# Anomalies as obstructions:
# from dimensional lifts to swampland

with Peng Cheng and Stefan Theisen

D=6 Super-Poincaré Representations with 8 supercharges:

| B.L.G. | representation | multiplet |
|---|---|---|
| | $(2,3;1)\times 2^2 = (3,3;1)+(1,3;1)+(2,3;1)$ | gravity |
| $SO(4)\times SU(2)$ | $(2,1;1)\times 2^2 = (3,1;1)+(1,1;1)+(2,1;1)$ | tensor |
| | $(1,2;1)\times 2^2 = (2,2:1)+(1,2;1)$ | Yang-Mills |
| | $2^2 = (2,1;1)+(1,1;2)$ | hyper |

Chiral bosonis and fermionic fields $\Rightarrow$ Anomalies

Anomaly cancelation possible if

- $I_8 = \frac{1}{2}\Omega_{\alpha\beta}\, X_4^\alpha X_4^\beta$
  - $\alpha,\beta = 0,1,...n_T$
  - $\Omega_{\alpha\beta}$ - symmetric inner product on the space of tensors with $(1,n_T)$ signature
- GSS couplings $\sim \Omega_{\alpha\beta}B_2^\alpha X_4^\beta$
- Anomalous BI $dH^\alpha = X_4^\alpha$

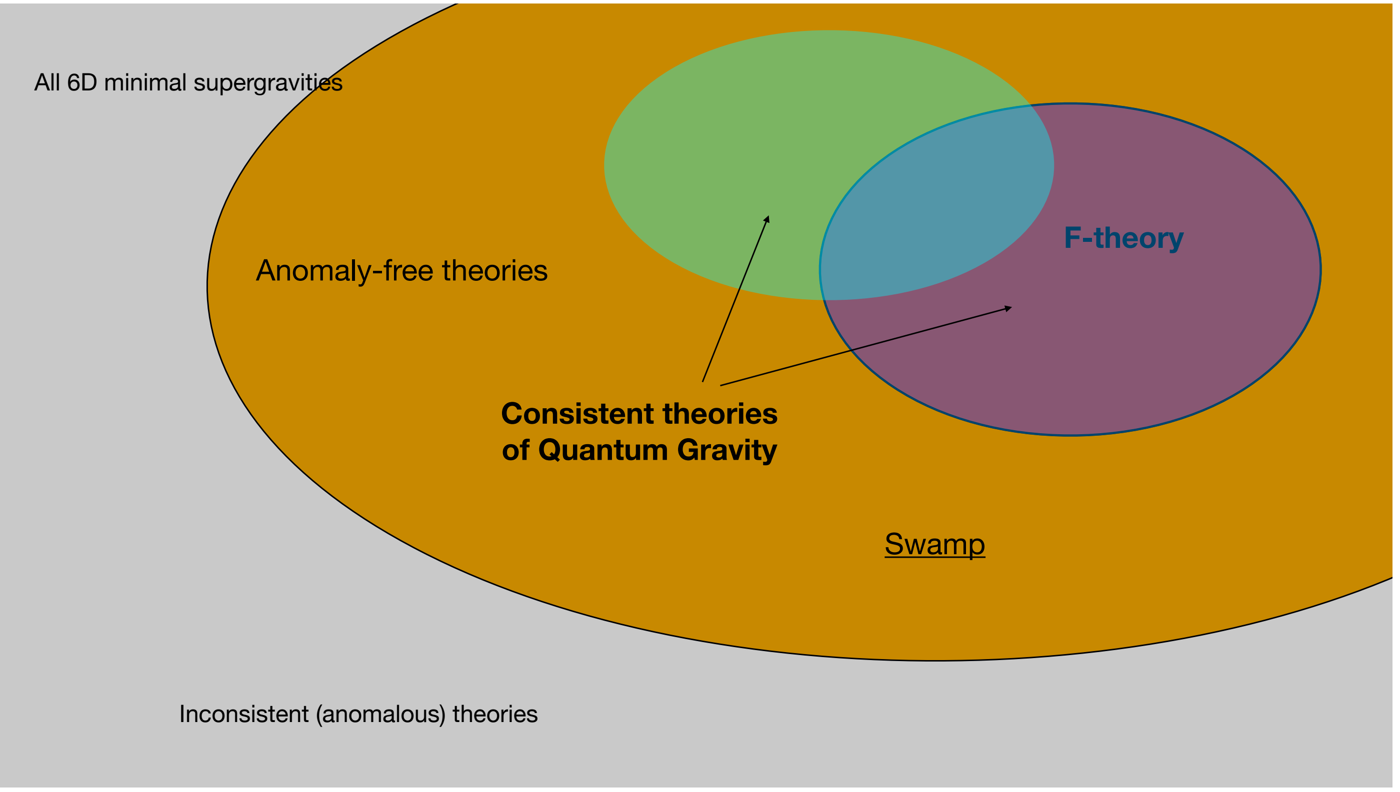
## Anomaly-free theories:

- Heterotic strings on K3
  - perturbative: $n_T = 1 \ (c_2 = 24)$
  - non-perturbative: $n_T > 1 \ (c_2 + N_{\text{NS5}} = 24)$
- Perturbative IIB constructions (K3 orientifolds)
- F-theory
  - ▷ Geometrisation of the necessary conditions for the anomaly cancellation
  - ▷ Kodaira condition for elliptic CY3's $\Rightarrow$ bound of physical couplings
- Anomaly-free supergravity models ( e.g. $n_T = 9 + 8k$ and $G = (E_8)^k$ )

## Questions:

- Extra consistency conditions?
  - ⋆ YES - unitarity of the worldsheet theory of the"supergraity strings" - according to H-C.Kim, G. Shiu and C. Vafa
- A (geometric) bound that all consistent theories should satisfy?
  - ⋆ The subject of this talk

A cartoon of the situation that can be imagined




The plan

- Review the unitarity argument in D=6 and ...
  - ⋆ Explain why we are we look answers to D=6 questions in D=5
- ... re-examine the unitary from D=5 point of view
- Establish a (geometric) bound for consistent theories

## Supergravity strings in D=6

Consider an anomaly free D=6 theory with 8 spuercharges with

- $n_T$ tensor multiplets
- Yang-Mills multiplets with a group $G = \prod_i G_i$
- hypermultiplets in different representations of the gauge group.

The anomaly polynomial:

- ⋆ $I_8 = \frac{1}{2}\Omega_{\alpha\beta}\, X_4^{\alpha} X_4^{\beta}$
  - ▷ $\alpha, \beta = 0, 1, ...n_T$
  - ▷ $\Omega_{\alpha\beta}$ - symmetric inner product on the space of tensors with $(1, n_T)$ signature
- ⋆ $X_4^{\alpha} = \frac{1}{8} a^{\alpha} \mathrm{tr} R^2 + \sum_i b_i^{\alpha} \frac{1}{4h_i^{\vee}} \mathrm{Tr}_{\mathrm{Adj}} F_i^2$
  - ▷ $a, b_i \in \mathbb{R}^{1,n_T}$ – determined by the field content of the theory

Dyonic BPS strings with (0,4) worldsheet supersymmetry:

- ⋆ $dH^{\alpha} = X_4^{\alpha} + Q^{\alpha} \prod_{a=1}^{4} \delta\left(x^a\right) dx^a$
  - ▷ $Q^{\alpha}$ – string charges

Anomaly inflow from $\Omega_{\alpha\beta} B_2^\alpha X_4^\beta$ to the BPS string $\quad\Rightarrow\quad$ (0,4) anomaly:

$$I_4 = -\Omega_{\alpha\beta}\, Q^\alpha \left( X_4^\beta (M_6)|_{W_2} + \frac{1}{2} Q^\beta \chi\,(N) \right)$$
$$= -\frac{1}{4}\,\Omega_{\alpha\beta} Q^\alpha \left(a^\beta p_1(TW_2) - 2\left(Q^\beta + a^\beta\right) c_2(SU(2)_1) + 2\left(Q^\beta - a^\beta\right) c_2(SU(2)_2) + ...\right)$$

