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Photons & Gravitons in a Casimir box

Massless scalar partition functions & Eisenstein series

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Physique théorique et
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in collaboration with $\left\{ \begin{array}{l} \text{Martin Bonte} \\ \text{Francesco Alessio} \end{array} \right.$

Motivation : BH partition function

operator quantization $\qquad Z(\beta) = \mathrm{Tr}\; e^{-\beta \hat{H}}$

PI representation $\qquad = \int \prod_{a,\tau} dq^a(\tau) \prod_{b,\tau} \frac{dp_b(\tau)}{2\pi\hbar}\; e^{-\frac{1}{\hbar} S_H^E}$

periodic paths in Euclidean time of period β

$$S_H^E[q,p] = \int_0^{\hbar\beta} d\tau' \left[-i\,\dot{q}^a p_a + H \right], \qquad H = \frac{1}{2} g^{ab} p_a p_b + V(q)$$

integrate out momenta $\qquad \tau' = i(\tau - t) \qquad t - t' = -i\hbar\beta$

$$Z(\beta) = \int \prod_{a,\tau} dq^a(\tau)\, \mathcal{M}\, e^{-\frac{1}{\hbar} S_L^E} \qquad S_L^E = \int_0^{\hbar\beta} d\tau' \left[\frac{1}{2} \dot{q}^a \dot{q}^b g_{ab} + V(q) \right]$$

$$\mathcal{M} = \left[\det(2\pi\hbar\, G) \right]^{-1/2} \qquad G^{ab}(\tau,\tau') = g^{ab}(q(\tau))\, \delta(\tau,\tau')$$

Gibbons & Hawking '77 compute Lagrangian PI for gravity

$$S_L^E = \int_V d^4x_E \sqrt{|g|}\, R + 2 \oint_{\partial V} \epsilon (K-K_0) \sqrt{|h|}\, d^3y$$

Einstein-Hilbert implements Dirichlet boundary conditions $g_{\mu\nu}|_{\partial V} = 0$

semi-classical expansion around black hole solution $g^0_{\mu\nu}$

$$\ln Z(\beta) = \underbrace{-\frac{1}{\hbar} S_L^E [g^0]}_{\text{contains all of black hole thermodynamics}} + O(\hbar^0)$$

contains all of black hole thermodynamics

$$\text{Entropy} = \frac{A}{4G\hbar}$$

Main questions :

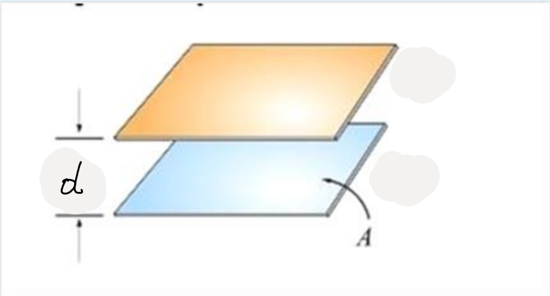
- What degrees of freedom are responsible for this contribution ?
- Can one construct the Hilbert space and do the trace in operator quantization ?
- Simpler models : linearized gravity $\longleftrightarrow$ electromagnetism
- What are good (spatial) boundary conditions ?
- Finite size effects in gauge, gravitational & scalar field theories

Perfectly conducting boundary conditions

Dirichlet / periodic boundary conditions ? — choice is irrelevant only in the large volume limit

undergraduate CEM course : perfectly conducting BC

$$\vec{E}^{\parallel}\big|_{\partial V} = 0 = \vec{B}^{\perp}\big|_{\partial V}$$



simplest geometry, box :

2 perfectly conducting parallel plates : set-up of

Casimir effect

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QUANTUM FIELD THEORY IN CURVED SPACETIME*

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Abstract:

Quantum field theory predicts a number of unusual physical effects in non-Minkowskian manifolds (flat or curved) that have no immediate analogs in Minkowski spacetime. The following examples are reviewed: (1) The Casimir effect; (2) Radiation from accelerating conductors; (3) Particle production in manifolds with horizons, including both stationary black holes and black holes formed by collapse. In the latter examples curvature couples directly to matter through the stress tensor and induces the creation of real particles. However, it also induces serious divergences in the vacuum stress. These divergences are analyzed, and methods for handling them are reviewed.

