Causality and effective theories

by Simon Caron-Huot, McGill

What is the space of consistent S-matrices?

↳ boundaries

↳ Is QCD at a boundary? under which assumptions?

Hard.

Today: Scale-separated setups EFTs

IR ⟷ UV

↑? $> M$

Plan: - $M_{2\to2}$ in EFTs

- Axioms: $M_{IR} \longleftrightarrow M_{UV}$

- Bounds: $G_N = 0$

($G_N \neq 0$)

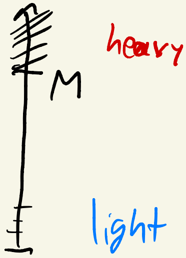
[SCH + V. Duong 2011.02957 ←

SCH + Mazáč + Rastelli + Simmons-Duffin 2008.04931

2102.08951 ←

2106.10274

+ many related!]



Effect of short-distance/heavy modes (known or unknown)

on long-distance/light observables: expand in $1/M^{\#}$

Ex: - Fluids at $\ell \gg \ell_{\text{mean free path}}$

- pions below $M = \Lambda_{QCD}$
- Standard Model below $M = M_{new}$ ¿?
- Gravity below $M = M_s$ or M_{pl}

...

Causality restricts the possible series.

Ex: Real scalar: $\mathcal{L} \supset g_2 \frac{(\partial\phi)^4}{4}$, $g_2 \sim \frac{1}{M^4}$

[Pham+Truong '85]
[Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi '05]

$\Rightarrow g_2 \geq 0$ → i) Manifest via forward dispersion relation

→ ii) Build time-machine if $g_2 < 0$.

i) is easier to systematize.

[Bellazzini, Miró, Rattazzi, Riembau, Riva, Zhou, Wang, Tolley, Trott, de Rham, Melville ......]

we'll learn that:

→ lots of info away from forward limits $(t \neq 0)$

→ It pays to use full crossing symmetry

EFTs: 1. Identify light DOFs and symmetries ["Landau"]

2. Write down most general L

3. Power-counting → Relevant (/marginal)
→ Irrelevant

Ex: real scalar ϕ.

$$L \to \phi\left(\partial^2 + m^2 + \frac{\partial^4}{M^4} + \cdots\right)\phi + g\phi^3 + g'\phi\partial\phi\partial\phi + \cdots + \lambda\phi^4 + \lambda\partial^2\phi^4 + \cdots$$

Today is about 2nd line.

Important: ϕ is not a 'physical obs' : $S[\phi] \simeq S[\phi + \text{field redefinition}]$

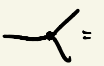
↳ with: $\phi \to \phi - \frac{1}{2}\frac{\partial^2}{M^4}\phi + \cdots$: trivialize kin term to all orders : $\partial^2 + m^2$.

↳ $+\partial^2\phi^2$: trivialize cubic : $g\phi^3$.

All Observable effects can be pushed into $\partial^{\#}\phi^4$ $\begin{cases}\text{mod EOM} \\ \text{mod total der.}\end{cases}$

⇕

polynomial S-matrices.

 $= g$

 : $\lambda\phi^4 \to -\lambda$

~~$\partial^2\phi^4 \to s+t+u = \text{const.}$~~

$\partial^4\phi^4 \to (s^2+t^2+u^2)$]

$\partial^6\phi^4 \to stu$

$$\mathcal{L}_{\text{low}} = \frac{R}{16\pi G} + \frac{1}{2}(D\phi)^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 + \frac{g_2}{2}\left[(D_\mu\phi)^2\right]^2 + \frac{g_3}{3}(D_\mu D_\nu\phi)^2(D_\sigma\phi)^2 + \frac{g_4}{4}\left[(D_\mu D_\nu\phi)^2\right]^2 + \ldots$$

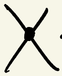
↕

$$\mathcal{M}_{\text{low}}(s,t) = 8\pi G\left[\frac{tu}{s} + \frac{su}{t} + \frac{st}{u}\right] - g^2\left[\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right] - \lambda + g_2(s^2+t^2+u^2) + g_3\, stu + g_4(s^2+t^2+u^2)^2 + \ldots + O(\text{loops})$$

= observables

Each contact int. has a scaling dimension and spin

$\uparrow$ = max angular mom. in any channel

Ex:  $\sim s^2 tu$,

dim 10, spin 3.

