SHADOWS AND SOFT EXCHANGE IN CELESTIAL CFT

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Celestial Amplitudes and Flat Space Holography
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References

- Based on work with Daniel Kapec [2109.00073]
- Closely related work:
  Nande, Pate, Strominger [1705.00608]
  Kapec, Mitra [1711.04371]
  Himwich, Narayanan, Pate, Paul, Strominger [2005.13433]
  Arkani-Hamed, Pate, Raclariu, Strominger [2012.04208]
  Magnea [2104.10254]
  Gonzáles, Rojas [2104.12979]
  Nguyen, Salzer [2105.10526]
  Kalyanapuram [2107.06660]
- Many more authors.

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- Why $D > 4$?
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- How?
  – Fix action using asymptotic symmetries.

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$$\mathcal{O}_i(\varphi) = U_i(\varphi_s) \mathcal{O}_i^h(\varphi_h).$$

- Integrate over the hard modes.

$$\begin{aligned}
\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_\mu &= \int [d\varphi_s] U_1 \cdots U_n \int [d\varphi_h] e^{iS_{bulk}[\varphi_s, \varphi_h]} \mathcal{O}_1^h \cdots \mathcal{O}_n^h \\
&= \int [d\varphi_s] U_1 \cdots U_n e^{-S_{soft}[\varphi_s]} \langle \mathcal{O}_1^h \cdots \mathcal{O}_n^h \rangle_\Lambda \\
&= \langle U_1 \cdots U_n \rangle_{soft} \langle \mathcal{O}_1^h \cdots \mathcal{O}_n^h \rangle_\Lambda.
\end{aligned}$$

This is the soft-hard factorization of the amplitude, $A_n(\mu) = e^{-\Gamma(\mu, \Lambda)} A_n(\Lambda)$.

- So far, we have assumed that all particles in the scattering amplitude are hard. If an amplitude contains soft external states, then we also have a factorization formula given by soft theorems

$$\langle\, S_1 \cdots S_m \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_\mu = \mathcal{J}_1 \cdots \mathcal{J}_m \langle\, \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_\mu .$$

Effective Soft Action - What about soft theorems?

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- It follows that

$$\langle\, S_1 \cdots S_m U_1 \cdots U_n \,\rangle_{soft} = \mathcal{J}_1 \cdots \mathcal{J}_m e^{-\Gamma(\mu,\Lambda)}.$$

The goal of this talk will be to find an effective soft action S_{soft} so that this equation is true.

Outline

I. Review
II. Boundary Action for Abelian Gauge Theory
III. Gravity
IV. Conclusions and open problems

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- $(d+2)$-dimensional momenta are parameterized as

$$p^\mu(\omega,x) = \omega \hat{p}^\mu\left(\frac{m}{\omega},x\right), \qquad \hat{p}^\mu\left(\frac{m}{\omega},x\right) = \hat{q}^\mu(x) + \frac{m^2}{\omega^2}n^\mu,$$

where

$$\hat{q}^\mu(x) = \left(\frac{1+x^2}{2}, x^a, \frac{1-x^2}{2}\right), \qquad n^\mu = \left(\frac{1}{2}, 0^a, -\frac{1}{2}\right).$$

I. Review - Momenta and Polarizations

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- The photon and graviton polarizations are

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$$\varepsilon_a^\mu(q)=\partial_a\hat{q}^\mu(x),\qquad \varepsilon_{ab}^{\mu\nu}(q)=\varepsilon^\mu_{\{a}(q)\varepsilon^\nu_{b\}}(q).$$

- Define

$$\mathcal{O}(\omega,x)\equiv a_{out}(p(\omega,x))\theta(\omega)+\bar{a}^\dagger_{in}(-p(\omega,x))\theta(-\omega).$$

- The S-matrix then takes the form

$$A_n = \langle\, \mathcal{O}_1(\omega_1, x_1) \cdots \mathcal{O}_n(\omega_n, x_n) \,\rangle \equiv \langle\, \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle.$$

