Entanglement Entropy in Expanding Backgrounds

Nikolaos Tetradis

University of Athens

Work with D. Giataganas

H.F.R.I. Hellenic Foundation for Research & Innovation
GSRT GENERAL SECRETARIAT FOR RESEARCH AND TECHNOLOGY

- We discuss the entanglement entropy of theories on time-dependent backgrounds.
- Having in mind possible applications to cosmology, we consider a background metric that takes the Friedmann-Robertson-Walker (FRW) form with an arbitrary scale factor.
- We are interested in the quantum entanglement between two subsystems classically confined within two regions A and B, separated by an entangling surface $\mathcal{A}$.
- The entaglement entropy is a measure of the quantum entanglement.
- The entanglement entropy associated with the region A is defined as the von Neumann entropy computed through the reduced density matrix, in which the degrees of freedom in the region B are traced out.
- The explicit calculation is difficult for time-dependent backgrounds.
- We use the holographic approach.

- In the framework of the AdS/CFT correspondence, we use appropriate coordinates for the bulk AdS space, such that the boundary metric takes the FRW form.
- We study the entanglement entropy for a CFT confined within a part of the AdS boundary delimited by an entangling surface $\mathcal{A}$ of fixed comoving size.
- In the static case, the entropy is proportional to the area of a minimal surface at fixed time, which starts from $\mathcal{A}$ and extends into the bulk (Ryu, Takayanagi 2006).
- When the background is time dependent, one must use the covariant formulation (Hubeny, Rangamani, Takayanagi 2007). The entanglement entropy is proportional to the area of an extremal surface, anchored on $\mathcal{A}$, which is obtained from the area functional by extremizing with respect to the timelike coordinate as well.

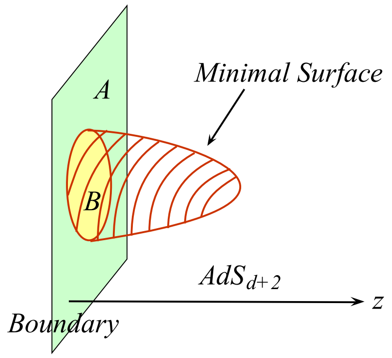

Figure: Nishioka, Ryu, Takayanagi (2009).

General expectation (Maldacena, Pimentel 2012)

- For $\mathrm{d} = 3$ the entanglement entropy of a CFT for a spherical entangling surface in a time-dependent background has the form

$$\mathrm{S} = \mathrm{C}_1 \frac{\mathcal{A}}{\epsilon^2} + (\mathrm{C}_2 + \mathrm{C}_3 \mathcal{A}) \log(\epsilon) + \mathrm{C}_4 \log(\mathcal{A}) + \mathrm{C}_5 \mathcal{A}.$$

  Proper area: $\mathcal{A}(\mathrm{T}, \mathrm{R}) = 4\pi \mathrm{a}^2(\mathrm{T}) \mathrm{R}^2$.
  Expansion rate: $\mathrm{H}(\mathrm{T}) = \mathrm{a}'(\mathrm{T})/\mathrm{a}^2(\mathrm{T})$.
- The terms involving C_1, C_2, C_4 would be present also for a static flat background.
- The terms involving C_3, C_5 have an explicit dependence on the expansion rate and the spatial curvature.
- We have neglected finite terms that are mere constants.

Plan

- Spatially flat FRW background
- Spatial curvature
- Temperature
- Cross-checks
- Conclusions

D. Giataganas, N. T.
arXiv:2105.12614 [hep-th], Phys. Lett. B 820 (2021) 136493

Spatially flat FRW

- We consider a slicing of (d + 2)-dimensional AdS with a boundary (d + 1)-dimensional FRW metric.

$$ds^2_{d+2} = \frac{1}{z^2}\left[dz^2 - N^2(\tau,z)d\tau^2 + A^2(\tau,z)\left(d\rho^2 + \rho^2\, d\Omega^2_{d-1}\right)\right]$$

$$\begin{aligned} N(\tau,z) &= a(\tau)\left(1 - \frac{-3a'^2(\tau) + 2a(\tau)a''(\tau)}{4a^4(\tau)}\, z^2\right) \\ A(\tau,z) &= a(\tau)\left(1 - \frac{a'^2(\tau)}{4a^4(\tau)}\, z^2\right). \end{aligned}$$

- All quantities are expressed in terms of the AdS length.
- The scale factor $a(\tau)$ is an arbitrary function of time.
- The metric is of the typical Fefferman-Graham form for an asymptotially AdS space.