Need to use:

- ▷ $\delta\,(x^a)\, dx^a$ – a particular representation of the Thom class $\Phi$ for $i: W_2 \hookrightarrow M_6$
  - $*$ Thom isomorphism: $i^*\Phi = \chi(N)$
- ▷ $\mathrm{tr} R^2|_{TW_2} = -2\, p_1(TW_2) - 2\, p_1(N)$
- ▷ $\chi(N) = c_2(SU(2)_1) - c_2(SU(2)_2)$ and $p_1(N) = -2(c_2(SU(2)_1) + c_2(SU(2)_2))$
- ▷ $SU(2)_2$ – $\mathcal{R}$-symmetry of the interacting part of the SCFT

Central charges (with c.o.m contribution):

$$c_L - c_R = -6\,\Omega_{\alpha\beta}\, a^\alpha\, Q^\beta \equiv -6 Q \cdot a$$
$$c_R = 3\,\Omega_{\alpha\beta}\, Q^\alpha\, Q^\beta - 6\,\Omega_{\alpha\beta}\, a^\alpha\, Q^\beta + 6 \equiv 3 Q \cdot Q - 6 Q \cdot a + 6$$

## Constraints on charges $Q$

- Well-defined moduli space:
  - $\diamond$ $j \cdot j > 0, \quad j \cdot b_i > 0, \quad j \cdot a < 0$
  - $\diamond$ $j \in \mathbb{R}^{1,n_T}$ – a $(1, n_T)$ vector on the tensor branch ( $SO(1,n_T)/SO(n_T)$ MS)
- Non-negative tension:
  - $\diamond$ $j \cdot Q \geq 0$
- Non-negative levels for $SU(2)_1$ and $G_i$ affine current algebras
  - $\diamond$ $k = \frac{1}{2}(Q \cdot Q + Q \cdot a + 2) \geq 0$ and $k_i = Q \cdot b_i \geq 0$

## Unitarity constraint on the worldsheet theory

▷ Left-moving current algebra for $G$ is bounded by $c_L$

$$\sum_i \frac{k_i \cdot \text{dim}\, G_i}{k_i + h_i^\vee} \leq c_L - 4 = 3\, Q \cdot Q - 9\, Q \cdot a + 2$$

▷ Allows to rule out anomaly-free supergravities without string-theoretic realisations

▷ Is not directly comparable with geometric bounds

## Why is it worthwhile to re-examine the question in D=5?

- Different way of packing the (same) information
  - ◇ Consider e.g. reduction on a smooth elliptic CY3
    - ▷ D=6: $L_{\text{GS}} \sim b_{\alpha ij} B_2^\alpha \wedge F_2^i \wedge F^j \quad \Leftarrow$ part of CY intersection form
    - ▷ D=5: - $-\frac{1}{6} C_{IJK} A^I \wedge F^I \wedge F^J \quad \Leftarrow$ entire CY intersection form

    $(C_{IJK} = \int \omega_I \wedge \omega_J \wedge \omega_K; \qquad I = 1, ..., n_T + n_V + 1)$
  - ◇ In the $S^1$ reduction from D=6 to D=5, a one-loop computation should reveal new info and ... “hide” the anomaly
- Different scaling of central charges w.r.t string charge $Q$
  - ◇ D=6: $c_L, c_R \sim \# Q \cdot Q + \#' Q \cdot a + \#''$
  - ◇ D=5: $c_L, c_R \sim \tilde{\#} Q \cdot Q \cdot Q + \#' Q \cdot a + \tilde{\#}''$
- ○ General questions about which theories are liftable
  - ◇ reductions with Wilson lines
  - ◇ reductions with discrete holonomies
- ○ Unitarity constraints for generic minimal D=5 theories...