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- Up to a (generalized) Mellin transform, the S-matrix transforms precisely as a correlation function of n conformal primary operators. For a massive scalar,

$$\widehat{\mathcal{O}}^{\pm}(\Delta, x) = m^{\Delta - d} \int d^d y \int_0^{\infty} d\omega \omega^{d-1} \mathcal{K}_{\Delta}(m/\omega; y|x) \mathcal{O}(\pm\omega, y)$$

where $\Delta \in \frac{d}{2} + i\mathbb{R}$ and

$$\mathcal{K}_{\Delta}(z, x|y) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})} \left(\frac{z}{(x-y)^2 + z^2} \right)^{\Delta}$$

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- We will NOT work in the Mellin basis.

- The soft-photon theorem is

$$\langle\, S_a(x)\mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{\mu,C=0} = \mathcal{J}_a(x)\langle\, \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{\mu,C=0}$$

where $S_a(x) = \frac{1}{e} \lim_{\omega\to 0}[\omega\mathcal{O}_a(\omega, x)]$ and

$$\mathcal{J}_a(x) = \sum_i Q_i \frac{\hat{p}_i \cdot \varepsilon_a(x)}{\hat{p}_i \cdot \hat{q}(x)} = \partial_a \sum_i Q_i \ln[-\hat{p}_i \cdot \hat{q}(x)].$$

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$Q_i \in \mathbb{Z}$ are the $U(1)$ charges of the particles.

- Soft theorems are Ward identities of asymptotic symmetries (in this case, large gauge symmetries).

- The conserved $U(1)$ current can be constructed from the soft operator

$$J_a(x) = \frac{1}{2c_{1,1}} \widetilde{S}_a(x)$$

where the $\sim$ denotes the shadow transform

$$\widetilde{\mathcal{O}}_{d-\Delta,\mathcal{R}}(x) = \int d^d y \frac{1}{[(x-y)^2]^{d-\Delta}} \mathcal{R}(\mathcal{I}(x-y)) \cdot \mathcal{O}_{\Delta,\mathcal{R}}(y).$$

Here, $\mathcal{I}_{ab}(x) = \delta_{ab} - 2\frac{x_a x_b}{x^2}$. The shadow transform satisfies

$$\widetilde{\widetilde{\mathcal{O}}} = c_{\Delta,\mathcal{R}}\mathcal{O}, \qquad c_{\Delta,s} = \frac{\pi^d(\Delta-1)(d-\Delta-1)\Gamma(\frac{d}{2}-\Delta)\Gamma(\Delta-\frac{d}{2})}{(\Delta-1+s)(d-\Delta-1+s)\Gamma(\Delta)\Gamma(d-\Delta)}.$$

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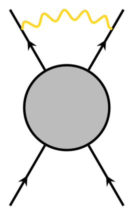
- The current Ward identity is

$$\langle \partial^a J_a(x) \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \sum_i Q_i \mathcal{K}_d(m_i/\omega_i, x_i|x) \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle.$$

Massless particles have a localized charge distribution since

$$\lim_{z\to 0} \mathcal{K}_d(z, x|y) = \delta^{(d)}(x-y).$$

I. Review - Soft Exchange/Infrared Divergences



- Scattering amplitudes factorize as

$$A_n(\mu) = e^{-\Gamma(\mu,\Lambda)} A_n(\Lambda).$$

where

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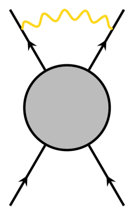

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- This integral can be evaluated (ref. Weinberg) explicitly, but the following form will is more convenient

$$\Gamma(\mu,\Lambda) = \alpha(A_1 + 2\pi i A_2), \qquad \alpha = \frac{e^2}{8\pi} \int_\mu^\Lambda d\omega \omega^{d-3}, \qquad A_1 = \int \frac{d^d x}{(2\pi)^d} [\mathcal{J}_a(x)]^2.$$

