- Vectors
  - definition as tangent to curves
- Tangent space at a point $T_p M$
  - coordinate bases
  - component xfms
- Vector fields
  - vectors smoothly defined all over M
  - integral curves

Vectors

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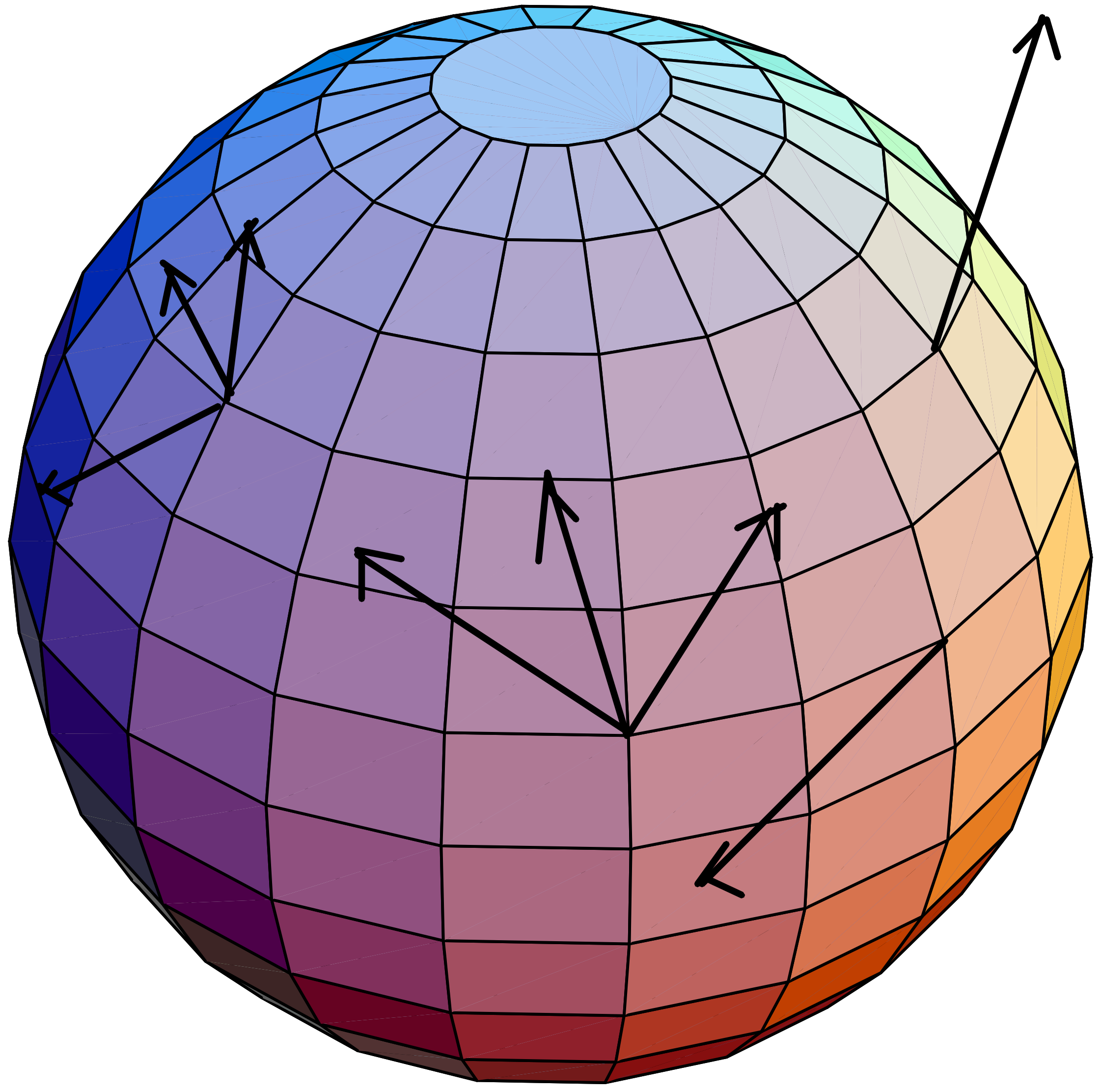
  - no reference to embeddings



Vectors are $\mathbb{R}^3$ vectors tangent to sphere embedded in $\mathbb{R}^3$!!!

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$\Rightarrow$ define them as objects tangent to curves...

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⇒ define them as objects tangent to curves...

- they form a different vector space at each point

→ no a priori natural association between vectors at each point

can't move vectors from one point to another: "stuck" at each point

Vectors

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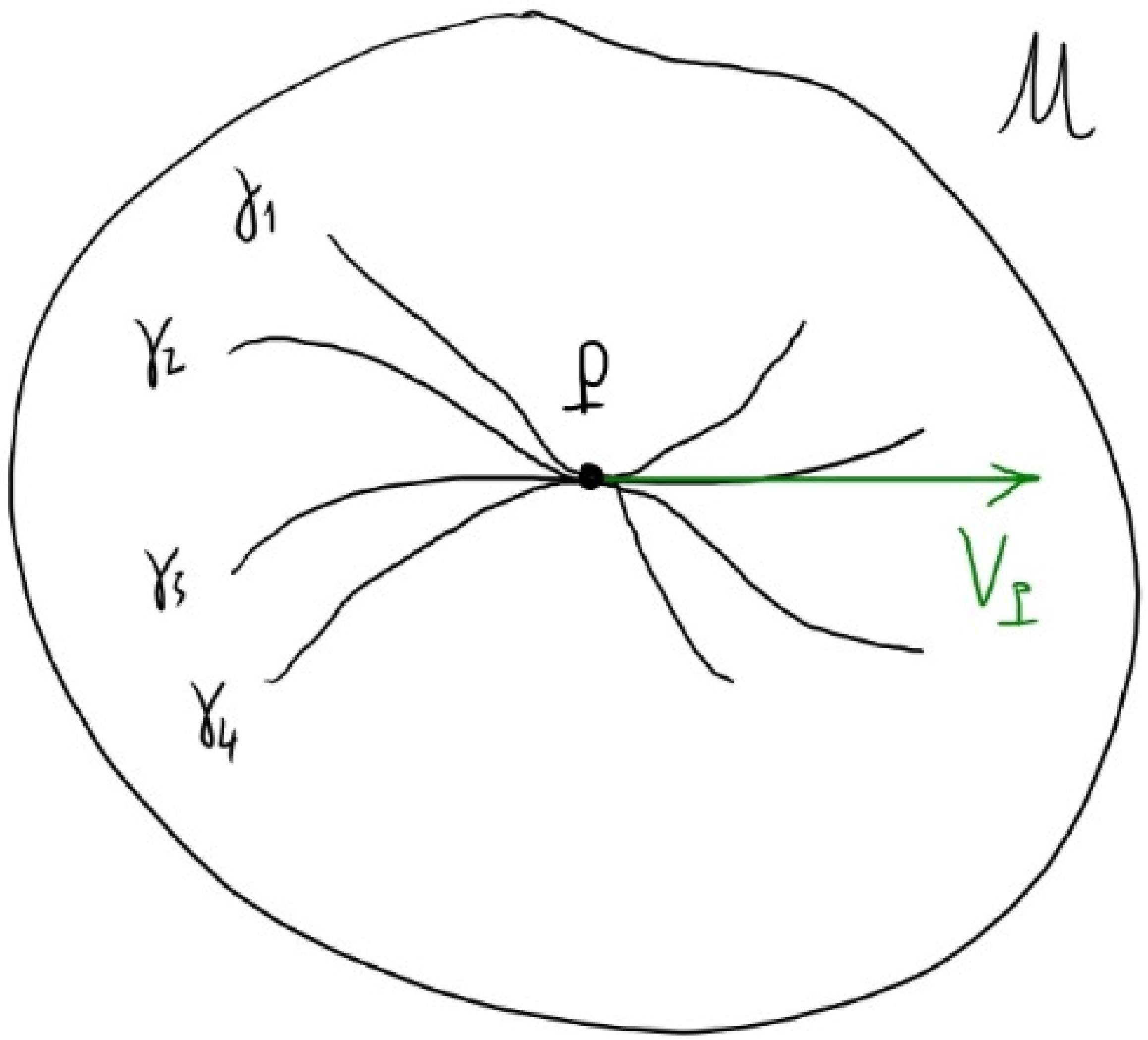
⇒ define them as objects tangent to curves...

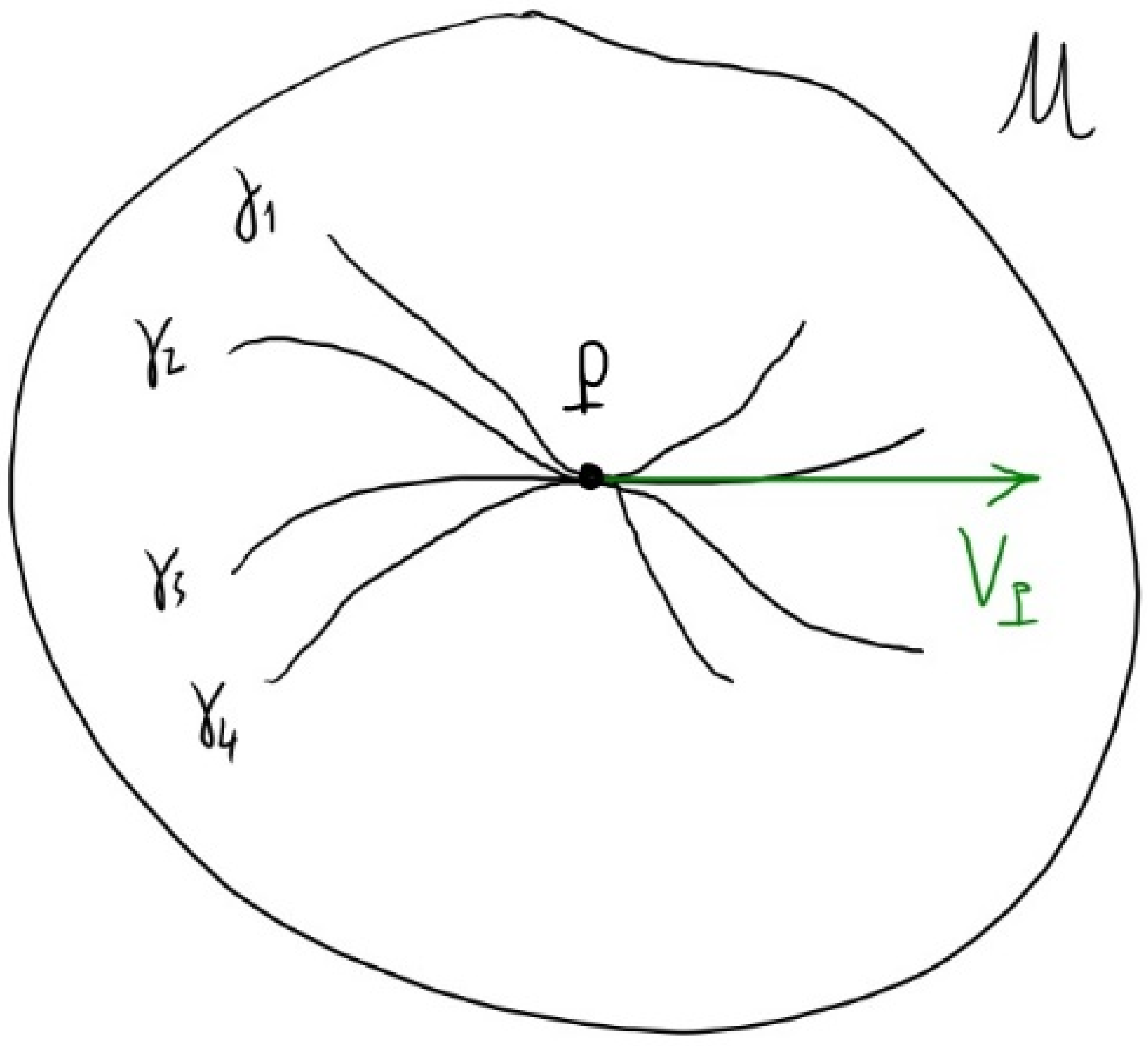
- they form a different vector space at each point
- they are the fundamental objects on which be build one-forms and higher rank tensors

(defined as linear maps on vectors + one-forms)



* Many curves passing through p have the same tangent vector

$$\text{vector} \equiv \left(\begin{array}{c} \text{equivalence class} \\ \gamma_i \sim \gamma_j \end{array} \right)$$



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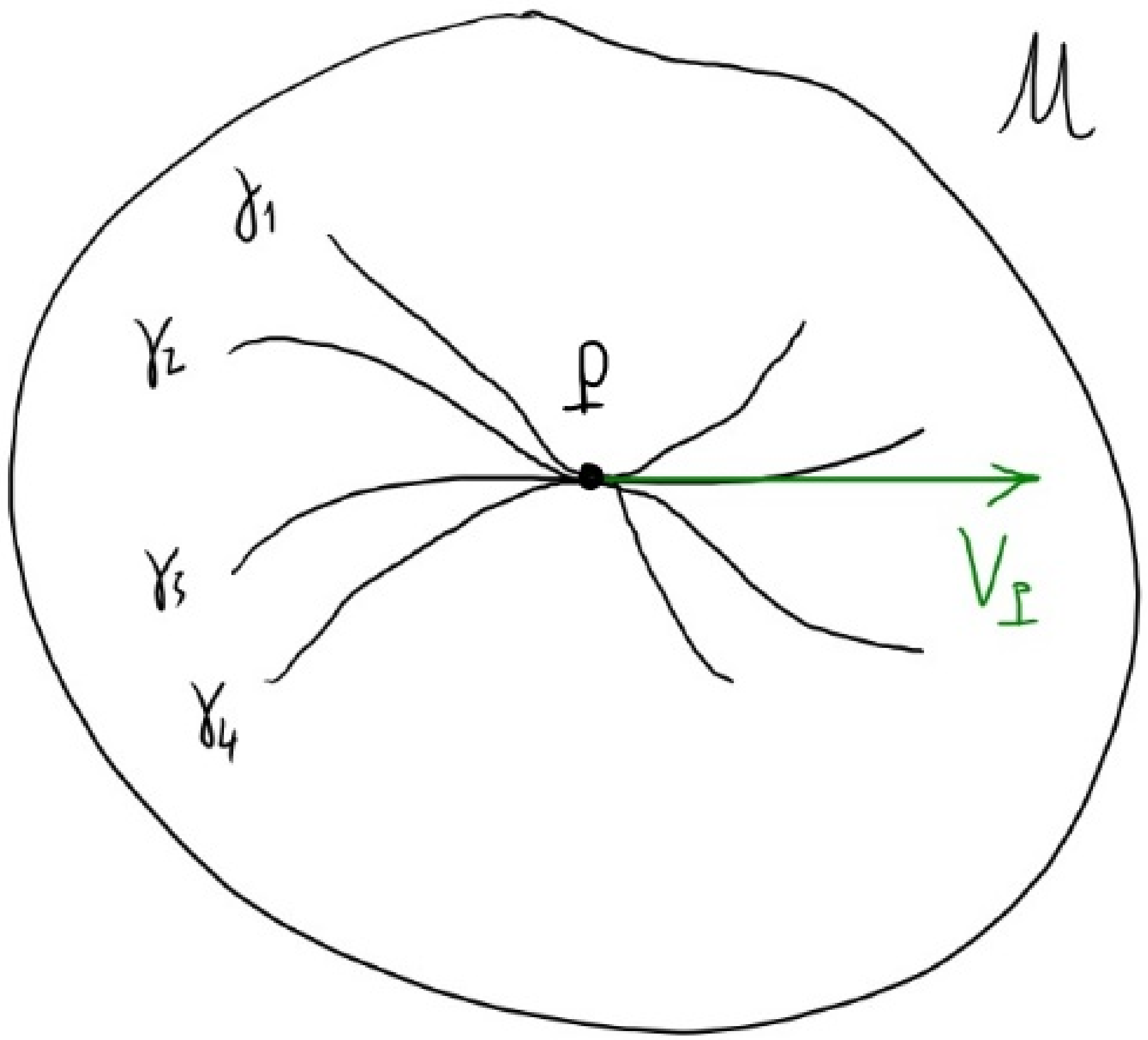
* The vector V_p ("velocity") depends on rate of change of "things" on γ_j



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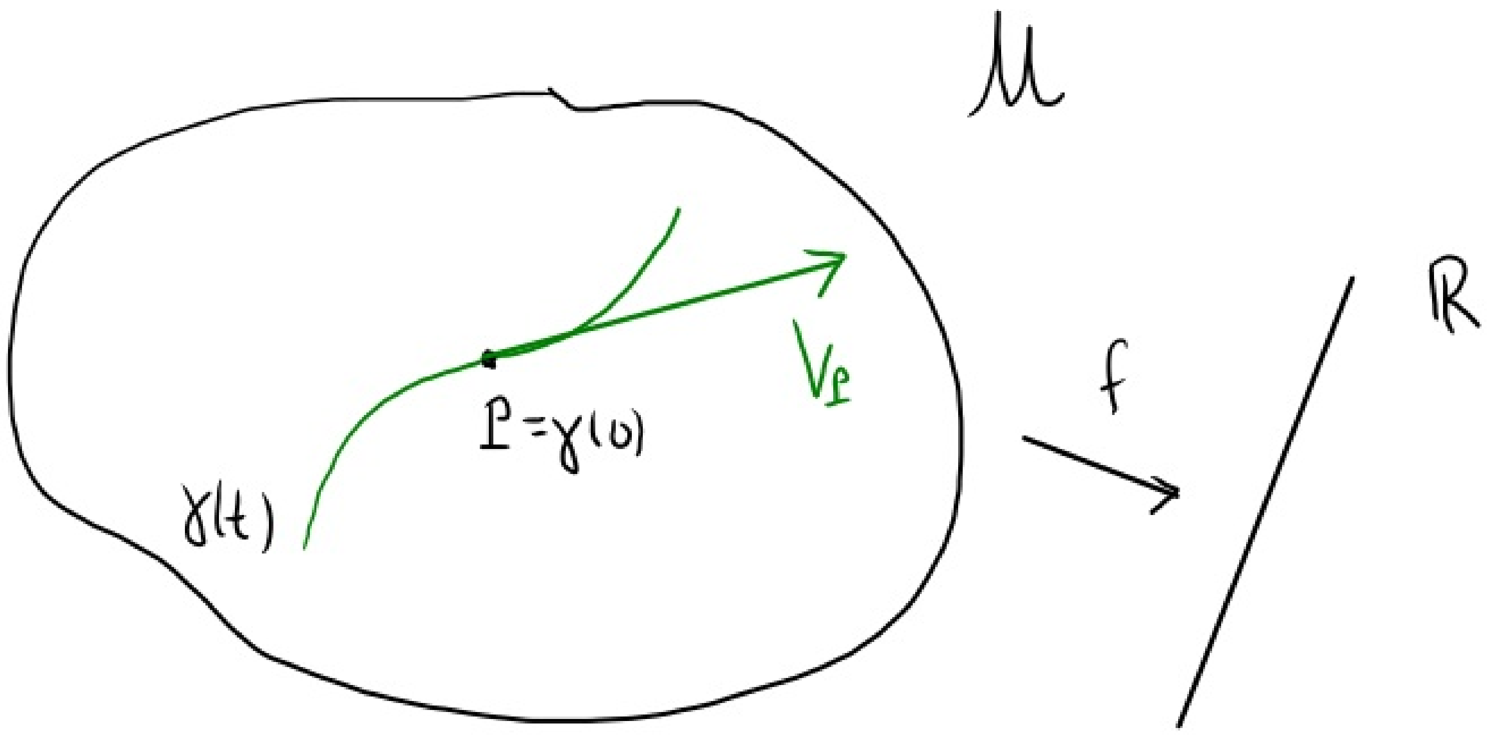
(for example, flat space vectors which are not "rates of change" cannot be generalized on curved spaces: position vector $\vec{r}$, $\vec{r} \times \vec{p}$, $\vec{r} \times \vec{F}$, ...)

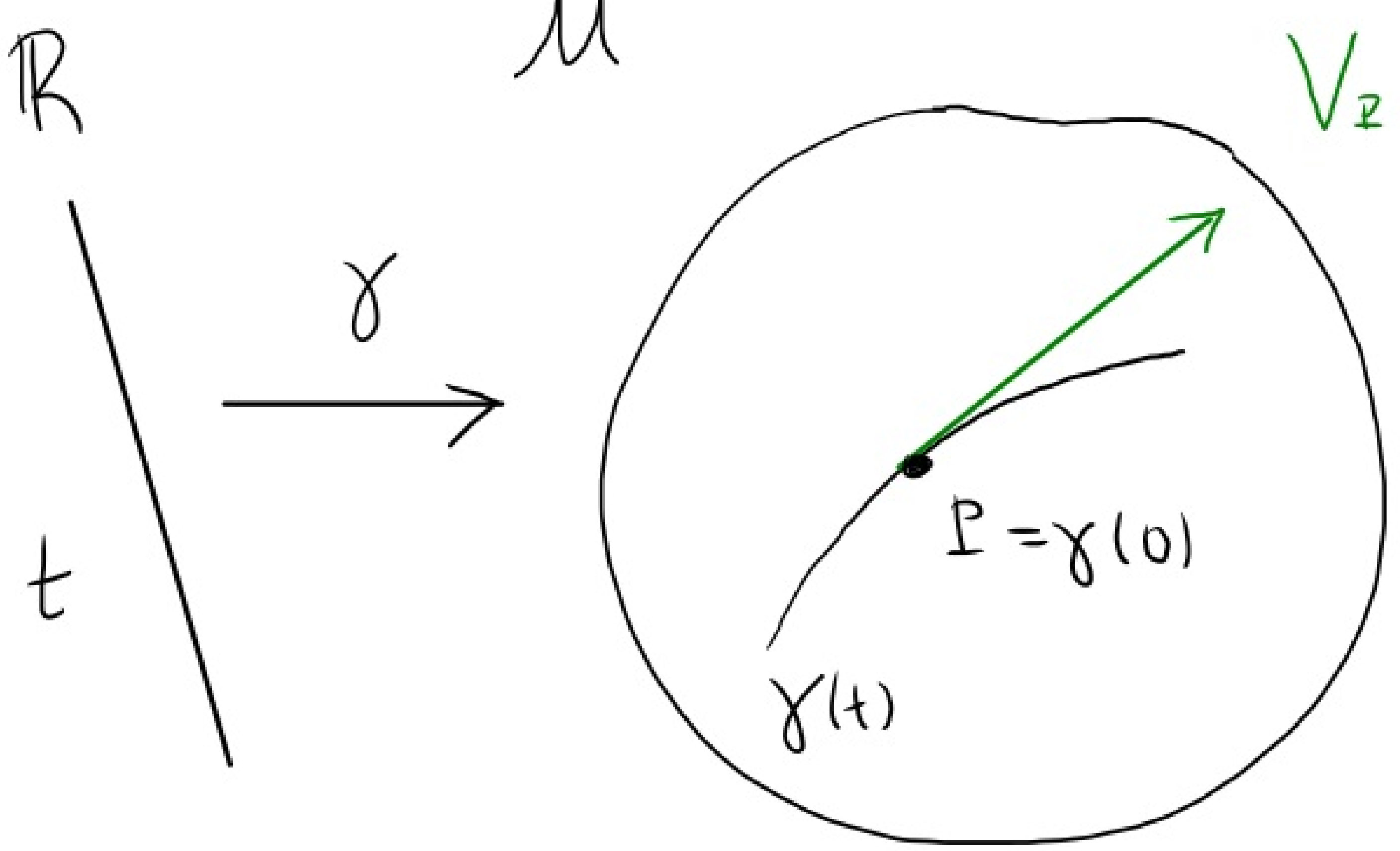


* Many curves passing through p have the same tangent vector

* The vector V_p ("velocity") depends on rate of change of "things" on γ;

* "things" are functions on M $\quad f: M \to \mathbb{R}$

(e.g. coordinate functions x^μ in a chart)



* a curve $\gamma : \mathbb{R} \to \mathcal{M}$
$\quad t \mapsto \gamma(t)$

e.g.: $P = \gamma(0)$

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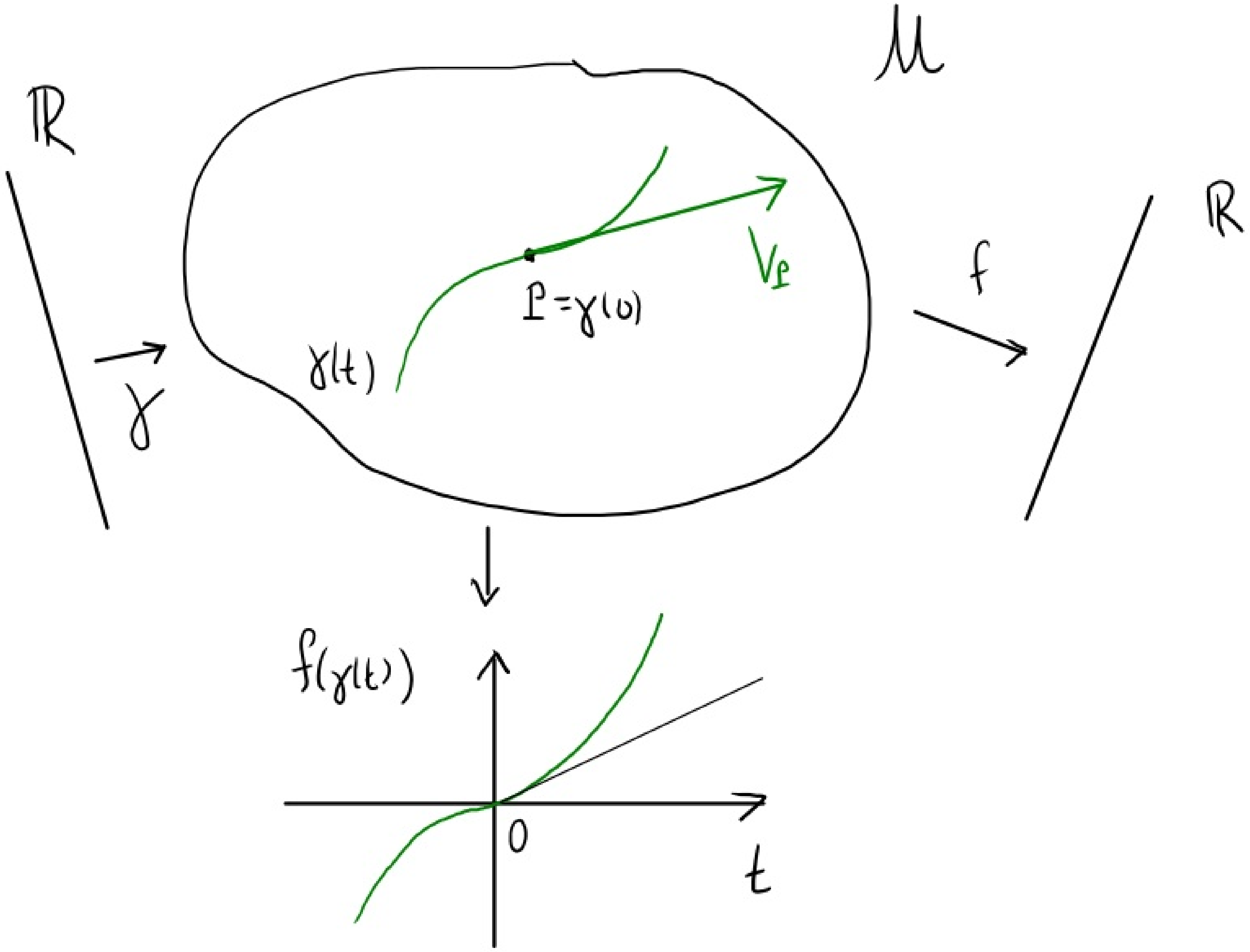

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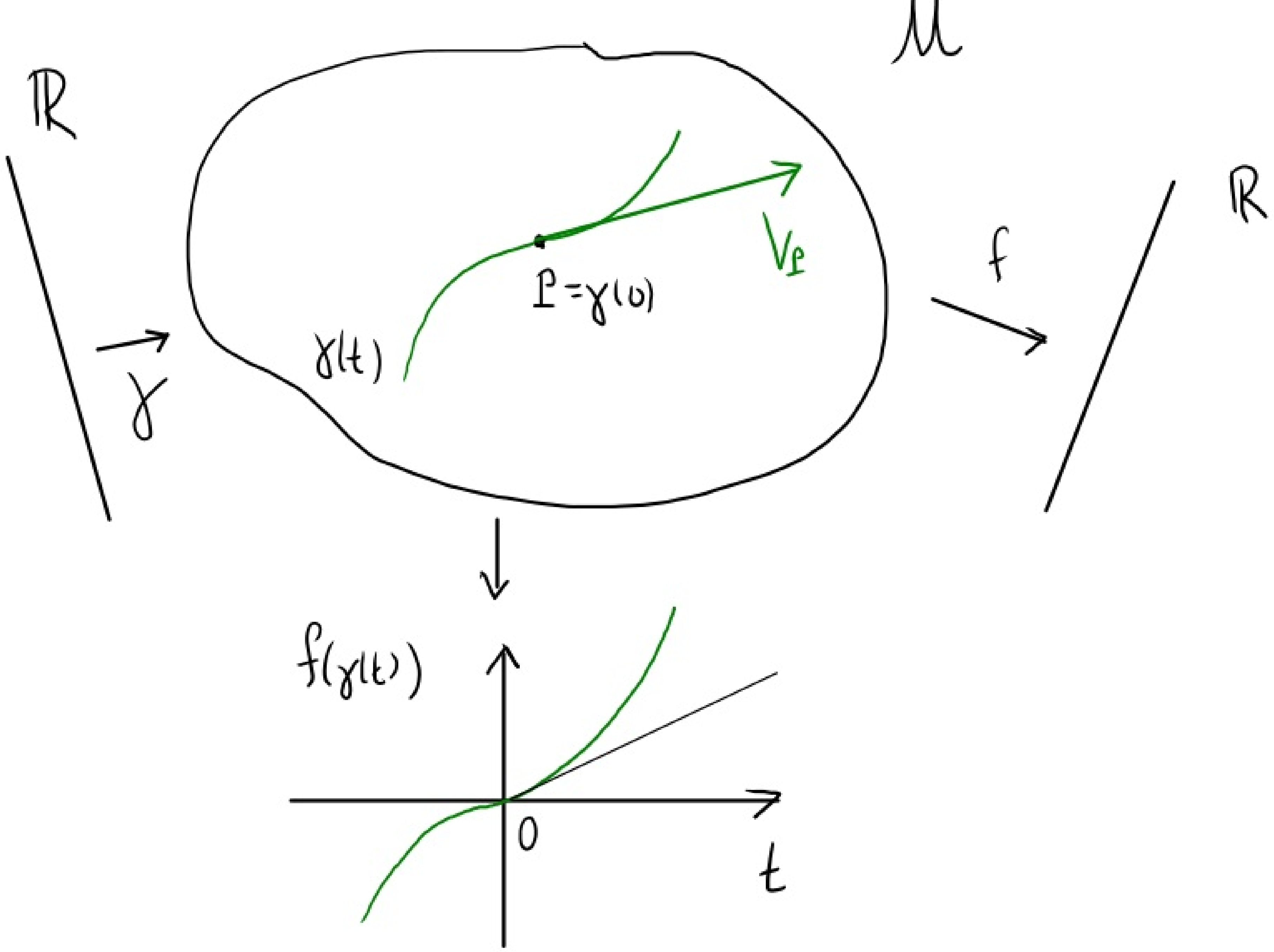

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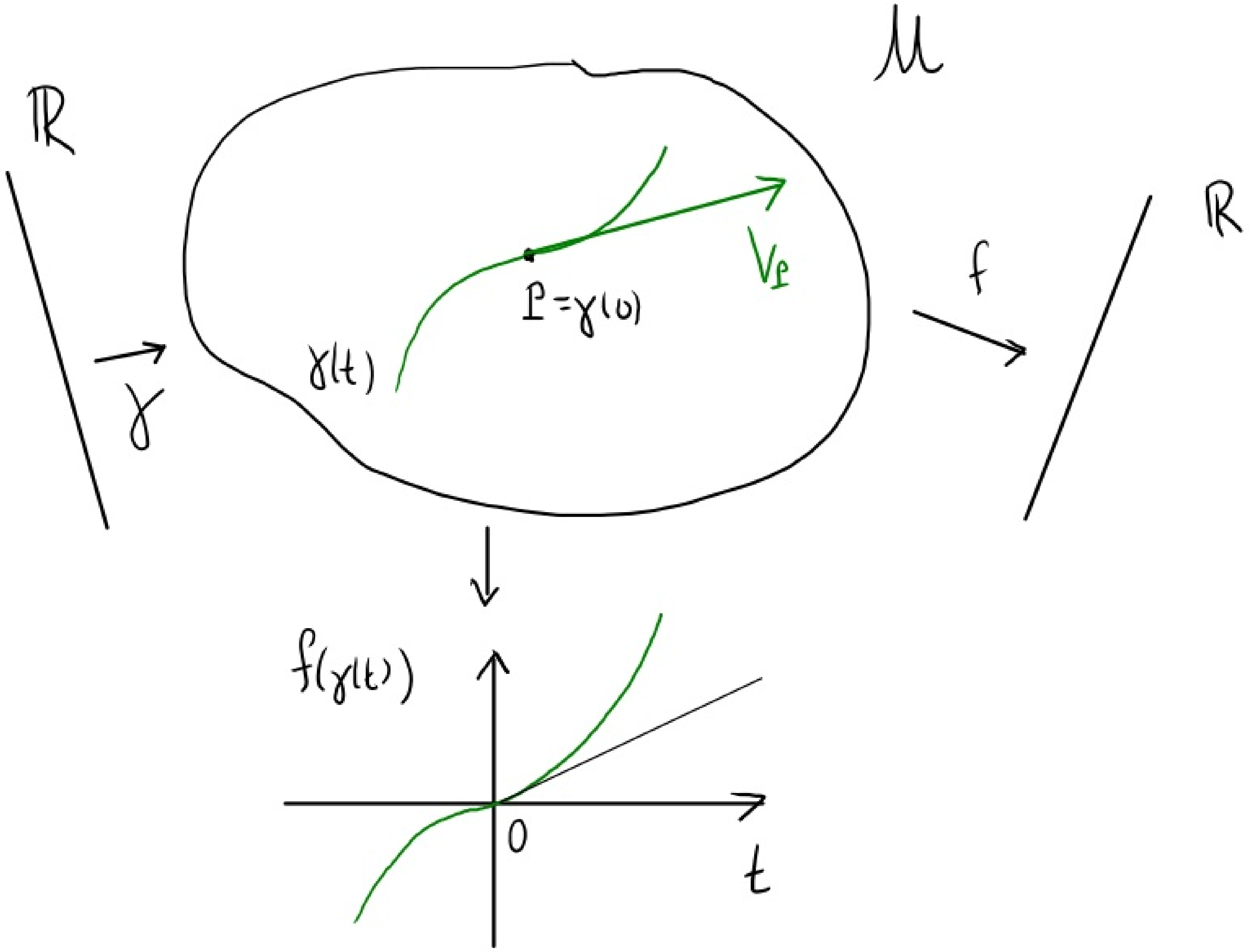

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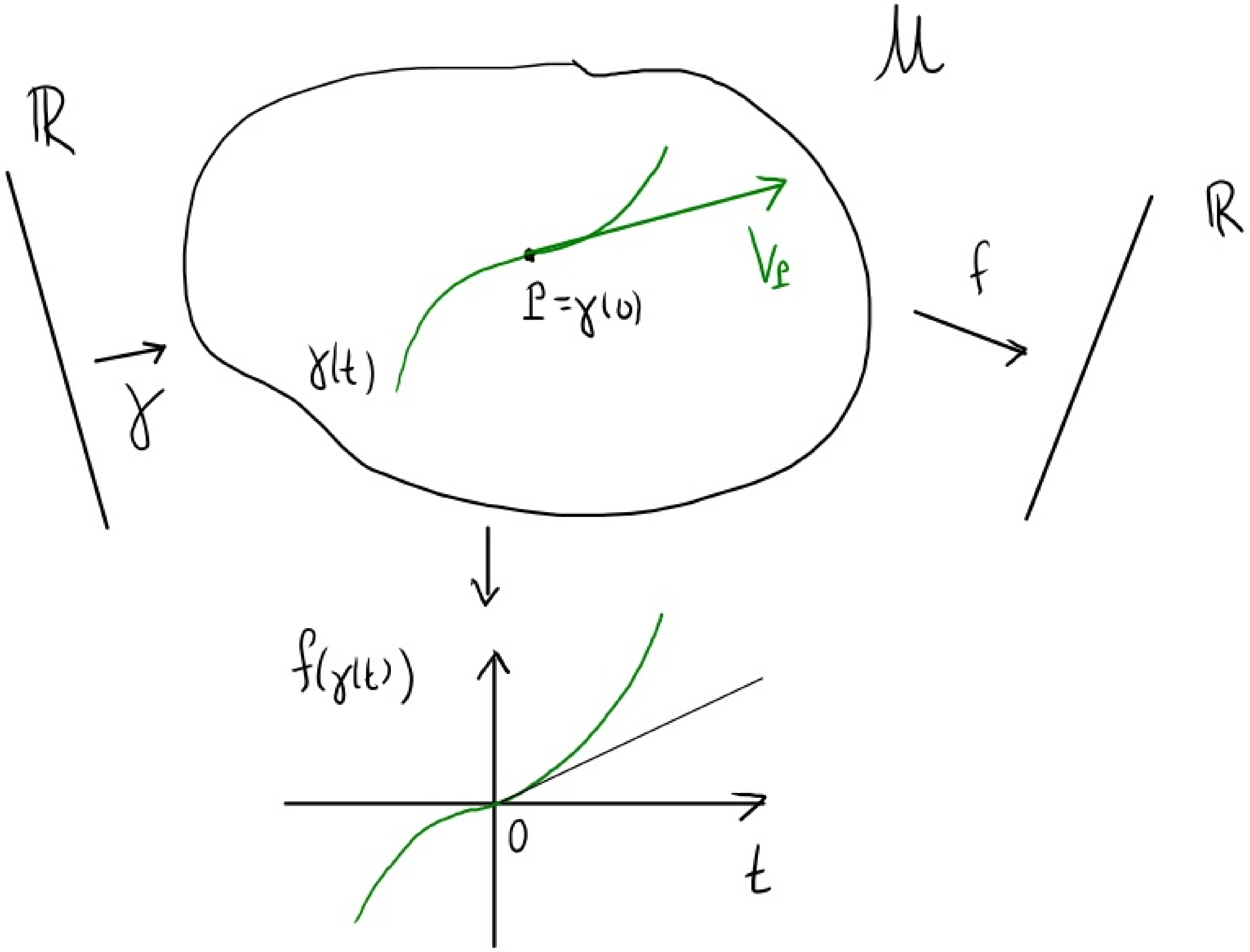
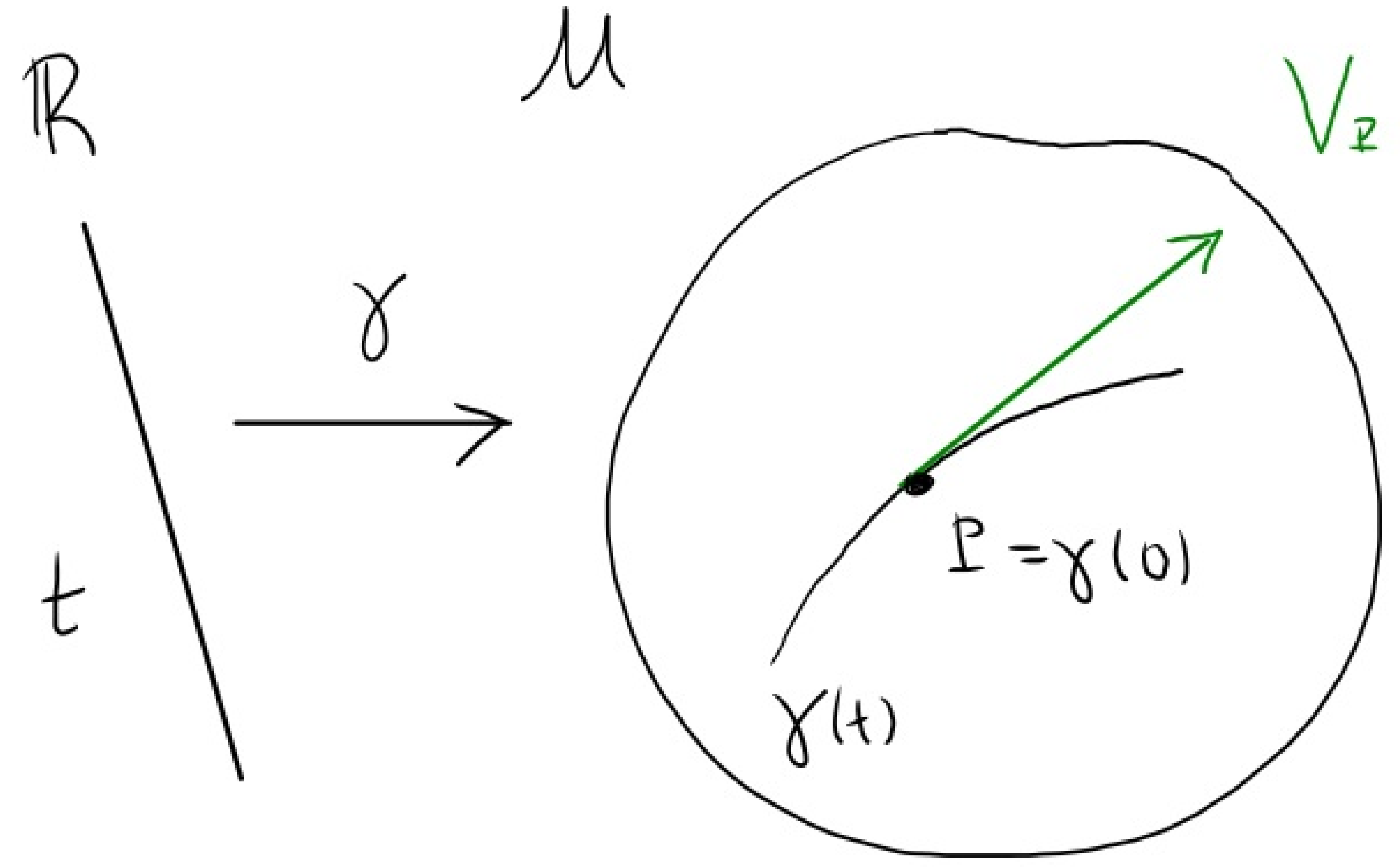