## Anomaly cancellation in D=6 (2n) $\Leftrightarrow$ gauge/diff invariance in D=5 (2n-1)

- $S^1$ resuction of the GS terms:

$$\delta(\iota_v L_{\text{GS}}) \neq 0 \quad !!!$$

- no D=5 anomaly to cancel it

Consider the simplest situation - $n_T = 1$ and $M_6 = M_5 \times S^1$ (no curvature):

▷ $I_8 = X_4 \wedge \tilde{X}_4; \qquad L_{\text{GS}} = \hat{B}_2 \wedge \tilde{X}_4; \qquad dH = X_4 \ \ (H = d\hat{B} + X_3^{(0)})$

▷ reduction: $\hat{B}_2 \mapsto (B_2, A_1); \qquad X_4 \mapsto (x_4, x_3)$

- $(dx_4, dx_3) = 0; \qquad (x_4 = dx_3^{(0)}, x_3 = dx_2^{(0)}); \qquad (\delta dx_3^{(0)} = dx_2^{(2)}, \delta x_2^{(0)} = 0)$

▷ $L_{\text{GS}} \mapsto A_1 \wedge x_4 + B_2 \wedge x_3 \ \longrightarrow \ dB_2 \wedge x_2^{(0)} \ \longrightarrow \ \tilde{F}_2 \llcorner x_2^{(0)} - x_3^{(0)} \wedge x_2^{(0)}$

- CS-like terms with field dependent coefficients - not gauge/diff invariant
- Can be cancelled by integrating out massive KK modes from chiral fields
- Conditions for cancellation - the same as for the anomaly cancelation in D=6
- many cases worked out by E. Poppitz, M, Unsal, F. Bonetti, T. Grimm, S. Hohenegger, P. Corvilain, D. Regalado ....

## Another (scheme-independent) way to look at the problem

▷ Reduction of the anomaly

$$\int_{M_{2n-1}\times S^1} I^1_{2n}(\epsilon,\hat{\mathcal{A}},\hat{\mathcal{F}}) = \delta_\epsilon \int_{M_{2n-1}} \Phi\cdot X(\mathcal{A},\mathcal{F}) + \ldots$$

◇ $\hat{\mathcal{A}}\,/\,\mathcal{A}$ and $\hat{\mathcal{F}}\,/\,\mathcal{F}$ $-$ fields and curvatures in D=2n/2n-1

◇ $\epsilon$ $-$ the variation (gauge or diffeomorphism) parameter,

◇ $\Phi$ $-$ Wilson line along the circle (for gravity $\Phi$ - graviphoton curvature)

◇ $\cdot$ $-$ trace over group indices

▷ $X(\mathcal{A},\mathcal{F})$ is derived from the Bardeen-Zumino polynomial

◇ ... indicate correction terms when $G \longrightarrow G'$ or Diff($M_{2n}$) $\longrightarrow$ Diff($M_{2n-1}$)

▷ Local counterterm $-\Phi\cdot X$ is *always* possible but can *never* be lifted to D=2n

▷ Liftability $\Rightarrow$ *different* counterterm

Obstruction to liftability

## New CS couplings in D=5

▷ involve reduced D=6 YM fields, and the graviphoton

$$\mathcal{L}_{\text{CS}} = -\frac{k_0}{6} A^{\text{KK}} \wedge F^{\text{KK}} \wedge F^{\text{KK}} + \frac{k_R}{96} A^{\text{KK}} \wedge \mathrm{tr} R^2$$

◇ $k_0 = 2(9 - n_T)$ and $k_R = 8(12 - n_T)$

▷ Anomaly inflow

$$c_R = k_0 Q_{\text{KK}}^3 + \frac{k_R}{2} Q_{\text{KK}} \qquad \text{and} \qquad c_L = k_0\, Q_{\text{KK}}^3 + k_R\, Q_{\text{KK}}$$

◇ The string source: $dF = d\rho(r) e_2/2$

- $d\rho(r) e_2/2$ – smooth representative of Thom class
- $e_2$ – global angular form
- $\int_{S^2} e_2 \wedge e_2 \wedge e_2 = 2 p_1(N)$

◇ $\mathrm{tr} R^2|_{TW_2} = -2\, p_1(TW_2) - 2\, p_1(N)$

▷ In D=5 there are strings with *cubic* central charges (not quadratic!)