Fixed-s: $-s^2 t(s+t) \sim t^2$: spin 2

Fixed-t: $-s^2 t(s+t) \sim s^3$: spin $\underline{3}$ "$\sim s^J$ in Regge limit".

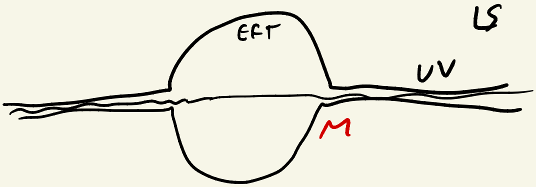
Fixed-u: $-s^2(s+u)u \sim s^3$: spin $\underline{3}$

$$M \sim s^2$$

$$\oint \frac{ds}{s^4} \to 0$$

Spin is important for dispersion relations.

Causality constraints relate UV+IR via analyticity: $\sim \frac{1}{M^{2k}}$

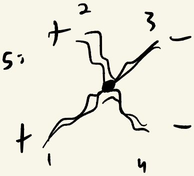


$$\oint_{|s|\sim m^2} \frac{ds}{s} \frac{M_{EFT}(s,t)}{s^k} = \int_{s>m^2} \frac{ds}{s} \frac{\mathrm{Im}\, M_{UV}}{s^k}$$

k-subtracted dispersive sum rule

A contact term of spin J is killed by all >J-subtracted sum rules!

For spinning external particles: contacts = poly in spinors:

gravitons: 

$\sim \underbrace{[12]^4 \langle 34\rangle^4}_{\sim s^4} F(s,t)$ $\leftarrow$ = polynomial at low energies

F = const is a spin 4 interaction. (Riem^4) $\qquad \oint \frac{ds}{s} F \Rightarrow$ spin 4 sum rule

We say that $\oint s\, ds\, F(s,t) = 0$ is a spin 2 sum rule

(note it is "anti-subtracted": superconvergence).

List of SM interactions

$J_{\max}$	dimension	interactions
0	dim. 3	$\cancel{\phi^{(a}\phi^b\phi^{c)}}$
	dim. 4	$\phi^{(a}\phi^b\phi^c\phi^{d)}$ $\sim H^4$
$\frac{1}{2}$	dim. 4	$\psi^{(i}\psi^{j)}\phi$ $\rightarrow$ Yuk.
	dim. 5	$\psi^{(i}\psi^{j)}\phi^{(a}\phi^{b)}$ $\leftarrow$ ν mass
1	dim. 5	$\cancel{F\psi^{[i}\psi^{j]}}$
	dim. 6	$F^{[a}F^bF^{c]}$, $\phi^{[a}D\phi^{b]}\phi^{[c}D\phi^{d]}$, $\psi^i\bar{\psi}_j\phi^{[a}D\phi^{b]}$, $\psi^{(i}\psi^{j)}\psi^{(k}\psi^{l)}$, $\psi^{(i}\psi^{j)}\bar{\psi}_{(k}\bar{\psi}_{l)}$, $F\psi^{[i}\psi^{j]}\phi$, $F^{(a}F^{b)}\phi^{(c}\phi^{d)}$
	dim. 7	$\cancel{F\phi^{[a}D\phi^bD\phi^{c]}}$, $\cancel{F^{[a}F^bF^{c]}\phi}$, $D\psi^{[i}\psi^j\psi^{k]}\bar{\psi}$
higher-points	dim. 6	ϕ^6, $\psi^2\phi^3$
	...	