* the derivatives $\frac{df}{dt}\Big|_0$ for any f, a measure of how fast things change along γ at P

* we define V_P to be the operator $\frac{d}{dt}\Big|_0$ acting on any f s.t.:

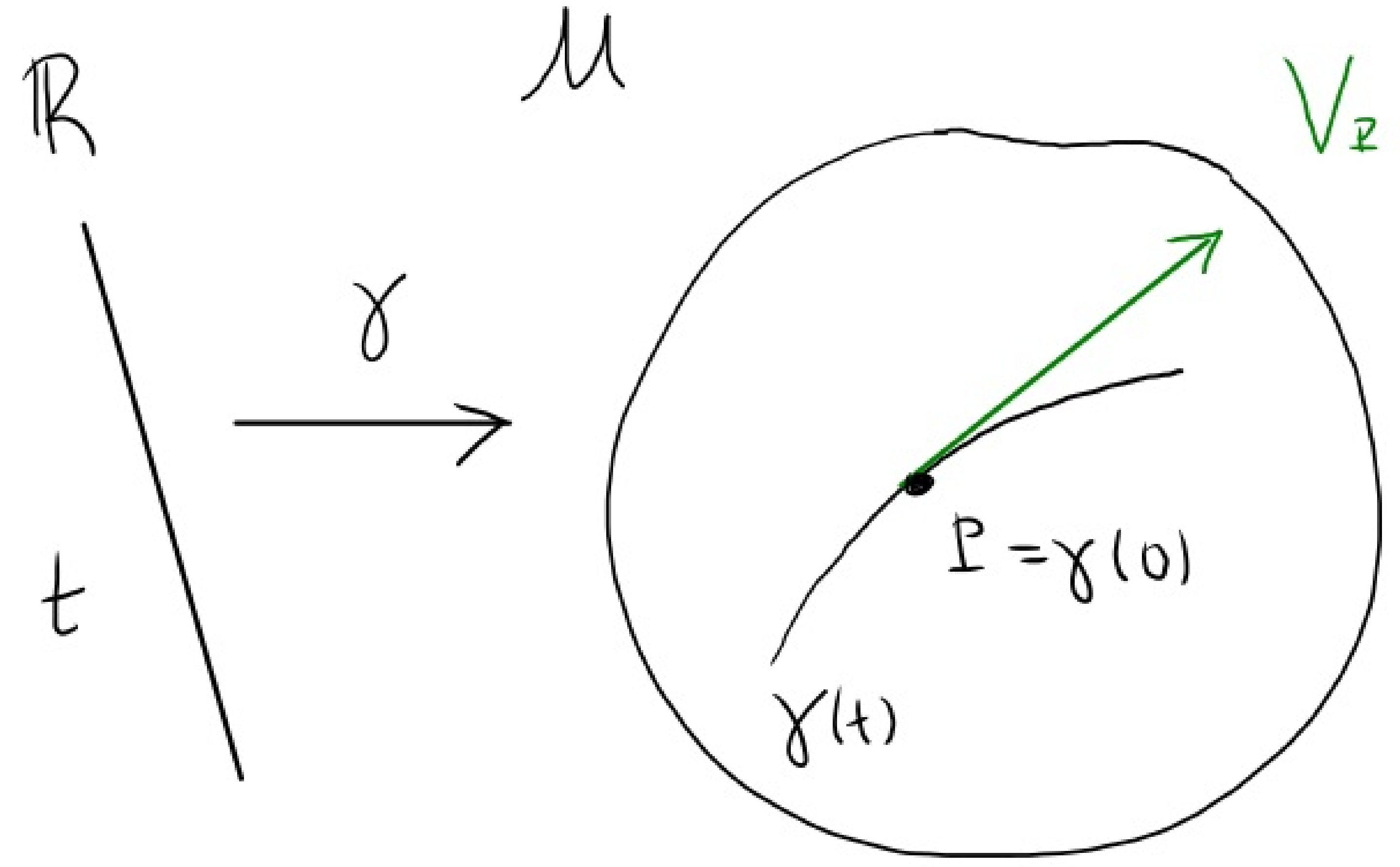
$$V_P(f) = \frac{df}{dt}\Big|_0$$

* $V_P(f)$ is the directional derivative of f along γ at P



* $V_P(f)$ is the directional derivative of f along γ at P
* Acts linearly on functions:

$$V_P(\alpha f + \beta g) = \alpha V_P(f) + \beta V_P(g) \quad \alpha, \beta \in \mathbb{R}$$



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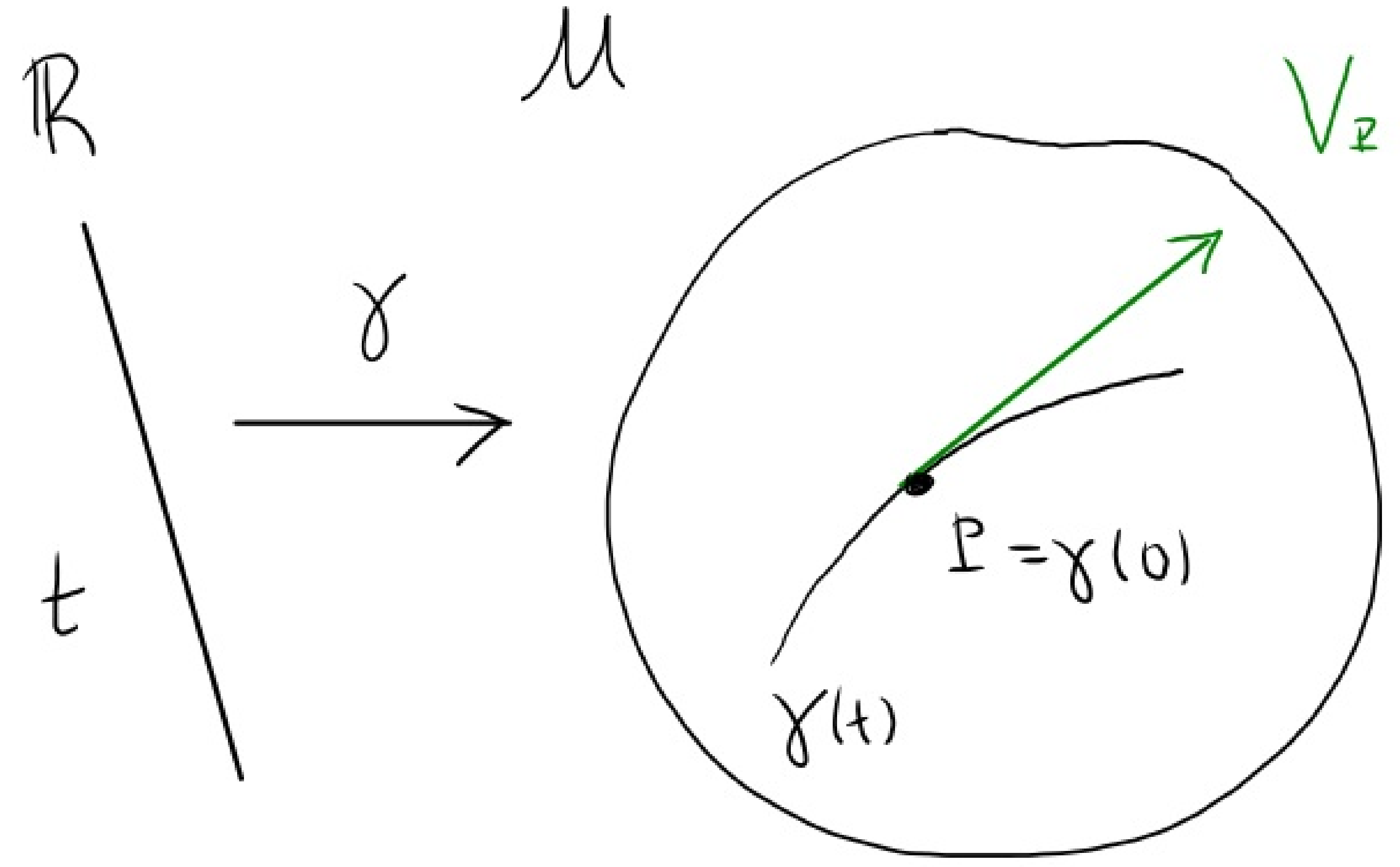

$\rightsquigarrow$ obvious:

$$\frac{d}{dt}(\alpha f + \beta g) = \alpha \frac{df}{dt} + \beta \frac{dg}{dt}$$

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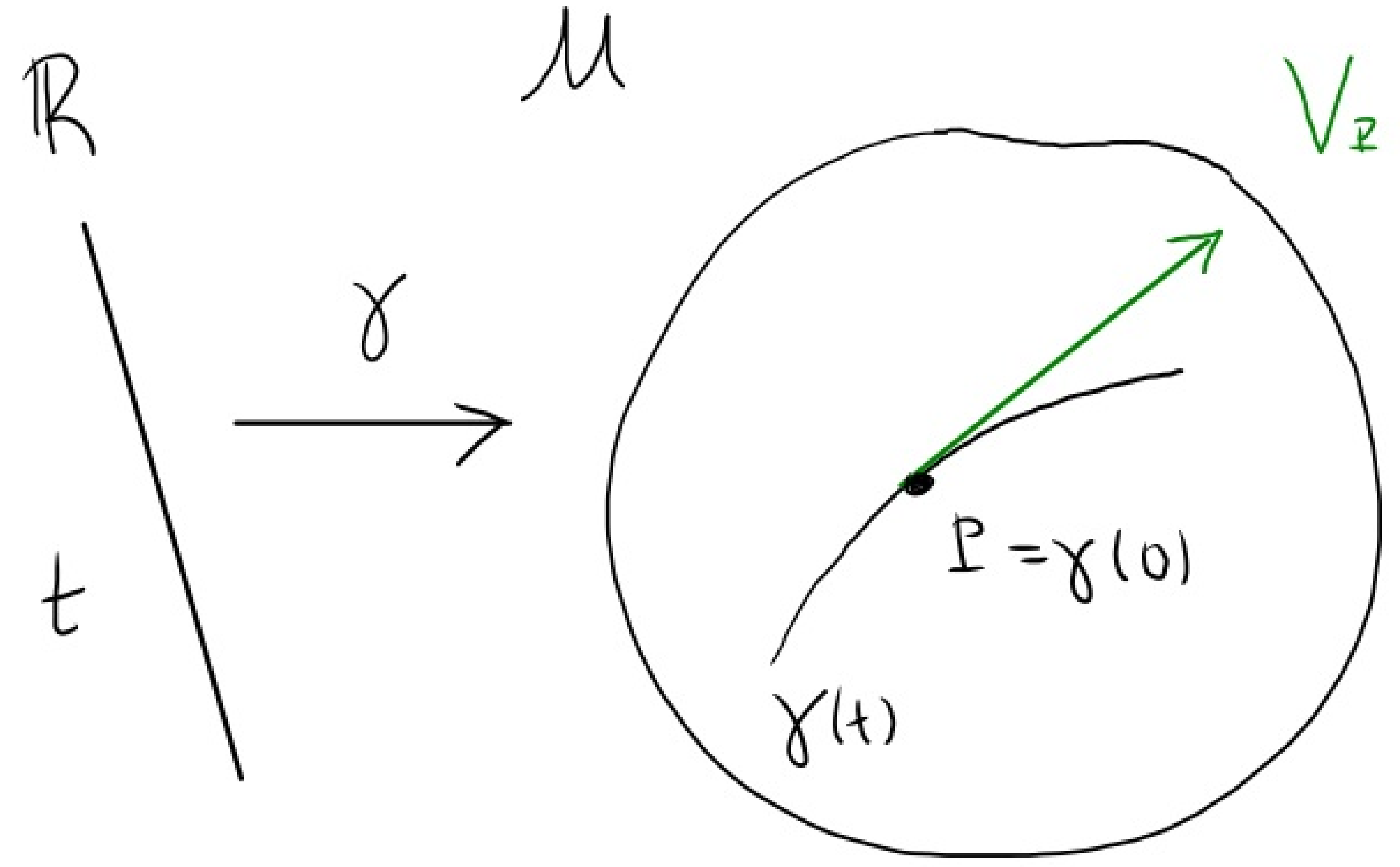

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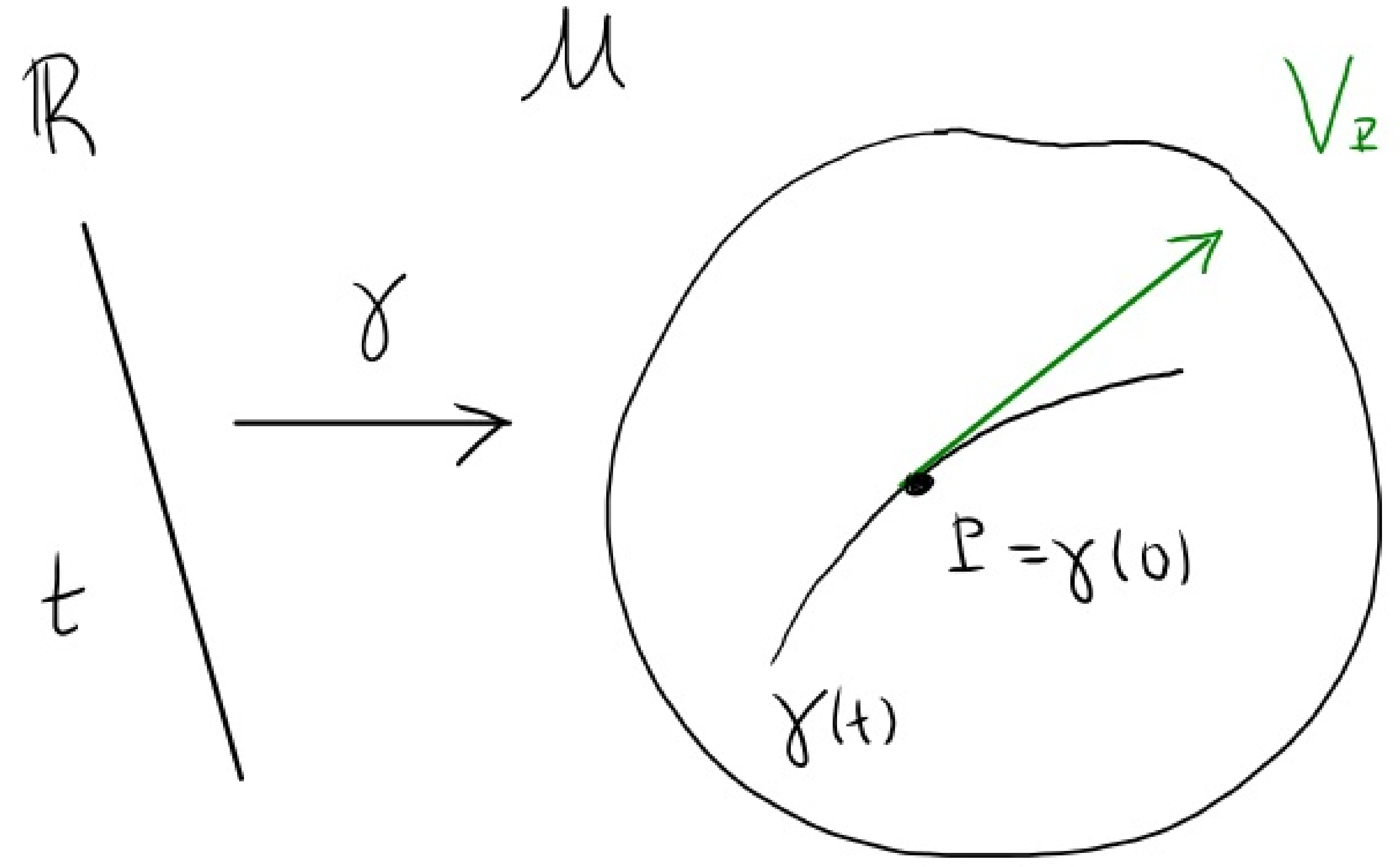
$$V_P(f \cdot g) = V_P(f) \cdot g + f \, V_P(g)$$

↳ ordinary product - not the composition

* $V_P(f)$ is the directional derivative of f along γ at P



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$\rightsquigarrow$ Indeed: $\displaystyle \frac{d}{dt}(f \cdot g)\Big|_0 = \frac{df}{dt} \cdot g\Big|_0 + f \, \frac{dg}{dt}\Big|_0$

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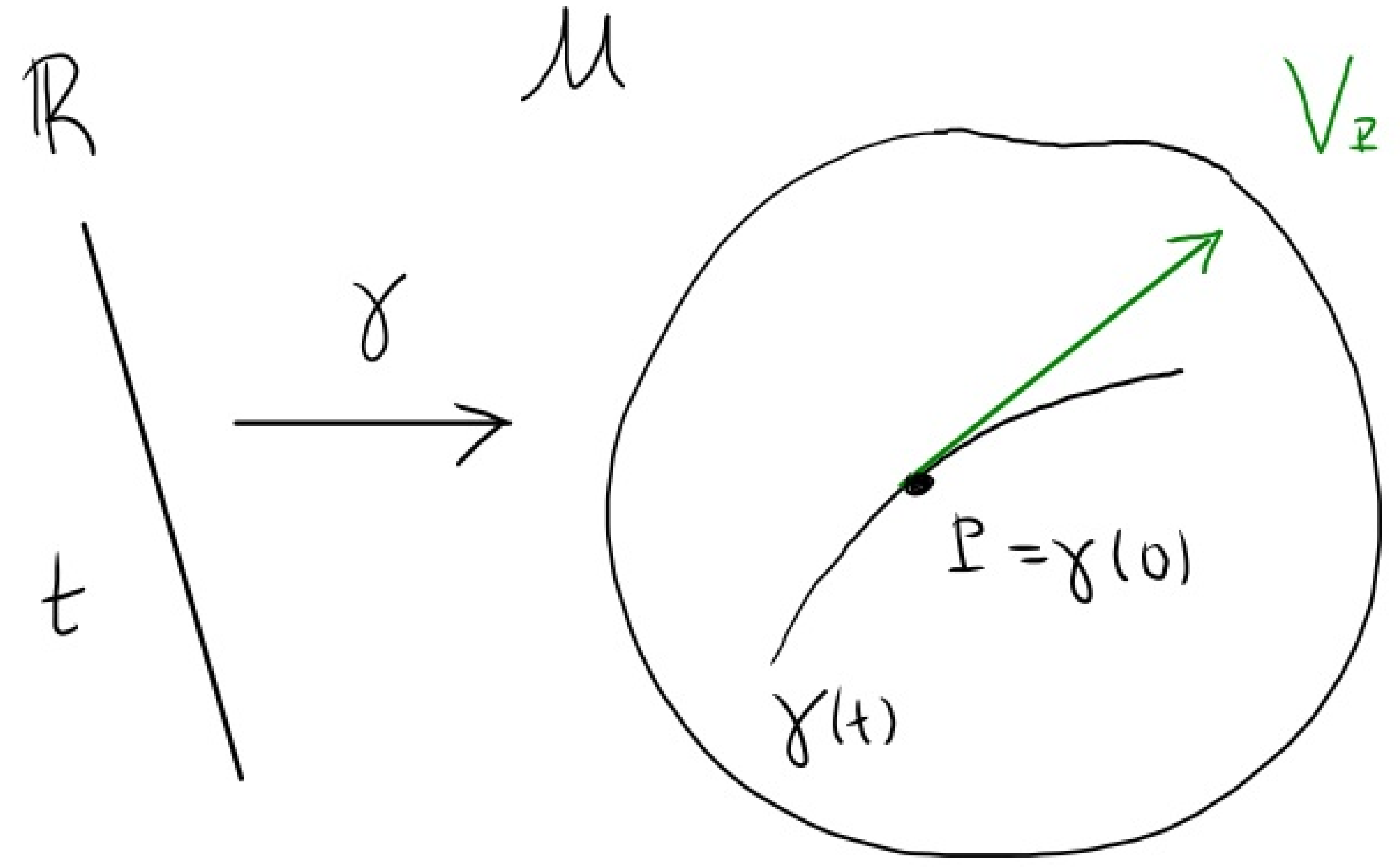

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$$\Rightarrow \quad V_P \text{ is a derivation on } F(\mathcal{M})$$

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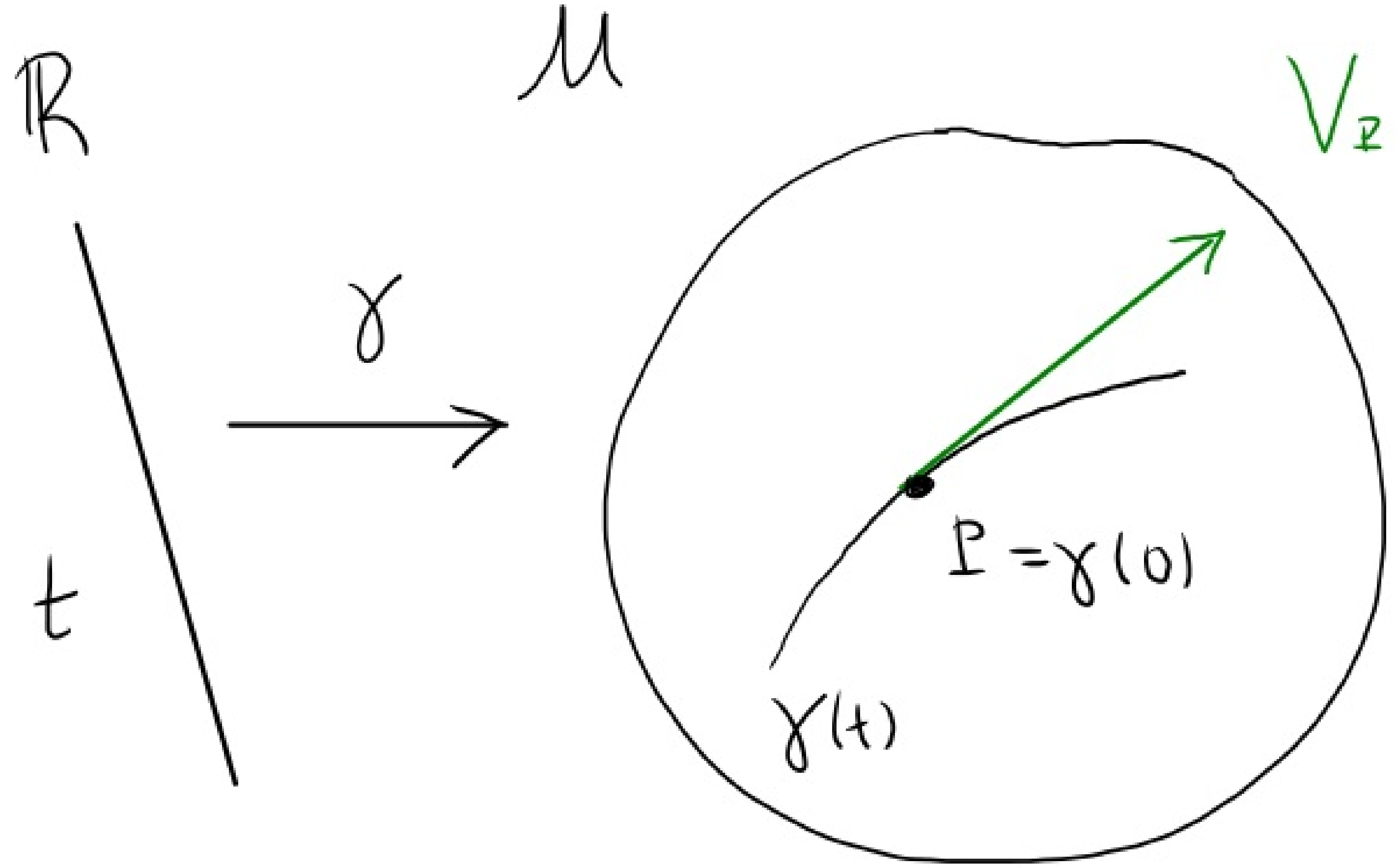

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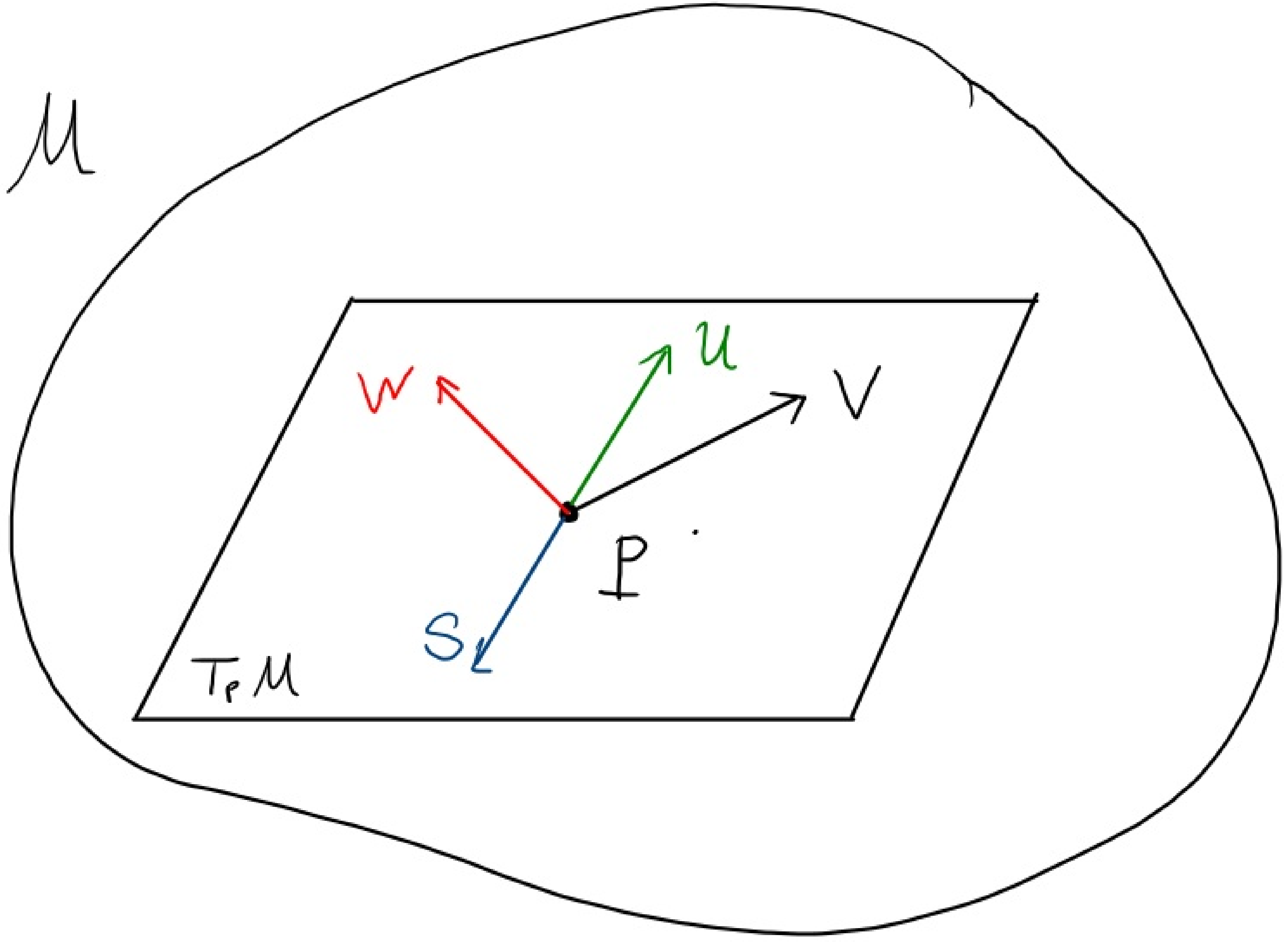
* Vectors at P are identified with all possible derivations on $\mathcal{F}(\mathcal{M})$

* Vectors at P form a vector space: $T_P\mathcal{M}$

The tangent space of $\mathcal{M}$ at P

$U, V \in T_P\mathcal{M} \Rightarrow$

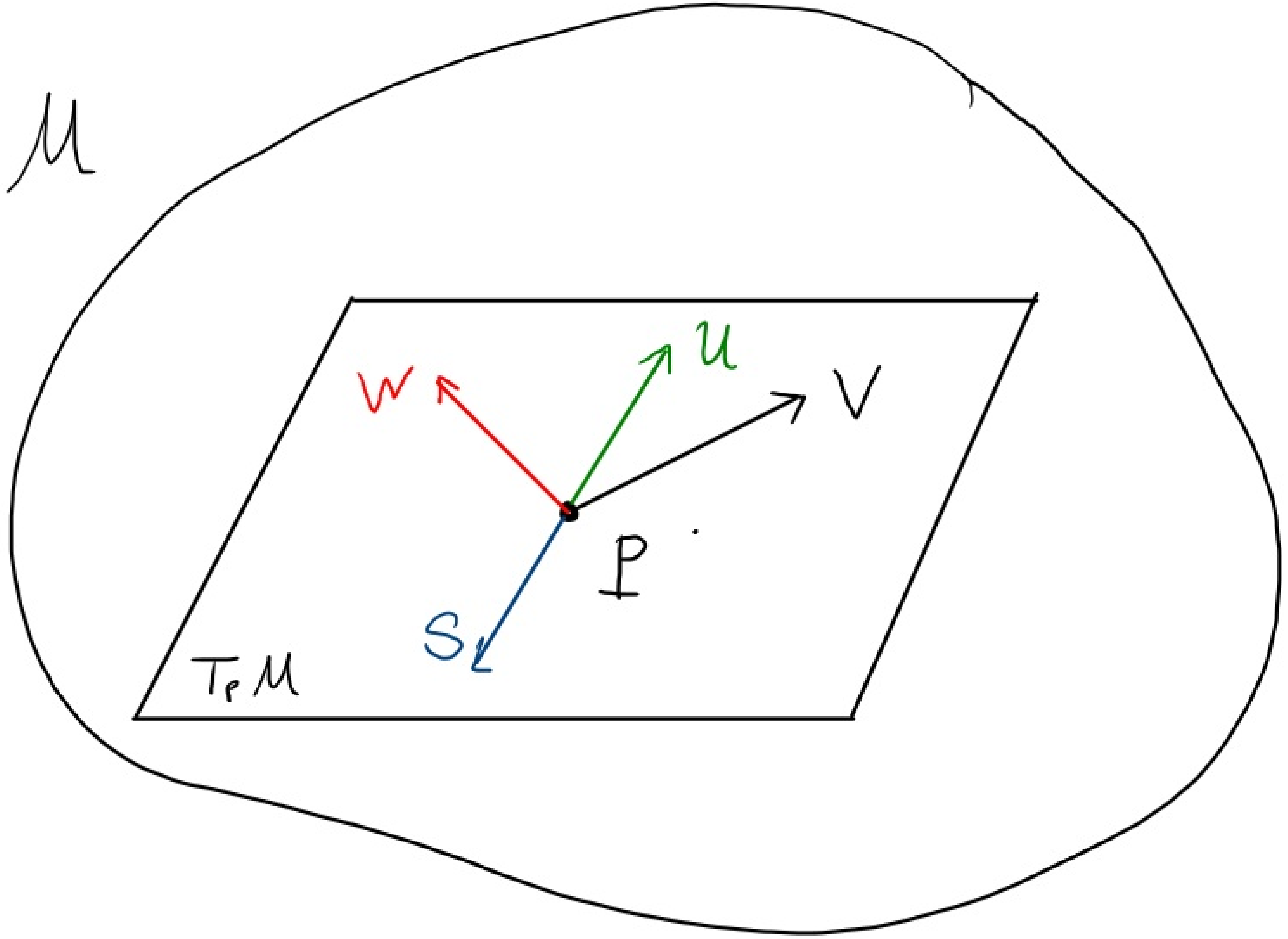
$\alpha V + \beta U \in T_P\mathcal{M}$



* Indeed $W = \alpha V + \beta U$ is a derivation: $\forall f, g \in \mathcal{F}(M)$

$$W(c_1 f + c_2 g) = c_1 W(f) + c_2 W(g)$$

$$W(f \cdot g) = W(f) \cdot g + f\, W(g)$$



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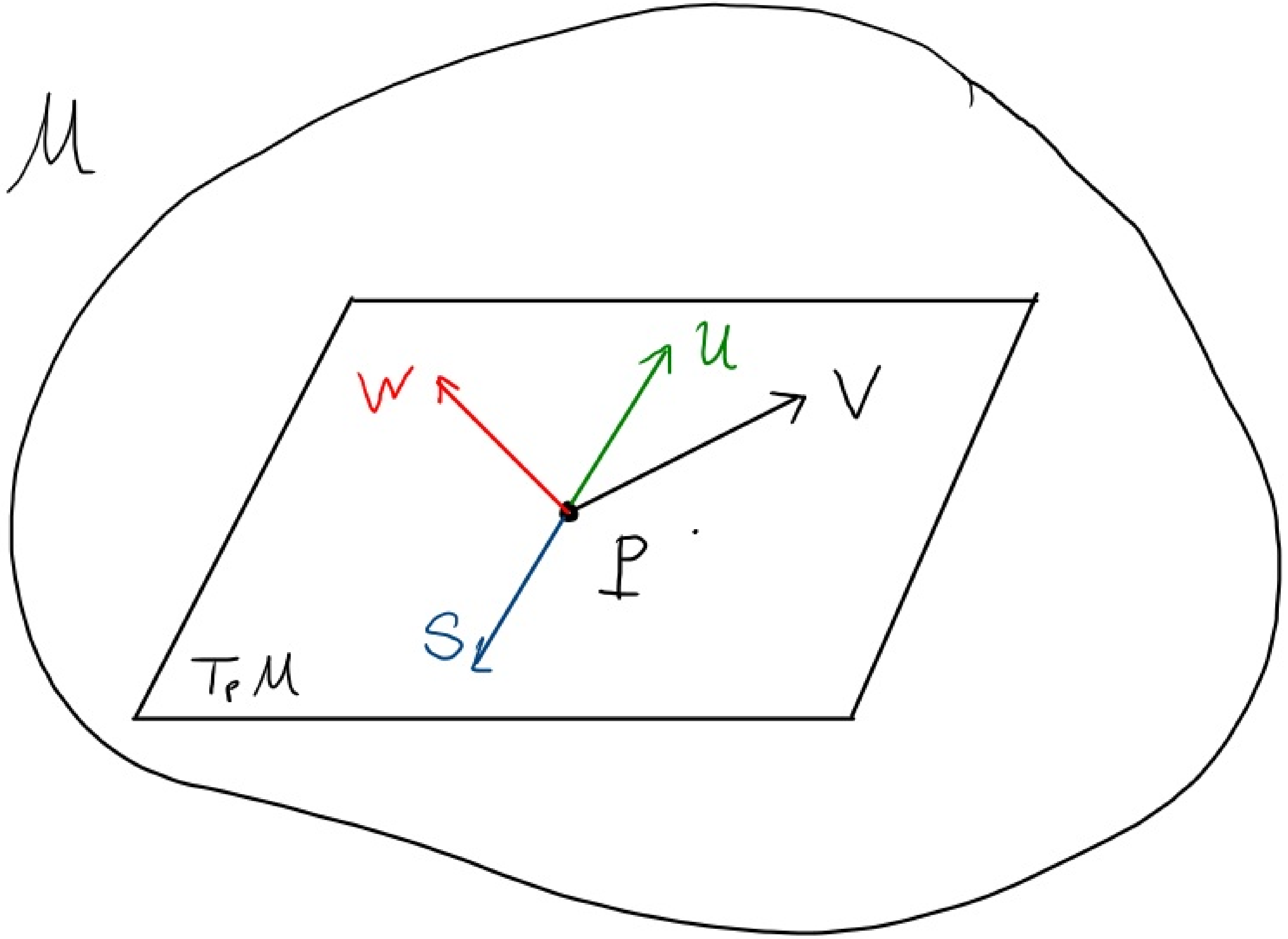

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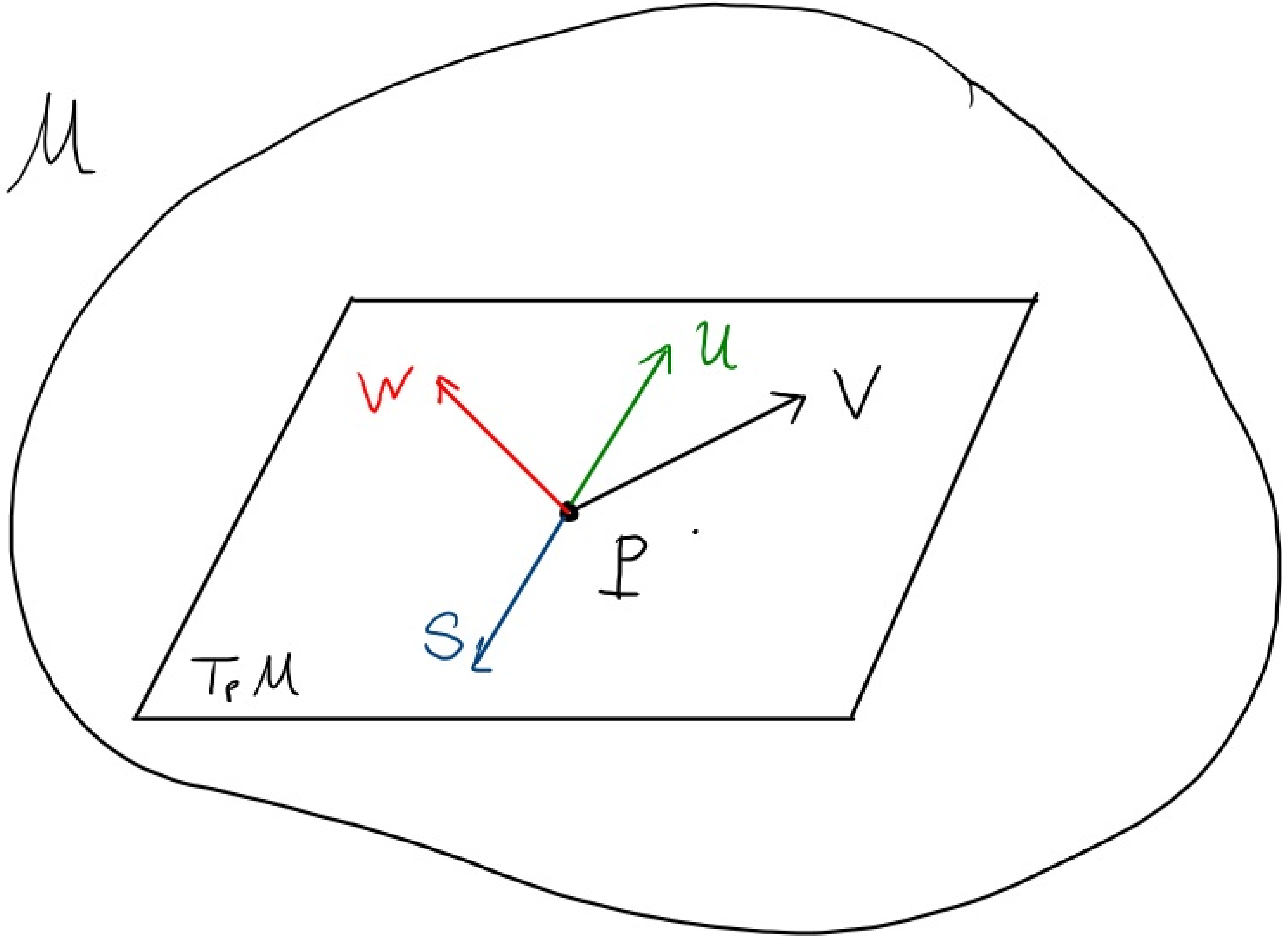