The effective soft action must satisfy

$$\langle\, S_{a_1}(y_1)\cdots S_{a_m}(y_m) U_1 \cdots U_n \,\rangle_{soft} = \mathcal{J}_{a_1}(y_1)\cdots \mathcal{J}_{a_m}(y_m) e^{-\Gamma(\mu,\Lambda)}$$

where

$$\mathcal{J}_a(x) = \partial_a \sum_i Q_i \ln[-\hat{p}_i \cdot \hat{q}(x)], \qquad \Gamma(\mu,\Lambda) = \alpha \int \frac{d^d x}{(2\pi)^d} [\mathcal{J}_a(x)]^2$$

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II. Boundary Action

- The boundary action depends on the soft modes in gauge theory - namely, S (soft photon mode) and $C = A|_{\partial \mathscr{I}}$ (edge mode) so we need to find $S_{soft}[S, C]$.

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- $S_{soft}[C]$ can be constructed by analyzing symmetries.

II. Boundary Action - Spontaneously Broken Large Gauge Symmetries

- The asymptotic symmetry group of Abelian gauge theories are large gauge transformations which are generated by a function ε which does not vanish at infinity $\varepsilon|_{\partial\mathscr{I}} \neq 0$. The order parameter for this symmetry is the Wilson line on the celestial sphere

$$W(x) = \exp\left(i\int_{x_0}^{x} C\right), \qquad W(x) \to e^{i[\varepsilon(x)-\varepsilon(x_0)]}W(x).$$

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- The Goldstone mode for the broken symmetry is the edge mode C which realizes the symmetry non-linearly, $C \to C + d\varepsilon$.

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- When the IR cutoff $\mu \neq 0$, the large gauge symmetries are EXPLICITLY broken. We must therefore also include explicit symmetry breaking term in the soft action. The simplest guess for such a term is a mass term

$$S_{soft}[C] = a(\mu) \int C \wedge \star\, C, \qquad C = d\theta.$$

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$$S_{soft}[C] = a(\mu) \int C \wedge \star C, \qquad C = d\theta.$$

- In this action, we have assumed locality. While the bulk theory must be local the boundary theory need not be! A more general version of this action is then

$$S_{soft}[C] = \int d^d x d^d y \, (P^{-1})^{ab}(x-y) C_a(x) C_b(y), \qquad C = d\theta.$$

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$$\mathcal{O}_i(\omega_i, x_i) \to \exp\left[iQ_i \int d^d x \varepsilon(x) \mathcal{K}_d(m_i/\omega_i, x_i; x)\right] \mathcal{O}_i(\omega_i, x_i).$$

For massless particles, $\mathcal{O}_i \to e^{iQ_i \varepsilon(x_i)} \mathcal{O}_i$.

II. Boundary Action - Operator Insertions

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For massless particles, $\mathcal{O}_i \to e^{iQ_i \varepsilon(x_i)} \mathcal{O}_i$.

- The Goldstone mode transforms as

$$C \to C + d\varepsilon \quad \Longrightarrow \quad \theta(x) \to \theta(x) + \varepsilon(x).$$

so we can take

$$U_i = \exp\left[iQ_i \int d^d x \theta(x) \mathcal{K}_d(m_i/\omega_i, x_i; x) \right].$$

- Note

$$U_1 \cdots U_n = \exp\left[i \int d^d x \theta(x) \sum_i Q_i \mathcal{K}_d(m_i/\omega_i, x_i; x) \right]$$
$$= \exp\left[-i \int d^d x \partial^a \theta(x) j_a(x) + i\theta_0 \sum_i Q_i \right].$$

where

$$j_a(x) = \frac{1}{2c_{1,1}} \widetilde{\mathcal{J}}_a(x), \qquad \partial^a j_a(x) = \sum_i Q_i \mathcal{K}_d(m_i/\omega_i, x_i; x).$$

- The soft path integral is therefore

$$\begin{aligned}\langle\, U_1 \cdots U_n \,\rangle_{soft} = \int [d\theta] \exp \Big[- \int d^d x d^d y \, (P^{-1})^{ab}(x-y) \partial_a \theta(x) \partial_b \theta(y) \\ - i \int d^d x \partial^a \theta(x) j_a(x) + i\theta_0 \sum_i Q_i \Big] \\ = \delta_{\sum_i Q_i, 0} \exp \left[-\frac{1}{4} \int d^d x d^d y P_{ab}(x-y) j^a(x) j^b(y) \right] .\end{aligned}$$