$(H^\dagger \partial H)^2$

unprobed by disp. rels.

potentially so: disp. rels marginal

Table 1. Interactions which have spin ≤ 1 in all channels and are thus not probed by dispersion relations; ϕ are scalars, ψ Weyl fermions, and F field strengths. Adding any further derivative or graviton coupling pushes these above the $J_{\max} = 1$ threshold. Struck-out interactions $\cancel{\phi\phi\phi}$ are incompatible with SM gauge invariance.

[Unpublished; but see: Durieux, Kitahara, Machado, Shadmi + Weiss '20]

All else is definitely constrained:

$J_{\max}$	dimension	interactions
$\frac{3}{2}$	dim. 7	$\psi\psi\phi\phi D^2$, $F\psi\bar{\psi}\phi D$, $FF\psi\psi$, $FF\bar{\psi}\bar{\psi}$, $\cancel{RF\psi\psi}$
	dim. 8	$\psi\bar{\psi}\phi\phi D^3$, $F\psi\psi\phi D^2$, $F\bar{\psi}\bar{\psi}\phi D^2$, $FF\psi\bar{\psi}D$
2	dim. 8	$\phi\phi\phi\phi D^4$, $\psi\psi\psi\psi D^2$, $F\bar{F}\psi\bar{\psi}D$, $FFFF$, $FF\bar{F}\bar{F}$
	dim. 9	$\psi\psi\phi\phi D^4$, $\psi\psi\psi\bar{\psi}D^3$, $F\phi\phi\phi D^2$, $F\psi\bar{\psi}\phi D^3$, $F\bar{F}\psi\psi D^2$, $FF\psi\psi D^2$, $FFF\phi D^2$, $FF\bar{F}\phi D^2$
	dim. 10	$\phi\phi\phi\phi D^6$, $\psi\psi\psi\psi D^4$, $\psi\psi\bar{\psi}\bar{\psi}D^4$, $FF\phi\phi D^4$, $F\bar{F}\phi\phi D^4$, $FFF\bar{F}D^2$, F^4D^2
2 w/ gravity	dim. ≤ 6	S_{GB}, S_{R^3}, $S'^{(D\geq 7)}_{R^3}$, RFF, $RR\phi\phi$,
	dim. 7	$RFF\phi$, $RRF\phi$, $RRR\phi$,
	dim. 8	$RF\phi\phi D^2$, $R\psi\psi\phi D^2$, $RFFF$, $RRFF$
	dim. 9	$\cancel{R\phi^3D^2}$, $\cancel{RFF\phi D^2}$, $\cancel{RRFFD^2}$

$(\phi R_{iem})^2$

expect $S = R + \text{small}$

gravity:

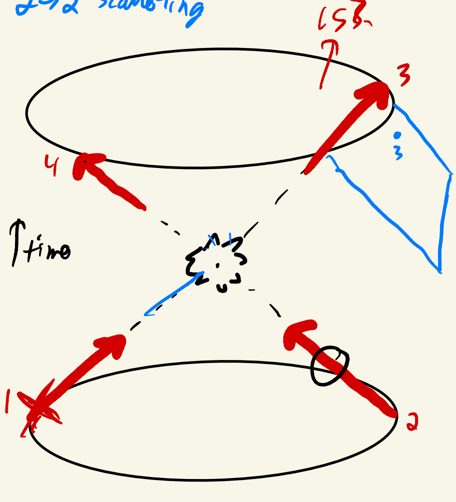
i) All contacts involving a graviton grow with spin ≥ 2!