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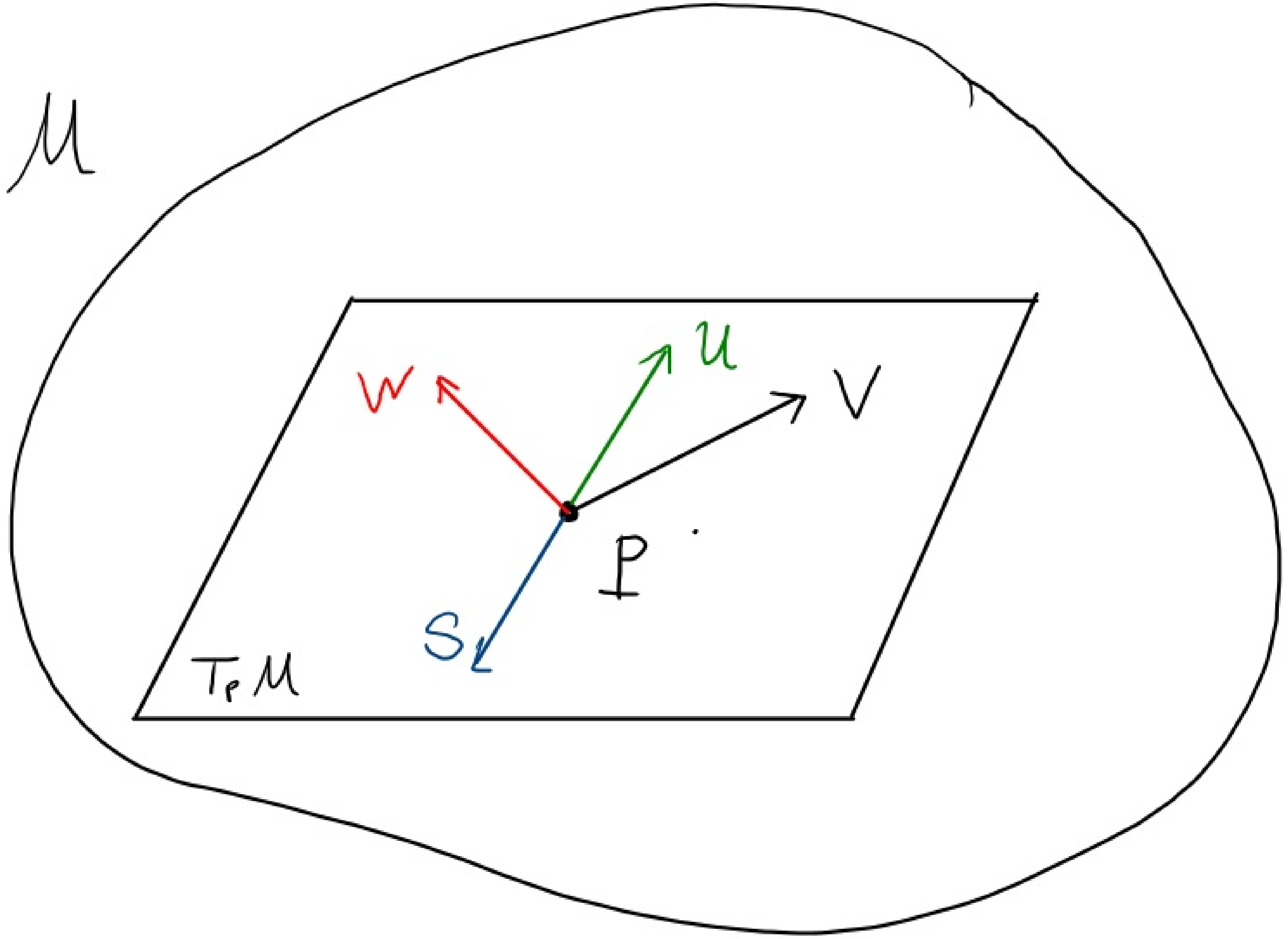

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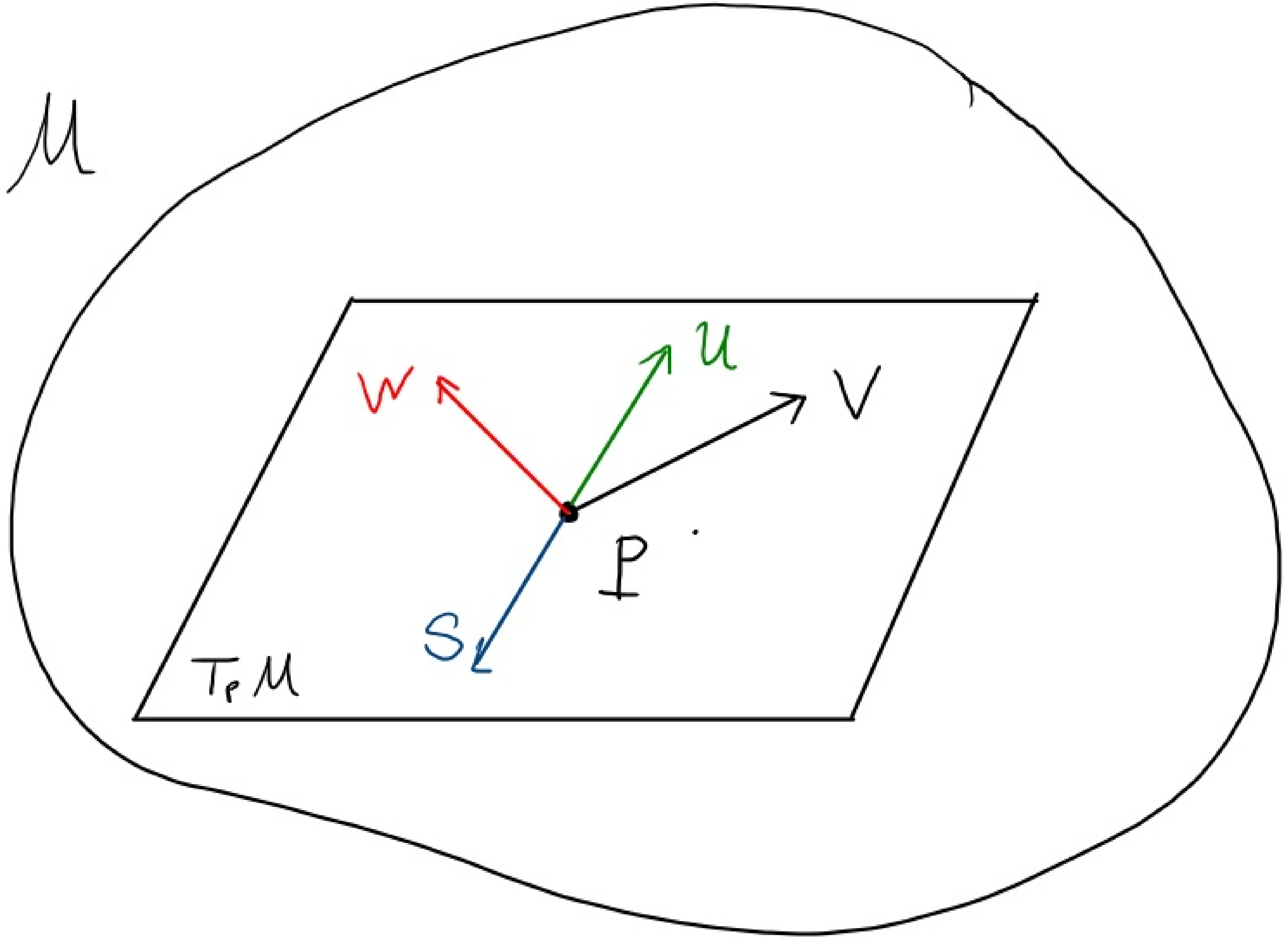

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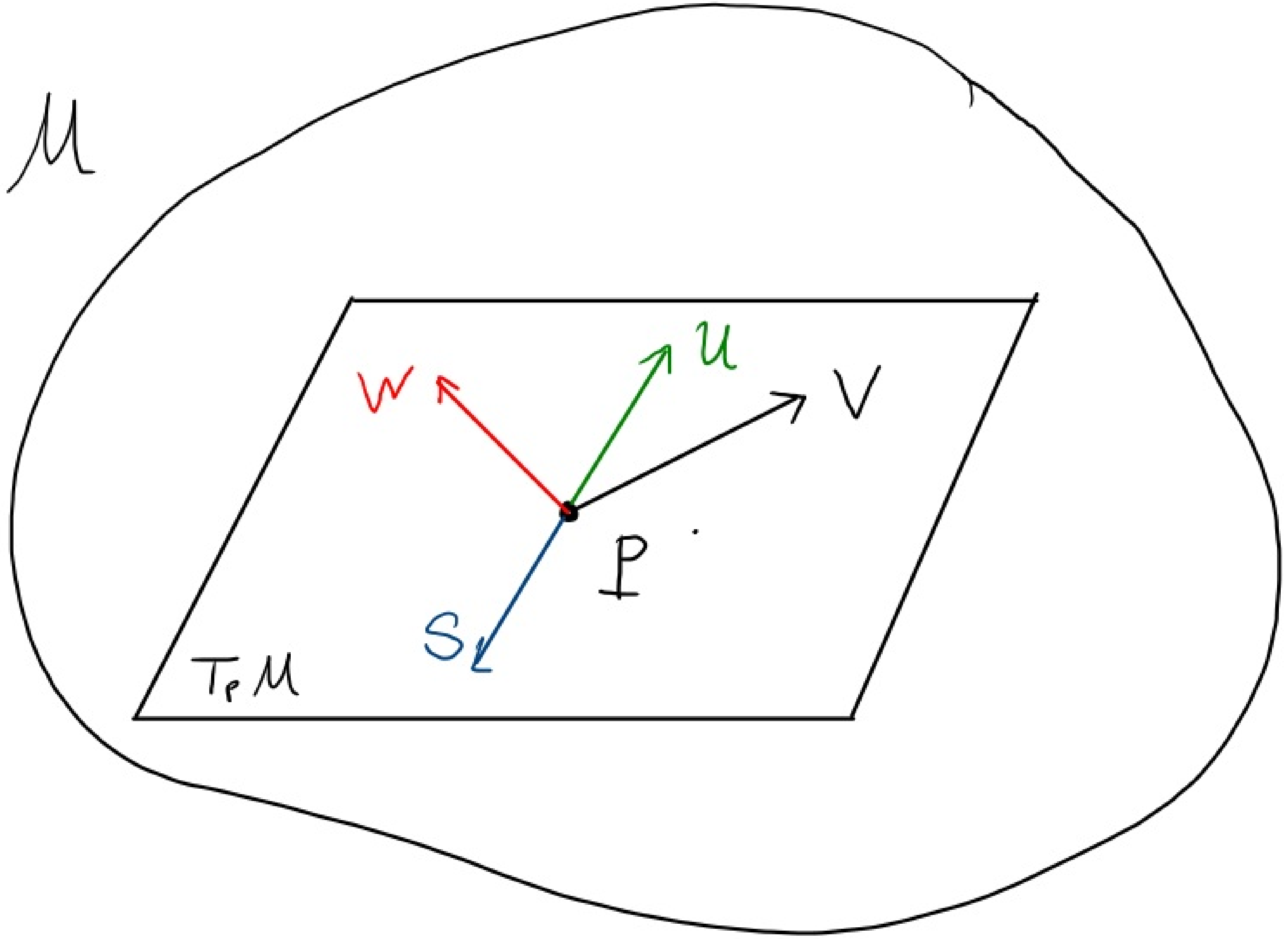

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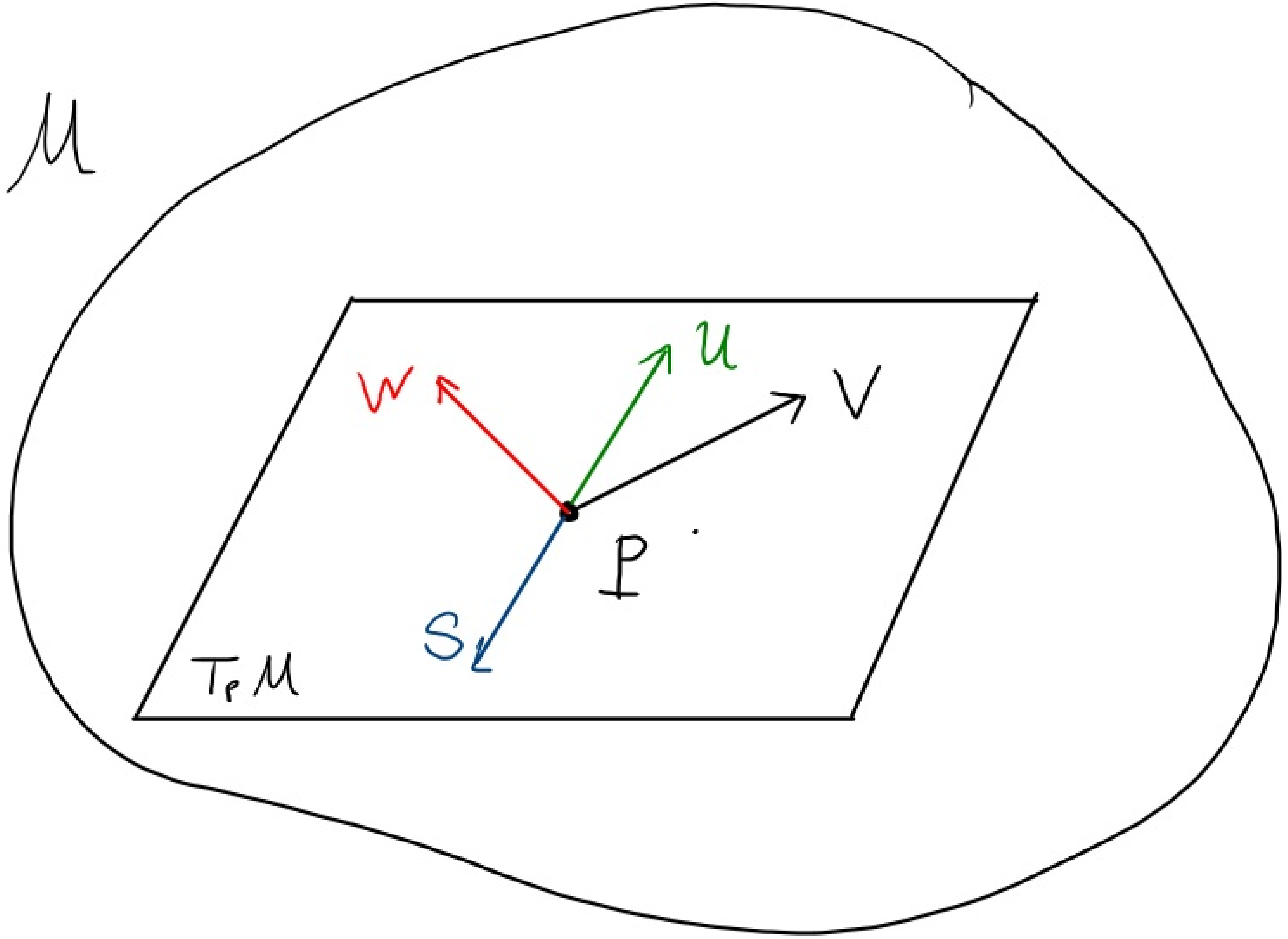

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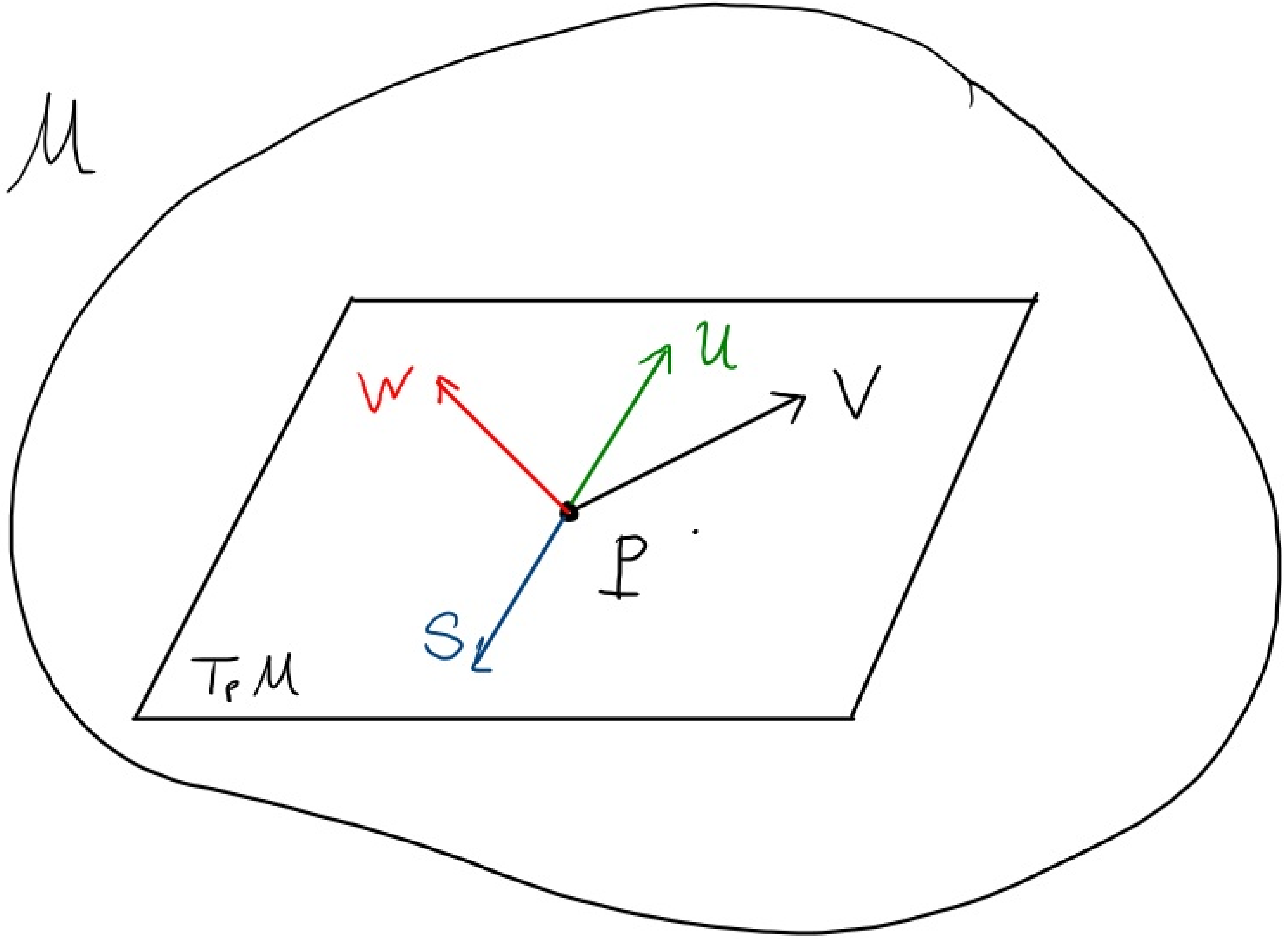

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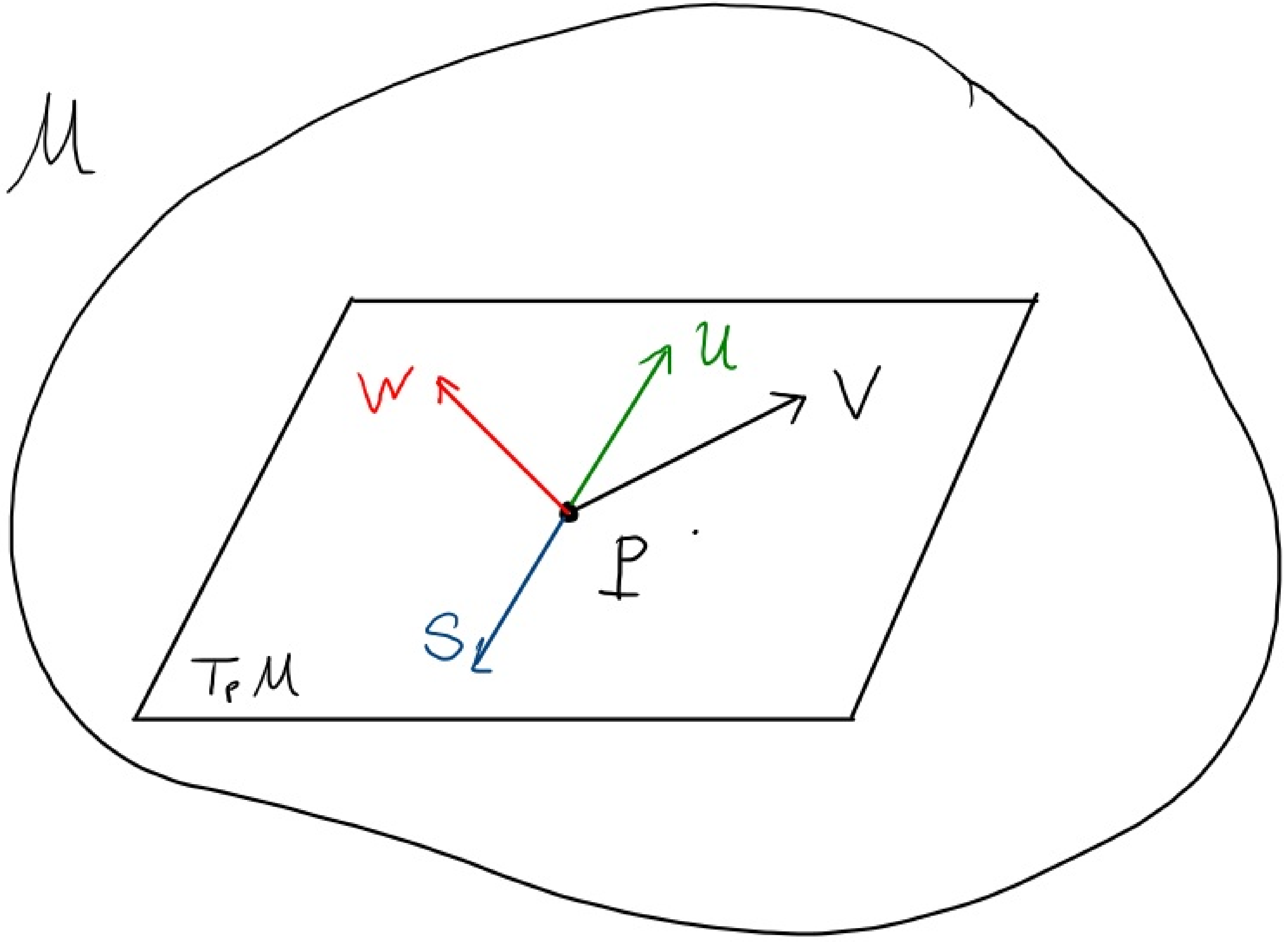

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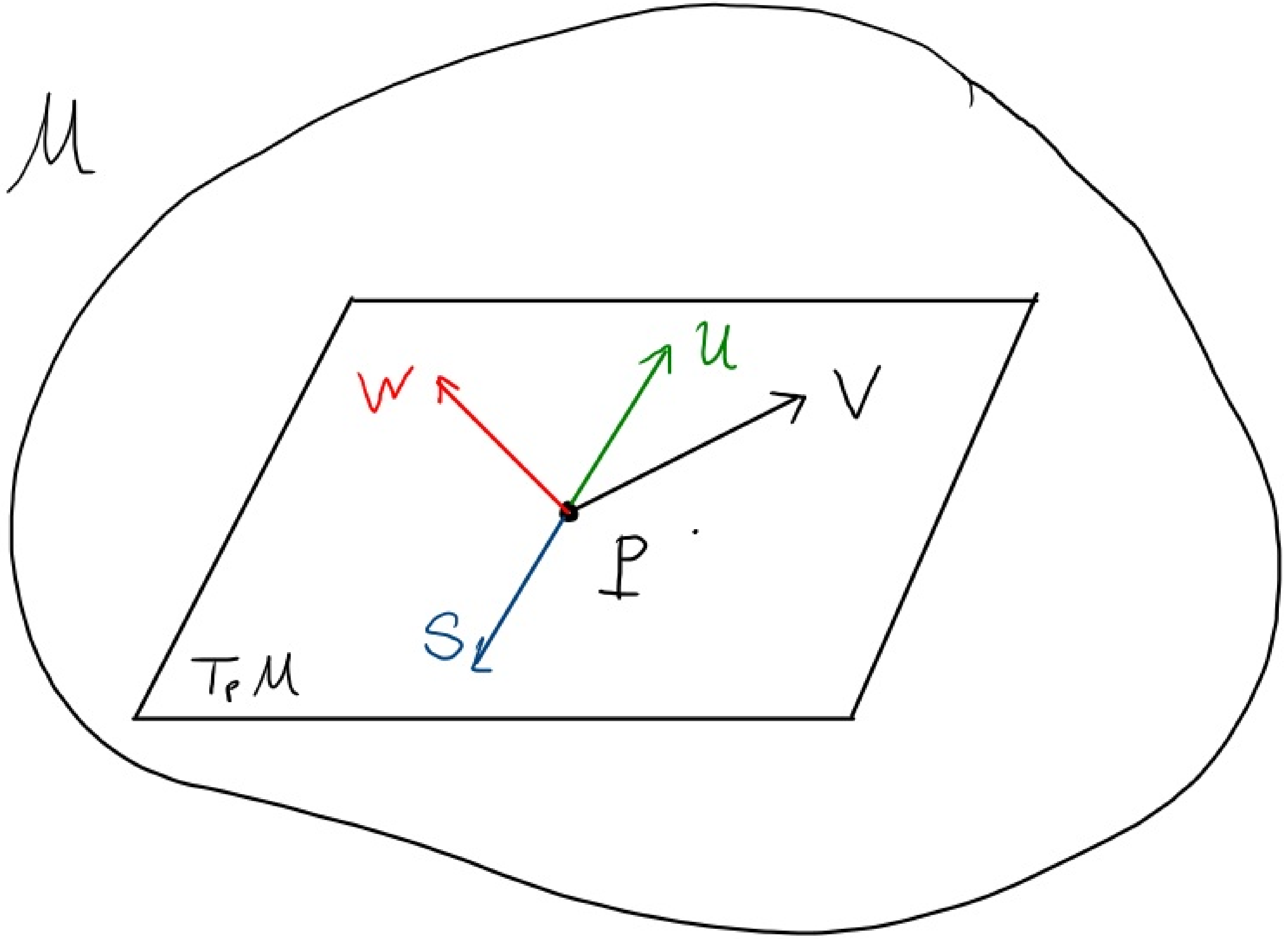

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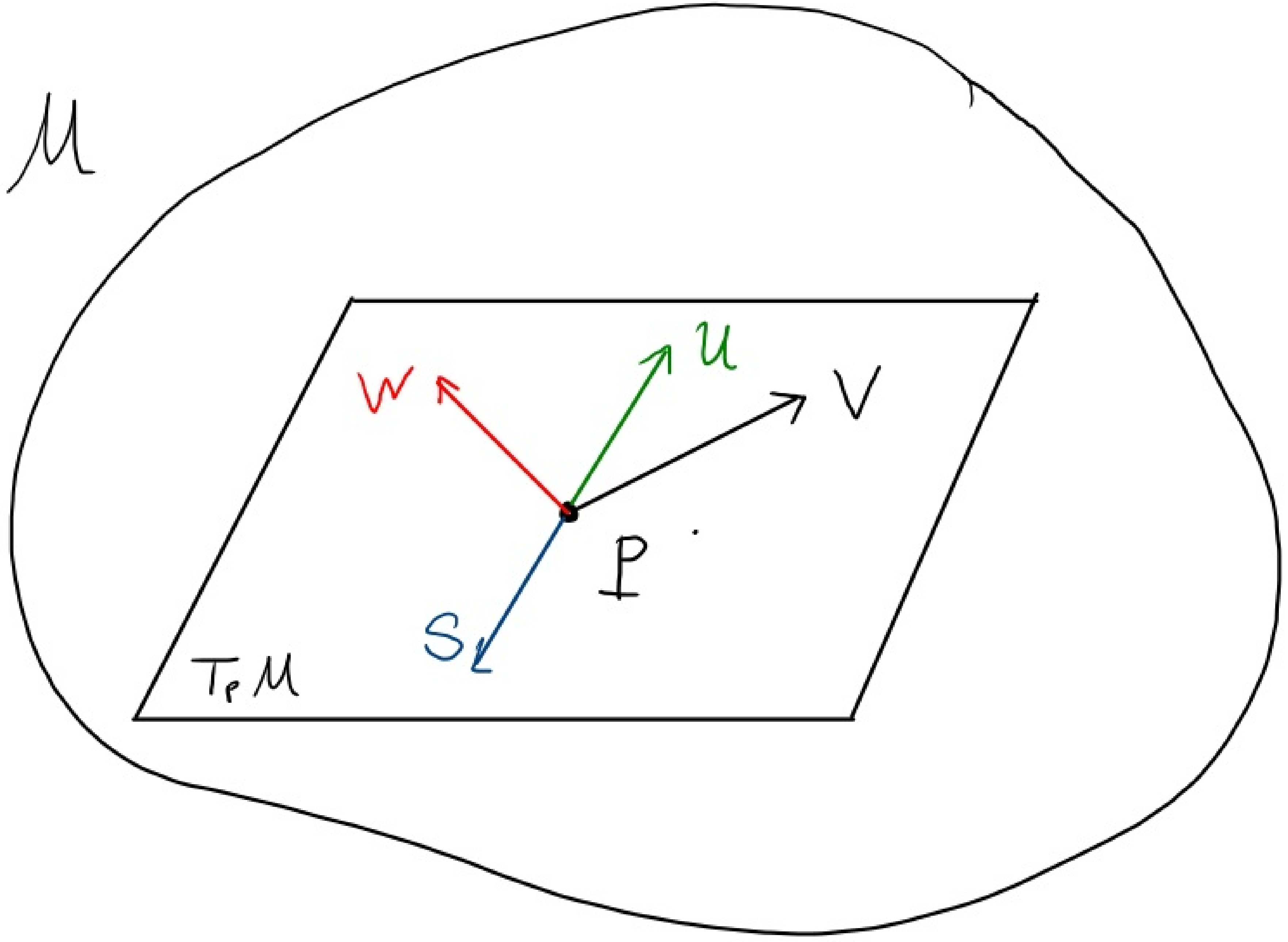

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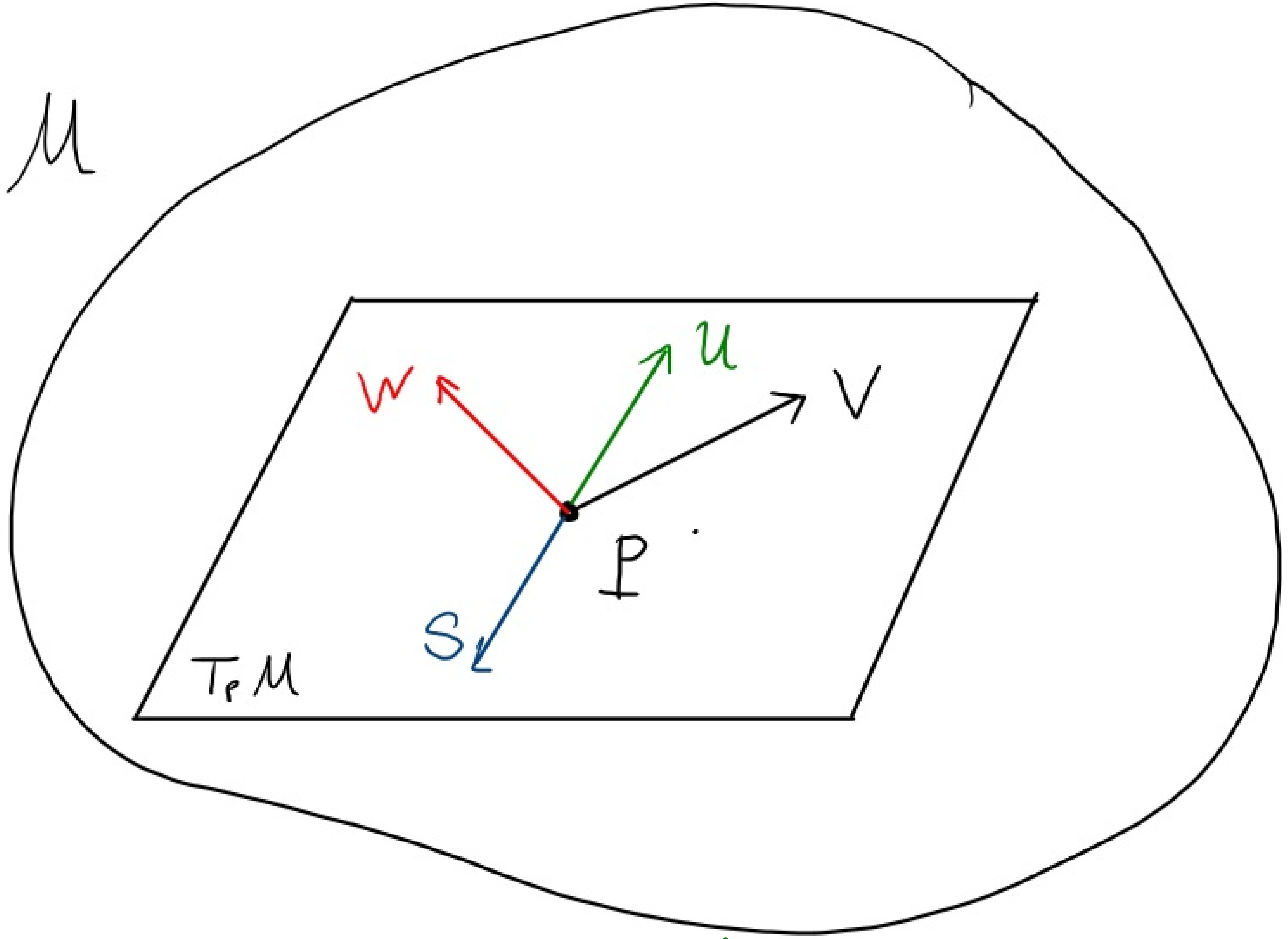
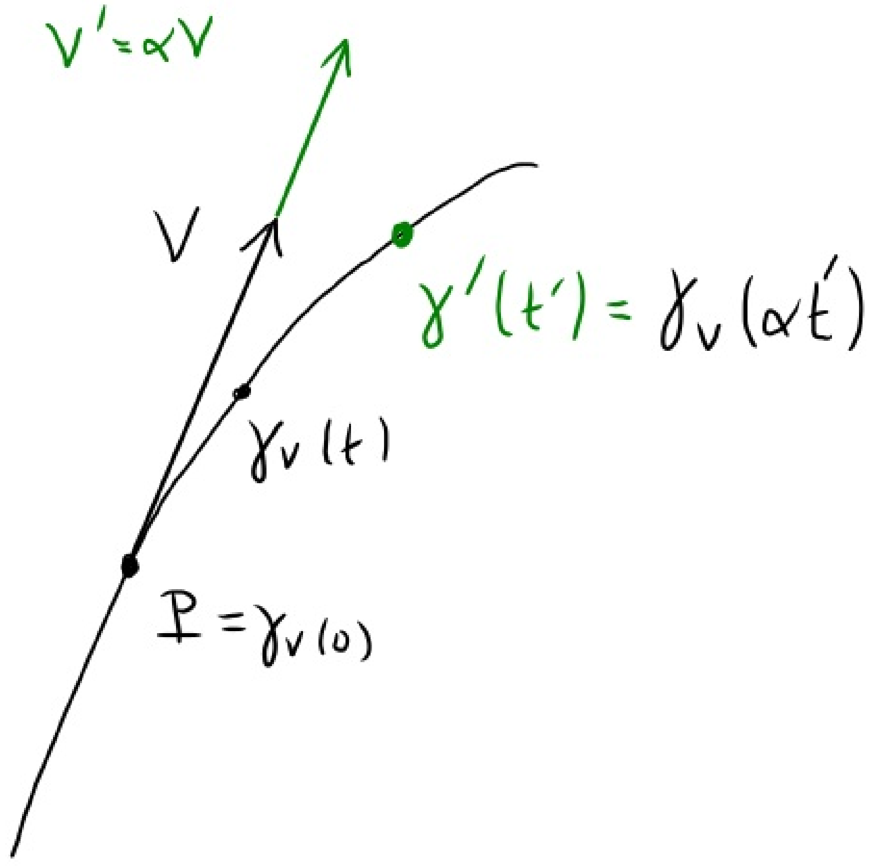
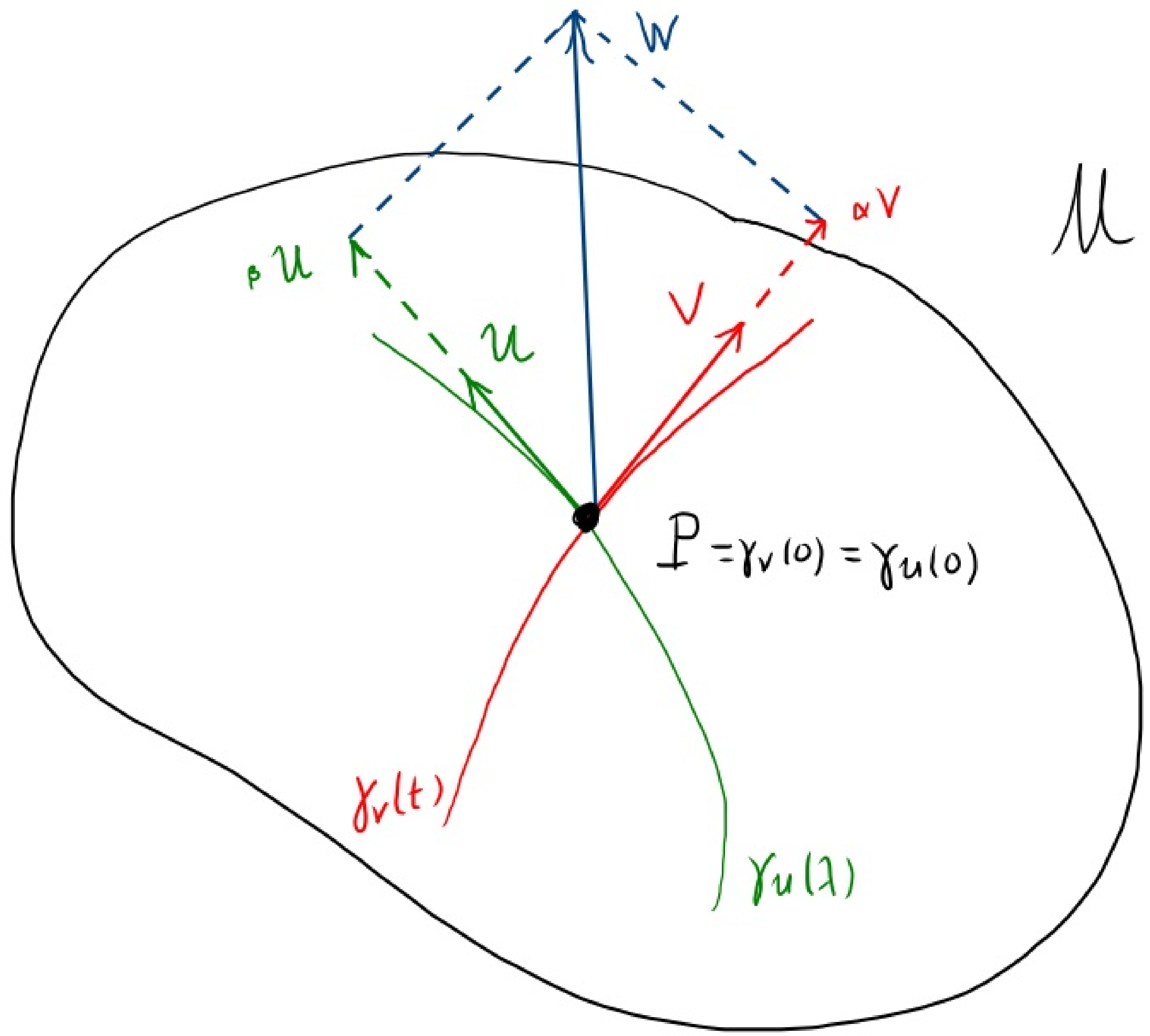


- How can we understand this geometrically?