▷ All strings with cubic central charges carry some magnetic KK charge

## Central charges for D=5 BPS strings

$$c_R = C_{IJK}Q^I Q^J Q^K + \frac{1}{2} a_I Q^I \qquad \text{and} \qquad c_L = C_{IJK}Q^I Q^J Q^K + a_I Q^I$$

◇ $I = 1, ..., n_T + n_V + 1$

▷ BPS strings in D=6 with transverse $S^1$ (normal bundle $\mathbb{R}^3 \times S^1$)

◇ Recall

$I_4 \sim \Omega_{\alpha\beta} Q^\alpha \left( a^\beta p_1(TW_2) - 2\left(Q^\beta + a^\beta\right) c_2(SU(2)_1) + 2\left(Q^\beta - a^\beta\right) c_2(SU(2)_2) + ...\right)$

◇ Take $c_2(SU(2)_1) = c_2(SU(2)_2) = c_2(N)$,

$$c_L = 2\, c_R = -12\, \Omega_{\alpha\beta}\, a^\alpha\, Q^\beta \equiv -12 Q \cdot a$$

▷ The interacting part of SCFT

$$c_R^{int} = -6\, Q \cdot a - 6 \qquad \text{and} \qquad c_L^{int} = -12\, Q \cdot a - 3$$

▷ The unitarity condition for linear strings

$$\sum_i \frac{(Q \cdot b_i) \cdot \dim G_i}{Q \cdot b_i + h_i^\vee} \leq c_L^{int} = -12\, Q \cdot a - 3$$

## Kodaira positivity and F-theory models

▷ In all F-theory models the following bound holds:

$$j \cdot (-12a - \sum_i x_i b_i) \geq 0$$

- ▷ $j \in \mathbb{R}^{1,n_T}$ – a $(1, n_T)$ vector on the tensor branch
- ▷ $a, b_i \in \mathbb{R}^{1,n_T}$ – determined by the field content of the theory
- ▷ $x_i$ – number of D7 needed for $G_i$ (multiplicity of respective singularity)

▷ Follows from the Kodaira condition - requirement that elliptic fibration over base B with singularities over divisors $S_i$ is CY:

$$-12K = \sum_i x_i S_i + Y$$

- ▷ $Y$ – residual divisor which must be *effective*
- ▷ For any nef divisor $D$ : $D \cdot Y = D \cdot (-12K - \sum_i x_i S_i) \geq 0$

▷ KPC ($j \cdot (-12a - \sum_i x_i b_i) \geq 0$) is not expected to be satisfied in any consistent D=6 theory

## The unitariry bound should hold in all consistent D=6 theories

▷ The strongest for of the constraint:

$$Q\cdot(-12a-\sum_i b_i(\frac{\dim G_i}{1+h_i^\vee}))\geq Q\cdot(-12a-\sum_i b_i(\frac{\dim G_i}{Q\cdot b_i+h_i^\vee}))\geq 3$$

- ◇ If the strong form is satisfied, it will hold also for $Q\cdot b_1>1$
- ◇ If it fails, need to check if $Q\cdot b_i=1$ is possible
- ◇ Impose: $Q\cdot Q+Q\cdot a+2\geq 0\,,\quad k_i=Q\cdot b_i\geq 0$ and $-Q\cdot a>0$