[Chowdhury, Gadde, Gopalka, Janagal, Minwalla '19]

ii) only 3 modifications to GR (in generic D) have spin 2

[Camanho, Edelstein, Maldacena + Zhiboedov '14]

2→2 scattering



Comments:

i) Fixed angle scattering can show time advances

[Giddings + Porto '09]

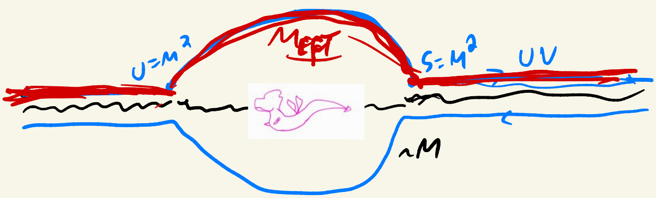
↳ causality controls Regge limit ($s \to \infty$, t or b fixed)

ii) Strongest statements involve crossing:

Particle $1 \to 3$ $\simeq$ antiparticle $3 \to 1$

$[1, 3] = 0$ spacelike.

Axioms



Analyticity (causality)
+ boundedness (unitarity)
= dispersion relation

i) Analyticity of $M(s,t)$ outside $(-M^2 \le t \le 0) \times$ (real axis with $s \geq M^2$ & $u \geq M^2$ + a crossing path from $s=M^2$ to $u=M^2$)

ii) Boundedness.

a) Mom. space $|M(s,t)/s^2| \to 0$, $t<0$. "spin ≥ 2 sum rules converge".

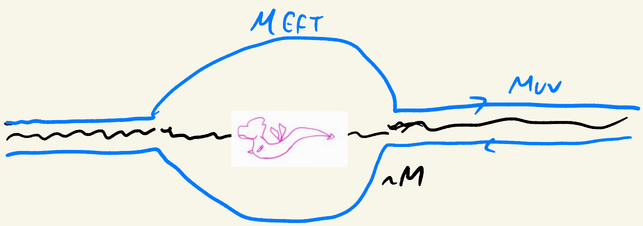
i.e.: - string theory: $M \sim s^{2+\alpha' t}$
- Froissart bound

b) Minimal: smeared $|M_\psi(s)/s| \le \text{const}$ as $|s| \to \infty$ "spin ≥ 1 converge".

where $\psi(p)$ = wavepacket $\begin{cases} \text{compact support in mom.} \\ \text{fast decay in impact param.} \end{cases}$

(stronger yet easier to prove! Rigorously valid in AdS/CFT, [SCH, Mazac, Rastelli, Simmons-Duffin '21])

Dispersive sum rules



("twice-subtracted")

B_2: $0 = \oint_{|s|=\infty} \frac{ds}{s} \frac{\mathcal{M}(s,t)}{s(s+t)}$ $(t<0)$

$$\oint \mathcal{M}_{EFT} = \int \mathrm{Im}\, \mathcal{M}_{UV} \qquad t<0.$$

$$= \sum_J \#_J \int_{M^2}^{\infty} \frac{dm^2}{m^2}\, m^{\#} \underline{\mathrm{Im}\, a_J(s)}\, P_J\left(1+\frac{2t}{m^2}\right)$$

$0 \le \mathrm{Im} \le 2$

? oscillatory.

$\mathcal{H}$ $= \sum$ Legendres with pos. coefficients.

$$\frac{8\pi G}{\underline{-t}} + 2g_2 - g_3 t + 8g_4 t^2 + \ldots = \left\langle \frac{(2m^2+t)\mathcal{P}_J(1+\frac{2t}{m^2})}{m^2(m^2+t)^2} \right\rangle_{m \ge M} \qquad -M^2 \le t \le 0$$

[Pham + Truong '85]
[Adams et al '05]
...

If $G_N = 0$: single out g's by expanding around forward limit.
$\Rightarrow g_2 = \langle \frac{1}{m^4} \rangle > 0$, etc.

Today: keep $G_N \neq 0$

Clearly, $\langle \cdots \rangle \geq 0 \Rightarrow G_N \geq 0$, Gravity is attractive!

How to bound other couplings? Laurent series around forward limit yields divergent sums!