* The vector αV is easy to understand:

- consider a reparametrization of $\gamma_V(t)$:





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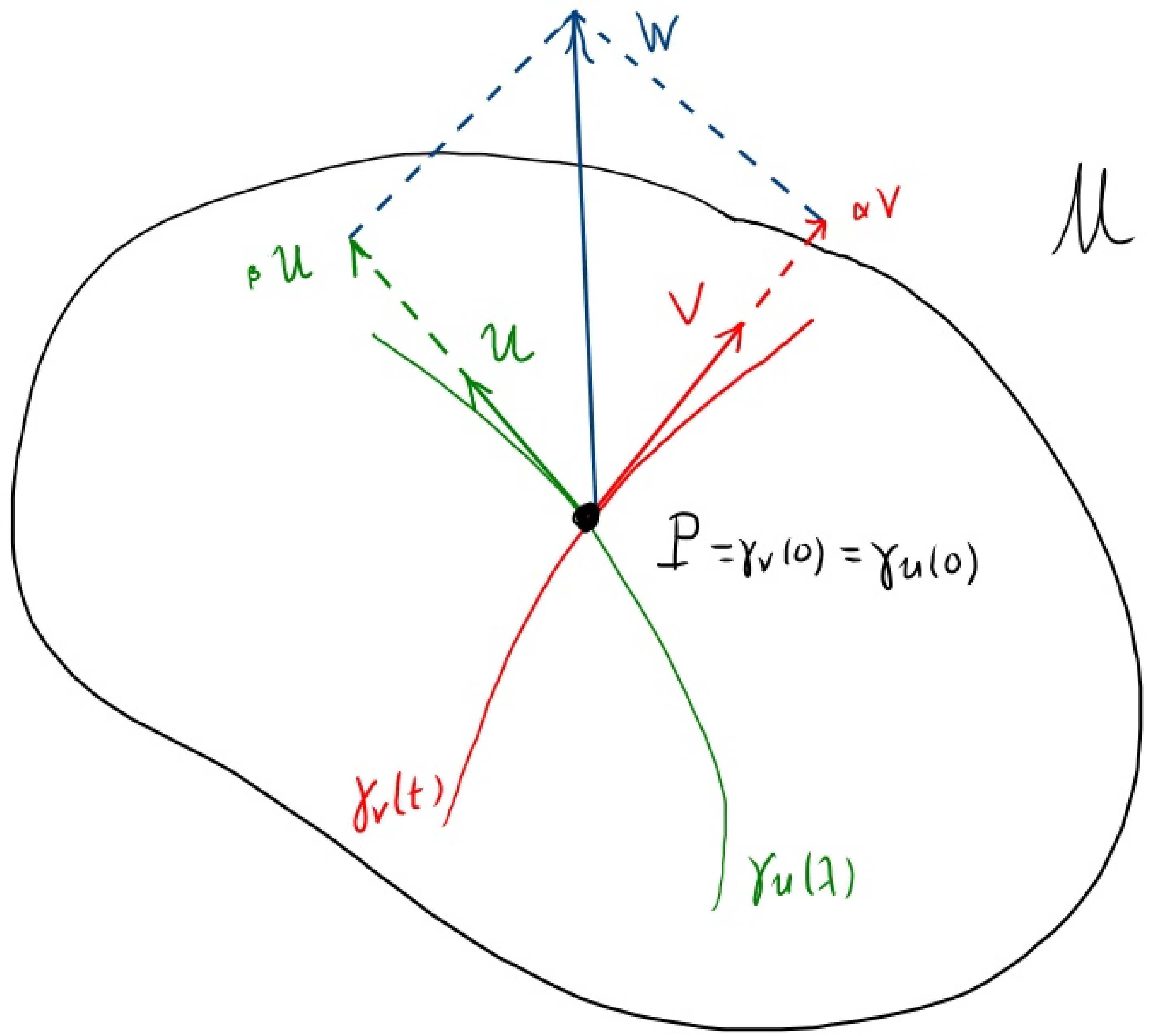

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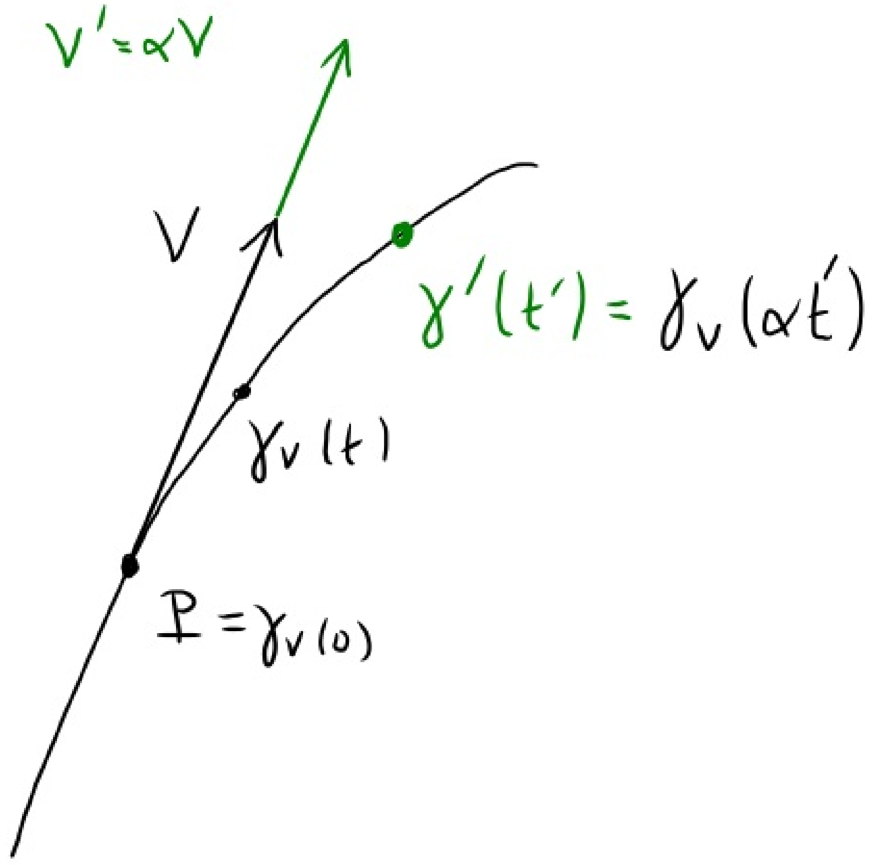

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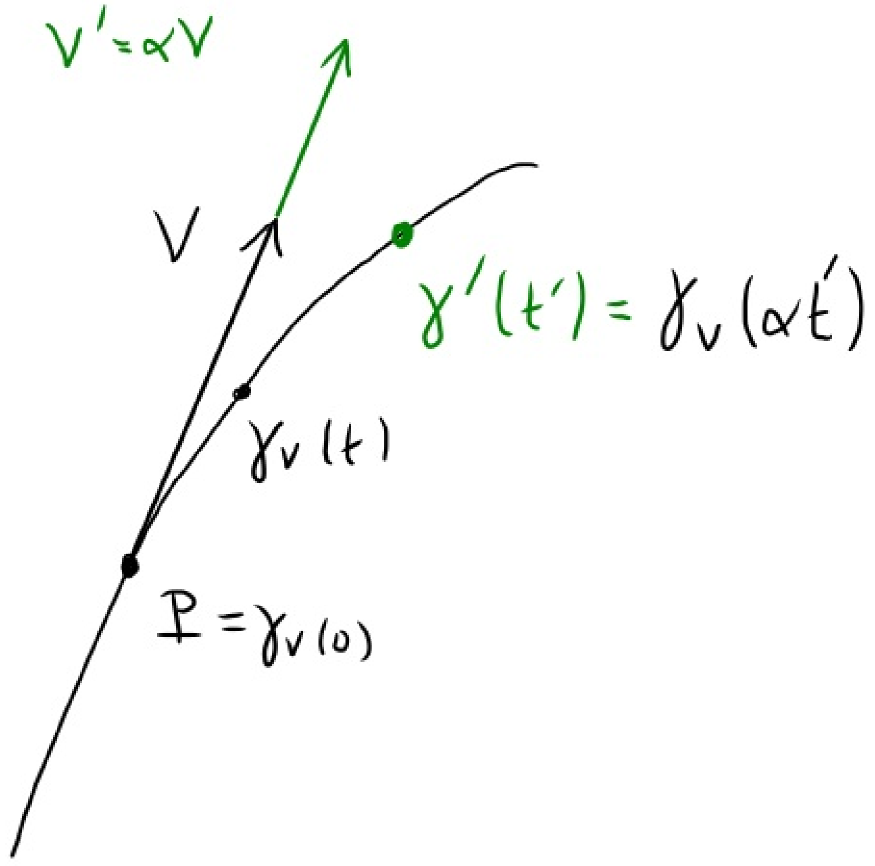

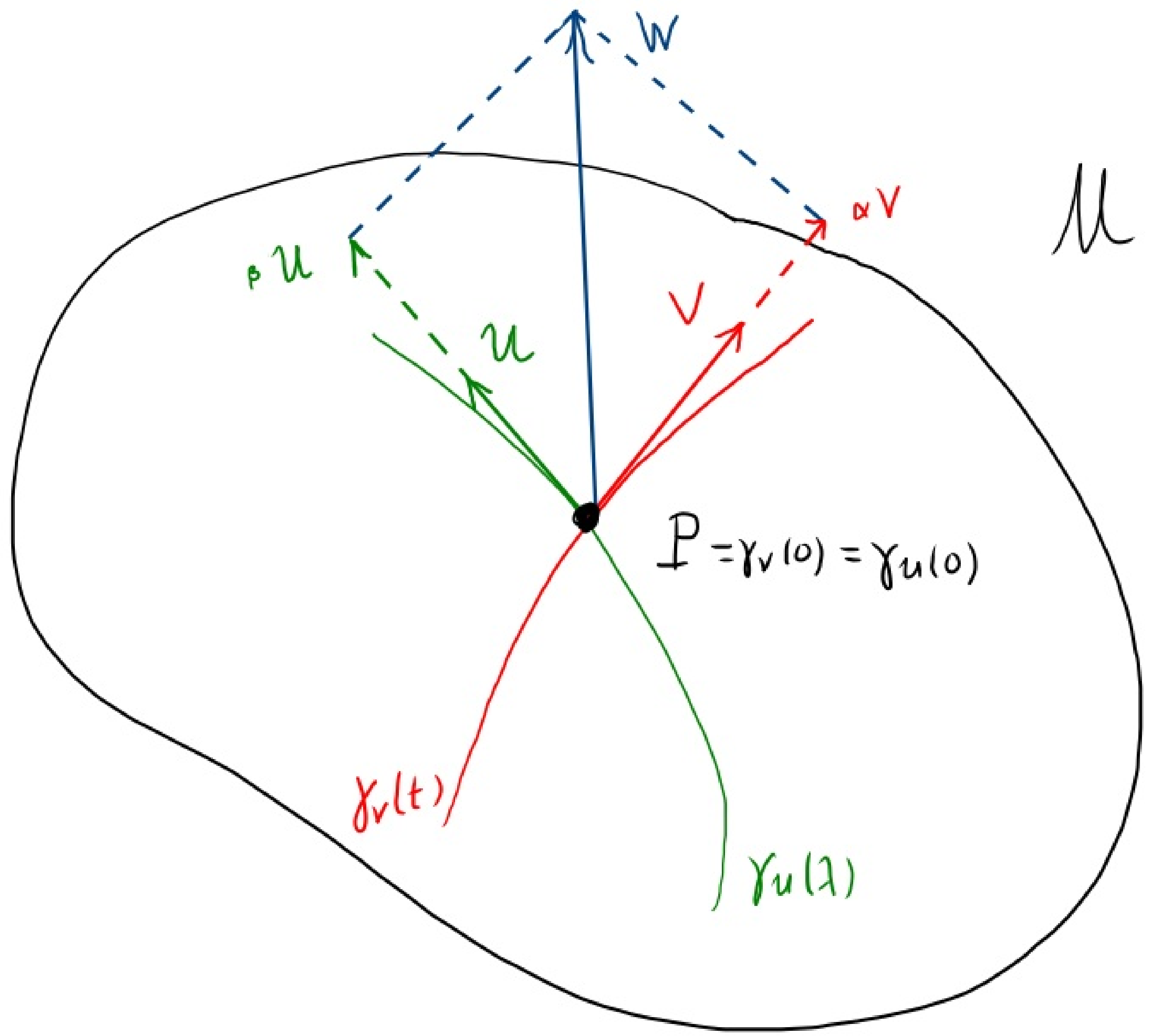

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$$V'_P(f) = \frac{d}{dt'} f \circ \gamma'(t') \Big|_0$$

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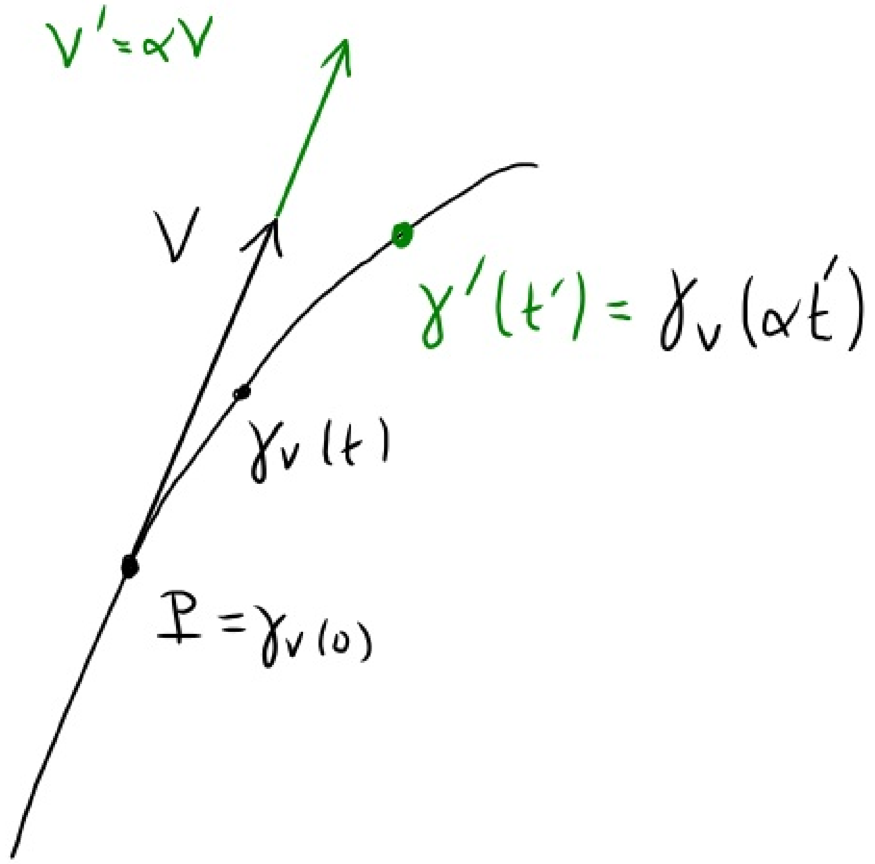

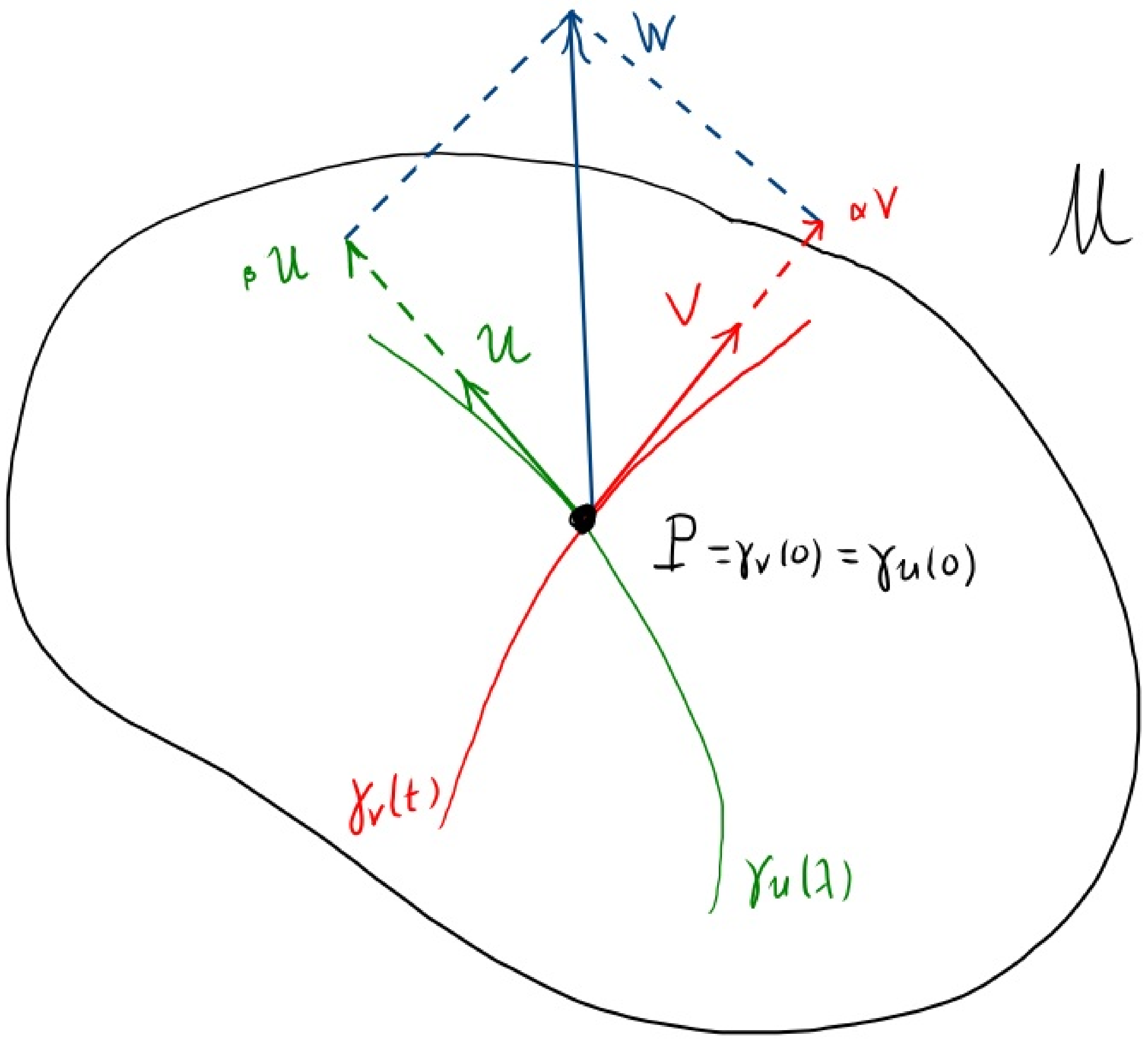

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$$V'_P(f) = \frac{d}{dt'} f \circ \gamma'(t')\Big|_0 = \frac{d}{dt'} f \circ \gamma_V(\alpha t')\Big|_0$$

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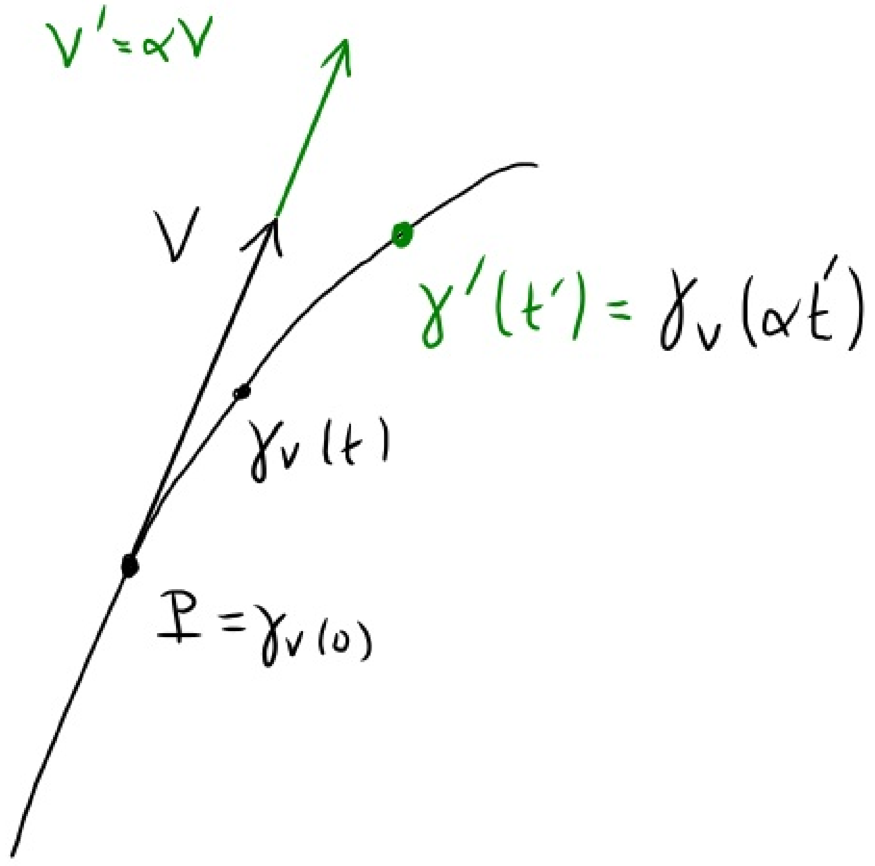

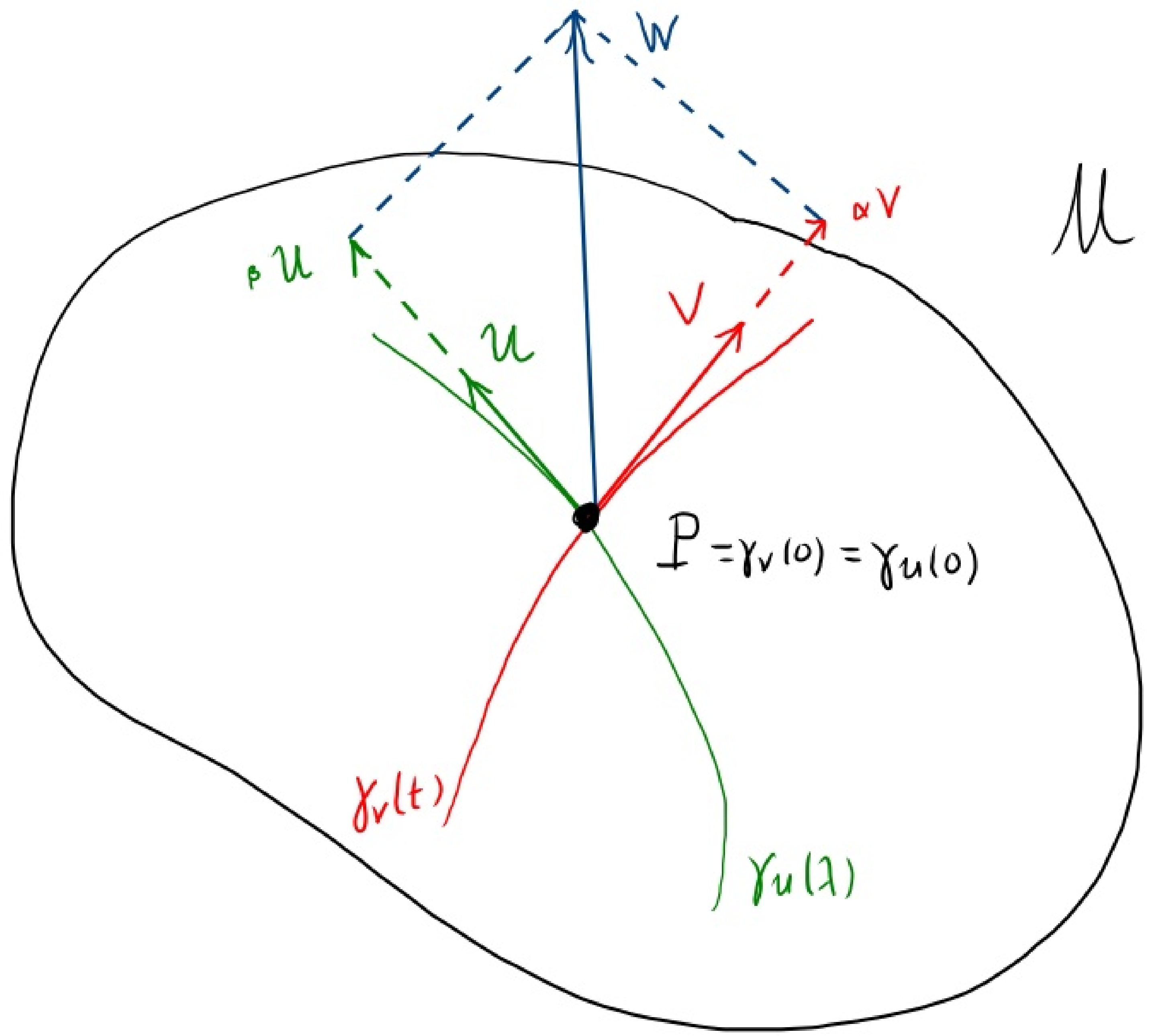

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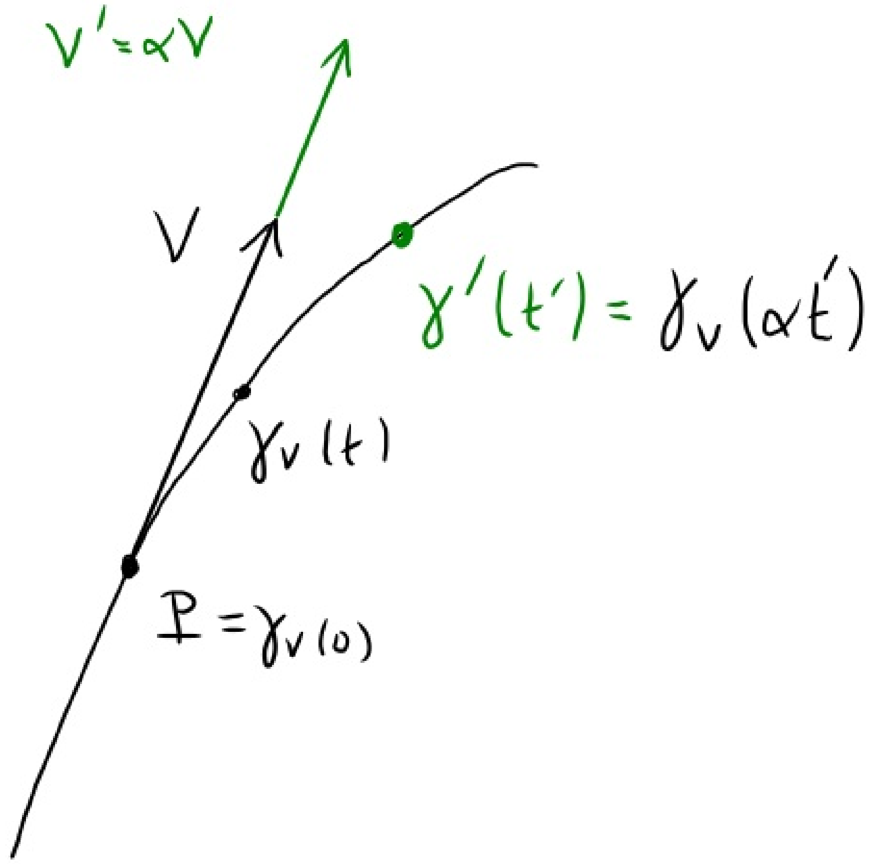
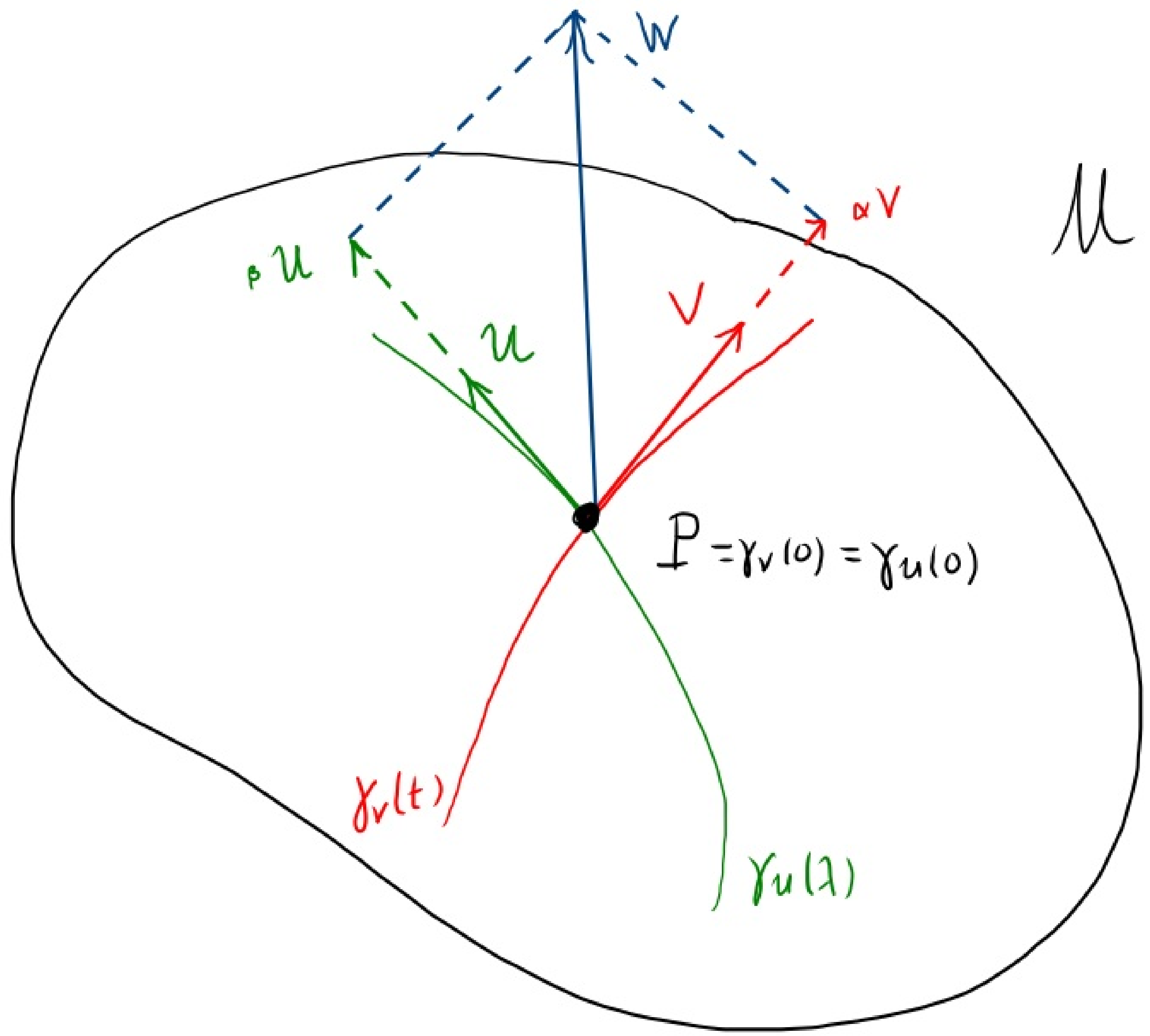
$$V'_P(f) = \frac{d}{dt'} f \circ \gamma'(t')\Big|_0 = \frac{d}{dt'} f \circ \gamma_V(\alpha t')\Big|_0 = \alpha \frac{d}{dt} f \circ \gamma_V(0)\Big|_0$$

* The vector αV is easy to understand:

- consider a reparametrization of $\gamma_v(t)$:





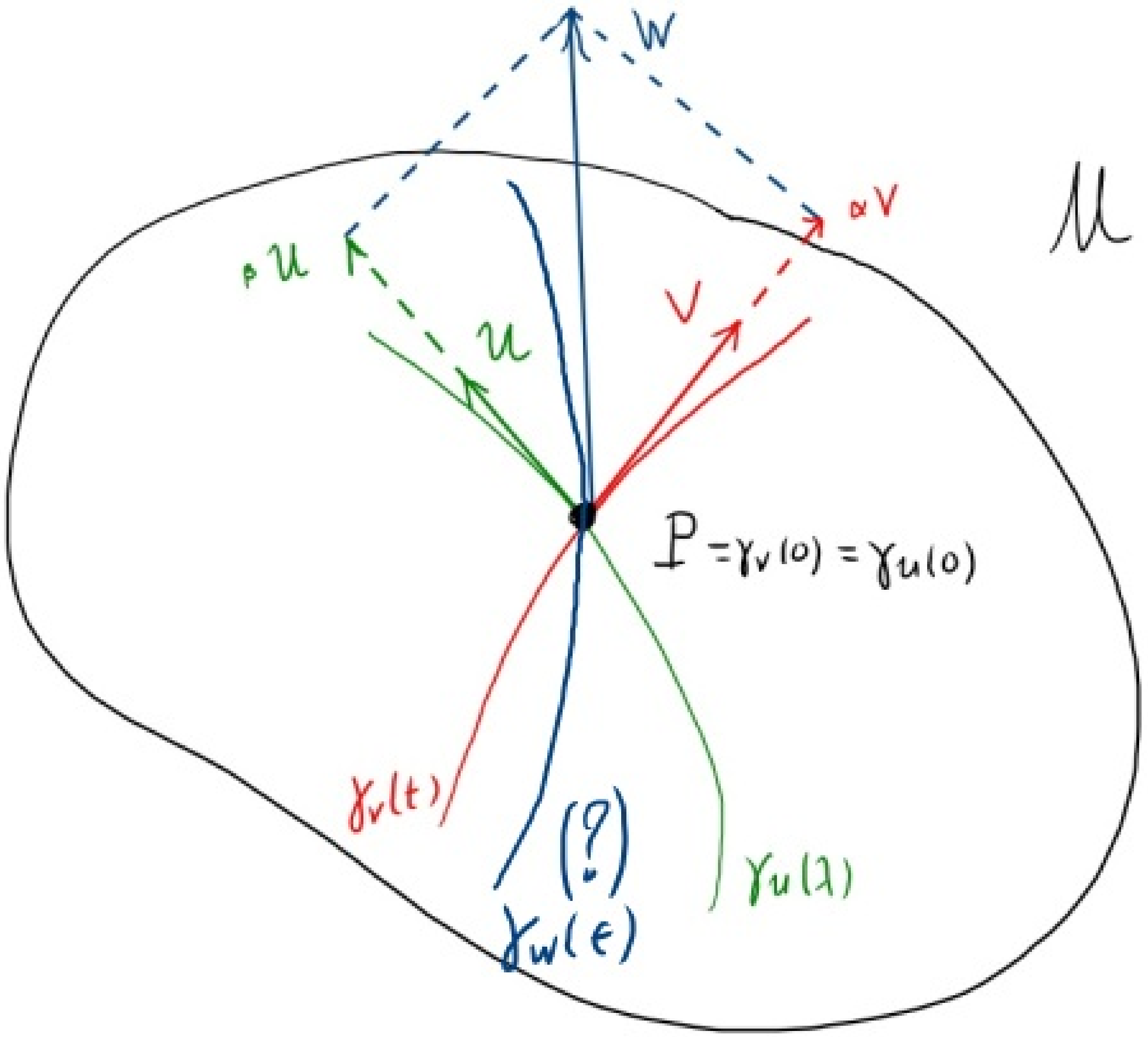
- define a new curve $\gamma'(t') = \gamma_v(\alpha t')$

$$V'_P(f) = \frac{d}{dt'} f\circ\gamma'(t')\Big|_0 = \frac{d}{dt'} f\circ\gamma_v(\alpha t')\Big|_0 = \alpha \frac{d}{dt} f\circ\gamma_v(0)\Big|_0$$

$$= \alpha\, V_P(f)$$

* The vector $\alpha V + \beta U$ is given by:

$$(\alpha V + \beta U)(f) = \alpha \left.\frac{df}{dt}\right|_0 + \beta \left.\frac{df}{d\lambda}\right|_0$$



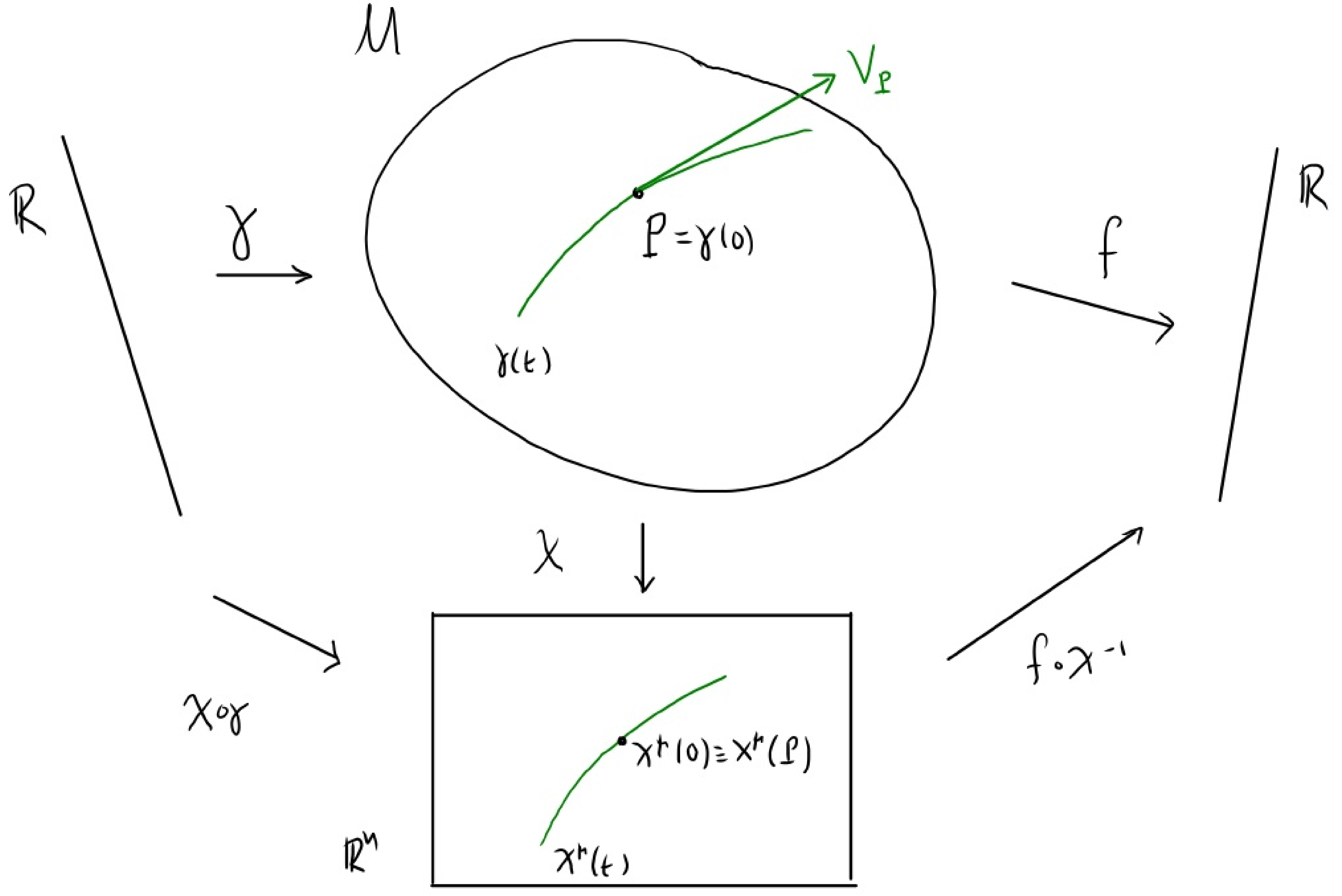
But is there a class of curves s.t.

$$\alpha \left.\frac{df}{dt}\right|_0 + \beta \left.\frac{df}{d\lambda}\right|_0 = \left.\frac{df}{d\epsilon}\right|_0 \quad \forall f \ ?$$

yes... but let's see why...