- The metric can also be written in Poincare coordinates

$$\mathrm{ds}^2_{\mathrm{d}+2} = \frac{1}{\zeta^2}\left[\mathrm{d}\zeta^2 - \mathrm{dt}^2 + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\Omega^2_{\mathrm{d}-1}\right].$$

- The two metrics are related through the coordinate transformation

$$\begin{aligned} \mathrm{t}(\mathrm{z},\tau) &= \tau + \frac{2\mathrm{a}'(\tau)\mathrm{a}(\tau)\,\mathrm{z}^2}{-4\mathrm{a}^4(\tau)+\mathrm{a}'^2(\tau)\,\mathrm{z}^2} \\ \zeta(\mathrm{z},\tau) &= \frac{\mathrm{z}}{\mathrm{a}(\tau)}\left(1-\frac{\mathrm{a}'^2(\tau)}{4\mathrm{a}^4(\tau)}\,\mathrm{z}^2\right)^{-1}. \end{aligned}$$

- There is a singularity at $\mathrm{z} = 2/\mathrm{H}(\tau)$, where $\mathrm{H}(\tau) = \mathrm{a}'(\tau)/\mathrm{a}^2(\tau)$.
- The transformation is well defined for $\mathrm{z} < 2/\mathrm{H}(\tau)$.

Extremal surface

- Consider a spherical entangling surface Σ on the boundary, with comoving radius $\rho = \mathrm{R}$.
- The entropy is proportional to the area

$$\mathrm{Area}(\gamma_A) = S^{d-1} I(\epsilon) = S^{d-1} \int_{\epsilon} d\rho\, \rho^{d-1} \frac{A^{d-1}(\tau(\rho), z(\rho))}{z^d(\rho)} \times \sqrt{A^2(\tau(\rho), z(\rho)) - N^2(\tau(\rho), z(\rho)) \left(\frac{d\tau(\rho)}{d\rho}\right)^2 + \left(\frac{dz(\rho)}{d\rho}\right)^2},$$

extremized with respect to the functions $\tau(\rho)$ and $z(\rho)$, with the boundary conditions $\tau(R) = T$ and $z(R) = 0$.

- The integral diverges near the boundary, so that a cutoff must be imposed on the bulk coordinate z at $z = \epsilon$.
- The entanglement entropy is given by the relation

$$S = \frac{S^{d-1}}{4G_{d+2}} I(\epsilon),$$

with G_{d+2} the bulk Newton's constant.

- The calculation is simplified if one switches to Poincare coordinates. The functional to be extremized is

$$\mathrm{Area}(\gamma_A) = S^{d-1} \int d\rho\, \rho^{d-1} \frac{1}{\zeta^d(\rho)} \sqrt{1 - \left(\frac{dt(\rho)}{d\rho}\right)^2 + \left(\frac{d\zeta(\rho)}{d\rho}\right)^2}.$$

- The solution for the function $t(\rho)$ is trivial: $t(\rho) = T =$constant.
- For a spherical entangling surface, the minimization with respect to $\zeta(\rho)$ is standard. The minimal surface is given by $\zeta(\rho) = \sqrt{R^2 - \rho^2}$.
- The extremal surface corresponding to an entangling surface of comoving radius R at a time T on the boundary is given by the implicit relations

$$\begin{aligned} T &= \tau + \frac{2a'(\tau)a(\tau)\, z^2}{-4a^4(\tau) + a'^2(\tau)\, z^2} \\ \sqrt{R^2 - \rho^2} &= \frac{z}{a(\tau)} \left(1 - \frac{a'^2(\tau)}{4a^4(\tau)}\, z^2\right)^{-1} \end{aligned}$$

for the functions $\tau(\rho)$ and $z(\rho)$.

- The integration for the area of the extremal surface can be performed by using ζ as an independent variable.

$$\mathrm{Area}(\gamma_A) = S^{d-1} I(\epsilon) = S^{d-1} \int_{\epsilon_\zeta(T)/R}^{1} dy \frac{(1-y^2)^{(d-2)/2}}{y^d},$$

where $y = \zeta/R$.