$1+2+3+4+\cdots = \frac{-1}{12}$

Warm-up: $G_N = 0 \Rightarrow$ Near-Forward sum rules

$t=0$

spin 2: $\quad s^2 t^2$

$$\cancel{\frac{8\pi G}{-t}} + 2g_2 - g_3 t + 8g_4 t^2 + \ldots = \left\langle \frac{(2m^2+t)\mathcal{P}_J(1+\frac{2t}{m^2})}{m^2(m^2+t)^2} \right\rangle$$

spin 4: $\sim s^4$

$$4g_4 + \ldots \qquad = \left\langle \frac{(2m^2+t)\,\mathcal{P}_J\left(1+\frac{2t}{m^2}\right)}{m^4\,(m^2+t)^3} \right\rangle$$

...

Just expand around $t=0$:

$$g_2 = \left\langle \frac{1}{m^4} \right\rangle_{m \geq M}, \qquad \overset{stu}{g_3} = \left\langle \frac{3 - \frac{4}{d-2}\mathcal{J}^2}{m^6} \right\rangle_{m \geq M}, \qquad 0 = \left\langle \frac{\mathcal{J}^2(2\mathcal{J}^2 - 5d + 4)}{m^8} \right\rangle$$

clearly (with $G=0$): $g_2 > 0$, $g_3 \leq \frac{3 g_2}{M^2}$. $\quad \left(b = \frac{2J}{m}\right)$

lower-bound? $\quad g_3 = \left\langle \frac{3}{m^6} \right\rangle - \# \left\langle \frac{b^2}{m^4} \right\rangle \leftarrow$ expected because stu is a spin 2 interaction.

Need to bound impact parameter of intermediate heavy states!

Key: IR-crossing: high-spin states can't couple too strongly!

Null constraints: $$0 = \left\langle \frac{\mathcal{J}^2(2\mathcal{J}^2 - 5d + 4)}{m^8} \right\rangle_{UV}$$

[Tolley, Wang + Zhou '20]
[SCH + van Rees '20]
[Huang^2 + Arkani-Hamed '20]

IR crossing symmetries constrains light-light-heavy couplings.

$$\left\langle \frac{1}{m^4} \frac{\overbrace{\mathcal{J}^2}^{\sim b^2}}{m^2} \right\rangle \leq \frac{\#}{\underline{M^2}} \left\langle \frac{1}{m^4} \right\rangle$$

As far as sum rules care, all heavy states have size $b \lesssim \frac{1}{M}$.

(ie. black holes, long strings, etc. can't couple strongly enough to be significant)

Result: 2-sided bounds on all coefficients (divided by first)!

EFT coefficient	Lower bound	Upper bound
$\tilde{g}_3$	-10.346	3
$\tilde{g}_4$	0	0.5
$\tilde{g}_5$	-4.096	2.5
$\tilde{g}_6$	0	0.25
$\tilde{g}'_6$	-12.83	3
$\tilde{g}_7$	-1.548	1.75
$\tilde{g}_8$	0	0.125
$\tilde{g}'_8$	-10.03	4
$\tilde{g}_9$	-0.524	1.125
$\tilde{g}'_9$	-13.60	3
$\tilde{g}_{10}$	0	0.0625
$\tilde{g}'_{10}$	-6.32	3.75

$\frac{g_k M^\#}{g_2}$

Compatible with geometric series!

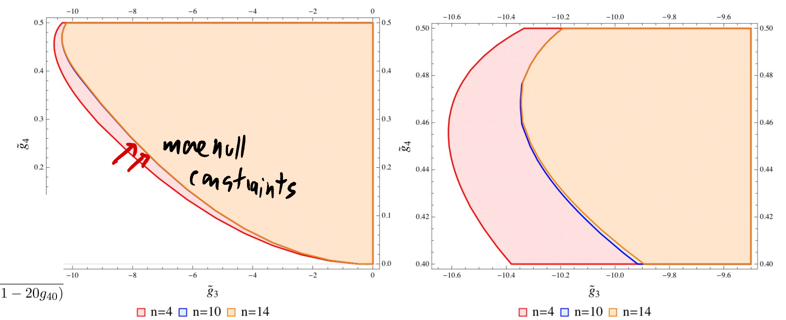
$$\frac{1}{M^2 - s} \to \frac{1}{M^2} + \frac{s}{M^4} + \cdots$$