* Consider first $V_p(f)$:

$$V_p(f) = \left.\frac{df}{dt}\right|_p$$



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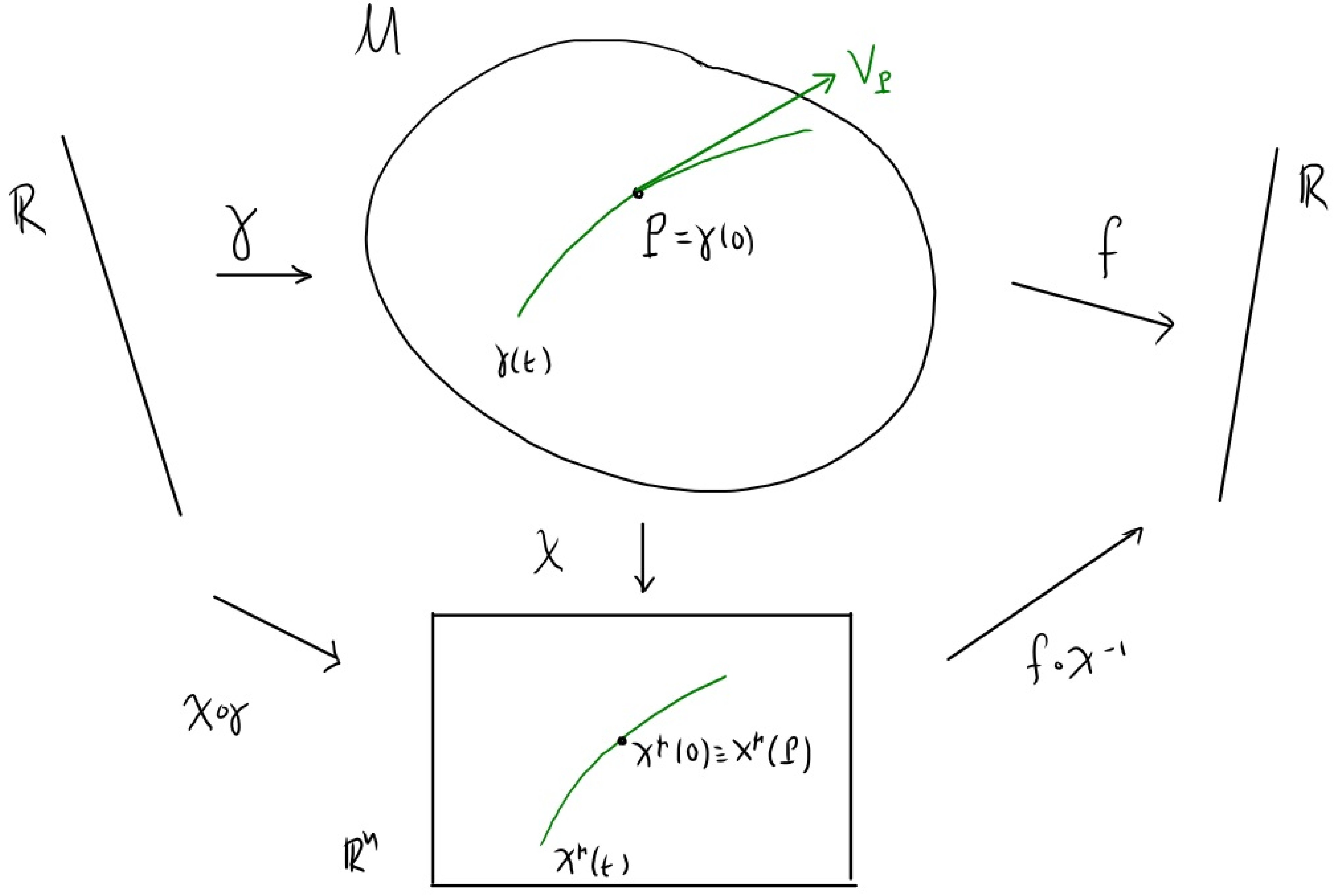


* Consider first $V_p(f)$:

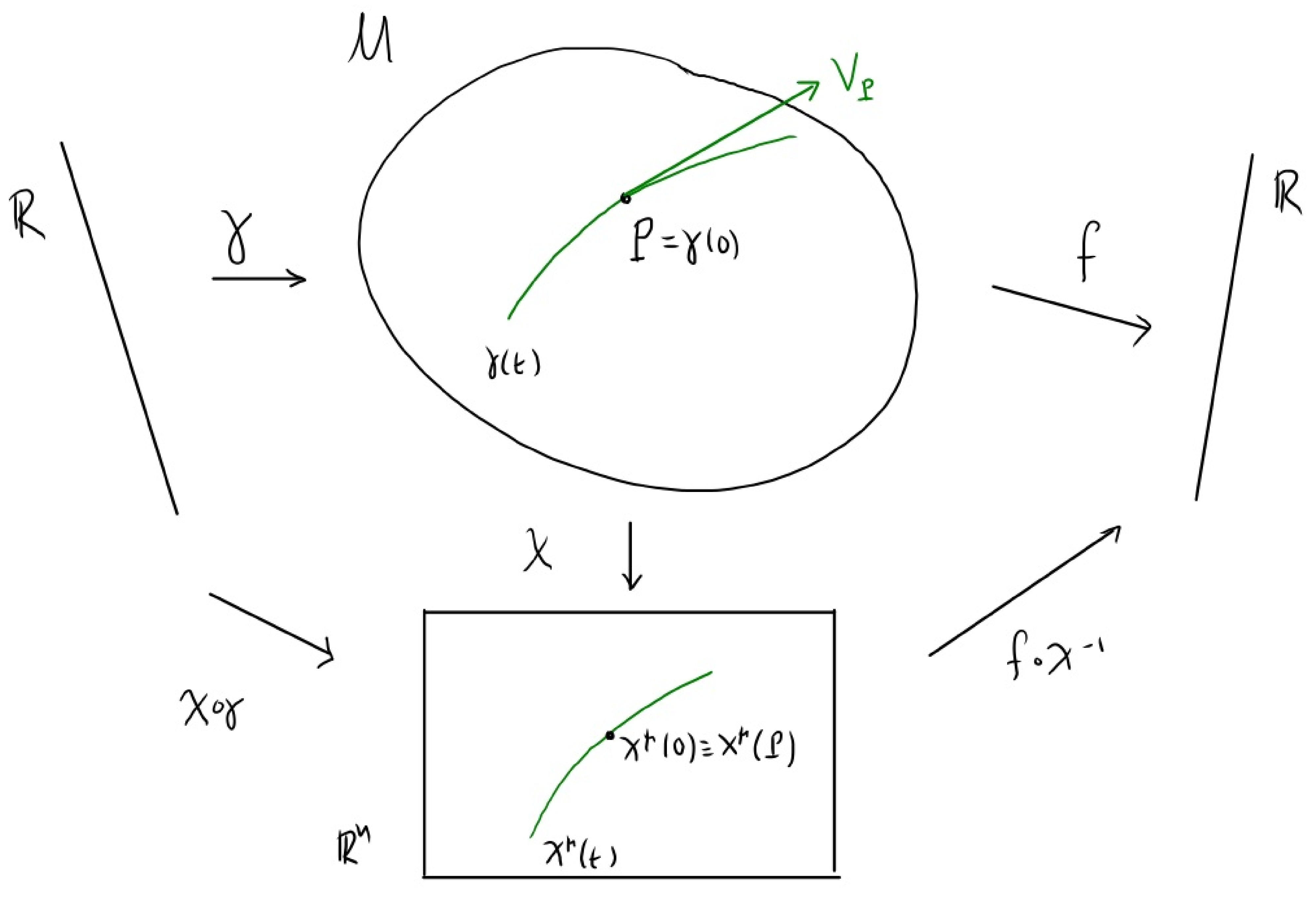
$$V_p(f) = \frac{df}{dt}\Big|_p$$

$$= \frac{d}{dt} f \circ \gamma(t)\Big|_0$$

$$= \frac{d}{dt} \underbrace{f \circ \chi^{-1}}_{f(x^\nu)} \circ \underbrace{\chi \circ \gamma(t)}_{x^\mu(t)}\Big|_0$$

$f(x^\nu)$ ↳ function of coordinates

$x^\mu(t)$ ↳ Image of curve in $\mathbb{R}^n$



<u>Notation:</u> $f(x^\nu) = f(x^0, x^1, x^2, \dots, x^{n-1})$ $\quad x^\mu(t) = \left(x^0(t), x^1(t), x^2(t), \dots, x^{n-1}(t)\right) \in \mathbb{R}^n$

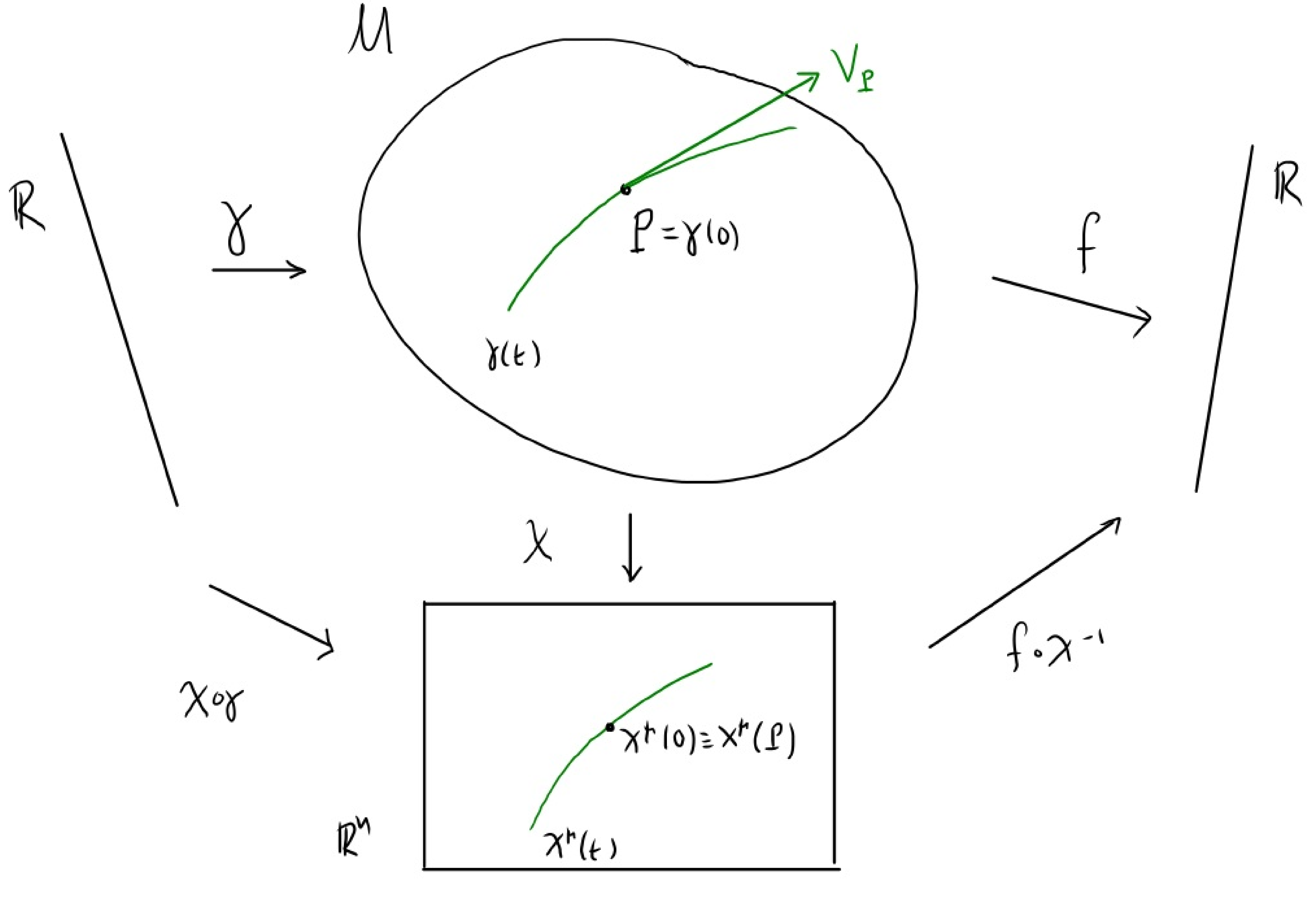
$\ast$ Consider first $V_p(f)$:

$$V_p(f) = \frac{df}{dt}\Big|_p$$

$$= \frac{d}{dt} f\circ\gamma(t)\Big|_0$$

$$= \frac{d}{dt} \underbrace{f\circ\chi^{-1}}_{f(x^\nu)} \circ \underbrace{\chi\circ\gamma(t)}_{x^\mu(t)}\Big|_0$$

$$= \frac{\partial f(x^\nu)}{\partial x^\mu} \cdot \frac{d x^\mu(t)}{dt}\Big|_0$$



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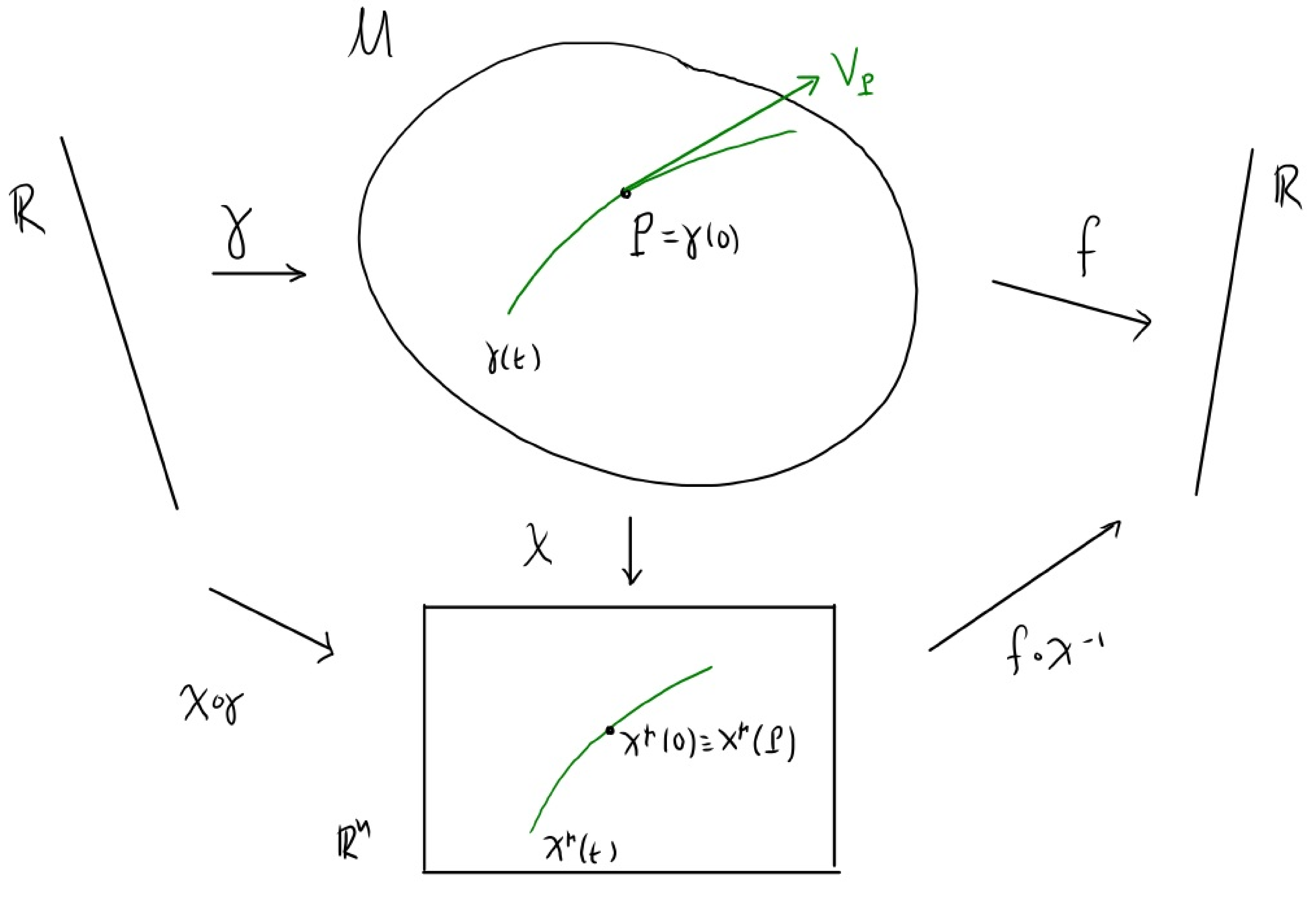


Einstein convention:
Sum over repeated indices!

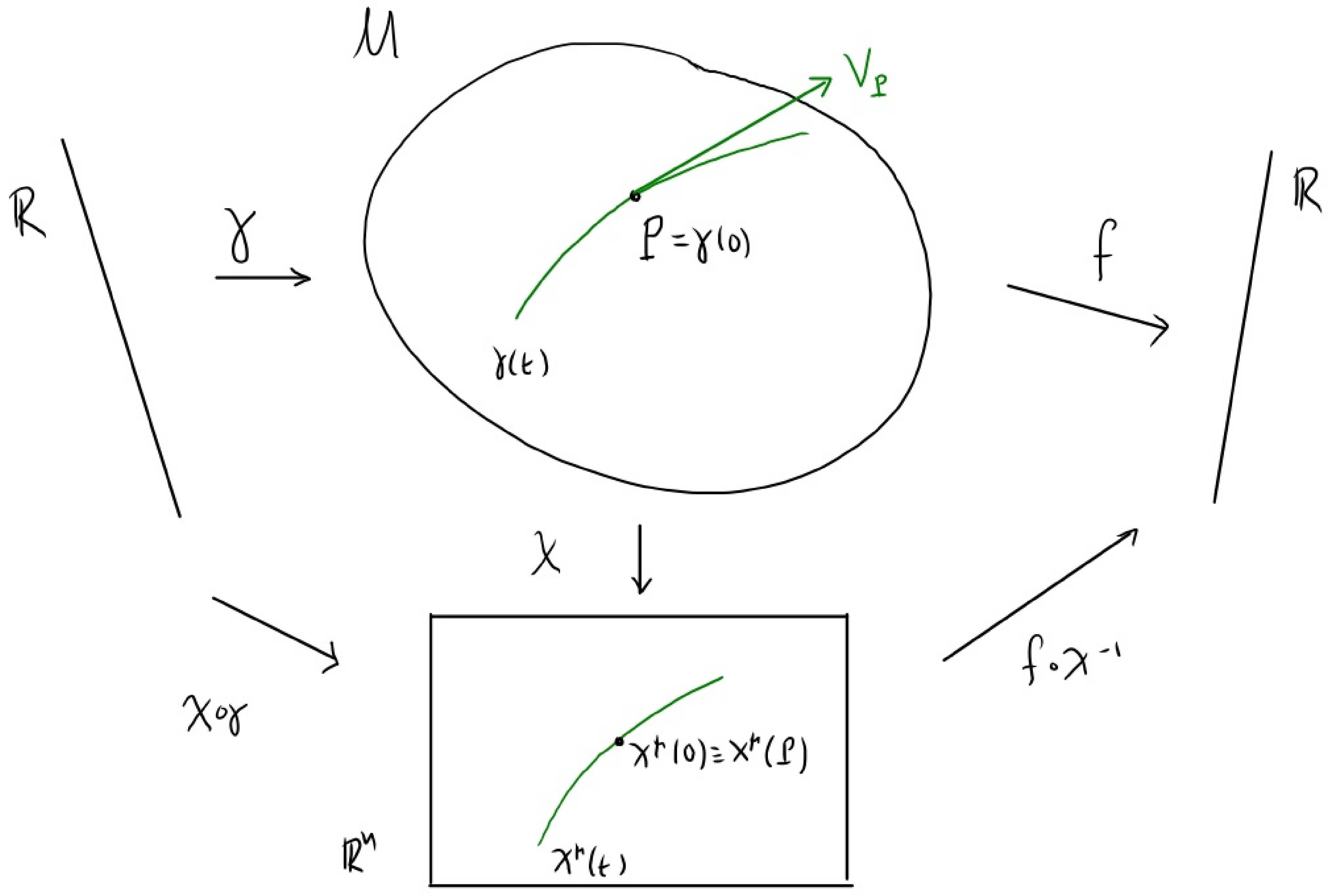
$$= \sum_{\mu=0}^{n-1} \frac{\partial f}{\partial x^\mu} \cdot \frac{dx^\mu}{dt}$$

* Consider first $V_p(f)$:

$$V_p(f) = \frac{\partial f(x^\nu)}{\partial x^\mu} \cdot \frac{dx^\mu}{dt}$$

$$= \frac{dx^\mu}{dt} \cdot \underbrace{\partial_\mu f}$$

Notation for $\frac{\partial f}{\partial x^\mu}$



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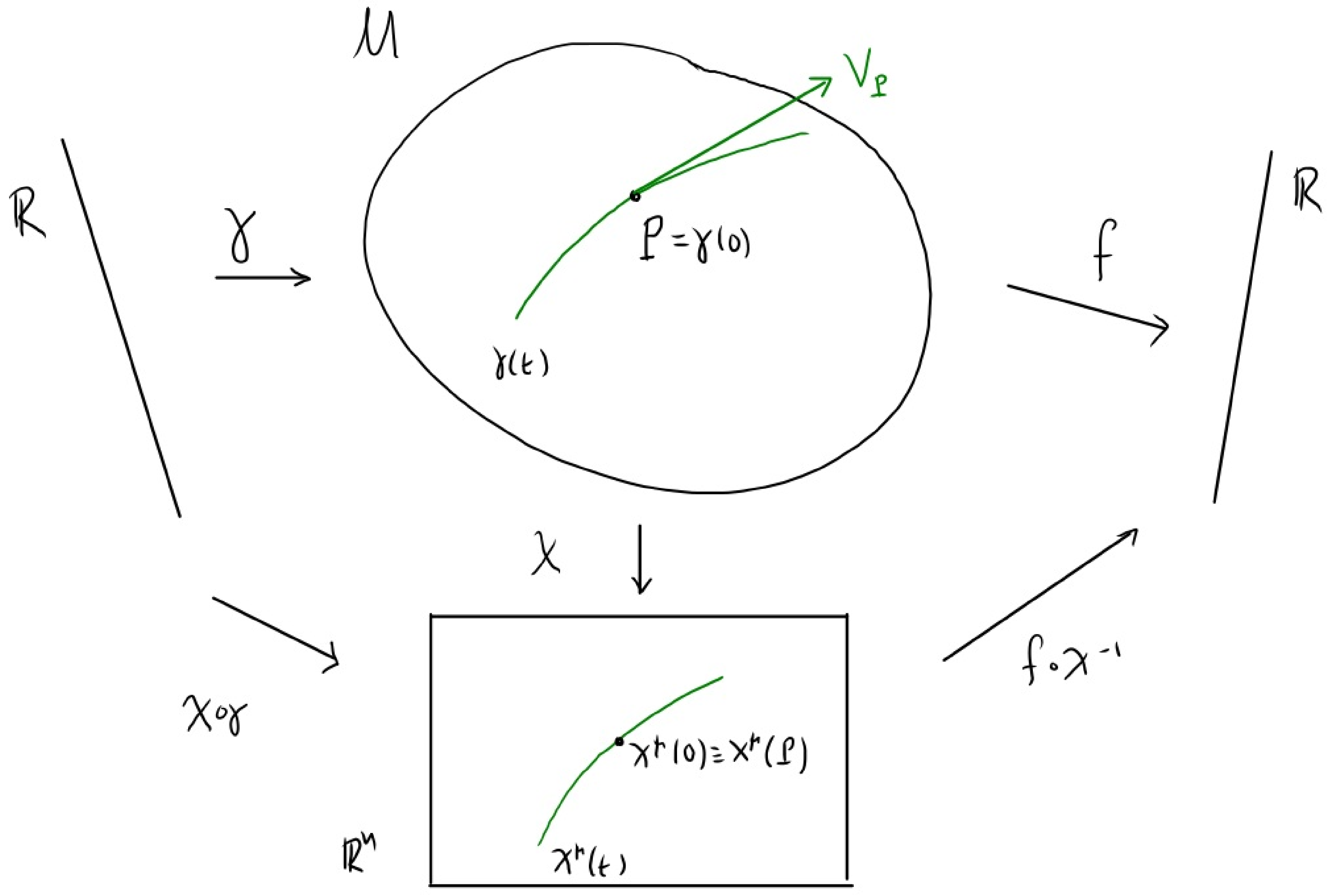
$$V_p(f) = \frac{\partial f(x^\nu)}{\partial x^\mu} \cdot \frac{dx^\mu}{dt}$$

$$= \frac{dx^\mu}{dt} \cdot \partial_\mu f$$

$$\Rightarrow V_p = \frac{dx^\mu}{dt} \cdot \partial_\mu \qquad !$$



* Consider $\gamma_v(t)$ s.t. $\gamma_v(0) = p$

$\gamma_u(t)$ " $\gamma_u(0) = p$

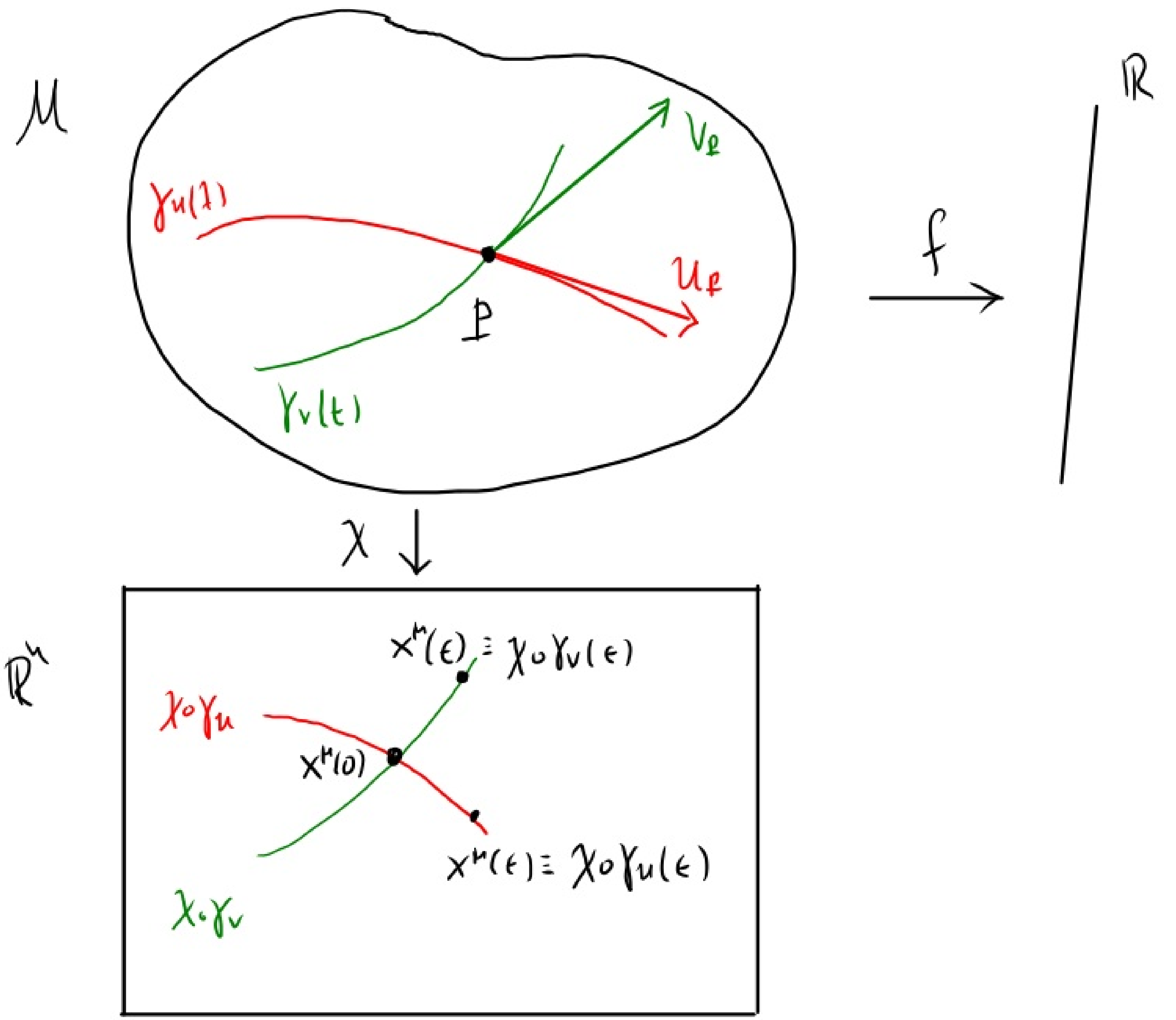
and

$$V_p(f) = \frac{df}{dt}\bigg|_p = \frac{dx^\mu}{dt}\frac{\partial f}{\partial x^\mu}\bigg|_p$$

→ rate of change on $\chi \circ \gamma_v(t)$

$$U_p(f) = \frac{df}{d\lambda}\bigg|_p = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}\bigg|_p$$

→ rate of change on $\chi \circ \gamma_u(\lambda)$



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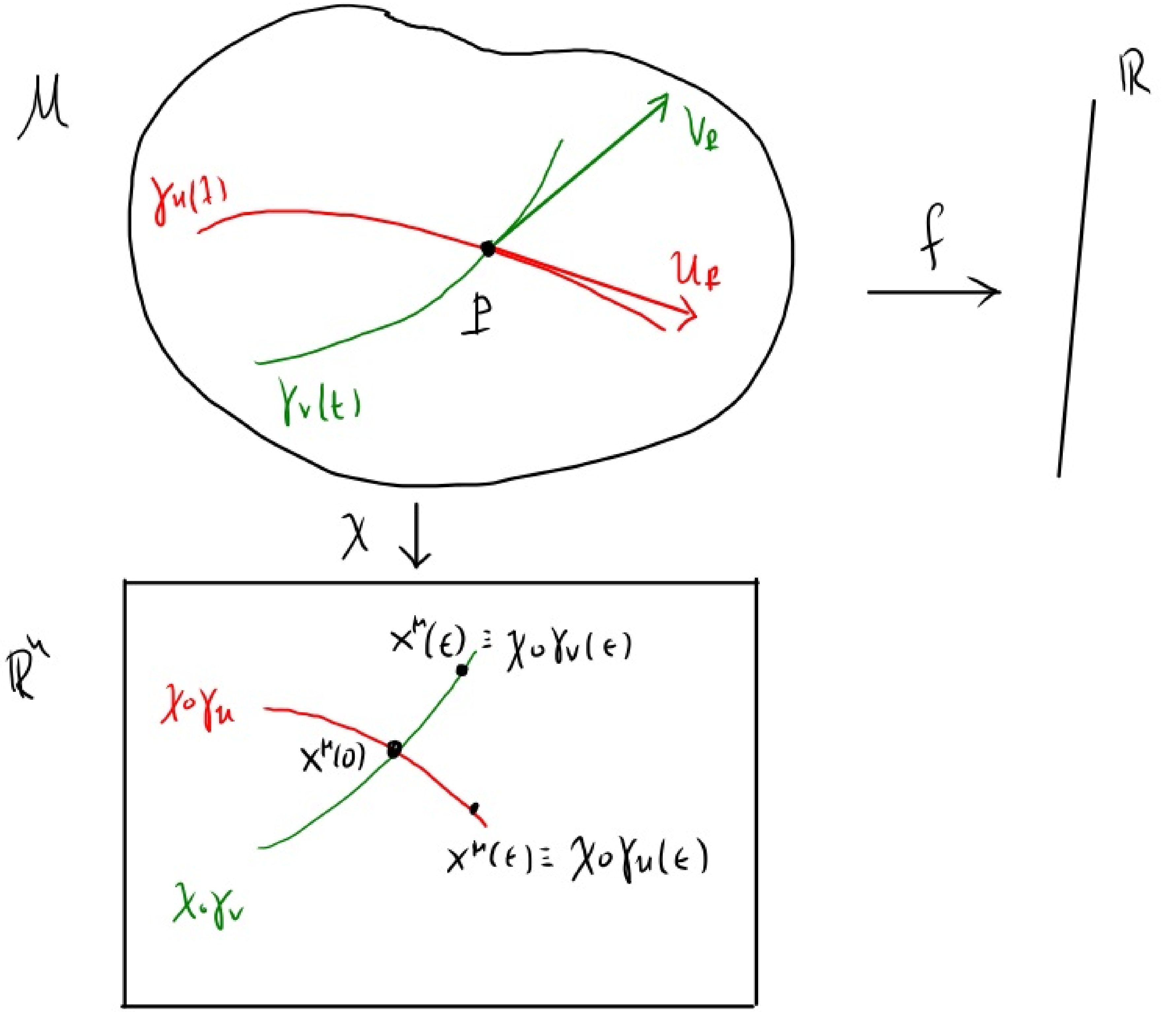

* for a coordinate system with $\chi: P \mapsto x^\mu(P)$ take a specific μ: then $P \mapsto x^\mu(P)$ is a real function on M

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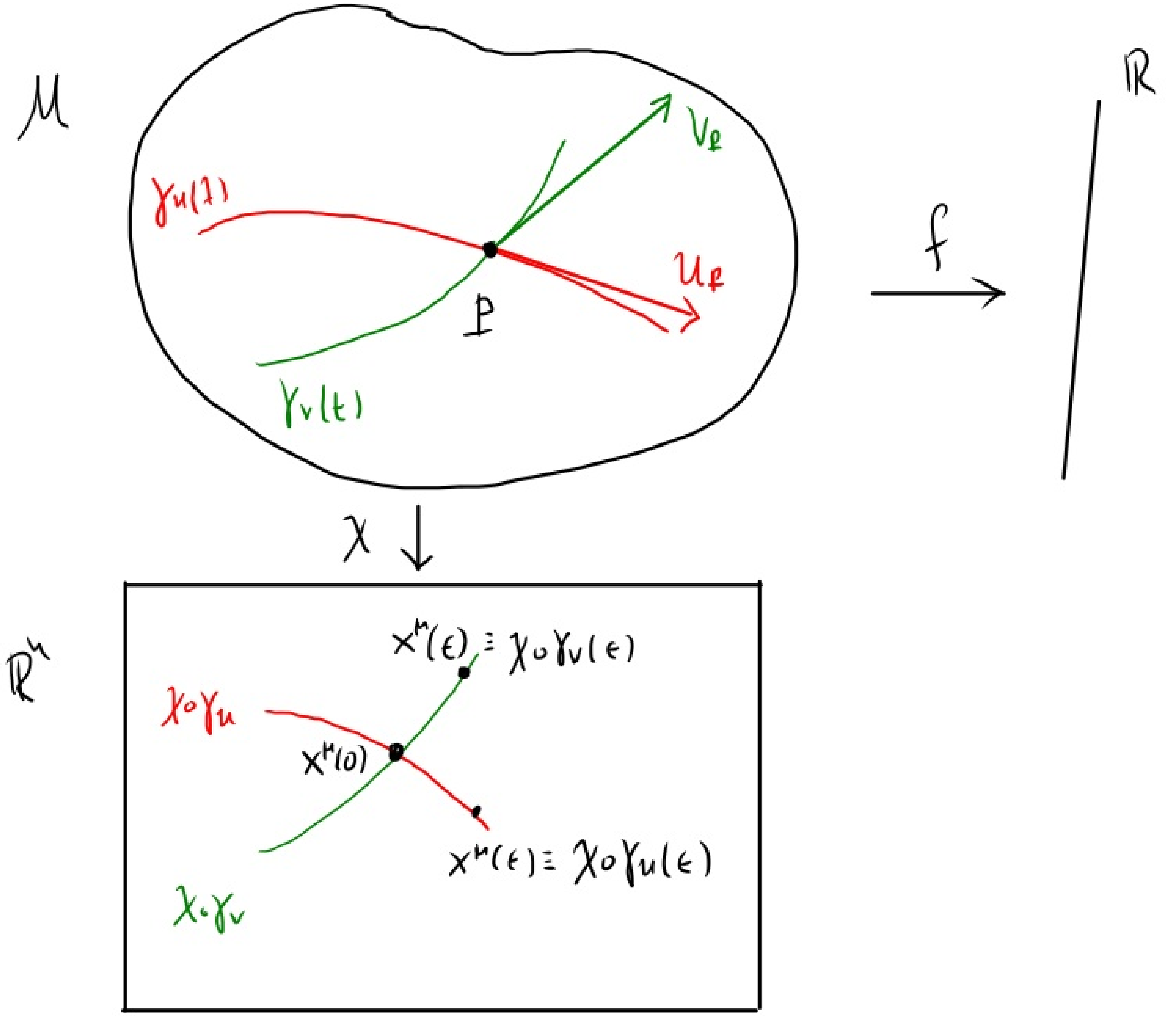

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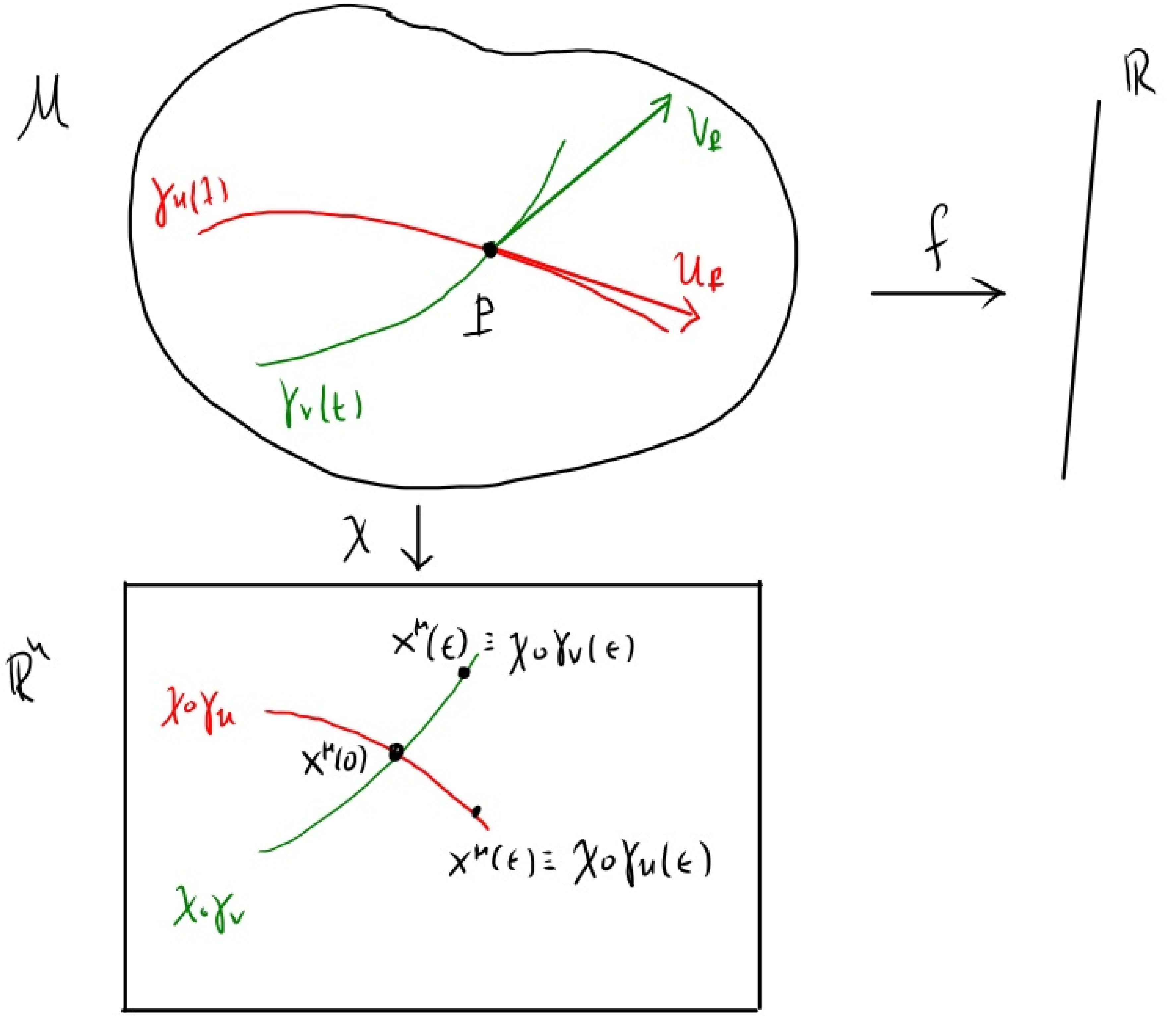

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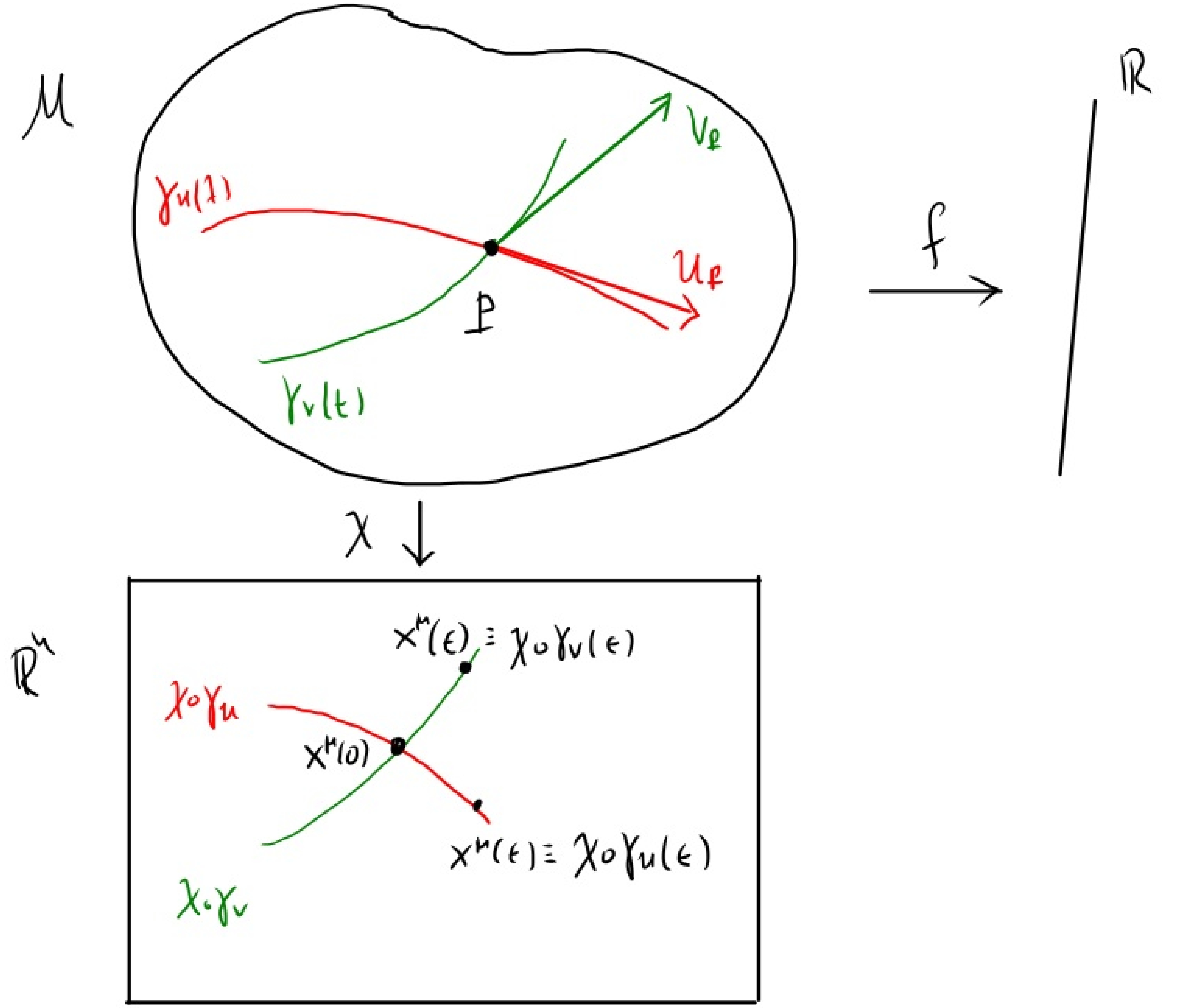
and



$$V_P(f) = \left.\frac{df}{dt}\right|_P = \frac{dx^\mu}{dt} \left.\frac{\partial f}{\partial x^\mu}\right|_P$$

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* In $\mathbb{R}^n$ we have the curves:

$$x^\mu(t) \equiv \chi \circ \gamma_v(t)$$

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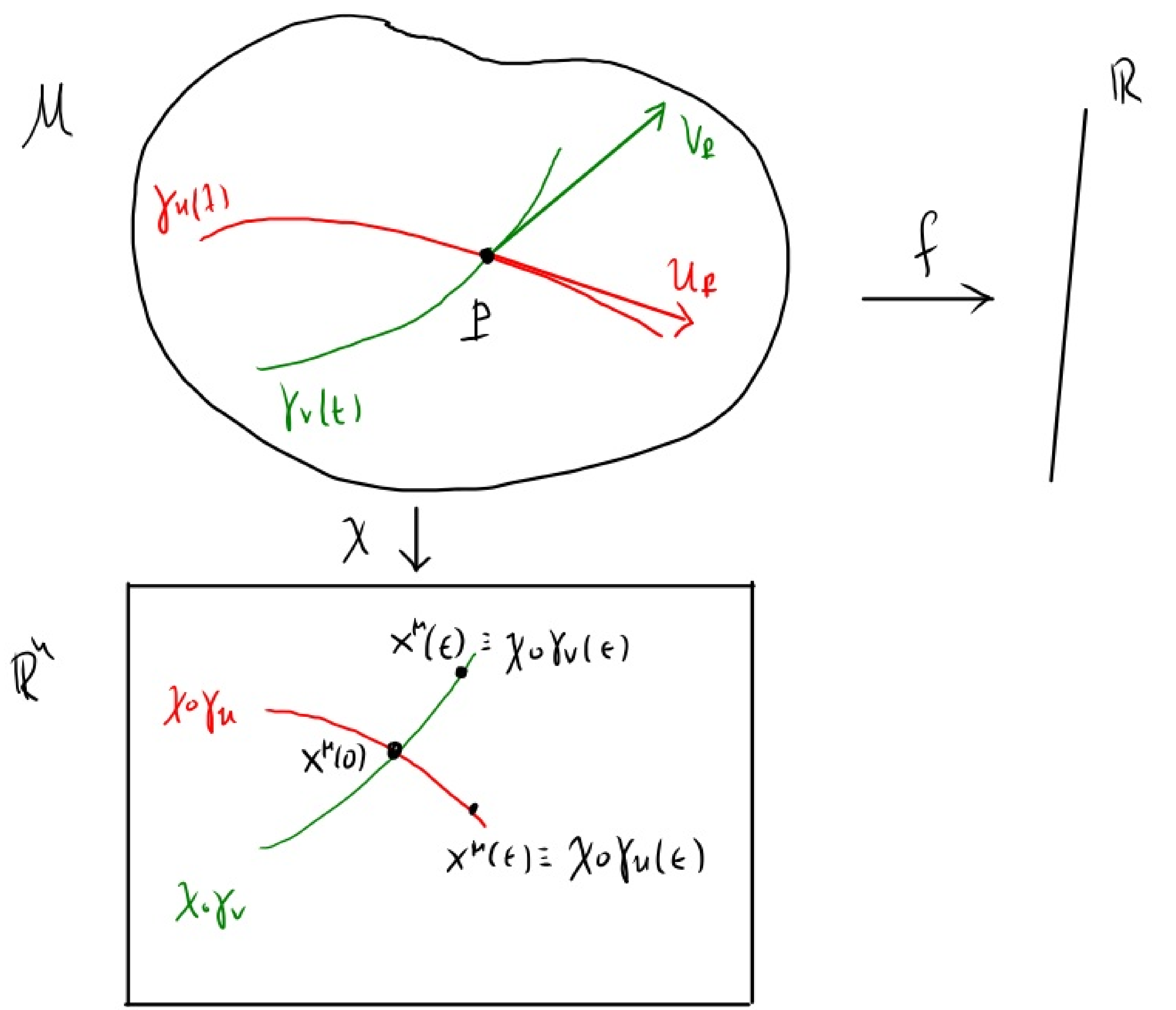