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$$\langle U_1 \cdots U_n \rangle_{soft} = \int [d\theta] \exp\left[-\int d^d x d^d y \, (P^{-1})^{ab}(x-y)\partial_a\theta(x)\partial_b\theta(y) \right.$$
$$\left. -i\int d^d x \partial^a\theta(x) j_a(x) + i\theta_0 \sum_i Q_i \right]$$
$$= \delta_{\sum_i Q_i, 0} \exp\left[-\frac{1}{4}\int d^d x d^d y P_{ab}(x-y) j^a(x) j^b(y) \right].$$

- Matching this result to soft exchange requires

$$\frac{1}{4}\int d^d x d^d y P_{ab}(x-y) j^a(x) j^b(y) = \alpha \int \frac{d^d x}{(2\pi)^d} [\mathcal{J}_a(x)]^2.$$

Using $j_a(x) = \frac{1}{2c_{1,1}} \widetilde{\mathcal{J}}_a(x)$, we can completely fix the propagator P and its inverse P^{-1}.

II. Boundary Action - Soft Action

- We find

$$(P^{-1})^{ab}(x-y) = \frac{(2\pi)^d}{16c_{1,1}^2\alpha}\int d^d w \frac{\mathcal{I}^{ac}(w-x)}{[(w-x)^2]^{d-1}}\frac{\mathcal{I}_c{}^b(w-y)}{[(w-y)^2]^{d-1}}.$$

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- The soft action then takes the form

$$S_{soft}[C] = \frac{(2\pi)^d}{16c_{1,1}^2\alpha}\int d^dx d^dy \int d^dw\, \frac{\mathcal{I}^{ac}(w-x)}{[(w-x)^2]^{d-1}}\frac{\mathcal{I}_c{}^b(w-y)}{[(w-y)^2]^{d-1}} C_a(x)C_b(y).$$

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- Written in terms of the shadow edge mode, the action is local

$$\boxed{S_{soft}[C] = \frac{(2\pi)^d}{16c_{1,1}^2\alpha}\int d^d x \widetilde{C}^a(x)\widetilde{C}_a(x)}$$

The operator insertions can also be written in terms of the shadow edge mode as

$$\boxed{U_1 \cdots U_n = \exp\left[-\frac{i}{2c_{1,1}}\int d^d x \widetilde{C}_a(x)\mathcal{J}^a(x) + i\theta_0 \sum_i Q_i\right]}$$

II. Boundary Action - Integrating 'in' $S_a(x)$

- The full soft action is

$$\boxed{S_{\text{soft}}[S,C] = \frac{\alpha}{(2\pi)^d}\int d^d x S^a(x) S_a(x) - \frac{i}{2c_{1,1}}\int d^d x \widetilde{C}_a(x) S^a(x)}$$

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- The full soft path integral can now be evaluated as

$$\begin{aligned} &\langle\, S_1 \cdots S_m U_1 \cdots U_n \,\rangle_{soft} \\ &= \int [dS][d\theta] \exp\left[-\frac{\alpha}{(2\pi)^d} \int d^d x S^a(x) S_a(x) \right. \\ &\qquad \left. + \frac{i}{2c_{1,1}} \int d^d x \widetilde{C}_a(x) [S^a(x) - \mathcal{J}^a(x)] + i\theta_0 \sum_i Q_i \right] S_1 \cdots S_m \\ &= \exp\left[-\frac{\alpha}{(2\pi)^d} \int d^d x \mathcal{J}^a(x) \mathcal{J}_a(x) \right] \mathcal{J}_1 \cdots \mathcal{J}_m . \end{aligned}$$