More on non-gravitational scalar EFT

The k=5 EFT-hedron gives the outer curve

$$\text{Region I:}\quad g_{31}^{\min} = -\frac{3}{2}\sqrt{g_{40}},$$
$$\text{Region II:}\quad g_{31}^{\max} = \frac{1}{2}\sqrt{\frac{427}{3}g_{40}},$$
$$\text{Region III:}\quad g_{31}^{\max} = \frac{30}{7}g_{40} + \frac{37}{42}\sqrt{g_{40}\left(21-20g_{40}\right)}$$

(see Huang's talk)

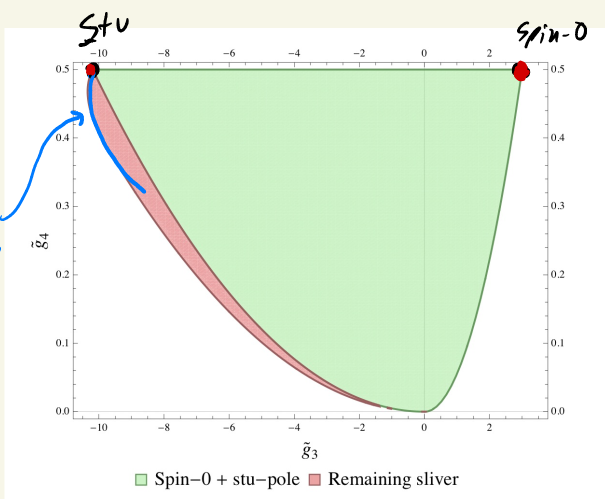


(a) Convergence is rapidly achieved by adding null constraints. (b) Close-up near the left kink at $(-10.19, 0.5)$.

Convergence with adding null constraints ($\simeq$ larger Hankel matrices) is fast in this example.

The kinks are simple S-matrices:

$$\mathcal{M}_{\text{spin-0}} = \frac{1}{m^2-s} + \frac{1}{m^2-t} + \frac{1}{m^2-u},$$
$$\mathcal{M}_{stu\text{-pole}} = \frac{m^4}{(m^2-s)(m^2-t)(m^2-u)} - \gamma(d)\mathcal{M}_{\text{spin-0}}.$$



What do we bound?

Ex: EFT of a single real scalar, below cutoff scale M

$$\mathcal{L}_{\rm low} = \frac{R}{16\pi G} + \frac{1}{2}(D\phi)^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 + \frac{g_2}{2}\left[(D_\mu\phi)^2\right]^2 + \frac{g_3}{3}(D_\mu D_\nu\phi)^2(D_\sigma\phi)^2 + \frac{g_4}{4}\left[(D_\mu D_\nu\phi)^2\right]^2 + \dots$$

$\leftarrow$ higher derivatives

Problem: Lagrangian isn't a physical observable! Rather, we focus on $\mathcal{M}$

$\mathcal{M}$

$$\mathcal{M}_{\rm low}(s,t) = 8\pi G\left[\frac{tu}{s} + \frac{su}{t} + \frac{st}{u}\right] - g^2\left[\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right] - \lambda + g_2(s^2+t^2+u^2) + g_3\, stu + g_4(s^2+t^2+u^2)^2 + \dots + O(\text{loops})$$

Naively: EFT parameters $\Leftrightarrow$ Taylor coefficients around $s,t \to 0$.

Problems: - Taylor series ill-defined since loops are non-analytic

- Contradicts EFT spirit: parameters should be matched through experiments at the scale $u \sim M$, NOT $u = 0$.