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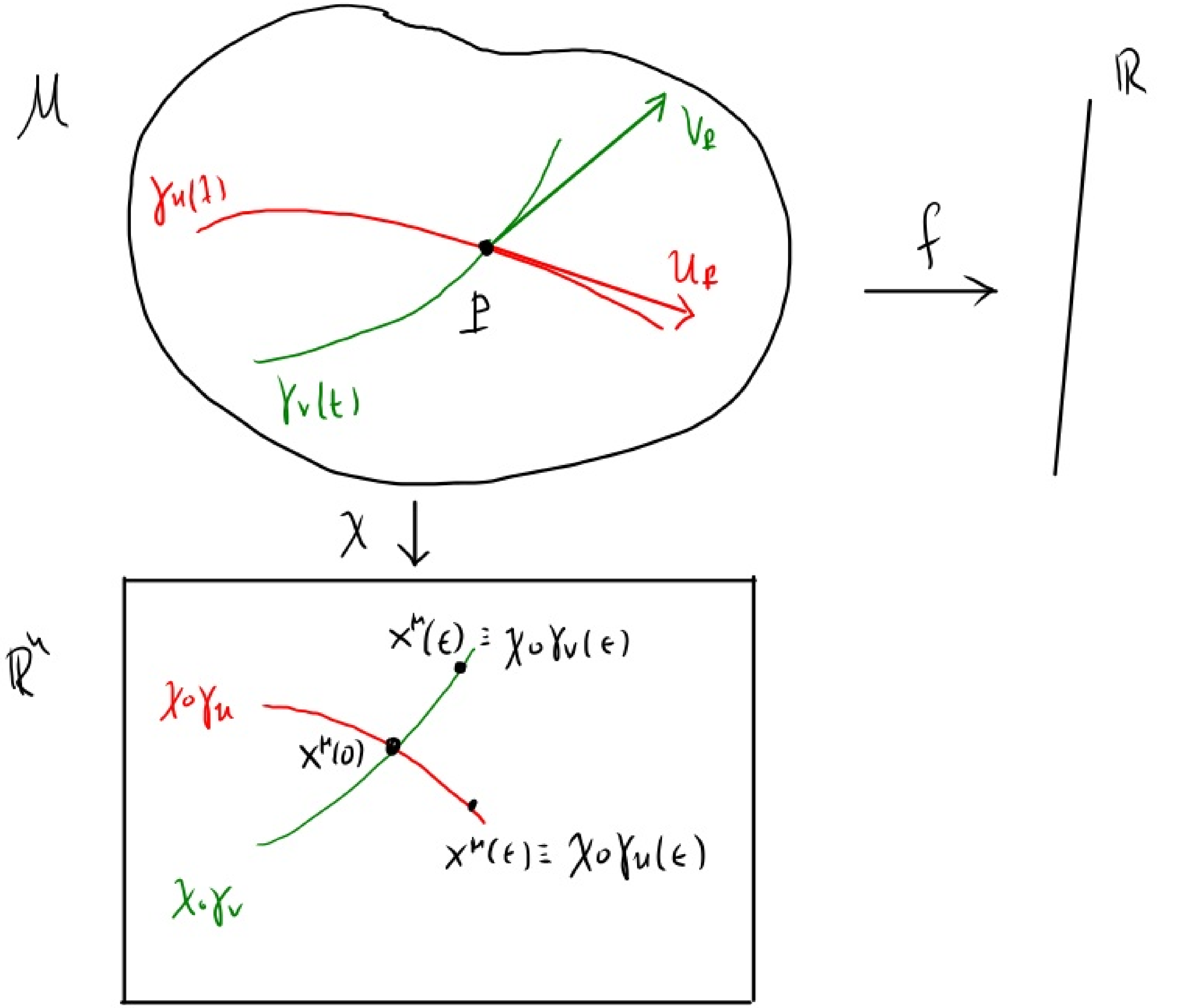
and

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \frac{dx^\mu}{dt}\Big|_0 + \mathcal{O}_v(\epsilon^2)$$

on $\chi \circ \gamma_v$

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \frac{dx^\mu}{d\lambda}\Big|_0 + \mathcal{O}_u(\epsilon^2)$$

on $\chi \circ \gamma_u$



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different points

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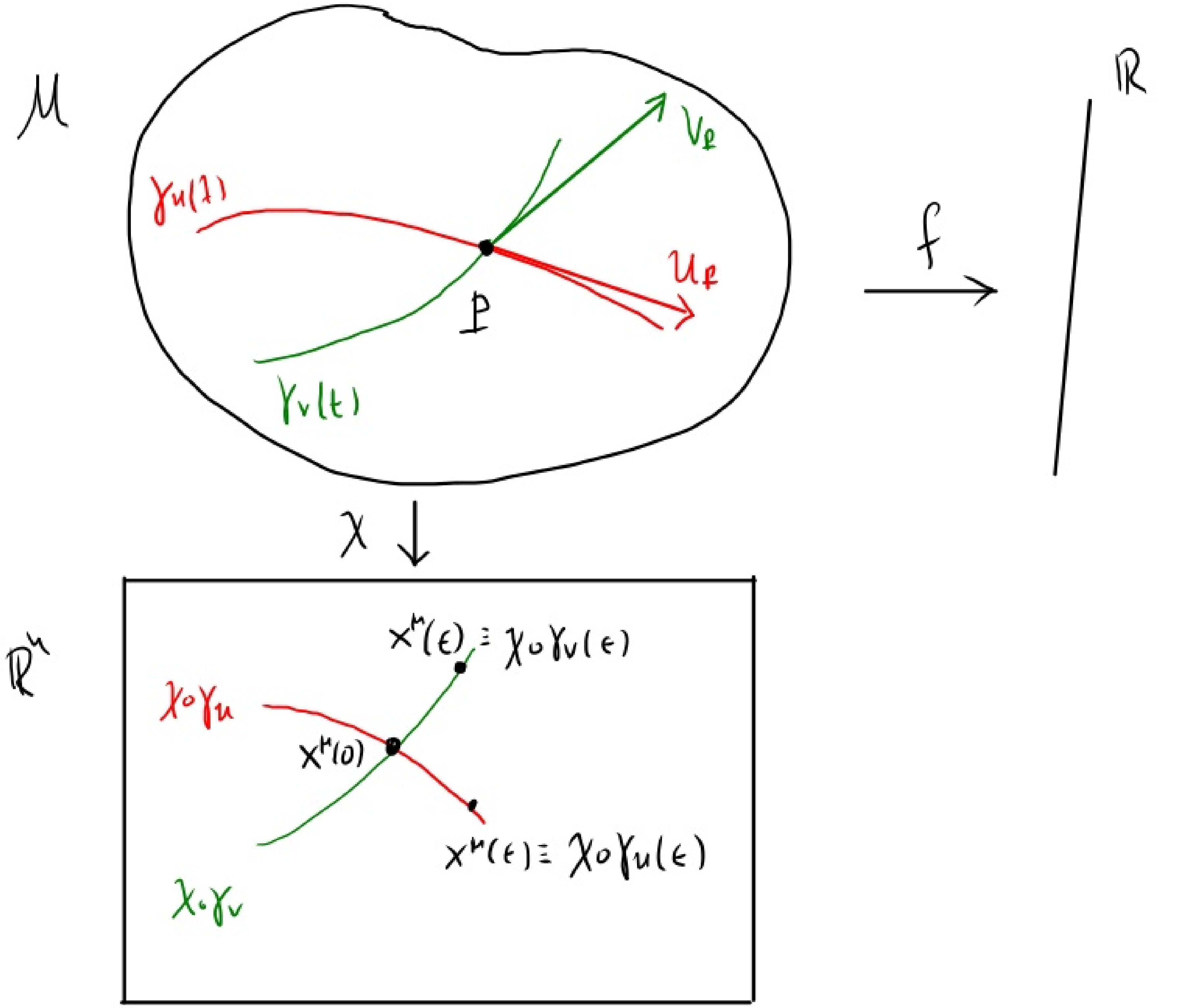

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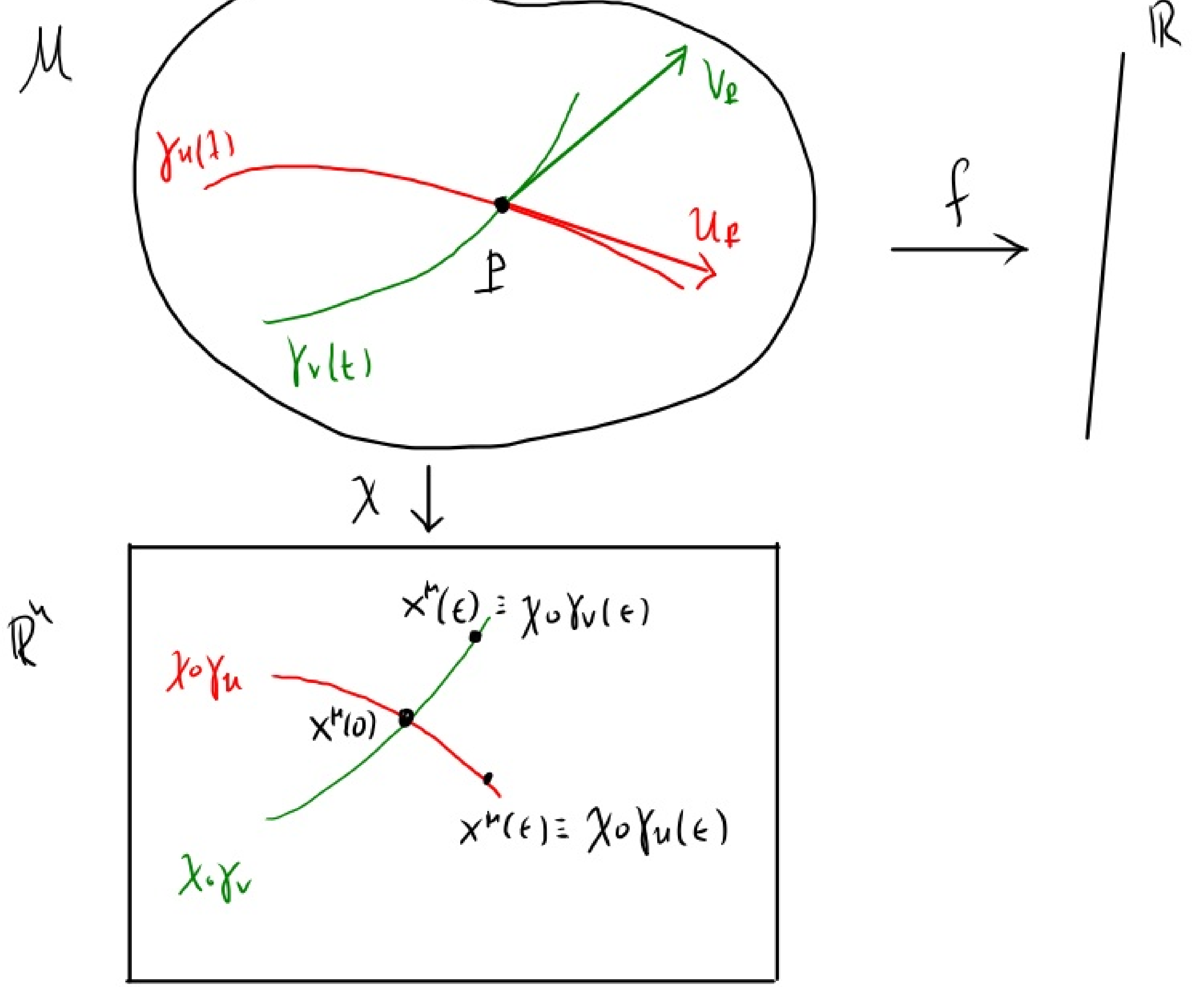


define <u>any</u> curve $\gamma_w(\epsilon)$, s.t.

$x^\mu(\epsilon) \equiv \chi \circ \gamma_w(\epsilon)$ is

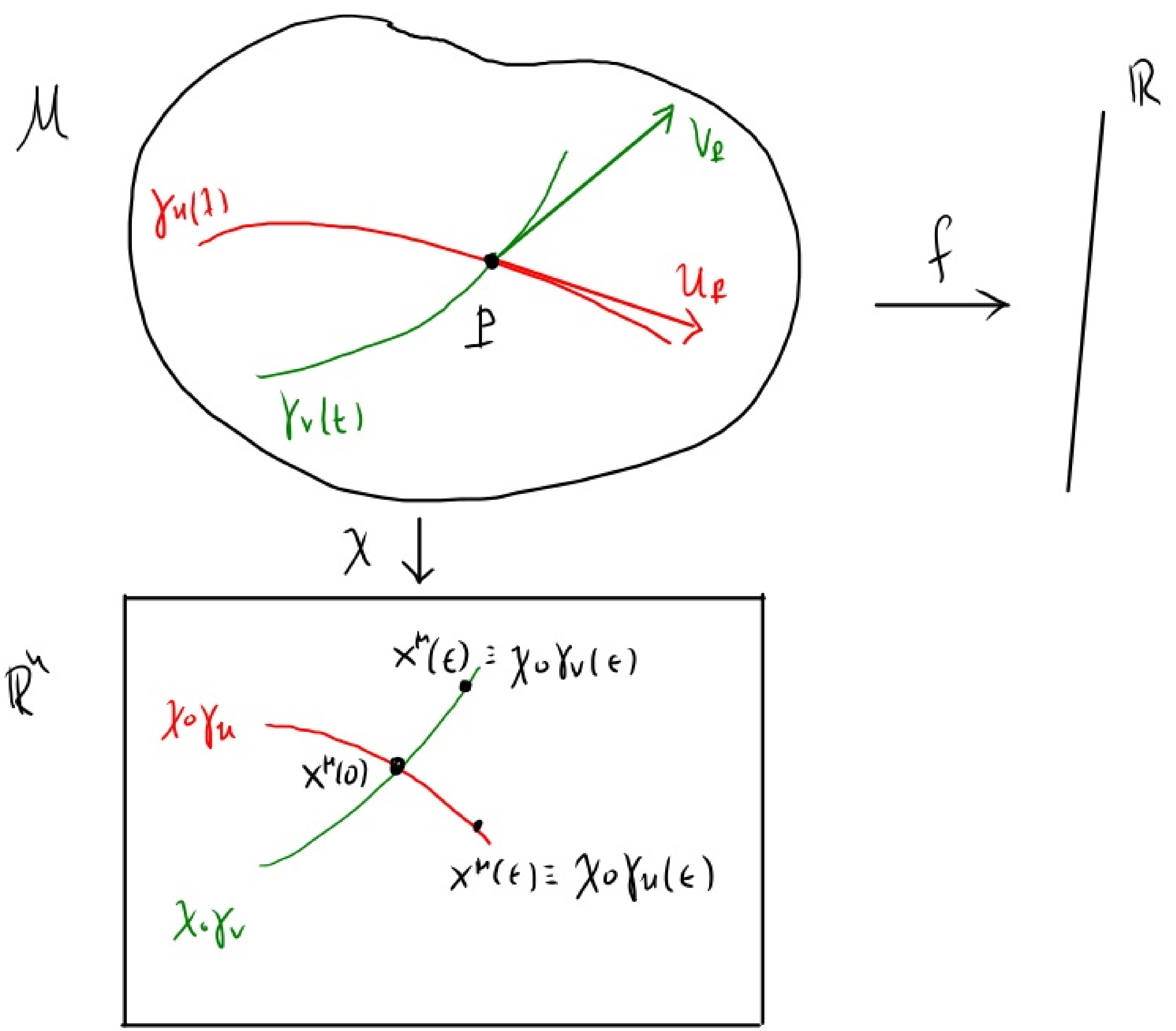
$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left[\alpha \left.\frac{dx^\mu}{dt}\right|_0 + \beta \left.\frac{dx^\mu}{d\lambda}\right|_0 \right] + \mathcal{O}_w(\epsilon^2)$$

numbers!

↳ any infinitesimal you like!

* for $\gamma_w(\epsilon)$, the tangent vector is:

$$W_p(f) = \frac{df}{d\epsilon}\Big|_0$$



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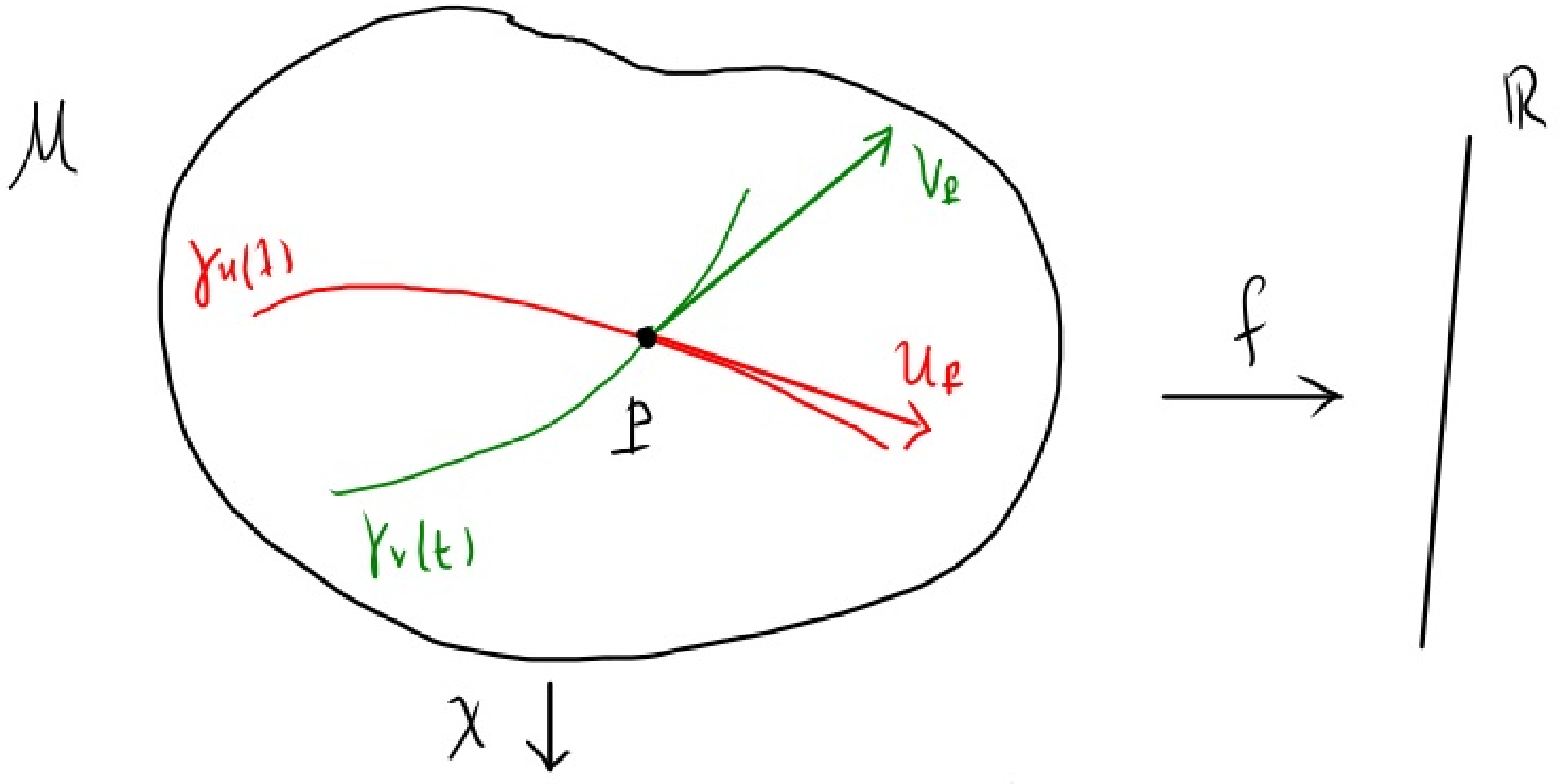

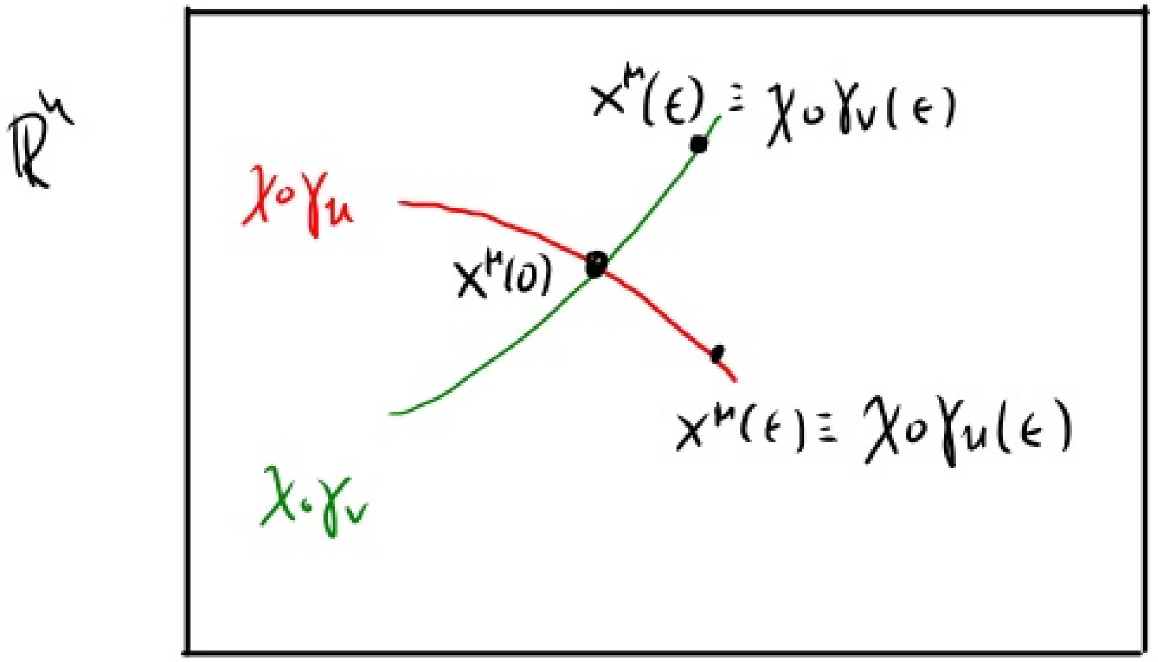

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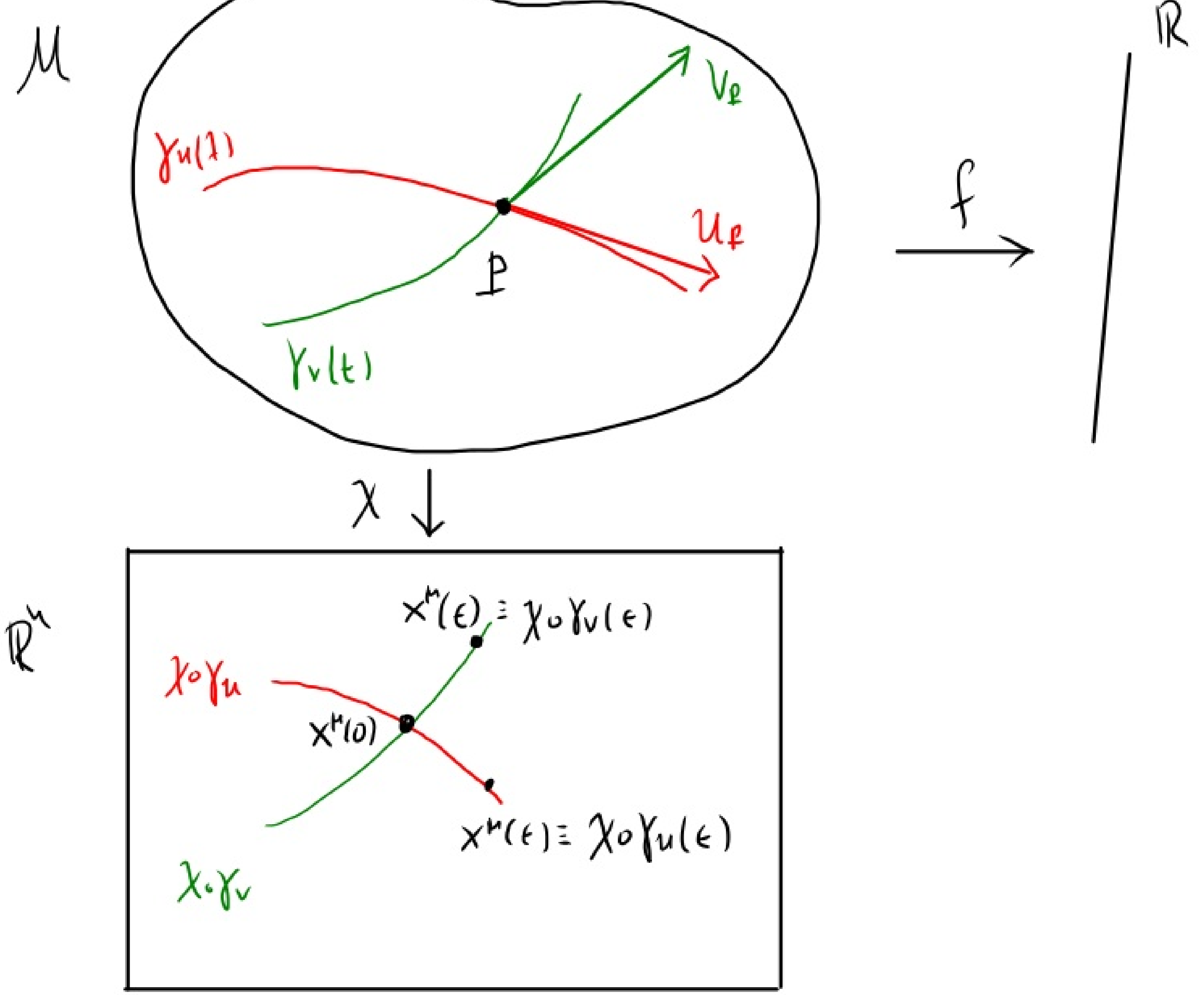

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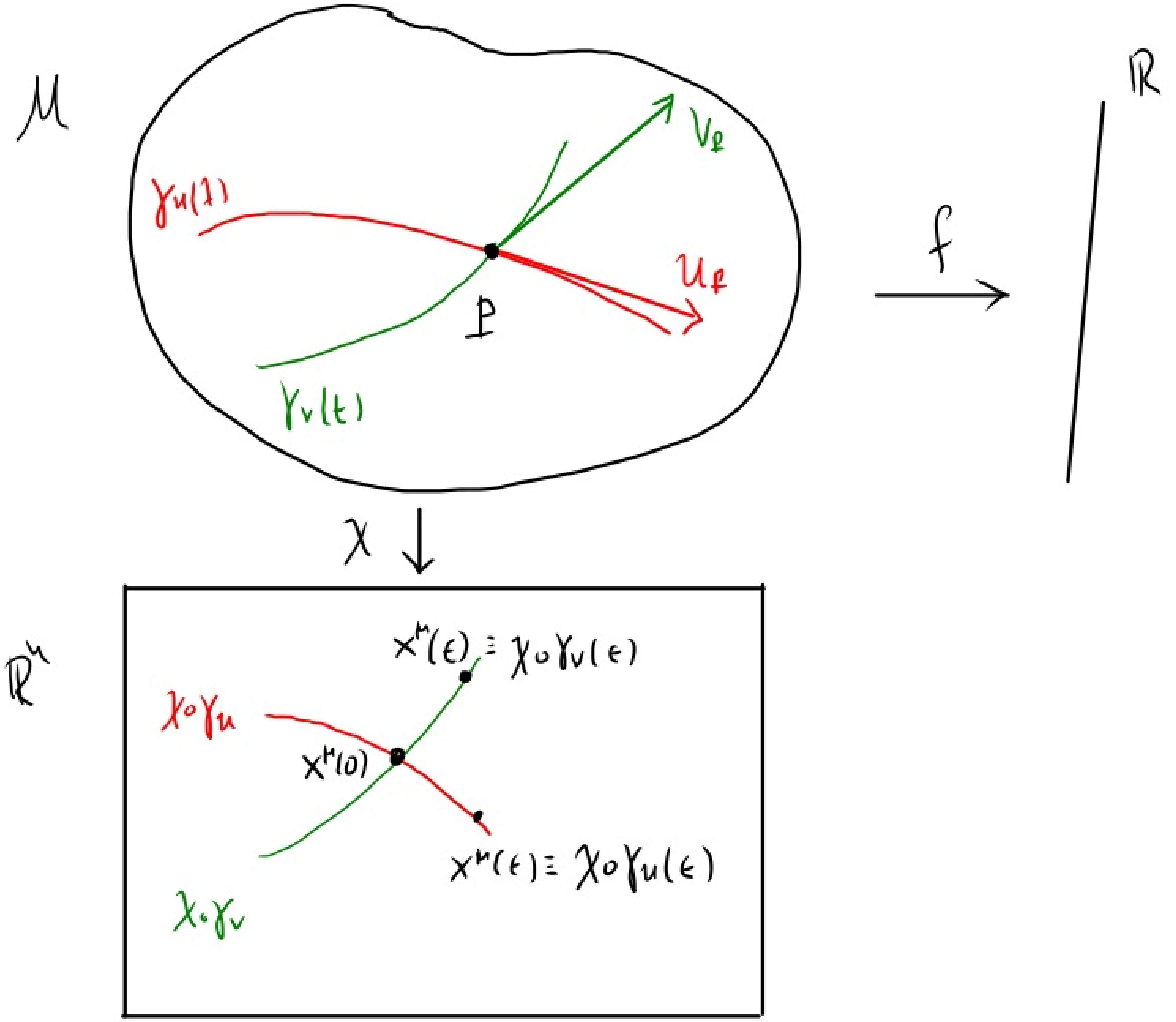
$$= \frac{\partial f}{\partial x^\mu} \cdot \frac{dx^\mu(\epsilon)}{d\epsilon}\Big|_0$$



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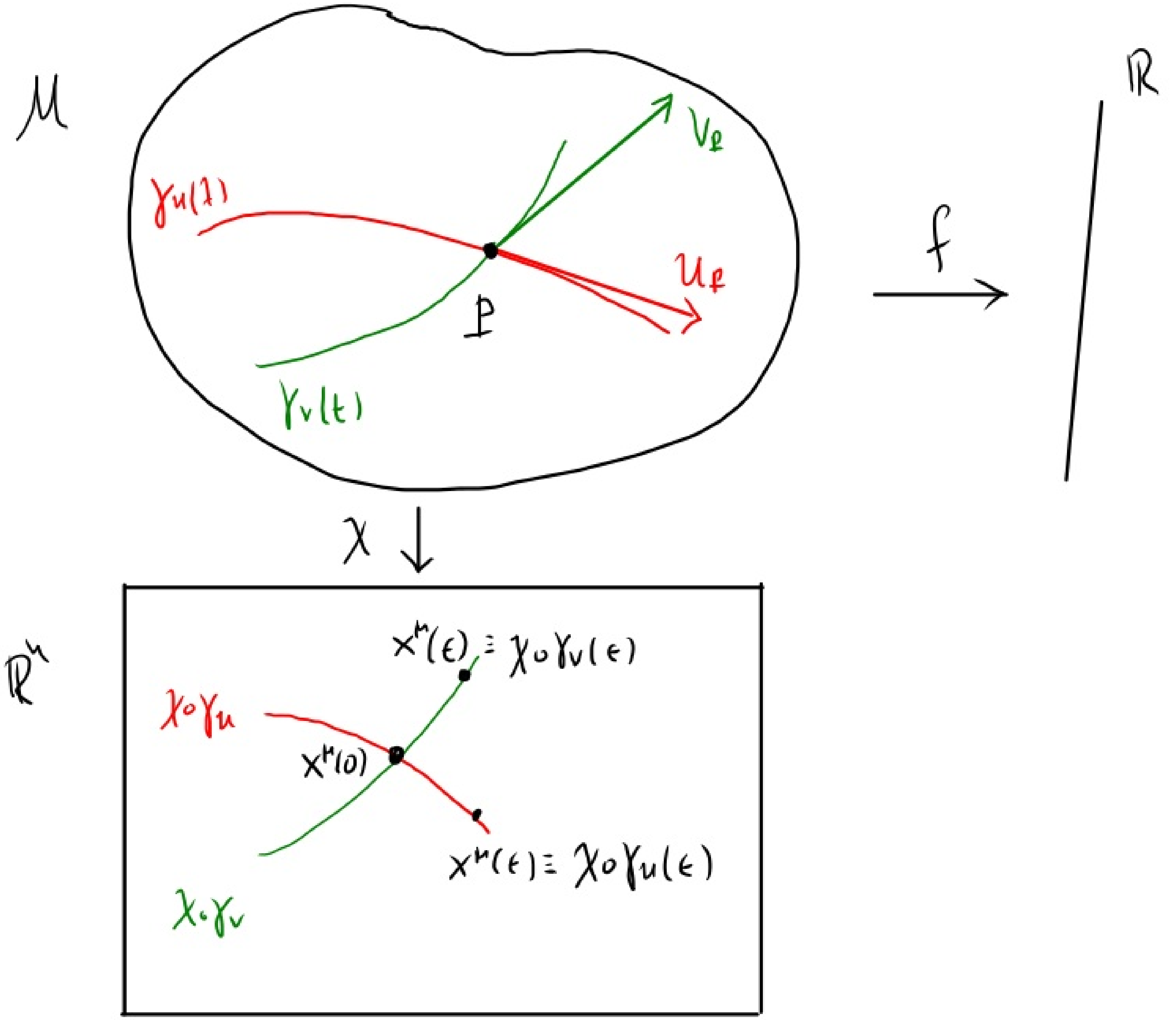

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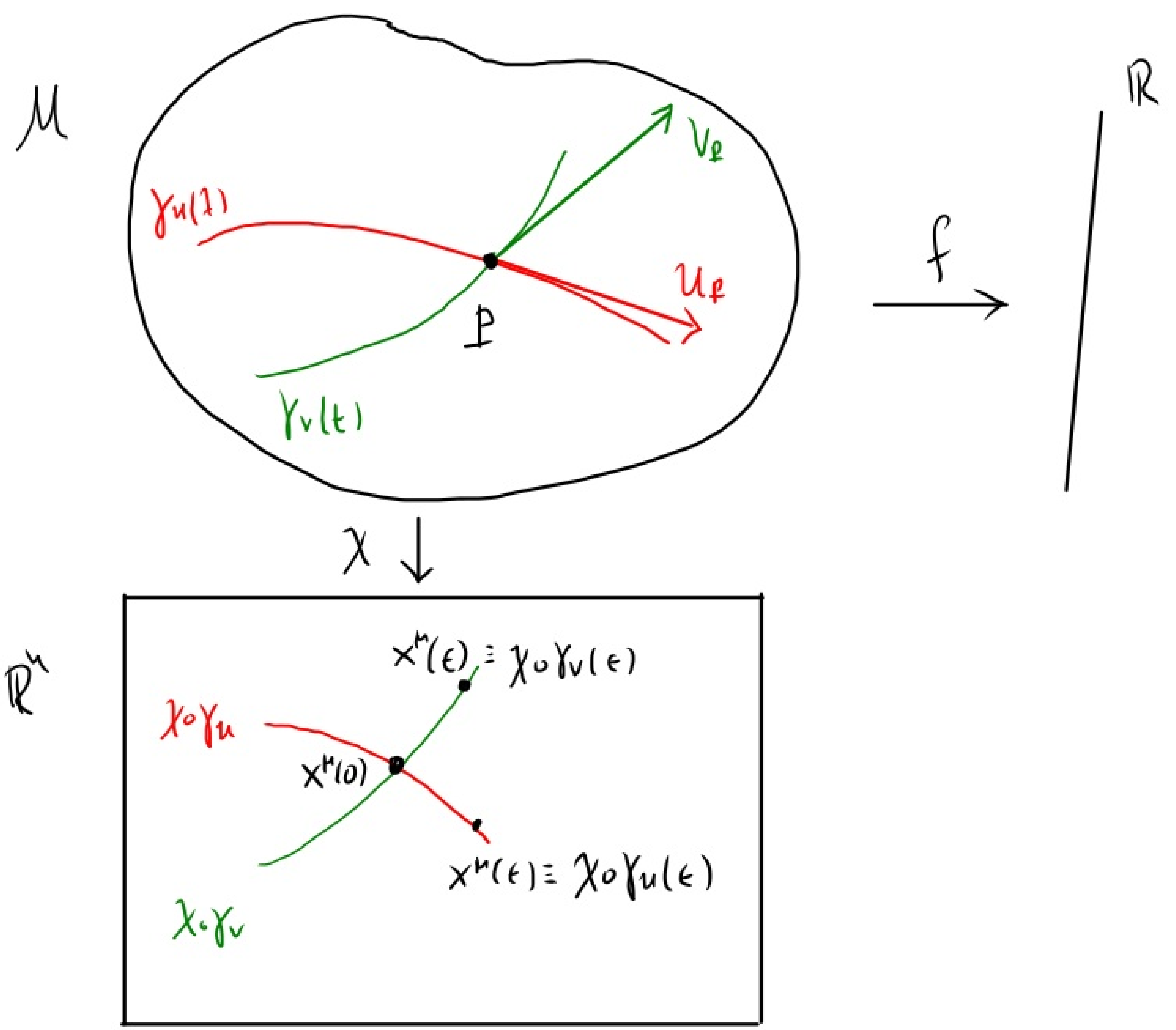
$$= \alpha V_p(f) + \beta U_p(f)$$



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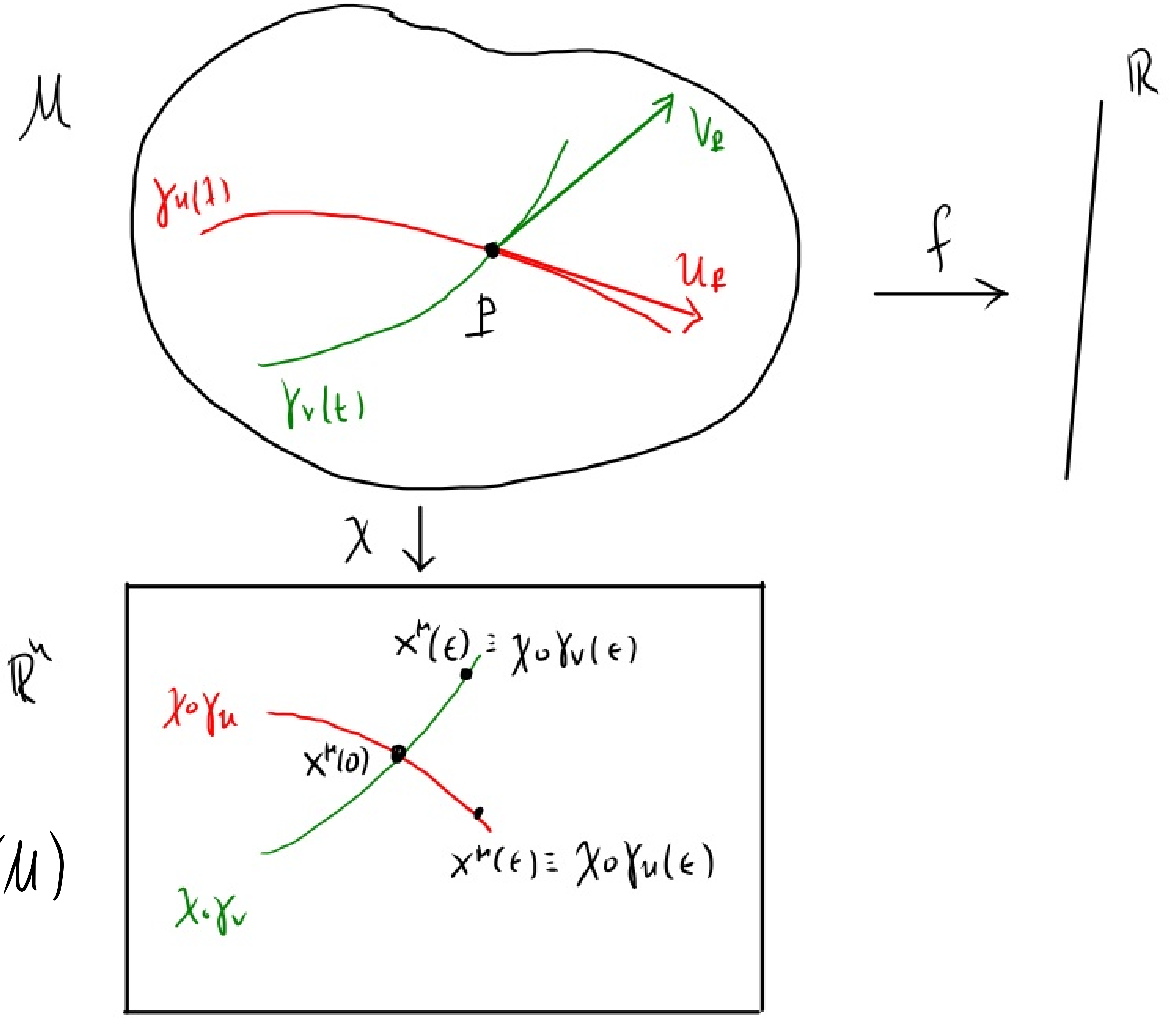
$$= \frac{\partial f}{\partial x^\mu} \cdot \left(\alpha \frac{dx^\mu}{dt}\Big|_0 + \beta \frac{dx^\mu}{d\lambda}\Big|_0 \right)$$

$$= \alpha V_p(f) + \beta U_p(f) \quad \forall f \in \mathcal{F}(M)$$

$$\Rightarrow \quad W_p = \alpha V_p + \beta U_p$$



* Coordinate basis:

Consider the curve:

$$\gamma^\mu : \mathbb{R} \to \mathcal{M}$$

$$x^\mu \mapsto \gamma^\mu(x^\mu)$$

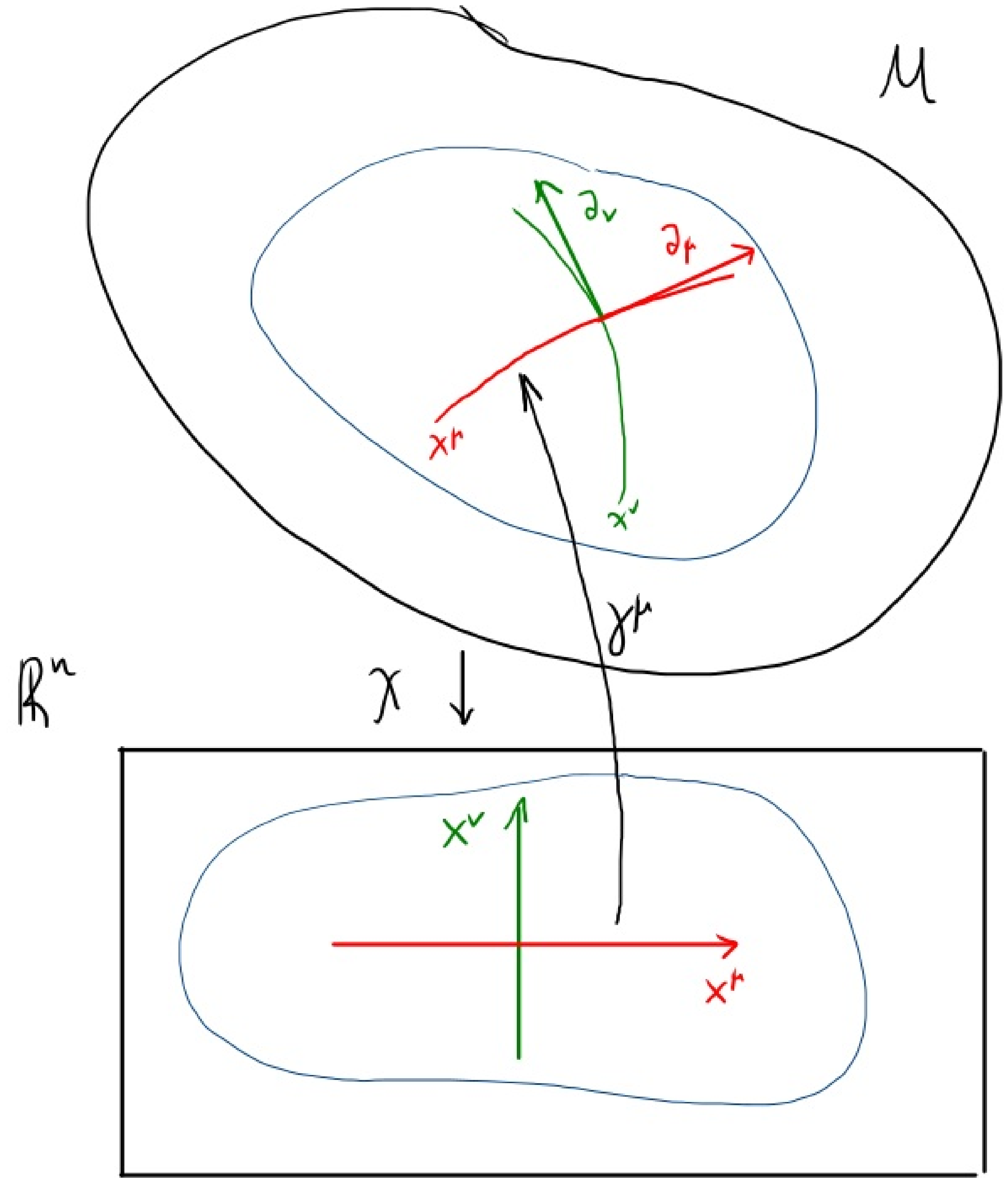
this is the parameter of the curve!

$$\gamma^\mu(x^\mu) = \chi^{-1}(\underbrace{c_0, c_1, \dots, c_{\mu-1}}_{}, x^\mu, \underbrace{c_{\mu+1}, \dots, c_{n-1}}_{})$$

constants!

fixed μ

(no summation - God forbid!)