- The area must be regulated by imposing a cutoff on ζ, resulting from the cutoff ϵ imposed on the Fefferman-Graham coordinate z:

$$\epsilon_\zeta(T) = \frac{\epsilon}{a(T)} \left(1 - \frac{1}{4} H^2(T) \epsilon^2\right)$$

We have defined the Hubble parameter $H(T) = a'(T)/a^2(T)$ in terms of the conformal time on the boundary.

- The whole dependence of the entropy on T enters through $\epsilon_\zeta(T)$. This cutoff results from the fundamental cutoff ϵ that regulates the UV divergences of the theory.

- For $\epsilon \to 0$ we have

$$\mathrm{d}=3\ \mathrm{I}(\epsilon)=\frac{\mathrm{a}^2(\mathrm{T})\mathrm{R}^2}{2\epsilon^2}+\frac{1}{2}\log\left(\frac{\epsilon}{2\mathrm{a}(\mathrm{T})\mathrm{R}}\right)+\frac{1}{4}\mathrm{a}^2(\mathrm{T})\mathrm{R}^2\mathrm{H}^2(\mathrm{T})-\frac{1}{4}+\mathcal{O}(\epsilon^2$$

$$\mathrm{d}=2\ \mathrm{I}(\epsilon)=\frac{\mathrm{a}(\mathrm{T})\mathrm{R}}{\epsilon}-1+\mathcal{O}(\epsilon^1)$$

$$\mathrm{d}=1\ \mathrm{I}(\epsilon)=-\log\left(\frac{\epsilon}{2\mathrm{a}(\mathrm{T})\mathrm{R}}\right)+\mathcal{O}(\epsilon^2).$$

- The time dependence of the scale factor $\mathrm{a}(\mathrm{T})$ is arbitrary.
- The boundary metric is not dynamical. However, the expressions are applicable to physical FRW cosmologies with metrics that have a dynamical origin.

Spatial curvature

- The bulk metric has the form

$$ds^2_{d+2} = \frac{1}{z^2}\left[dz^2 - N^2(\tau,z)d\tau^2 + A^2(\tau,z)\left(\frac{d\rho^2}{1-\frac{k}{R_0^2}\rho^2} + \rho^2\, d\Omega^2_{d-1}\right)\right],$$

with

$$\begin{aligned} N(\tau,z) &= a(\tau)\left(1 - \frac{1}{4}\left(-3\frac{a'^2(\tau)}{a^4(\tau)} - \frac{k}{a^2(\tau)R_0^2} + 2\frac{a''(\tau)}{a^3(\tau)}\right) z^2\right) \\ A(\tau,z) &= a(\tau)\left(1 - \frac{1}{4}\left(\frac{a'^2(\tau)}{a^4(\tau)} + \frac{k}{a^2(\tau)R_0^2}\right) z^2\right), \end{aligned}$$

with $k = 0, \pm 1$, depending on the spatial curvature of the boundary. The parameter R_0 sets the scale of the spatial curvature.

- This metric can also be rewritten as

$$\mathrm{ds}^2_{\mathrm{d}+2} = \frac{1}{\zeta^2}\left[\mathrm{d}\zeta^2 - \left(1 + \frac{1}{4}\frac{\mathrm{k}}{\mathrm{R}_0^2}\zeta^2\right)^2 \mathrm{dt}^2 \right. \\ \left. + \left(1 - \frac{1}{4}\frac{\mathrm{k}}{\mathrm{R}_0^2}\zeta^2\right)^2 \left(\frac{\mathrm{d}\rho^2}{1 - \frac{\mathrm{k}}{\mathrm{R}_0^2}\rho^2} + \rho^2\, \mathrm{d}\Omega^2_{\mathrm{d}-1}\right)\right].$$

- The two metrics are related through a coordinate transformation that does not involve ρ and the angular variables.

- The extremal surface corresponds to the minimization of the functional

$$\mathrm{Area}(\gamma_A) = \mathrm{S}^{d-1} \int_{\epsilon_\zeta(T,R)} d\rho\, \rho^{d-1} \frac{\left(1 - \frac{1}{4}\frac{k}{R_0^2}\zeta^2\right)^{d-1}}{\zeta^d} \times \sqrt{\frac{\left(1 - \frac{1}{4}\frac{k}{R_0^2}\zeta^2\right)^2}{1 - \frac{k}{R_0^2}\rho^2} + \left(\frac{d\zeta(\rho)}{d\rho}\right)^2}.$$

- The cutoff is the same as in the spatially flat case:

$$\epsilon_\zeta(T,R) = \frac{\epsilon}{a(T)R}\left(1 - \frac{1}{4}H^2(T)\epsilon^2\right)$$

where ϵ is the cutoff imposed on the bulk coordinate z.