II. Boundary Action - Final Comments

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- In $D = 4$, $\alpha \to \infty$ as $\mu \to 0$ so the first term is dominant and gives us the infrared divergence.
- Another specialty feature of $D = 4$ is that the shadow transform of an exact 1-form is local

$$\widetilde{C}_a(x) = 2\pi C_a(x) \quad \text{if} \quad C_a = \partial_a \theta(x).$$

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III. Gravity
IV. Conclusions and open problems

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$$\langle N_{ab}(x)\mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \mathcal{J}_{ab}(x) \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle .$$

where $N_{ab}(x) = \frac{1}{\sqrt{8\pi G}} \lim_{\omega\to 0}[\omega \mathcal{O}_{ab}(\omega, x)]$ and

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- The conserved current is defined by $P^+_a(x) = \frac{1}{4c_{1,2}}\partial^b \widetilde{N}_{ab}(x)$ with

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- The soft exchange amplitude is

$$\Gamma_{\text{gr}}=\alpha_{\text{gr}}(A_1^{\text{gr}}+2\pi i A_2^{\text{gr}}),\qquad \alpha_{\text{gr}}=G\int_\mu^\Lambda d\omega\,\omega^{d-3},\qquad A_1^{\text{gr}}=\int\frac{d^dx}{(2\pi)^d}[\mathcal{J}_{ab}(x)]^2.$$

III. Gravity - Soft Action

- The soft action in GR is depends on the soft graviton mode N_{ab} and the gravitational edge mode $C_{ab} \sim r^{-1}(g_{ab} - r^2\delta_{ab})|_{\partial\mathscr{I}}$. C satisfies a higher dimensional analogue of the CK constraint (magnetic part of Weyl tensor is zero) which solves to $C_{ab} = 2\partial_{\{a}\partial_{b\}}C(x)$. Under supertranslations $C(x) \to C(x) + f(x)$.

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- The operators U_i are

$$U_i = \exp\left[\frac{i}{2}m_i\int d^dx C(x)\mathcal{K}_{d+1}(m_i/\omega_i, x_i; x)\right]$$

with

$$U_1\cdots U_n = \exp\left[\frac{i}{2}\int d^dx C(x)\sum_i m_i\mathcal{K}_{d+1}(m_i/\omega_i, x_i; x)\right]$$
$$= \exp\left[\frac{i}{16c_{1,2}}\int d^dx \widetilde{C}^{ab}(x)\mathcal{J}_{ab}(x) - i\xi_\mu\sum_i p_i^\mu\right]$$

Note that the zero modes have the form $C(x) = \xi^0(1+x^2) - 2\xi^a x_a - \xi^{d+1}(1-x^2)$.

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  - ☐ There are other soft modes – Superrotations, subleading soft graviton theorem and the stress tensor.

THANK YOU

Imaginary Part of Γ

- The imaginary part of Γ has the form

$$A_2 = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int \frac{d^d x}{(2\pi)^d} [|\mathcal{J}_a^+(\nu, x)|^2 + |\mathcal{J}_a^-(\nu, x)|^2]_r,$$

where

$$\mathcal{J}_a^{\pm}(\nu, x) \equiv \partial_a \sum_{i \in \substack{\mathbf{out}(+) \\ \mathbf{in}(-)}} Q_i \frac{[-\hat{p}_i \cdot \hat{q}(x)]^{i\nu}}{i\nu}.$$

The $[\ ]_r$ symbol removes any $i = j$ terms in the integrand.

- One way to reproduce this in a CCFT is to introduce two fields, $C_a^{\pm}(\nu, x)$ with $C_a^{\pm}(\nu, x)^* = C_a^{\pm}(-\nu, x)$ with action

$$S_{soft} \sim i \int_{-\infty}^{\infty} d\nu \int d^d x \left(\left|\widetilde{C}_a^+(\nu, x)\right|^2 + \left|\widetilde{C}_a^-(\nu, x)\right|^2 + \mathrm{Re}\left[\widetilde{C}_a^+(\nu, x)\mathcal{J}_a^+(\nu, x) + \widetilde{C}_a^-(\nu, x)\mathcal{J}_a^-(\nu, x)\right]\right)$$