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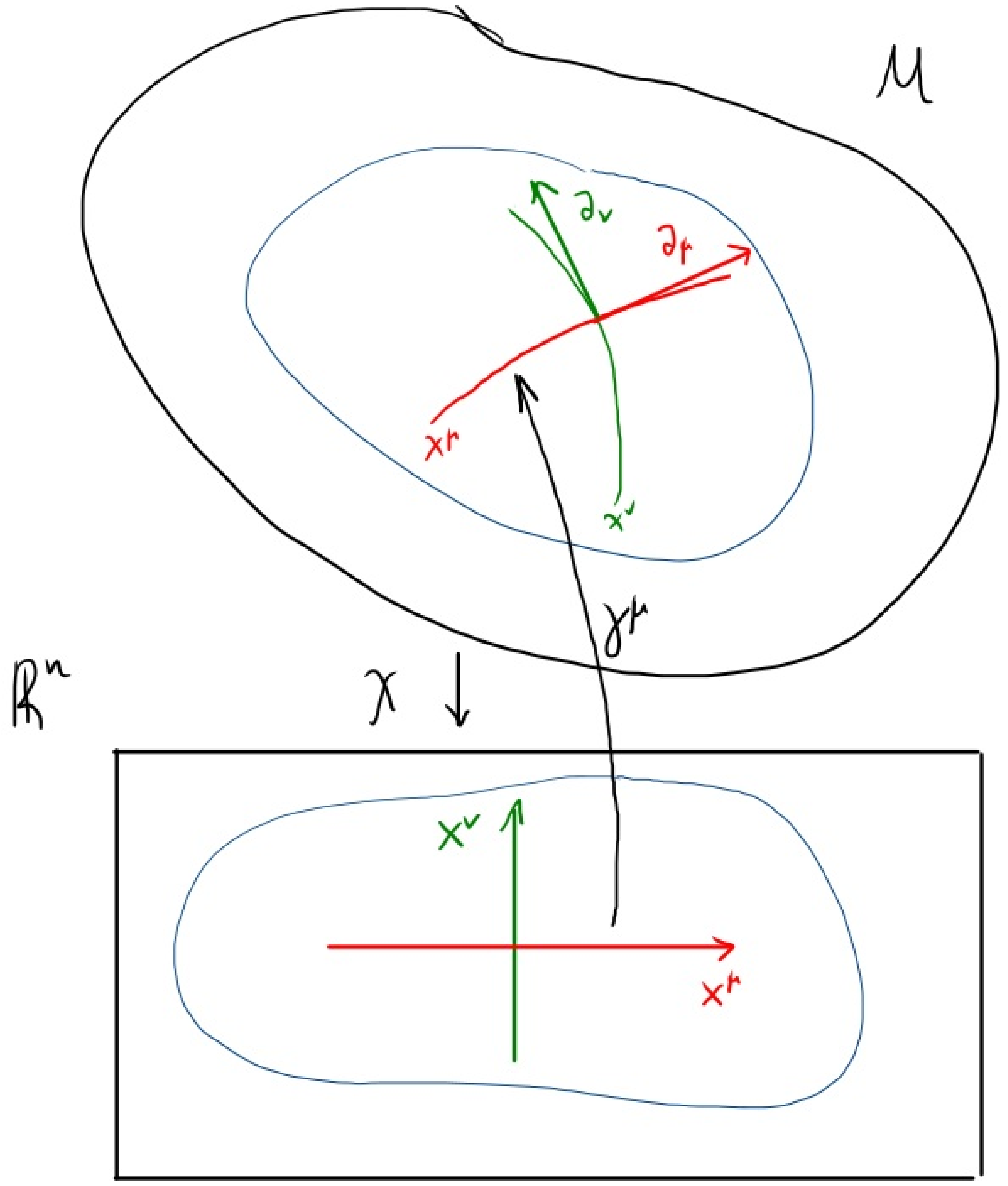
- in $\mathbb{R}^n$ we move parallel to the x^μ-axis (all other x^ν are fixed)



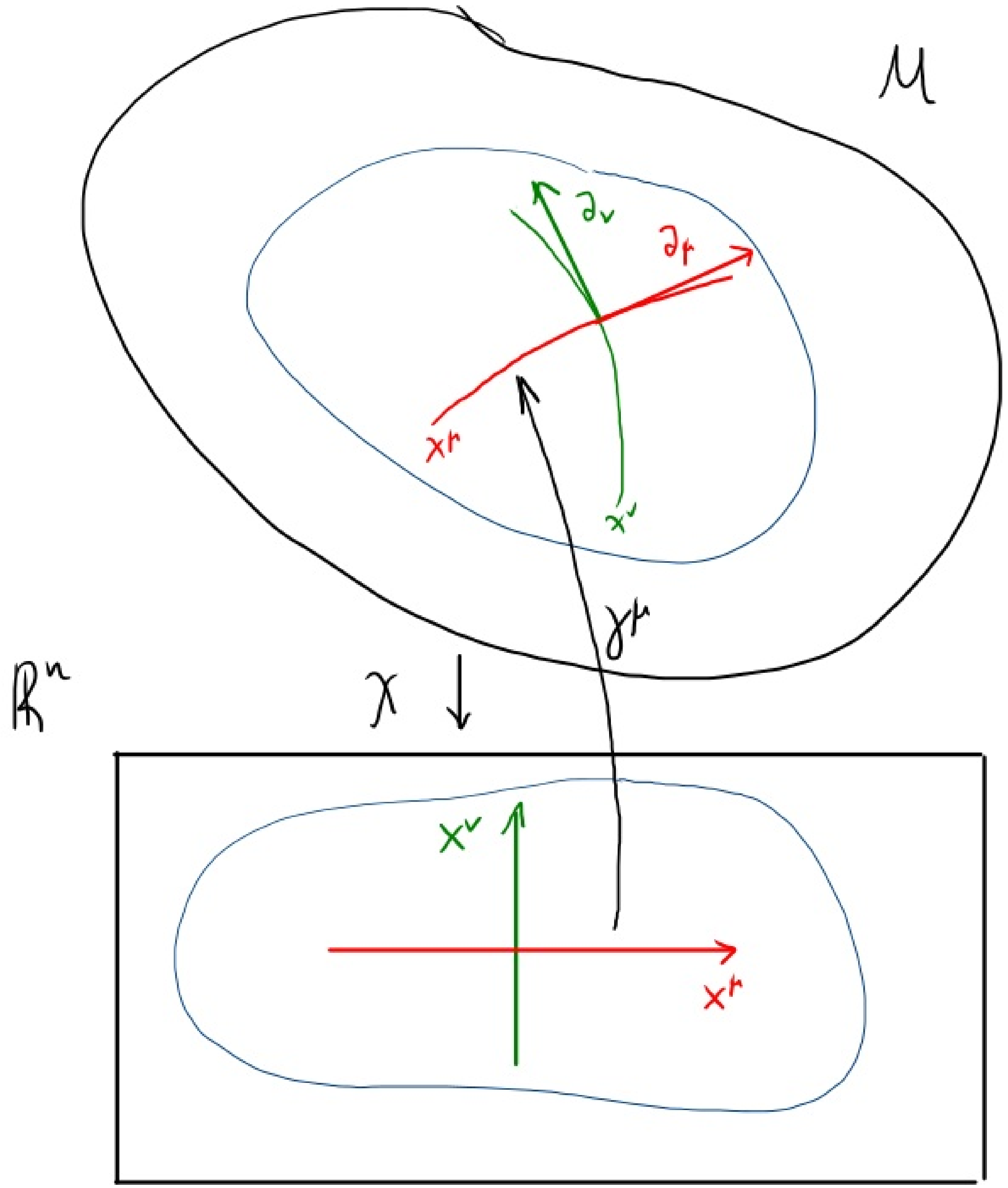
* Coordinate basis:

Consider the curve:

$\gamma^\mu : \mathbb{R} \to M$

$x^\mu \mapsto \gamma^\mu(x^\mu)$ this is the parameter of the curve!

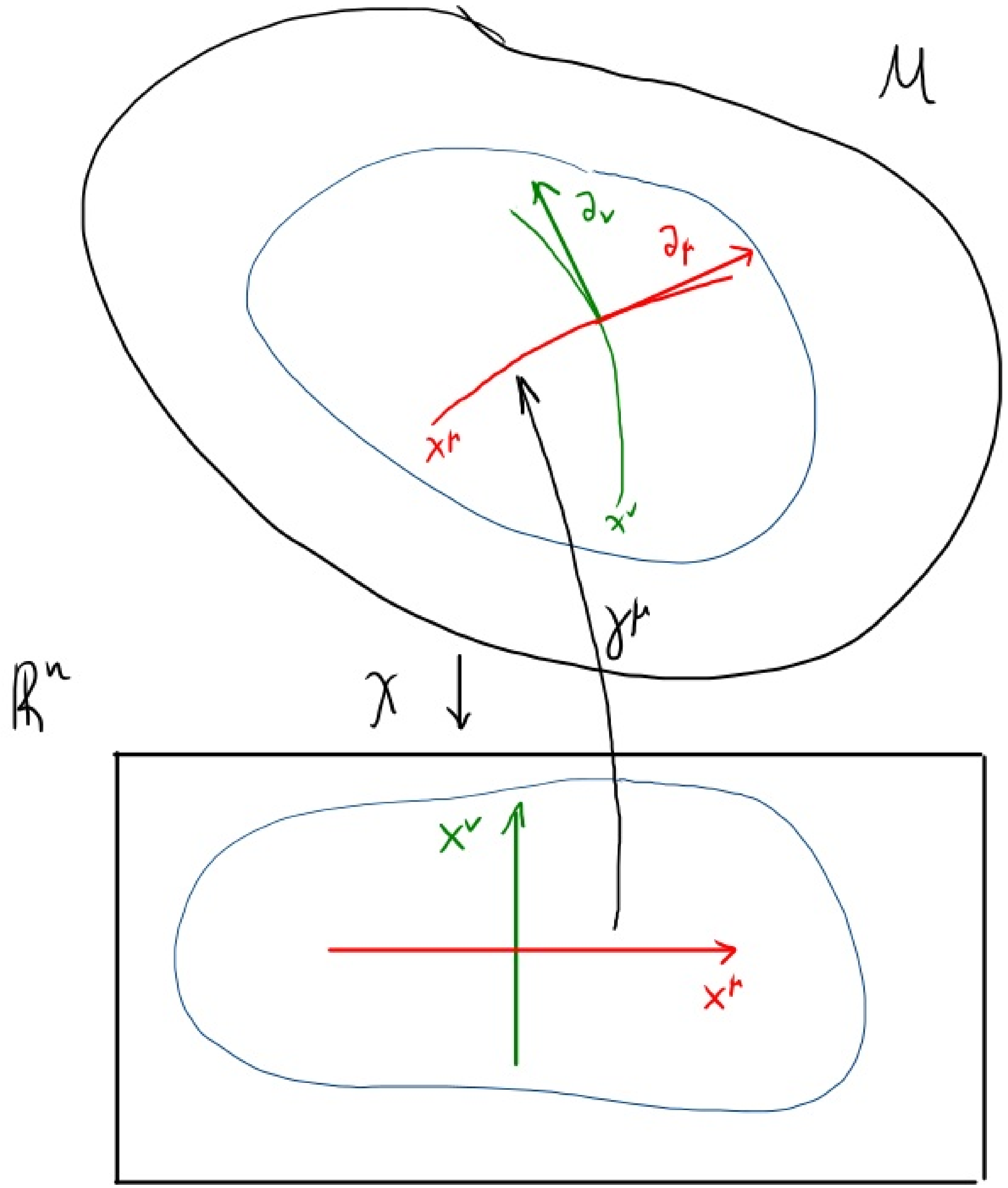
- in $\mathbb{R}^n$ we move parallel to the x^μ-axis (all other x^ν are fixed)
- the tangent vector of γ^μ at P is ∂_μ



* Coordinate basis:

Then: $\frac{df}{dx^\mu}\Big|_p = \frac{d}{dx^\mu} f \circ \gamma^\mu(x^\mu)\Big|_p$

the parameter of the curve!



_ the tangent vector of γ^μ at P is ∂_μ

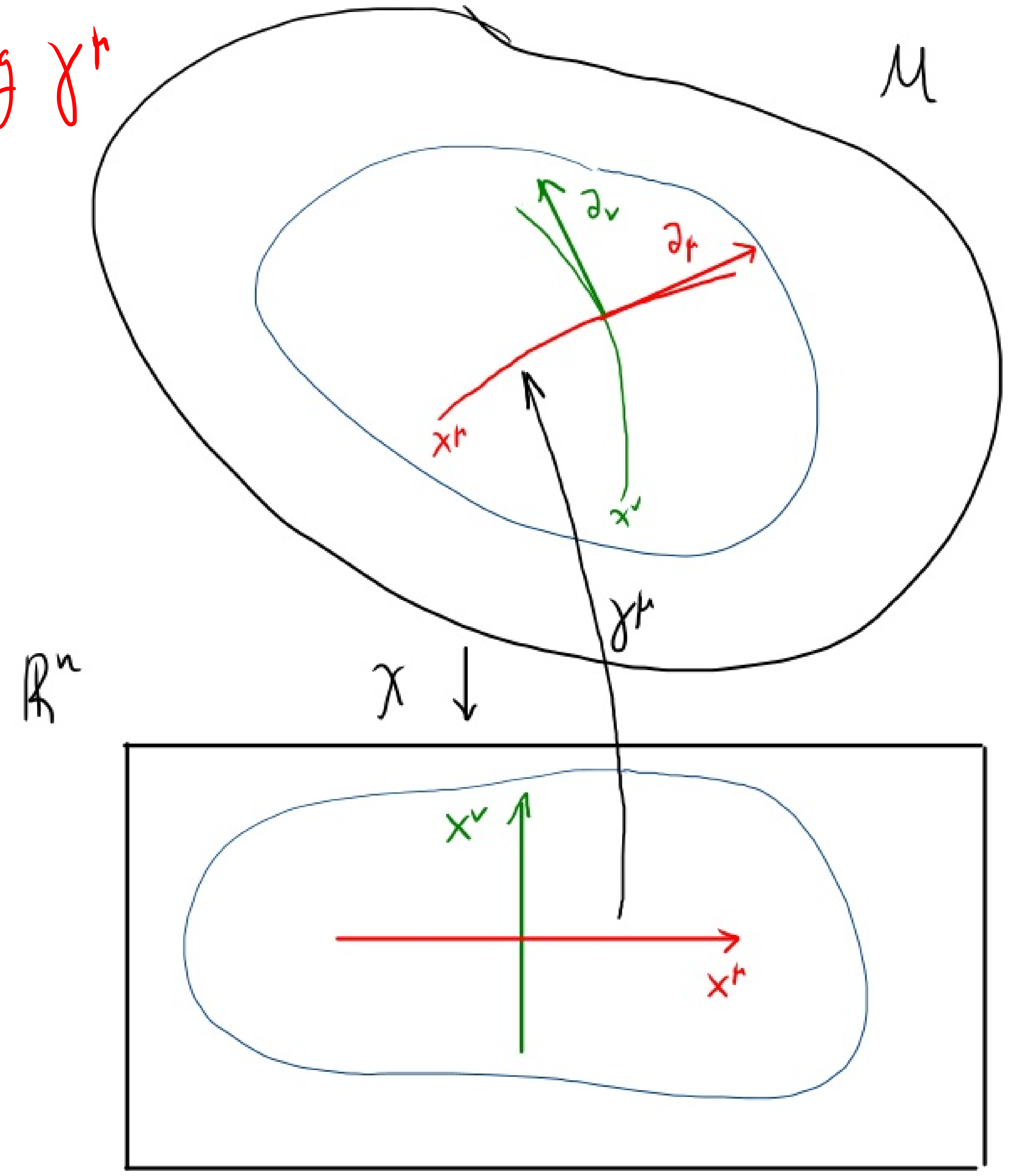
* Coordinate basis:

Then: $\dfrac{df}{dx^\mu}\Big|_p = \dfrac{d}{dx^\mu} f \circ \gamma^\mu(x^\mu)\Big|_p$ directional derivative along γ^μ

$= \dfrac{\partial}{\partial x^\mu} f \circ \chi^{-1}(x^\nu)\Big|_p$

definition of partial derivative:

we vary x^μ, hold x^ν fixed!



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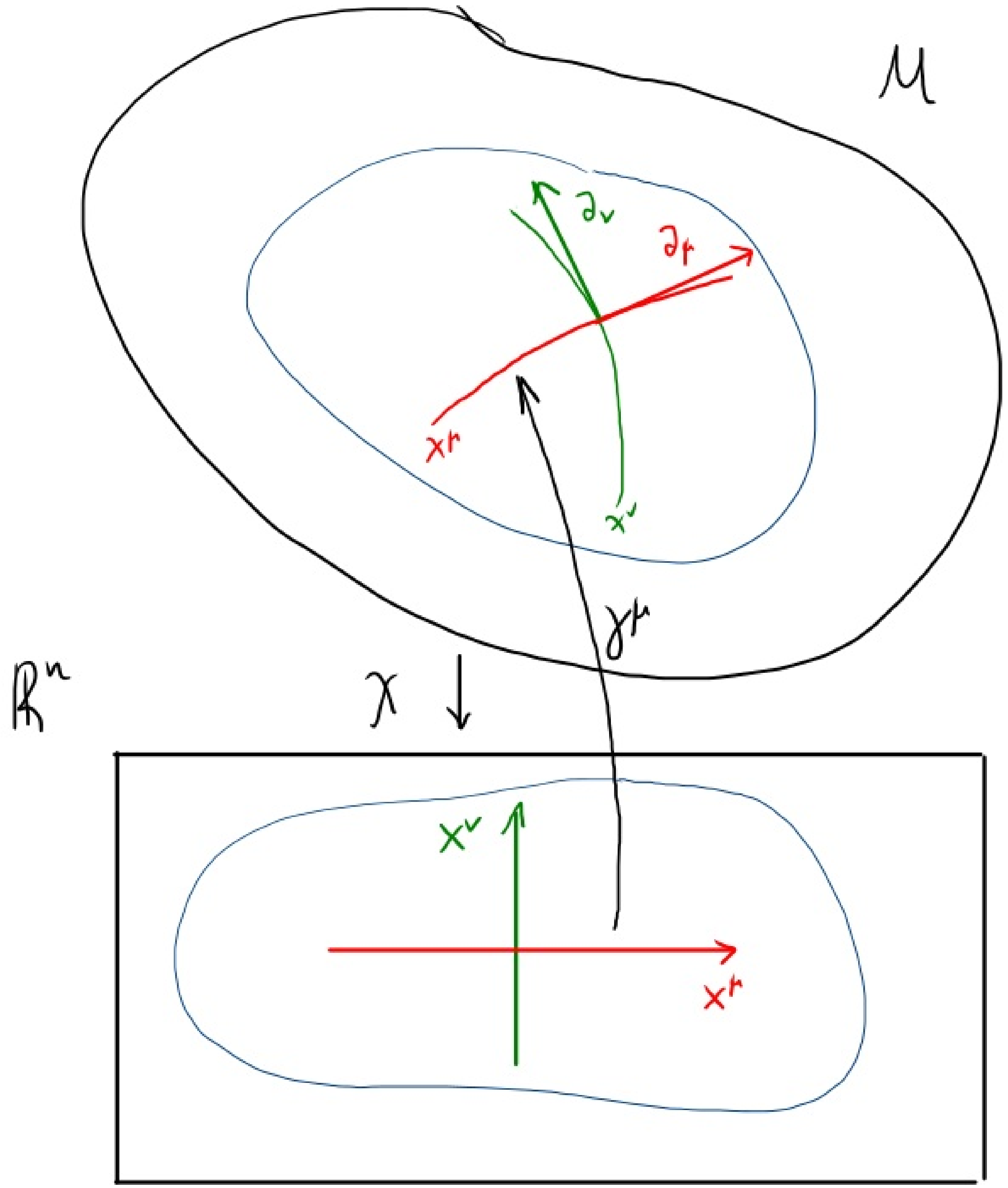


We define:

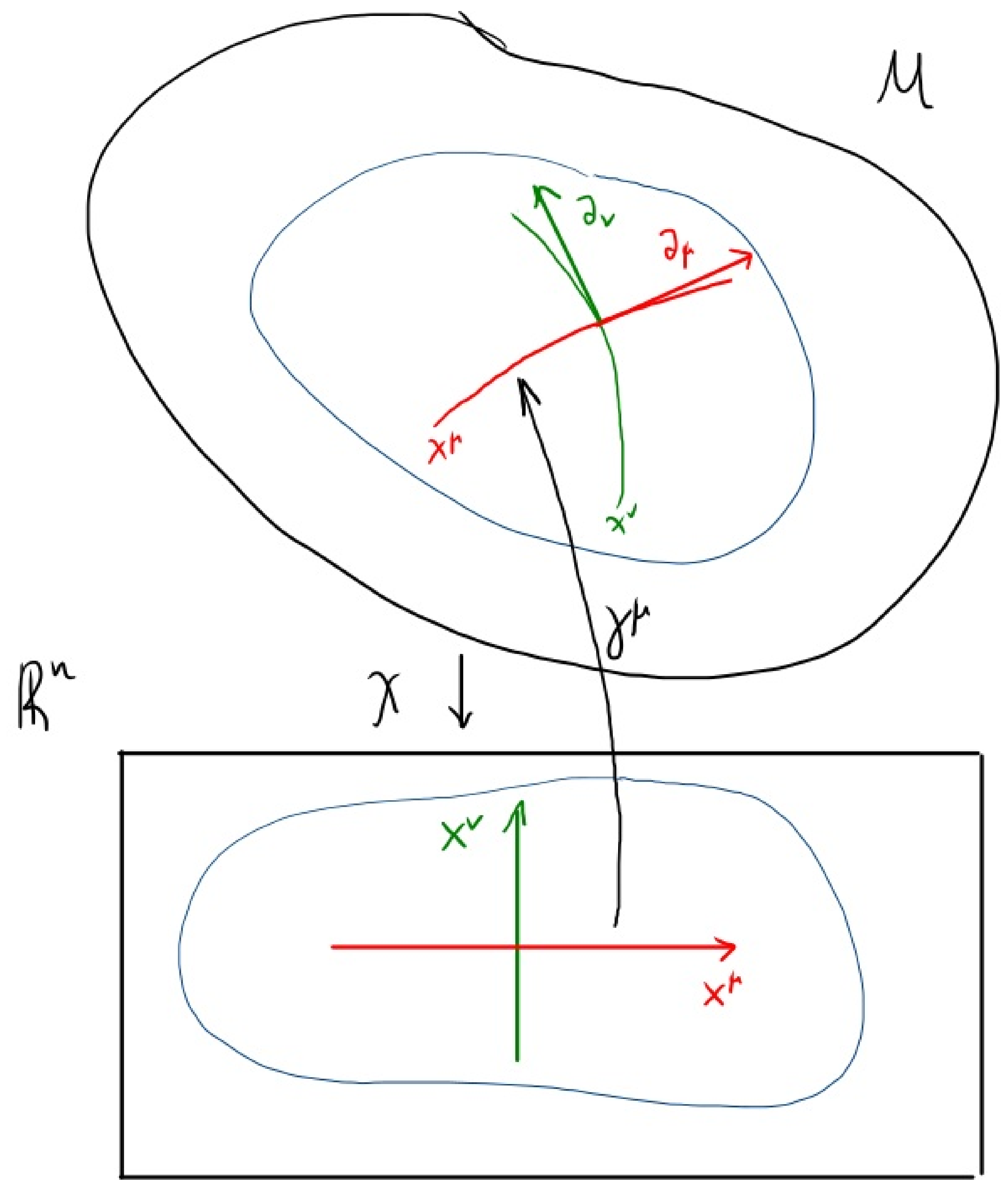
$\displaystyle \partial_\mu\big|_p = \frac{d}{dx^\mu}\Big|_p$ s.t. $\displaystyle \partial_\mu f\big|_p = \frac{\partial f\circ x^{-1}}{\partial x^\mu}\Big|_p$

* Coordinate basis:

We have shown that $\forall f$

$$V_p(f) = \frac{dx^\mu}{dt}\bigg|_0 \frac{\partial f}{\partial x^\mu}\bigg|_p$$

$$\equiv \frac{dx^\mu}{dt} \cdot \partial_\mu f \;\Big|_p$$



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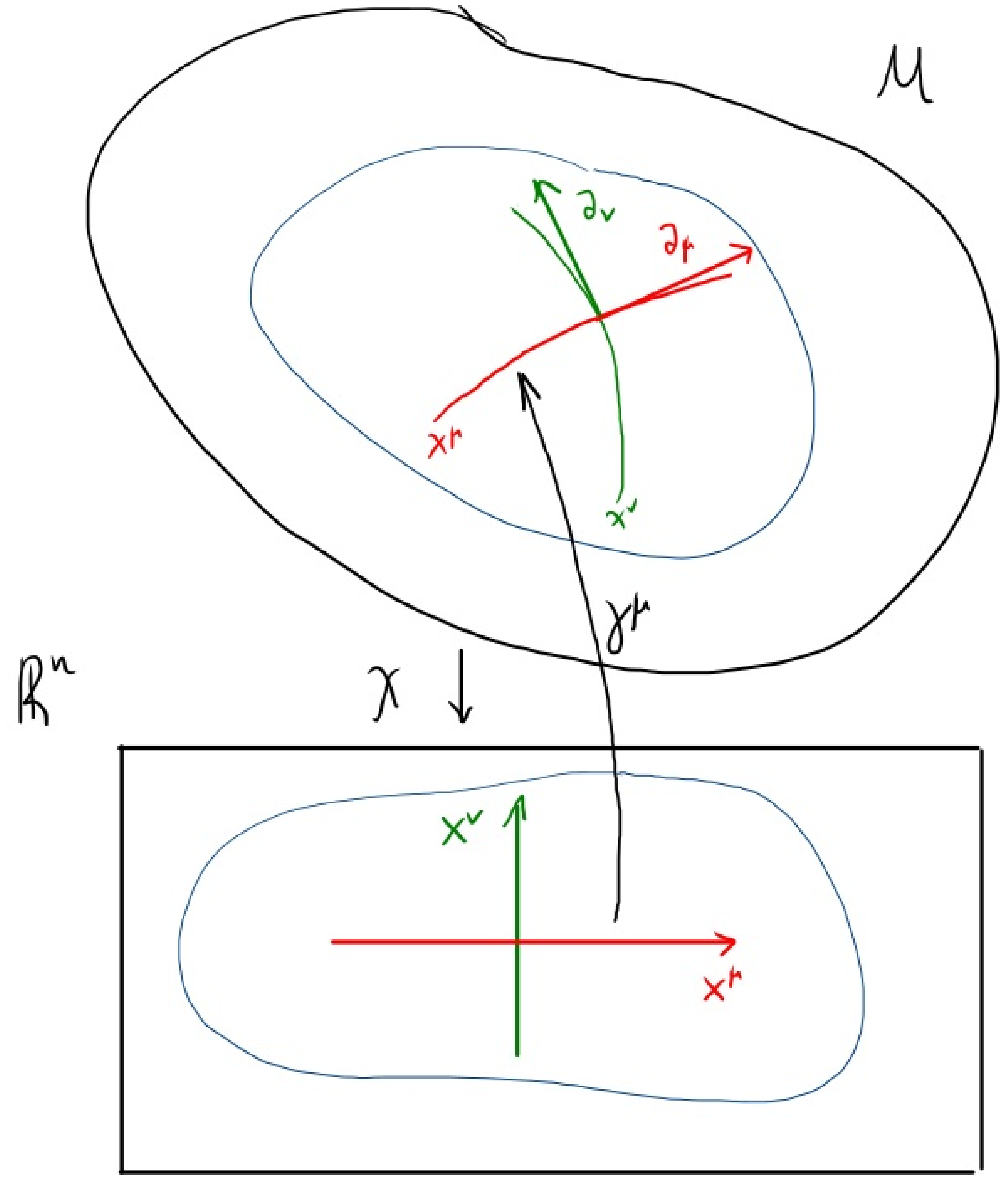
$$\Rightarrow \quad V_p = \frac{dx^\mu}{dt}\, \partial_\mu \Big|_p$$



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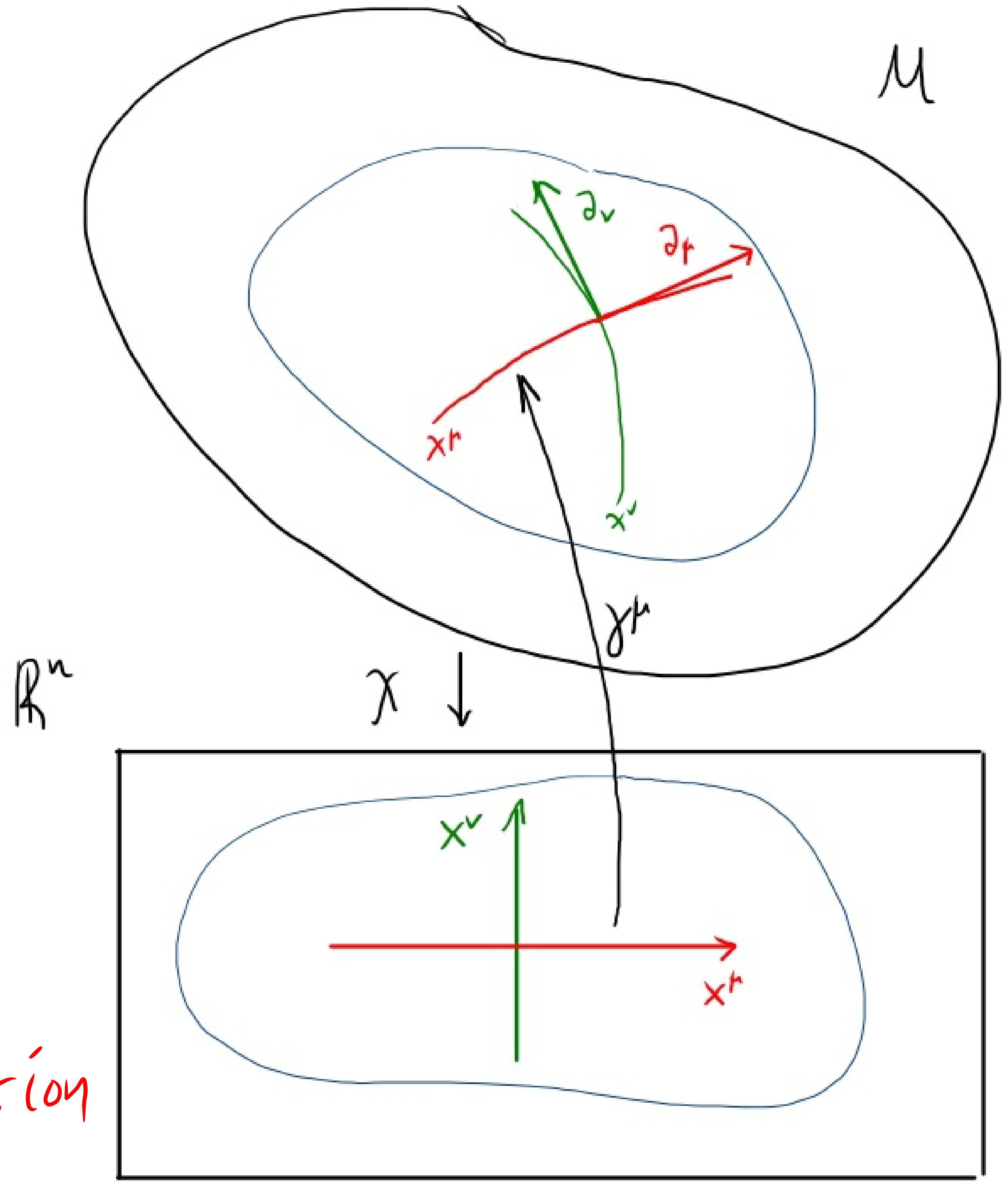
$$\Rightarrow \underbrace{V_p}_{\text{an operator acting on any } f} = \frac{dx^\mu}{dt} \, \partial_\mu \, \Big|_p$$

coordinate vectors $\forall \mu$

a linear combination of all ∂_μ



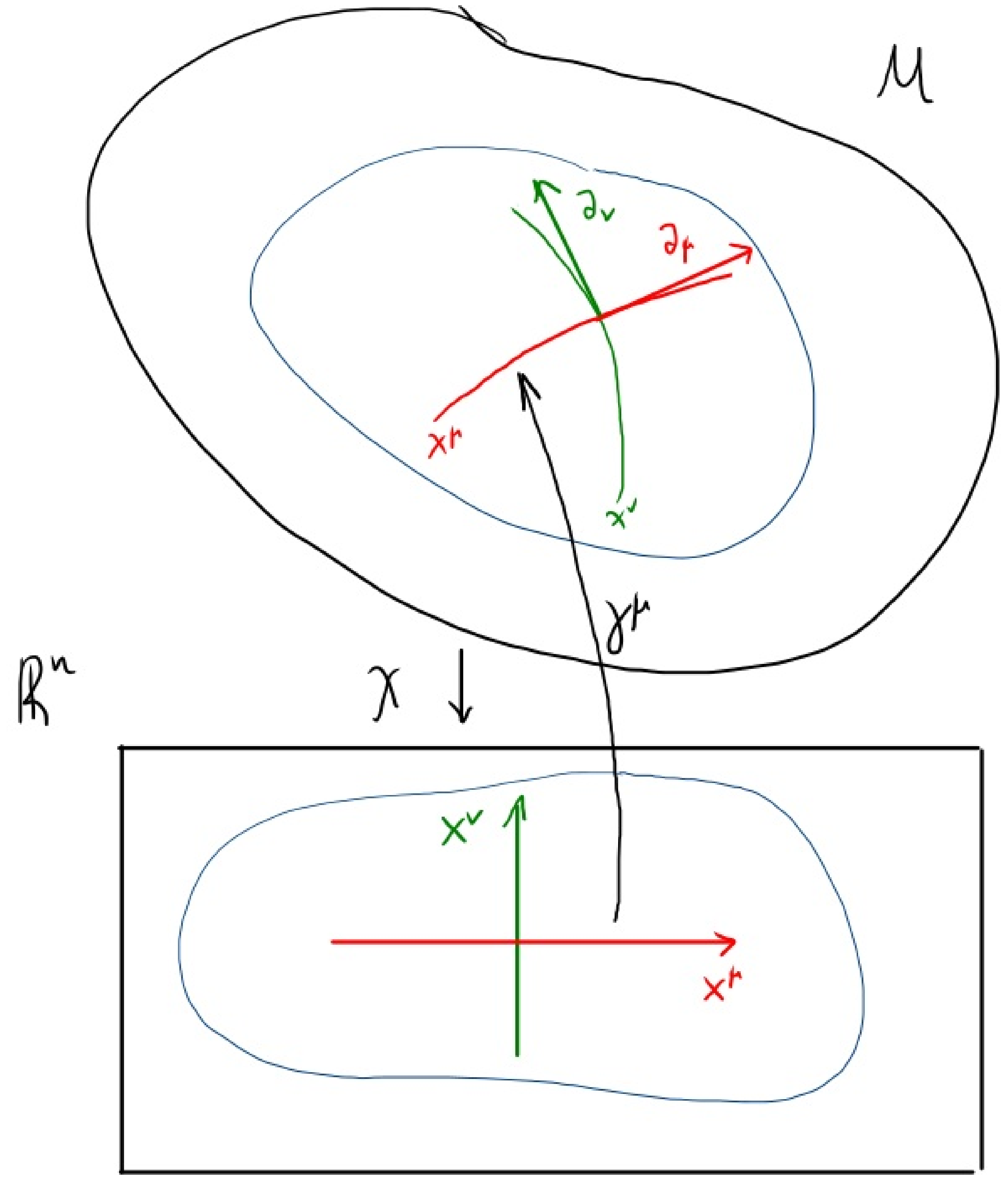
* Coordinate basis:

$$V = \underbrace{\frac{dx^\mu}{dt}} \partial_\mu = V^\mu \partial_\mu$$

$\to$ true $\forall$ p

$\to$ remove p from notation

components of V in the $\{\partial_\mu\}$ basis



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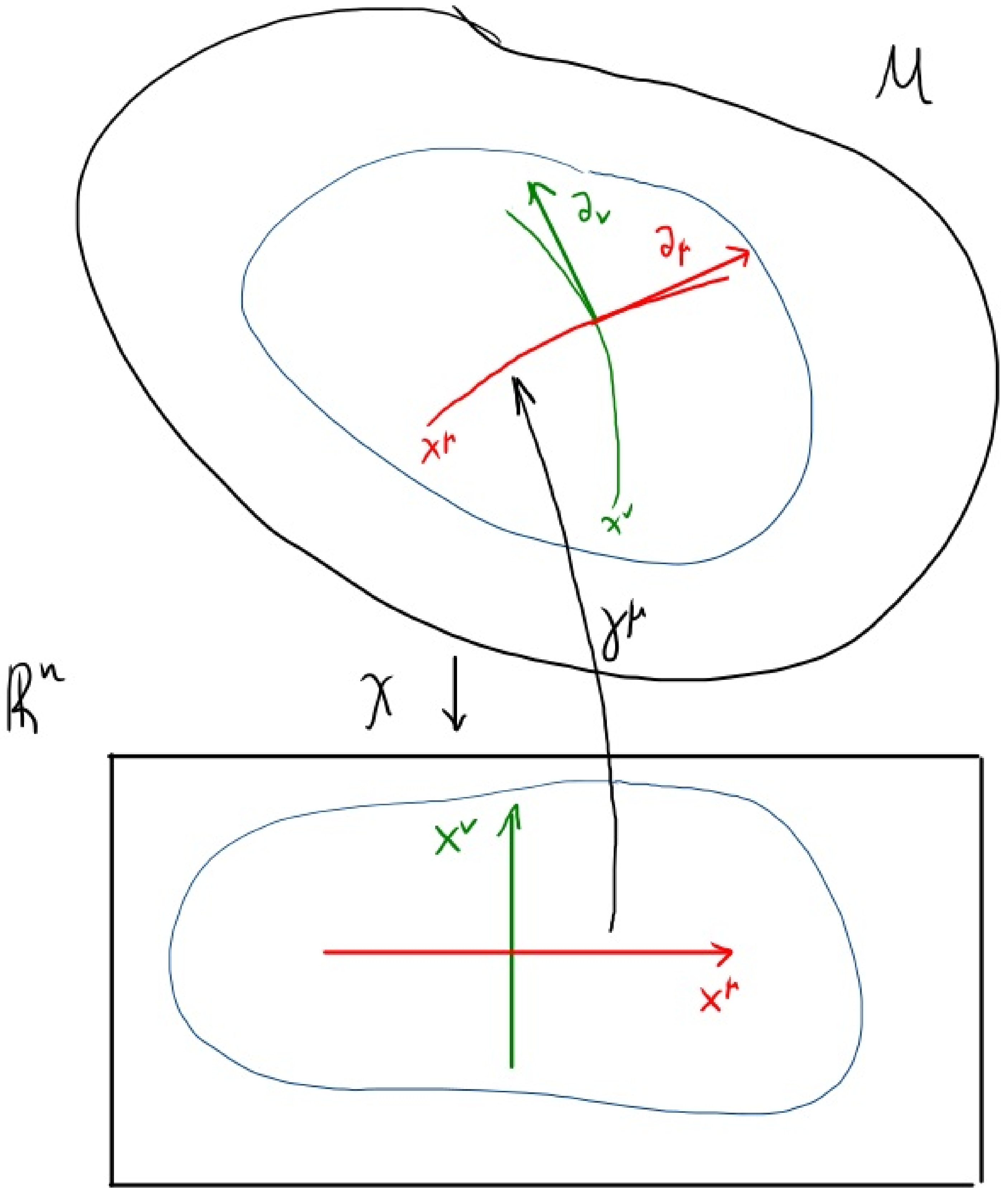
* $\{\partial_\mu\}$ is a basis $\Rightarrow$ $T_p M$ n-dimensional