▷ (Assuming D=6 theory is F-theoretic) UC can be converted into geometric form:

$$D\cdot(-12K-\sum_i y_i S_i)\geq 3 \qquad \text{with} \quad y_i=\frac{\dim G^i}{1+h_i^\vee}$$

▷ $x$ is always larger than $y$:

| Type of gauge algebra | $x_i-y_i$ | Gauge algebra |
|---|---|---|
| $K_1$ | $<2$ | $su(m)$, $sp(1)$, $sp(2)$, $sp(3)$ in Kodaira type $I$ |
| $K_2$ | $\geq 2$ | All other groups |

## Comparing UC and KPC

$$D \cdot Y \geq 3 - \sum_i (x_i - y_i) D \cdot S_i$$

- In most of the cases the bound is automatic given KPC (KPC is stronger than UC)
  - ▷ At least 3 gauge group factors (gauge divisors $S_{1,2,3}$ ($D \cdot S_{1,2,3} > 0$ holds))
  - ▷ At least 2 gauge groups and at least 1 is type $K_2$ ($x_i - \frac{\dim G_i}{D \cdot S_i + h_i^\vee} \geq x_i - y_i \geq 2$ )
- In other cases, KPC may be satisfied while UC is violated if

$$12\,n - 3 < \sum_i \mu_i D \cdot S_i \leq 12\,n - \sum_i (x_i - \mu_i) D \cdot S_i$$

  - ▷ $Y = -12K - \sum_i x_i S_i$ – **NOT** numerically $0$ (GDs $S_i$ do not sweep $-12K$)
- 3 cases when UC imposes stronger constraints
  - ▷ $\exists S_1 \in \{S_i\}$ and nef $D$ : $D \cdot S_1 = 1,\ D \cdot S_i = 0\,(i \neq 1)\ \&\ -D \cdot K \in \mathbb{Z}_+$
    $\Rightarrow D \cdot Y \geq 1$ for $SU(12n)$ and $D \cdot Y \geq 2$ for $SU(12n-1)$
  - ▷ $\exists S_1 \in \{S_i\}$ for $D \cdot Y \geq 1 \ \Rightarrow\ SO(24n-5)$, $SO(24n-4)$ or $Sp(6n)$ ( $I_{12n}$ type)
  - ▷ $\exists S_1, S_2 \in \{S_i\}$ for $D \cdot Y \geq 1 \ \Rightarrow SU(a) \times SU(12n-a)$, $Sp(1) \times SU(12n-2)$, $Sp(2) \times SU(12n-4)$ or $SU(12n-6) \times Sp(3)$ ( $I_2$, $I_4$ and $I_6$ type)

Example: $SU(N) \times SU(N)$, $n_H = 2$ ( bifundamentals) and $n_T = 9$

$$\Omega = \text{diag}\left(+1, (-1)^9\right), \qquad a = \left(-3, (+1)^9\right)$$
$$b_1 = \left(1, -1, -1, -1, 0^6\right), \quad b_2 = \left(2, 0, 0, 0, (-1)^6\right)$$

▷ $Q = (1, 0, 0, 0, -1, 0.., 0)$

▷ $Q \cdot Q = 0$, $Q \cdot a = -2$ and $Q \cdot b_1 = Q \cdot b_2 = 1$

▷ UC: $2(N-1) \leq 24 - 3 \;\Rightarrow\; N \leq 11$ (stronger bound in D6 UC)

