$(\nabla \times E)^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\, \partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla \cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\partial_j F_{k0} + \partial_0\left(\frac{1}{2}\epsilon_{ijk}F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \frac{1}{2}\partial_0 F_{jk}\right] = 0 \qquad (5)$$

For $i=1$, (5) becomes $(\Rightarrow j,k = 2,3)$:

$$\epsilon_{123}\partial_2 F_{30} + \epsilon_{132}\partial_3 F_{20} + \frac{1}{2}\epsilon_{123}\partial_0 F_{23} + \frac{1}{2}\epsilon_{132}\partial_0 F_{32} = 0$$

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij}F_{ij}$$

$$(\nabla\times B)^i - \partial_t \mathcal{E}^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0\mathcal{E}_i = J_i \quad (1)$$

$$\nabla\cdot\mathcal{E} = \rho \qquad \partial_i\mathcal{E}_i = \rho \equiv J^0 \quad (2)$$

$$\left.\begin{matrix}(1)\\(2)\end{matrix}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\Leftrightarrow$$

$$(\nabla\times\mathcal{E})^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\partial_j\mathcal{E}_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\, \partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk} F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \tfrac{1}{2}\partial_0 F_{jk}\right] = 0 \qquad (5)$$

For $i=1$, (5) becomes ($\Rightarrow j,k = 2,3$):

$$\epsilon_{123}\,\partial_2 F_{30} + \epsilon_{132}\,\partial_3 F_{20} + \tfrac{1}{2}\epsilon_{123}\,\partial_0 F_{23} + \tfrac{1}{2}\epsilon_{132}\,\partial_0 F_{32} = 0 \Rightarrow$$

$$\partial_2 F_{30} - \partial_3 F_{20} + \tfrac{1}{2}\,\partial_0 F_{23} - \tfrac{1}{2}\,\partial_0 F_{32} = 0$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij} F_{ij}$$

$$(\nabla\times B)^i - \partial_t \mathcal{E}^i = J^i \qquad \epsilon_{ijk}\,\partial_j B_k - \partial_0 \mathcal{E}_i = J_i \quad (1)$$

$$\nabla\cdot\mathcal{E} = \rho \qquad \partial_i \mathcal{E}_i = \rho \equiv J^0 \quad (2)$$

$$\Leftrightarrow \qquad \left.\begin{matrix} (1) \\ (2) \end{matrix}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$(\nabla\times\mathcal{E})^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\,\partial_j \mathcal{E}_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\,\partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk} F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \tfrac{1}{2}\partial_0 F_{jk}\right] = 0 \qquad (5)$$

For $i=1$, (5) becomes $(\Rightarrow j,k = 2,3)$:

$$\epsilon_{123}\,\partial_2 F_{30} + \epsilon_{132}\,\partial_3 F_{20} + \tfrac{1}{2}\epsilon_{123}\,\partial_0 F_{23} + \tfrac{1}{2}\epsilon_{132}\,\partial_0 F_{32} = 0 \Rightarrow$$

$$\partial_2 F_{30} - \partial_3 F_{20} + \tfrac{1}{2}\,\partial_0 F_{23} - \tfrac{1}{2}\,\partial_0 F_{32} = 0 \Rightarrow$$

$$\partial_2 F_{30} + \partial_3 F_{02} + \partial_0 F_{23} = 0$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij} F_{ij}$$

$$(\nabla\times B)^i - \partial_t \mathcal{E}^i = J^i \qquad \epsilon_{ijk}\,\partial_j B_k - \partial_0 \mathcal{E}_i = J_i \quad (1)$$

$$\nabla\cdot\mathcal{E} = \rho \qquad \partial_i \mathcal{E}_i = \rho \equiv J^0 \quad (2)$$

$$\Leftrightarrow \qquad \left.\begin{matrix}(1)\\(2)\end{matrix}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$(\nabla\times\mathcal{E})^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\,\partial_j \mathcal{E}_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\partial_j F_{k0} + \partial_0\left(\frac{1}{2}\epsilon_{ijk}F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \frac{1}{2}\partial_0 F_{jk}\right] = 0 \qquad (5)$$