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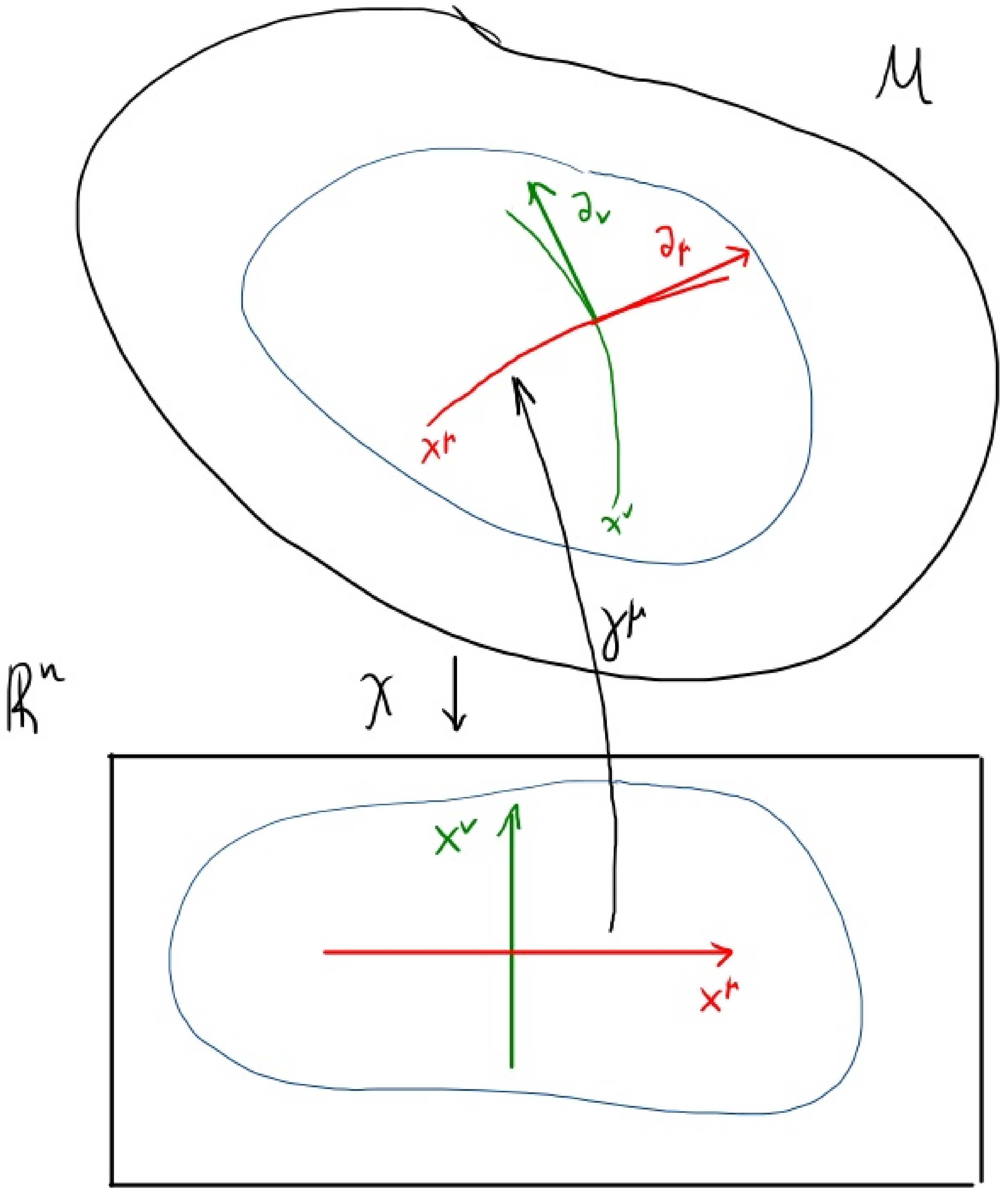

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  $\longrightarrow$ coordinate basis

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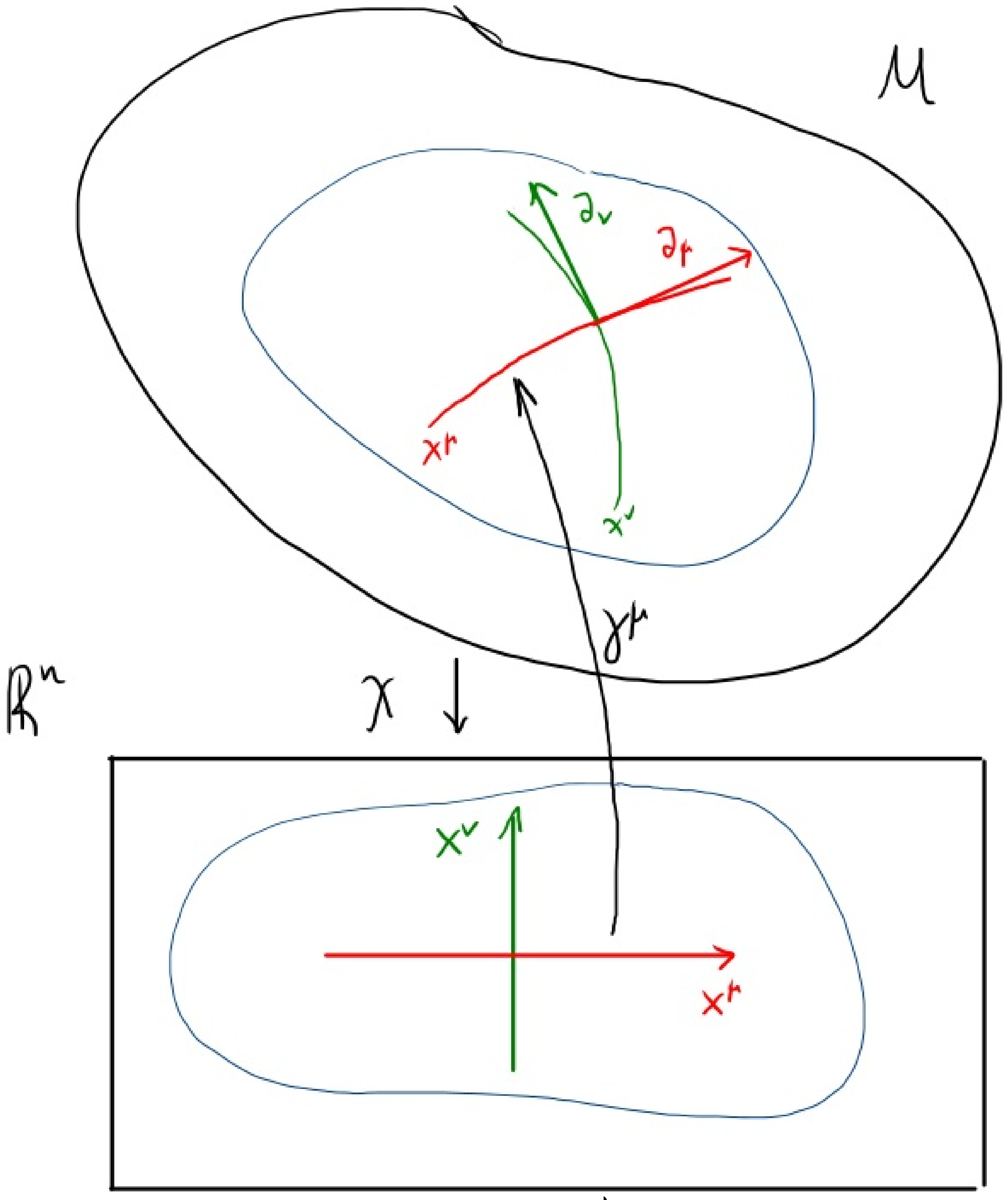

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* not all bases in $T_p M$ are coordinate bases: any $e_\alpha = \Lambda_\alpha{}^\mu \partial_\mu$, rank $\Lambda_\alpha{}^\mu = n$

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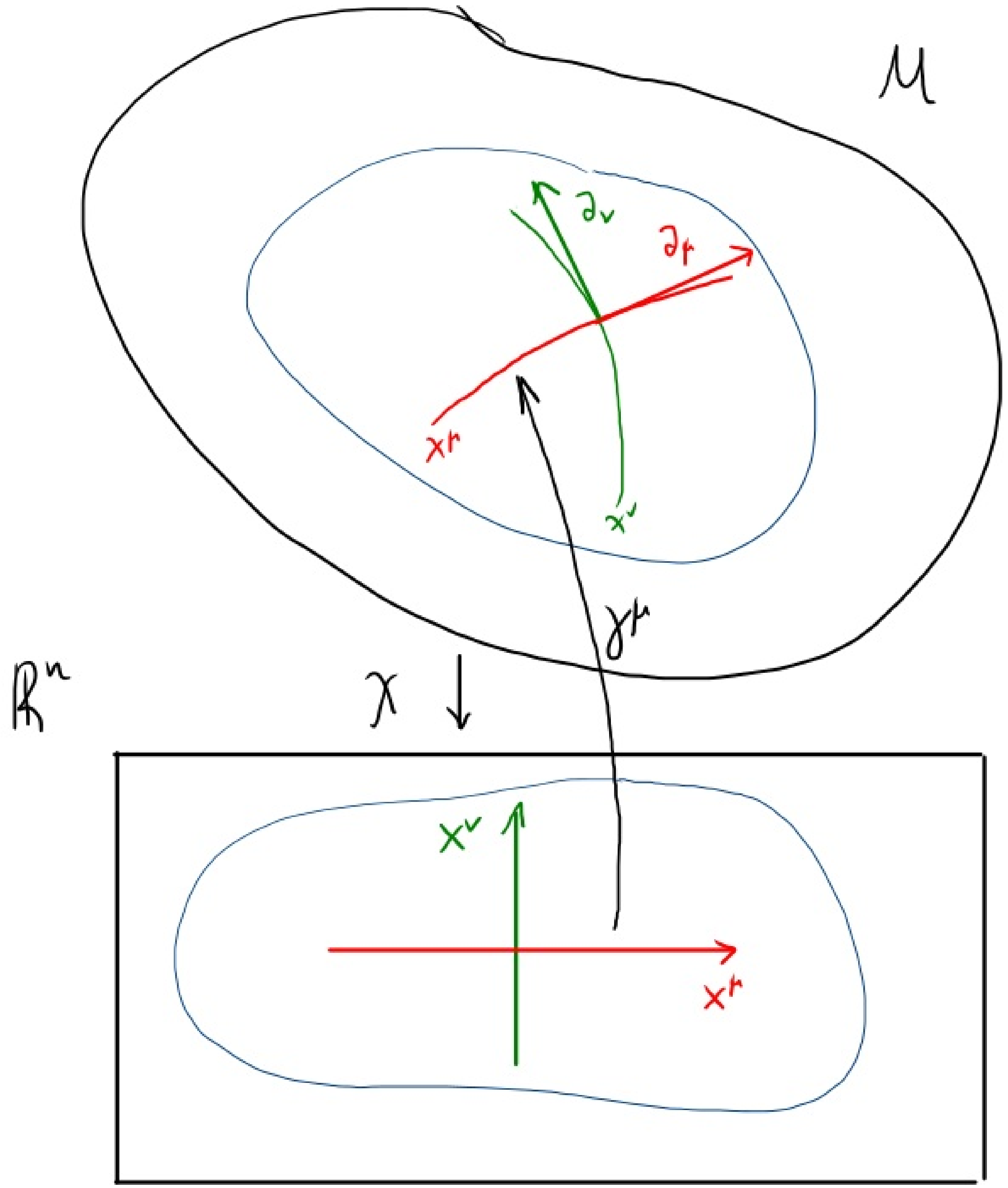

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* not all bases in $T_p M$ are coordinate bases: any $e_\alpha = \Lambda_\alpha{}^\mu \partial_\mu$, rank $\Lambda_\alpha{}^\mu = n$
* If there is a metric $\rightarrow$ inner product in $T_p M$, $\{\partial_\mu\}$ may not be orthonormal

* Change a basis $\Rightarrow$ change of components (of the <u>same</u> vector)

- coordinate basis

  Due to chain rule, for any f : $\partial_{\mu'} f(x^{\nu'}) = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial f(x^{\nu})}{\partial x^{\mu}}$

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Notice placement of indices:

$$\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu}}$$

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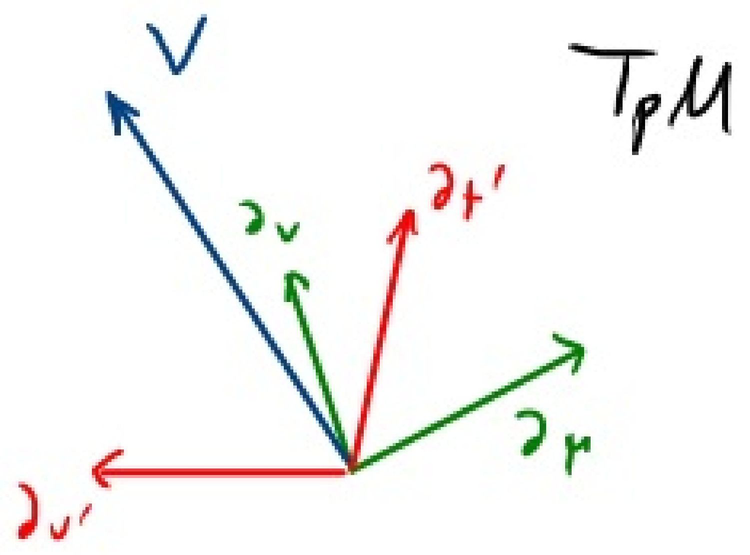

$$\left.\begin{aligned} V &= V^\mu \partial_\mu \\ V &= V^{\mu'} \partial_{\mu'} \end{aligned}\right\} \begin{aligned} &\text{same vector} \\ &\text{different basis} \end{aligned}$$

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$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'}$$

Inverting the above linear system of equations, and using

$$\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}} \right) = \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}} \right)^{-1}$$

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* Change a basis $\Rightarrow$ change of components (of the <u>same</u> vector)
  - coordinate basis

$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'} \quad \Rightarrow \quad \frac{\partial x^{\nu'}}{\partial x^{\mu}} V^{\mu} = \frac{\partial x^{\nu'}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'} = \delta^{\nu'}{}_{\mu'} V^{\mu'} = V^{\nu'}$$

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we obtain: (renamed $\nu' \to \mu'$)

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}$$

* Change a basis $\Rightarrow$ change of components (of the <u>same</u> vector)

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* Change a basis ⇒ change of components (of the <u>same</u> vector)
  - coordinate basis

$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'} \quad \Longleftrightarrow \quad V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}$$

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$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'} \quad \Leftrightarrow \quad V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}$$

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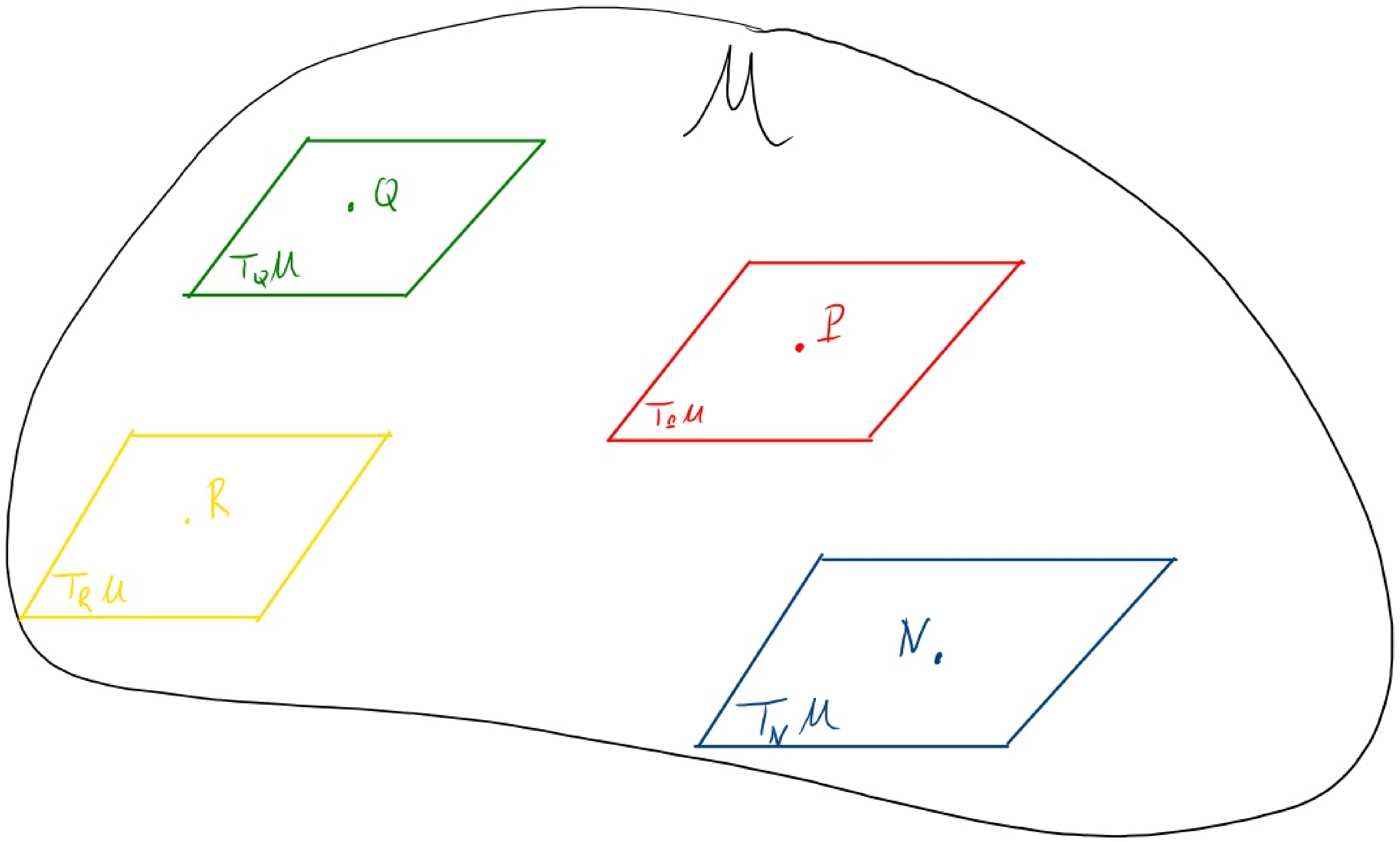
$$\left. \begin{aligned} V &= V^{\alpha} e_\alpha = V^{\alpha} \Lambda_\alpha{}^{\mu} \partial_\mu \\ V &= V^{\mu} \partial_\mu \end{aligned} \right\} \Rightarrow V^{\mu} = V^{\alpha} \Lambda_\alpha{}^{\mu}$$

Vector fields:

* Repeat the construction of vectors foreach $P \in \mathcal{M}$

Then, each point P has a $T_P\mathcal{M}$ "hanging above" it ("tangent bundle")



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$\forall f \in F(M)$, the $V(f) = \frac{df}{dt}$ is a smooth function on M

- if $V(f)$ is smooth $\forall f$, then V is smooth

Vector fields:

- $\{\partial_\mu\}$ are smooth vector fields
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- $\{\partial_\mu\}$ are smooth vector fields
  a coordinate basis of $T_p M$ $\forall\, p$ in the chart

- $V = V^\mu \partial_\mu$ with $V^\mu = V^\mu(p)$ a smooth function on the chart

  $V(x^\mu) = V^\mu$

  $\Rightarrow$ a smooth V has smooth components in a coordinate basis

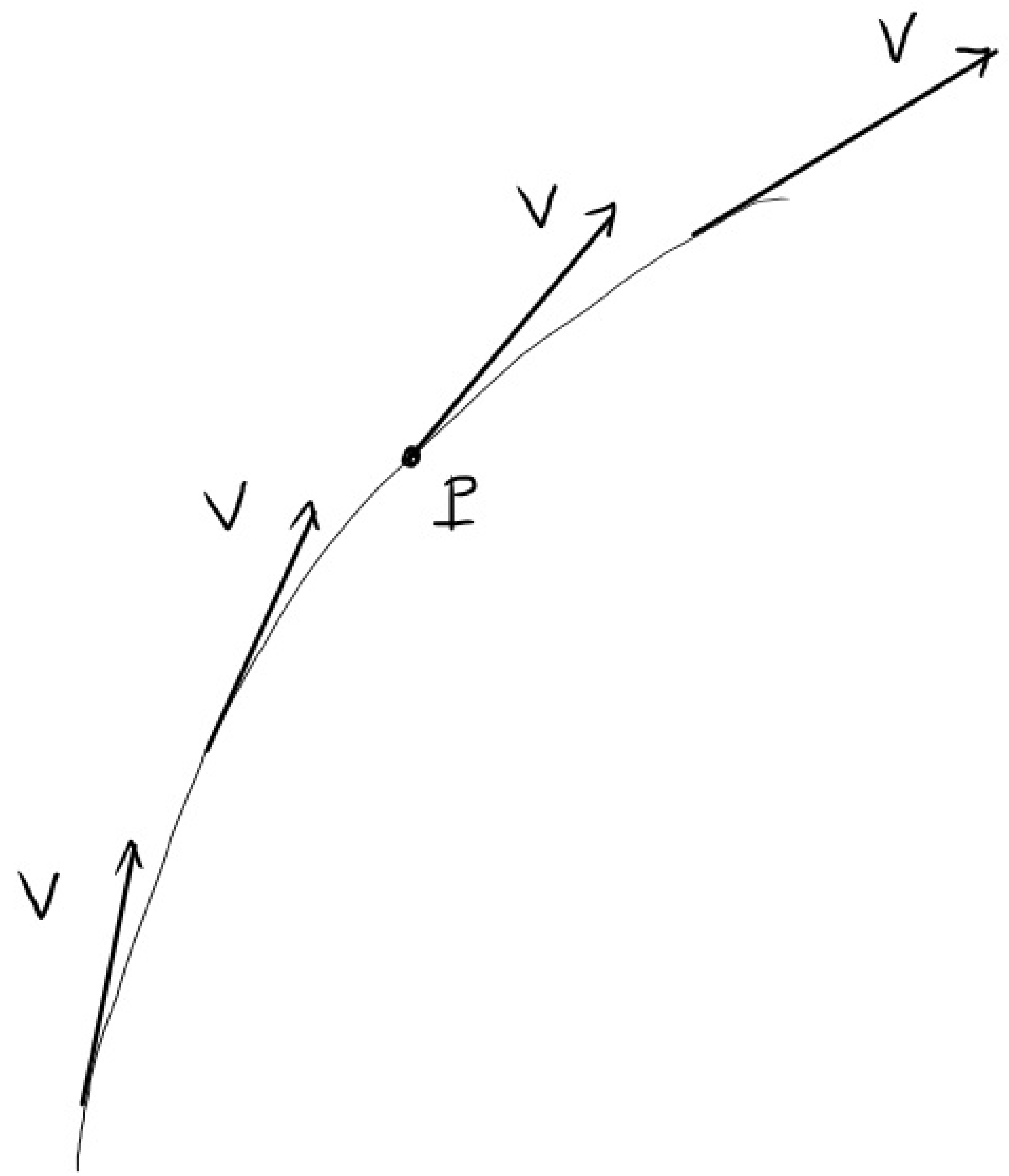
Integral curves:

At each point P in a chart

$$\frac{dx^\mu}{dt} = V^\mu(x^\nu)$$

↳ value of component V^μ at point with coordinates $\{x^\nu\}$

t: parameter of curve, to which V is tangent at all its points



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At each point P in a chart

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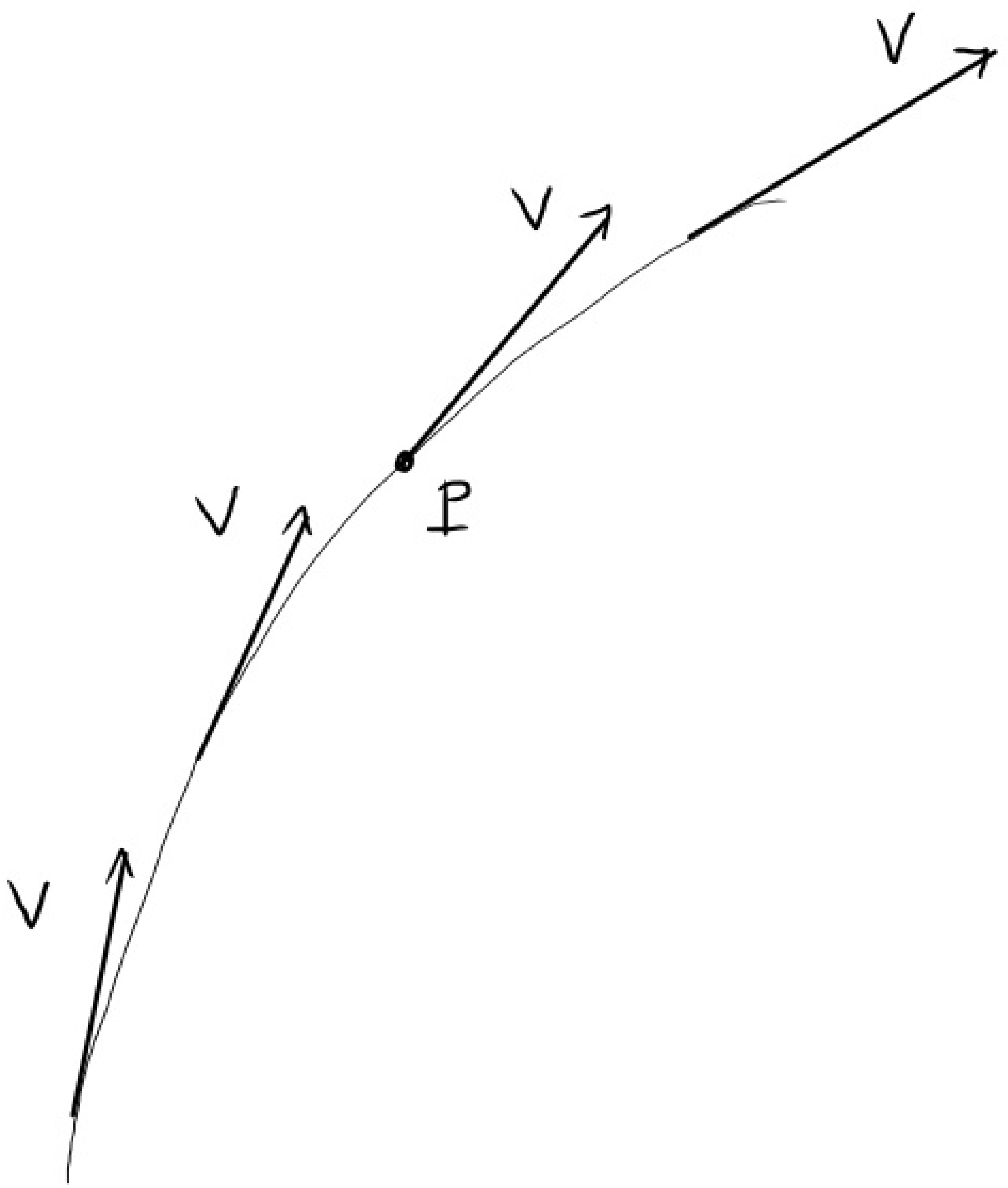
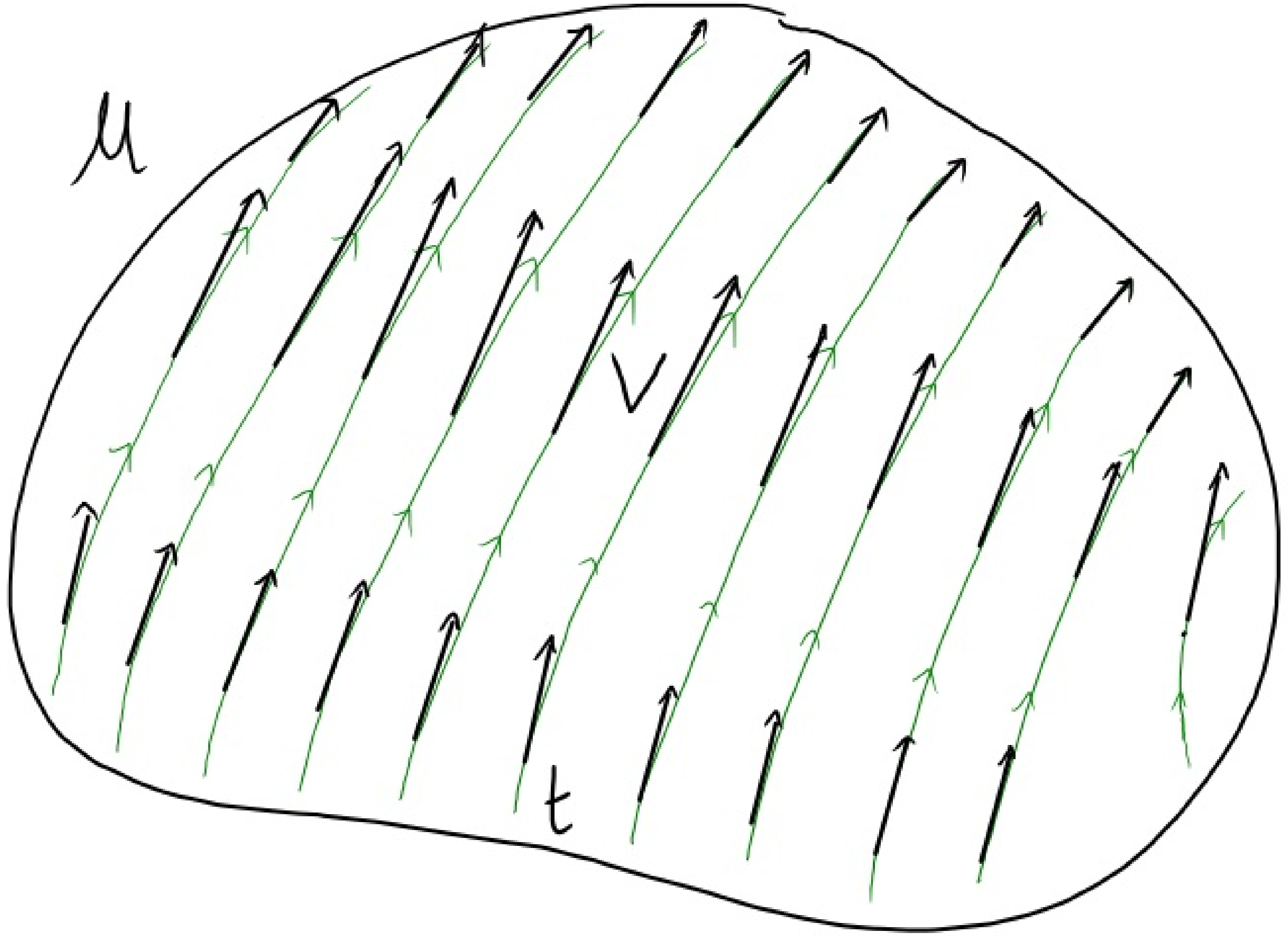


t: parameter of curve, to which V is tangent at all its points

If $x^\mu(0)$ are the coordinates of P, then (1) has unique solution

$\Rightarrow$ $\exists$ unique integral curve of vector field V going through P

* The integral curves of a nonvanishing vector field on $U \subseteq M$, "fill" U



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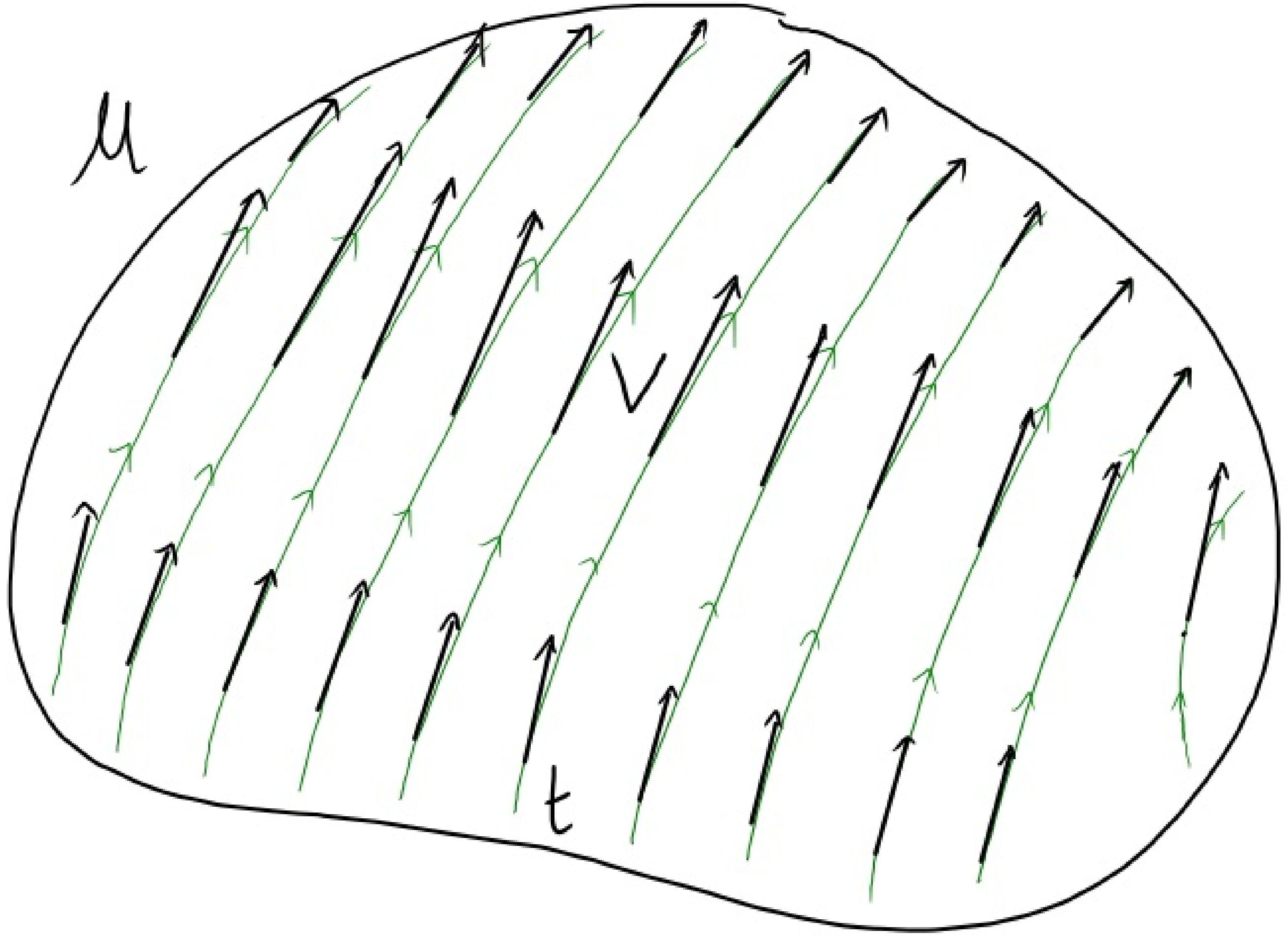

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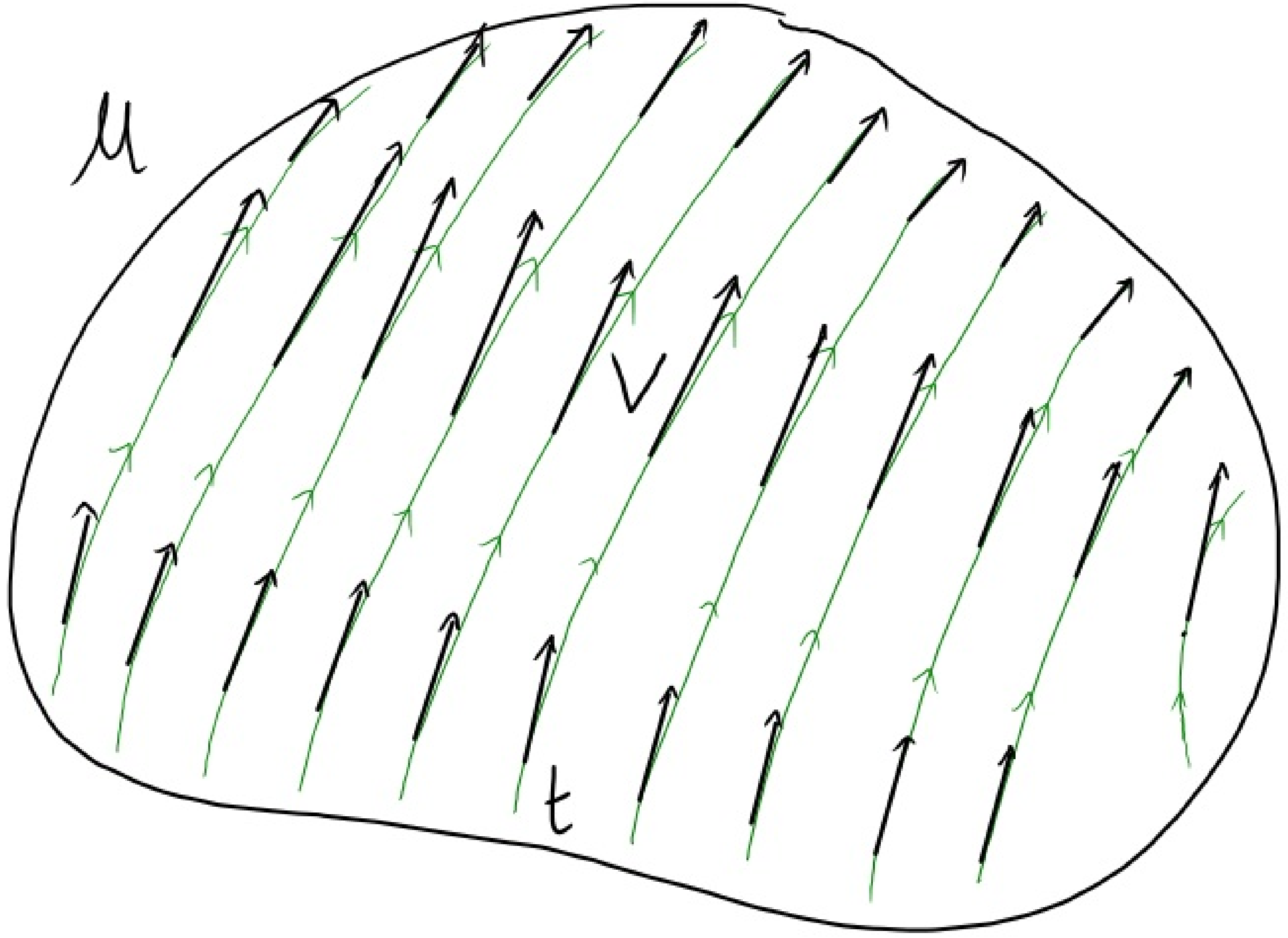

* The integral curves of a nonvanishing vector field on $U \subseteq M$, "fill" U

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$\Rightarrow$ they form a "congruence"