- The minimization of the area can be performed in a straightforward manner.
- The curvature gives a nonzero contribution for $d = 3$:

$$I(\epsilon) = \frac{a^2(T)R^2}{2\epsilon^2} + \frac{1}{2}\log\left(\frac{\epsilon}{2a(T)R}\right) + \frac{1}{4}a^2(T)R^2H^2(T) + \frac{1}{4}\left(k\frac{R^2}{R_0^2} - 1\right) + \mathcal{O}(\epsilon^2)$$

Temperature

- The simplest way to introduce an energy scale is by considering the theory at nonzero temperature. The dual picture includes a bulk black hole, whose Hawking temperature is identified with the CFT temperature.
- An explicit calculation is possible for a (2+1)-dimensional bulk with a BTZ black hole. The metric can be written in Schwarzschild coordinates as

$$\mathrm{ds}^2 = -\mathrm{f(r)dt}^2 + \frac{\mathrm{dr}^2}{\mathrm{f(r)}} + \mathrm{r}^2\mathrm{d}\phi^2, \qquad \mathrm{f(r)} = \mathrm{r}^2 - \mu.$$

- The Hawking temperature of the black hole is $\theta_0 = \sqrt{\mu}/(2\pi)$.

- The above metric can also be expressed as

$$ds^2 = \frac{1}{z^2}\left[dz^2 - N^2(\tau,z)d\tau^2 + A^2(\tau,z)d\phi^2\right],$$

where

$$\begin{aligned} N(\tau,z) &= a(\tau)\left(1 - \frac{\mu\, a(\tau)^2 - 3a'^2(\tau) + 2a(\tau)a''(\tau)}{4a^4(\tau)}\, z^2\right) \\ A(\tau,z) &= a(\tau)\left(1 + \frac{\mu\, a(\tau)^2 - a'^2(\tau)}{4a^4(\tau)}\, z^2\right). \end{aligned}$$

The metrics are related through the coordinate transformation

$$\begin{aligned} t(z,\tau) &= \tau + \frac{1}{2\sqrt{\mu}} \log\left[\frac{4a^4 - \left(\sqrt{\mu}\, a(\tau) + a'(\tau)\right)^2 z^2}{4a^4 - \left(\sqrt{\mu}\, a(\tau) - a'(\tau)\right)^2 z^2}\right] \\ r(z,\tau) &= \frac{a(\tau)}{z}\left(1 + \frac{\mu\, a(\tau)^2 - a'^2(\tau)}{4a^4(\tau)}\, z^2\right). \end{aligned}$$

- The stress-energy tensor of the dual CFT on the time-dependent boundary, as determined via holographic renormalization, is

$$\begin{aligned} \rho &= -\langle T^t{}_t \rangle = \frac{1}{16\pi G_3}\left(\frac{\mu}{a^2} - \frac{a'^2}{a^4}\right) \\ P &= \langle T^\phi_\phi \rangle = \frac{1}{16\pi G_3}\left(\frac{\mu}{a^2} + \frac{-3a'^2 + 2aa''}{a^4}\right). \end{aligned}$$

- The terms proportional to μ/a^2 can be interpreted as the thermal energy density and pressure of a CFT at a temperature $\theta(T) = \theta_0/a(T)$.

- We consider the entanglement entropy of the CFT on a segment of comoving length $\Delta\phi = 2\mathrm{R}$ on the boundary.
- The minimization of the area functional can be performed by switching to the coordinates (t, r).
- The minimal curve has a trivial time dependence: $\mathrm{t}(\phi) = \mathrm{T}$=constant. T corresponds to the value that the function $\tau(\phi)$ on the minimal surface takes on the boundary.
- The minimal area is (Hubeny, Rangamani, Takayanagi 2007)

$$\mathrm{S} = \frac{1}{2\mathrm{G}_3} \log\left(\frac{1}{\pi\,\theta_0\,\epsilon_\mathrm{r}} \sinh(2\pi\,\theta_0\,\mathrm{R})\right).$$

- The cutoff ϵ_r has been imposed on the bulk coordinate at $\mathrm{r} = 1/\epsilon_\mathrm{r}$.
- This is the standard result for the static case (Calabrese, Cardy 2004).