For $i=1$, (5) becomes $(\Rightarrow j,k = 2,3)$:

$$\epsilon_{123}\partial_2 F_{30} + \epsilon_{132}\partial_3 F_{20} + \frac{1}{2}\epsilon_{123}\partial_0 F_{23} + \frac{1}{2}\epsilon_{132}\partial_0 F_{32} = 0 \Rightarrow$$

$$\partial_2 F_{30} - \partial_3 F_{20} + \frac{1}{2}\partial_0 F_{23} - \frac{1}{2}\partial_0 F_{32} = 0 \Rightarrow$$

$$\partial_2 F_{30} + \partial_3 F_{02} + \partial_0 F_{23} = 0 \Rightarrow \partial_{[0}F_{23]} = 0$$

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij}F_{ij}$$

$$(\nabla\times B)^i - \partial_t \mathcal{E}^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0\mathcal{E}_i = J_i \quad (1)$$

$$\nabla\cdot\mathcal{E} = \rho \qquad \partial_i\mathcal{E}_i = \rho \equiv J^0 \quad (2)$$

$$\left.\begin{matrix}(1)\\(2)\end{matrix}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\Leftrightarrow$$

$$(\nabla\times\mathcal{E})^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\partial_j\mathcal{E}_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\,\partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk} F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \tfrac{1}{2}\partial_0 F_{jk}\right] = 0 \qquad (5)$$

For $i=1$, (5) becomes $\partial_{[0} F_{23]} = 0$

$i=2$ $\qquad \partial_{[0} F_{13]} = 0$

$i=3$ $\qquad \partial_{[0} F_{12]} = 0$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$

$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij} F_{ij}$

$(\nabla\times B)^i - \partial_t \mathcal{E}^i = J^i \qquad \epsilon_{ijk}\,\partial_j B_k - \partial_0 \mathcal{E}_i = J_i \quad (1)$

$\nabla\cdot\mathcal{E} = \rho \qquad \partial_i \mathcal{E}_i = \rho \equiv J^0 \quad (2)$

$\Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$

$\Leftrightarrow$

$(\nabla\times\mathcal{E})^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\,\partial_j \mathcal{E}_k + \partial_0 B_i = 0 \quad (3)$

$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$

$$(3) \Rightarrow \epsilon_{ijk} \partial_j F_{k0} + \partial_0 \left(\tfrac{1}{2} \epsilon_{ijk} F_{jk} \right) = 0 \Rightarrow$$

$$\epsilon_{ijk} \left[\partial_j F_{k0} + \tfrac{1}{2} \partial_0 F_{jk} \right] = 0 \qquad (5)$$

For $i=1$, (5) becomes $\partial_{[0} F_{23]} = 0$

$i=2$ $\qquad \partial_{[0} F_{13]} = 0$

$i=3$ $\qquad \partial_{[0} F_{12]} = 0$

$$\epsilon_{kij} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$

$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2} \epsilon_{kij} F_{ij}$

$$(4) \Rightarrow \partial_i \left(\tfrac{1}{2} \epsilon_{ijk} F_{jk} \right) = 0 \Rightarrow \epsilon_{ijk} \partial_i F_{jk} = 0 \Rightarrow \partial_{[i} F_{jk]} = 0$$

$$(\nabla \times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk} \partial_j B_k - \partial_0 E_i = J_i \quad (1)$$

$$\nabla \cdot E = \rho \qquad \partial_i E_i = \rho \equiv J^0 \quad (2)$$

$$\left. \begin{matrix} (1) \\ (2) \end{matrix} \right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\Leftrightarrow$$

$$(\nabla \times E)^i + \partial_t B^i = 0 \qquad \epsilon_{ijk} \partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla \cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk}F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \tfrac{1}{2}\partial_0 F_{jk}\right] = 0 \qquad (5)$$

For $i=1$, (5) becomes $\partial_{[0}F_{23]} = 0$

$i=2$ $\quad \partial_{[0}F_{13]} = 0$

$i=3$ $\quad \partial_{[0}F_{12]} = 0$

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$

$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij}F_{ij}$

$$(4) \Rightarrow \partial_i\left(\tfrac{1}{2}\epsilon_{ijk}F_{jk}\right) = 0 \Rightarrow \epsilon_{ijk}\partial_i F_{jk} = 0 \Rightarrow \partial_{[i}F_{jk]} = 0$$

$$(\nabla\times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 E_i = J_i \quad (1)$$

$$\nabla\cdot E = \rho \qquad \partial_i E_i = \rho \equiv J^0 \quad (2)$$

$$\Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$\Leftrightarrow$

$$(\nabla\times E)^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\Rightarrow \partial_{[\mu}F_{\nu\lambda]} = 0$$