▷ KPC : $2N \leq 24 \to N \leq 12$

▷ Assuming F-theoretic realisation: $-12K = NS_1 + NS_2 + Y$

▷ For $N \geq 4$, the singular divisors are of type $I_N$

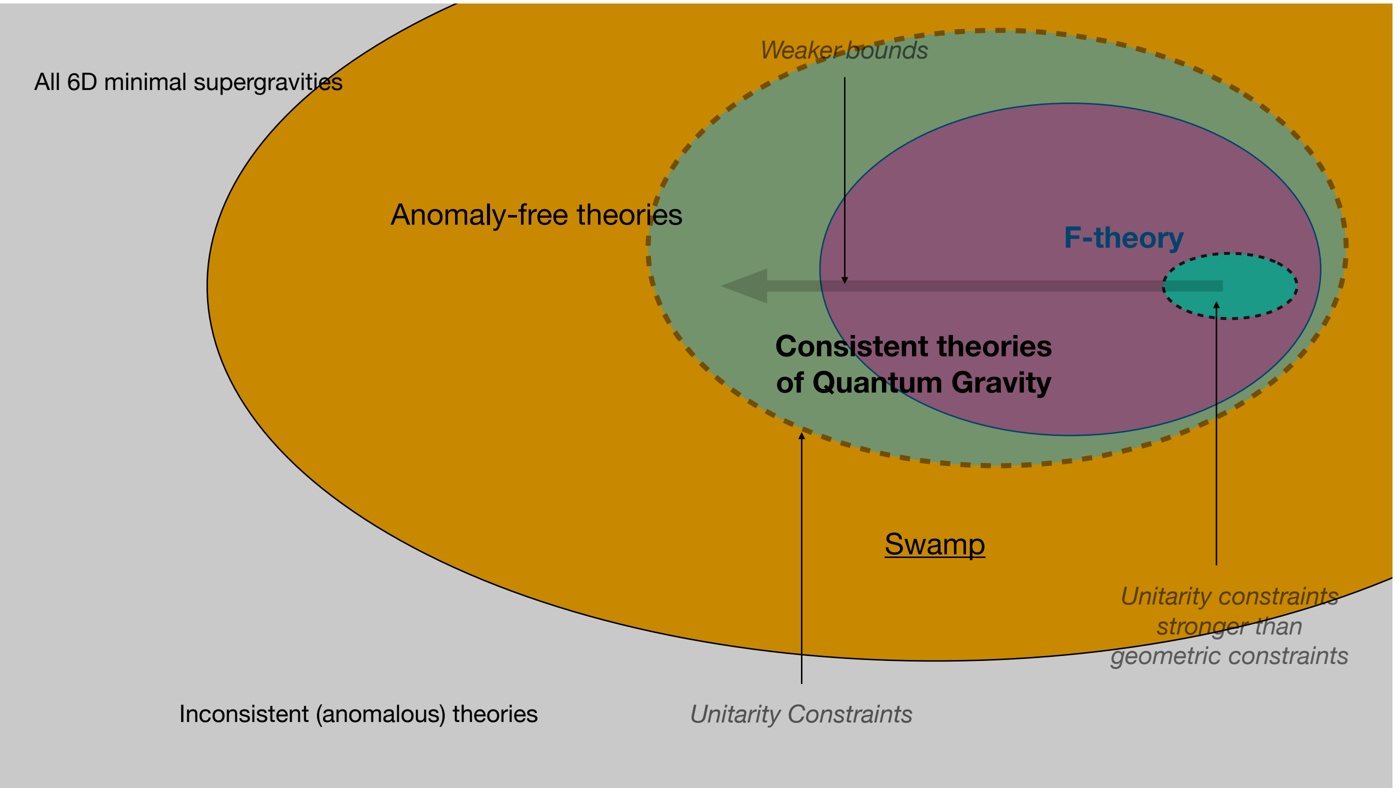
◇ $S_1 \cdot K = S_2 \cdot K = 0$

◇ 2 bifundamental hypers: $S_1 \cdot S_1 = -2 = S_2 \cdot S_2$ and $S_1 \cdot S_2 = 2$

◇ $n_T = 9$ translates into $K \cdot K = 0$.

▷ Can verify that $Y = -12K - 12S_1 - 12S_2$ has to be numerically non-trivial ($-12K = 12S_1 + 12S_2$ cannot be realised on the base $B$ of an elliptic Calabi-Yau threefold with the required singularity structure)

# A refined cartoon of the space of D=6 theories




... and much more left to be understood