- The cutoff ϵ_r can be related to the cutoff ϵ that we must impose on the Fefferman-Graham coordinate z for the metric with a time-dependent boundary.
- We find $\epsilon_r = \epsilon/a(T) + \mathcal{O}(\epsilon^3)$.
- The higher-order corrections can be neglected for a $(1+1)$-dimensional boundary. However, this is not the case in higher dimensions.
- We obtain

$$\begin{aligned} S &= \frac{1}{2G_3} \log\left(\frac{a(T)}{\pi\,\theta_0\,\epsilon} \sinh(2\pi\,\theta_0\,R)\right) \\ &= \frac{1}{2G_3} \log\left(\frac{1}{\pi\,\theta(T)\,\epsilon} \sinh(2\pi\,\theta(T)\,a(T)R)\right) \end{aligned}$$

for the time-dependent background.

(3+1)-dimensional boundary (preliminary)

- In 3+1 dimensions, the entanglement entropy of a thermalized CFT can be computed in closed form for a static strip geometry (Erdmenger, Miekley 2018).
- The bulk is asymptotically AdS and includes a planar black hole.
- This result can be used in order to compute the entanglement entropy for a FRW boundary.
- The necessary coordinate transformations that set the boundary metric in a FRW form are known (Apostolopoulos, Siopsis, Tetradis 2009).
- The effective UV cutoff can still be written as

$$\epsilon_{\rm eff}(\mathrm{t}) = \frac{\epsilon}{\mathrm{a}(\mathrm{t})}\left(1 - \frac{1}{4}\mathrm{H}^2(\mathrm{t})\epsilon^2\right).$$

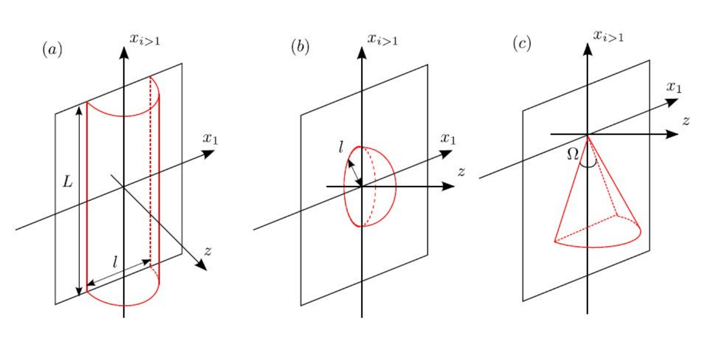

Figure: Nishioka, Ryu, Takayanagi (2009).

- For a large width of the strip, the maximal extension of the minimal surface z_* coincides with the black hole horizon z_h.
- In this limit, the entanglement entropy is

$$S = \frac{\tilde{l}^2 a^2(t)}{4G_5 \epsilon^2} + \frac{\tilde{l}^2 l}{4G_5 z_h^3} + G \frac{\pi \tilde{l}^2}{16\sqrt{6} G_5 z_h^2} + \frac{\tilde{l}^2 a^2(t) H^2(t)}{8G_5},$$

  where the AdS length has been set equal to 1, and $G \simeq -2.07678$ is a certain value of the Meijer G-function.
- The parameter l is the width of the strip, and $\tilde{l}$ stands for the size of the two dimensions parallel to the strip in comoving coordinates.
- The parameter $z_h = 1/(\pi T_h)$ is related to the Hawking temperature of the black hole, whose mass is $\mu = 1/z_h^4$ in five dimensions.