$\hookrightarrow$ Our approach: bound observables that are:

i) Linear in $2\to2$ S-matrix

ii) Dominated by $|s|, |t| \sim M^2$

iii) Reduce to g_k if S-matrix $\to$ tree-level EFT

These observables will be bounded nonperturbatively, but harder to interpret if EFT is strongly interacting

(physics above M could be strongly coupled)

$$\frac{8\pi G}{-t} + 2g_2 - g_3 t + 8g_4 t^2 + \ldots = \left\langle \frac{(2m^2+t)\mathcal{P}_J(1+\frac{2t}{m^2})}{m^2(m^2+t)^2} \right\rangle$$

Method to bound other terms:

i) Use higher-subtracted sum rules to eliminate all couplings with $s^{\geq 4}$ Regge growth.

Ex: $$B_4: \quad 4g_4 + \ldots = \left\langle \frac{(2m^2+t)\,\mathcal{P}_J\left(1+\frac{2t}{m^2}\right)}{m^4\,(m^2+t)^3} \right\rangle$$

"Null constraints": IR crossing relates the coefficients of s^2t^2 and s^4.

$\Rightarrow$ Improved sum rules

$$\underbrace{\frac{8\pi G}{-t} + 2g_2 - g_3 t}_{\text{Finite sum!!}} = \left\langle C^{\text{improved}}_{2,t}[m^2, J] \right\rangle \qquad -m^2 < t < 0$$

[SCH, Mazac, Rastelli + Simmons-Duffin '21]

ii) construct wavefunctions $\psi(p)$ with positive heavy action:

IF: $\int_0^M \psi(p)\, C^{\text{improved}}_{2,-p^2}[m, J] \geq 0 \quad \forall\, m \geq M,\ J$

$\Downarrow$

Then, $\int_0^M \psi(p) \left(\frac{8\pi G}{-p^2} + 2g_2 + g_3 p^2 \right) \geq 0$. bound on light interactions

Linear programming: optimize $\psi(p)$ to get optimal constraint on $g_2, g_3,$ etc.

Such $\psi(p)$ exist!

Among other properties:

- compact support in p
- positive in b

$\left(\text{ex: } \int_0^1 (1-p)\,dp\cos(pb) = \frac{1-\cos(b)}{b^2} \geq 0\right)$

We do not know a general basis for generic d.

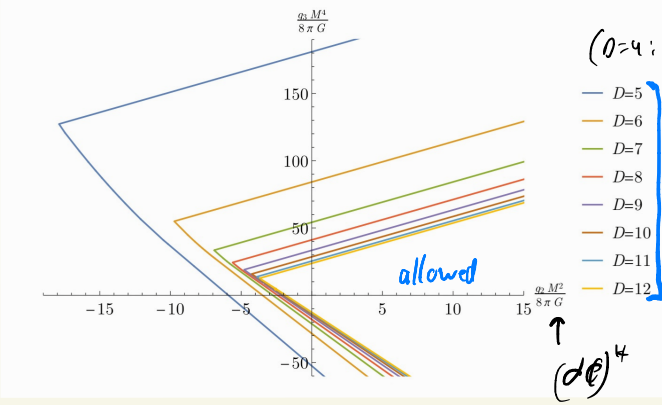
In practice, we make a polynomial ansatz and use linear programming.

For the scalar EFT, we find an allowed core:

$$-8.15\frac{g_2}{M^2} - 28.8\frac{8\pi G}{M^4} \leq g_3 \leq 3\frac{g_2}{M^2} + 93.0\frac{8\pi G}{M^4} \qquad (D=6)\,,$$

with $g_2 \geq -\#\frac{8\pi G}{M^2}$

stu $(\partial^6\phi^4)$



(D=4: logs)

$(\partial\phi)^4$

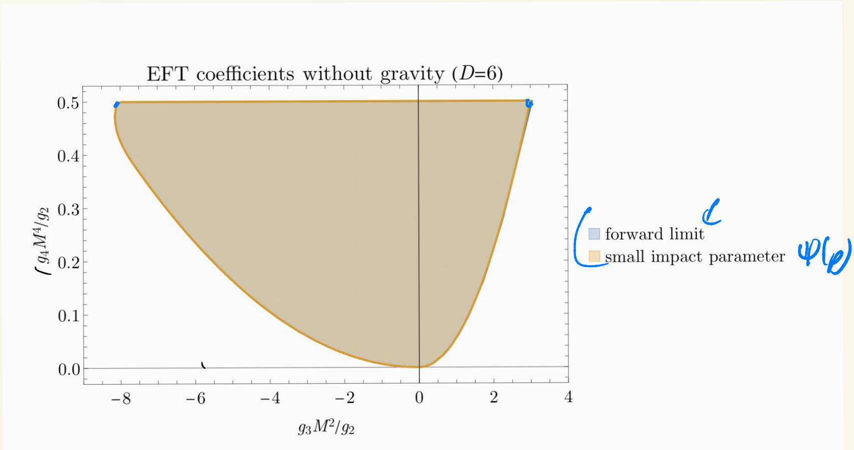
$g_2 \geq -\frac{-G_N}{M^2}\log\left(\frac{M^2}{m_{IR}^2}\right)$

negative g_2 allowed by gravity.

resolves puzzle in QED+gravity?

[deRham + spring]

The $G \to 0$ limit of the "wavepacket" bounds give the same:



wavepacket sum rules → Seamlessly deal with graviton pole (perhaps also loops?)

→ Much easier to justify physically $\left(\left|\frac{M_F(s)}{s}\right| < C \text{ vs } \frac{M(s,t)}{s^2} \to 0 \right)$

What to expect for graviton scattering?

preliminary: $\overset{10D}{\vee}$max SUSY: $\mathcal{M}_{low} = \delta^{16}(Q)\cdot\left(\frac{8\pi G_N}{stu} + \overset{R^4}{g_0} + \dots\right)$

"anti-subtracted" sum-rules exist: $|\mathcal{M}_4 s^3| \le$ const. $\Rightarrow$ leading sum rules measure G_N only !!

i) g_0 is subleading in Regge limit $\Rightarrow$ expect upper bound on $\frac{g_0}{G_N}$.

ii) UV spectral density can't vanish $\Rightarrow$ lower bound (use $0 \le \mathrm{Im}\, a_J \le 2$)

Indeed: $0 < 0.14\,\frac{8\pi G}{M_{pl}^6} \le g_0 \le 3{,}000\,\frac{8\pi G}{M^6}$

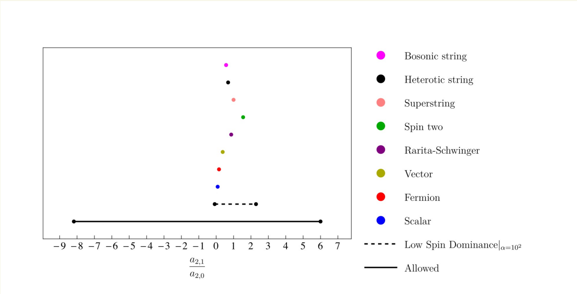
[Guerrieri, Penedones + Vieira '21]

[SCH, Mazac, Rastelli + Simmons-Duffin '21]

The lower bound seems saturated in IIB moduli space!!!...

The upper bound is easily satisfied by Veneziano-Shapiro ($2\zeta_3 \approx 2.4 < 3{,}000$)

$\Rightarrow$ Is it saturated in some theory?



[Bern, Kosmopoulos, Zhiboedov '21]

Are we missing some constraints?

Conclusion

$$\boxed{\text{Causality} \implies \text{two-sided bounds on generic EFT coefficients}}$$

"causal EFT" is a pleonasm! (= "causal whatever")

↳ Limits on causal modifications of GR? (ongoing)

↳ Lots to explore: $0 \le I_n \le 2$? Kinks? dim 6? loops?

↳ Is AdS more constraining than flat space?

Causality certainly holds more surprises...