- The physical temperature $T(t)$ of the CFT redshifts as $T_h/a(t)$, as can be checked by computing the dual stress-energy tensor

$$-T^0{}_0 = \rho = \frac{3}{64\pi G_5}\frac{4\mu + \dot{a}^4}{a^4}$$

$$T^i{}_i = P = \frac{1}{64\pi G_5}\frac{4\mu + \dot{a}^4 - 4a\dot{a}^2\ddot{a}}{a^4}$$

- The physical lengths are $a(t)l$ and $a(t)\tilde{l}$.
- The entanglement entropy can be rewritten as

$$S = \frac{\tilde{l}^2 a^2(t)}{4G_5\epsilon^2} + \frac{\pi^3\tilde{l}^2 l a^3(t)T^3(t)}{4G_5} + G\frac{\pi^3\tilde{l}^2 a^2(t)T^2(t)}{16\sqrt{6}G_5} + \frac{\tilde{l}^2 a^2(t)H^2(t)}{8G_5}.$$

- Using the standard AdS/CFT relation $\pi/(2G_5) = N^2$, the second term can be identified with the thermal entropy.

Cross-checks

- For $d = 3$, the expectation for the entanglement entropy is

$$S = C_1 \frac{\mathcal{A}}{\epsilon^2} + (C_2 + C_3 \mathcal{A}) \log(\epsilon) + C_4 \log(\mathcal{A}) + C_5 \mathcal{A},$$

with $\mathcal{A}(T, R) = 4\pi a^2(T) R^2$ the proper area.

- Our result is

$$S = \frac{1}{8G_5 \epsilon^2} \mathcal{A} + \frac{\pi}{2G_5} \log\left(\frac{\epsilon}{\sqrt{\mathcal{A}}}\right) + \frac{1}{16G_5}\left(H^2 + \frac{k}{a^2 R_0^2}\right) \mathcal{A}.$$

- The first term is the standard area term.
- The coefficient of the second term involves the central charge of the dual CFT, in this case the $\mathcal{N} = 4$ supersymmetric SU(N) gauge theory in the large-N limit: $\pi/(2G_5) = N^2$. For this CFT, the coefficients C_2 and C_4 are related: $C_2 = -2C_4 = N^2$.
- The last term depends explicitly on the expansion rate and the spatial curvature.
- The logarithmic term proportional to C_3 is absent for this particular CFT (Solodukhin 2008, 2011).

- Consistency with Maldacena, Pimentel 2012 for a conformal theory in dS.
- For d = 1 the entanglement entropy was computed in Chu, Giataganas 2017 for a dS boundary, using holography.
- A first-principle calculation of the entanglement entropy was performed in Berges, Floerchinger, Venugopalan 2018. It concerns the massless Schwinger model of quantum electrodynamics in 1+1 spacetime dimensions with an expanding geometry. The entanglement entropy in a finite rapidity interval $\Delta\eta$ is equivalent to that of a (1+1)-dimensional conformal field theory at a finite temperature T that scales as $1/\tau$ in terms of proper time. The entanglement entropy agrees with our result with $\mathrm{R} = \Delta\eta/2$, $\mathrm{a}(\tau) = \tau$ and $\theta_0 = 1/(2\pi)$.

Comments

- The divergent terms have the same form as for a static background, with the radius of the entangling surface corresponding to the physical radius a(T)R that determines the proper area of the surface. The divergences are associated with the entanglement of UV degrees of freedom very close to the entangling surface.
- A possible logarithmic divergence that would depend on the curvature of the background related to the expansion is absent for the particular theory that we considered.
- The spatial curvature gives a finite contribution that depends quadratically on the physical radius, so that the effect is proportional to the proper area. This contribution vanishes when the ratio of the physical radius to the curvature radius of the background goes to zero.

- The finite term involving the expansion rate accounts for contributions to the entanglement entropy from regions not confined in the vicinity of the entanglement surface.
- The dependence of the finite term on the square of the expansion rate means that it has the same value both for expanding and contracting cosmologies.
- The finite term is still proportional to the area of the entangling surface. There is no effect proportional to the volume in the zero-temperature case.
- For a cosmological expansion driven by matter with $\mathrm{p} = \mathrm{w}\varepsilon$ with $\mathrm{w} > -1/3$, the scale factor evolves as $\mathrm{a(T)} \sim \mathrm{T}^{2/(1+3\mathrm{w})}$. At very early times $\mathrm{T} \to 0^+$, the finite contribution to the entanglement entropy scales as $\mathrm{a}^2(\mathrm{T})\mathrm{R}^2\mathrm{H}^2(\mathrm{T}) \sim \mathrm{T}^{-2}$. Its growth at early times could be attributed to physical distances between adjacent points being small in this limit.
- As a final remark, we emphasize the general applicability of our results for any form of the scale function a(T) and any number of dimensions.