2.4. *The Casimir effect as a problem in manifold structure. The massless scalar field*

The method of computation in the above examples, in which we simply pick a set of basis functions appropriate to the desired boundary conditions, underscores the fact that even the Casimir effect is very much a problem of Riemannian manifold structure. In each case we pick a different Riemannian manifold – a slab, a half-space, or Minkowski space – and the properties of the vacuum depend on our choice. This prompts us to ask whether the properties we have found depend primarily on the manifold or are peculiar to the electromagnetic field. To answer this question in general would require the opening up of a whole new line of research. I can only report here on what I have found in the case of one other field, the massless scalar field.

What boundary conditions should one impose at the edges of a slab-manifold in the case of a scalar field? Setting the field equal to zero there would seem to be a natural procedure. And yet this leaves one with an uneasy feeling. What is the analog of a conductor in the case of a scalar field? In electromagnetic theory we know what a conductor is, both from years of experiment and years of model building. We do not hesitate to impose the standard boundary conditions for the electric and magnetic fields, because we know that the theory is consistent on many levels. Indeed, Boyer [5], in his study of the Casimir effect, has suggested that the electromagnetic field is unique – that there is no calculable analog of the Casimir effect for fields of other spin. Well, what are the facts?

Spectrum of photons in a Casimir box

$$S = \int dx^0 \left[\int_V d^3x \; \partial_0 A_i \pi^i - H + \int_V d^3x \; A_0 \partial_i \pi^i \right], \qquad H = \frac{1}{2} \int d^3x \left(\pi^i \pi_i + B^i B_i \right)$$

$x^i = (x^a, x^3) \qquad V = L_1 L_2 L_3 = A d \quad , \quad L_a$ large $\qquad \pi^i = -E^i , \quad B^i = \epsilon^{ijk} \partial_j A_k$

BC : $\vec{1}_{x_3} \times \vec{E} = 0 = \vec{1}_{x_3} \cdot \vec{B}$ at $x^3 = 0$ & $x^3 = d \qquad k_3 = \frac{\pi n_3}{d} , \quad k_a = \frac{2\pi n_a}{L_a}$

compatible* with**

$$\pi^a = i \sum_{n_a, n_3 > 0} \pi^a_k \psi^H_k , \quad \psi^H_k = \sqrt{\frac{2}{V}} \, e^{i k_a x^a} \sin k_3 x^3 \quad \text{Dirichlet}$$

$$\pi^3 = \sum_{n_a, n_3 \geq 0} \pi^3_k \psi^E_k \quad \text{Neumann} \quad \text{additional } k_3 = 0 \text{ mode}$$

$$= \sum_{n_a, n_3 = 0} \pi^3_{k^a, 0} \sqrt{\frac{1}{V}} \, e^{i k_a x^a} + \sum_{n_a, n_3 > 0} \pi^3_k \sqrt{\frac{2}{V}} \, e^{i k_a x^a} \cos k_3 x^3$$

idem for A_a, A_3

* not the most general solution
$$\left(\begin{array}{l} A_a^a = \partial_a \phi(x^0, x^a) + A_a \\ A_0^a = \partial_0 \phi(x^0, x^a) + A_0 \end{array} \right)$$

** disregard particle zero mode

"soft modes"

$$\Pi^3_{k_a=0,0} = p \,, \quad A^3_{k_a=0,0} = q \,, \quad H = \frac{1}{2} p^2$$

Adapted polarization vectors $e^i_{\parallel} = \frac{k^i}{k}$ $e^i_H = \frac{1}{k_\perp}\begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix}$ $e^i_E = \frac{1}{k_\perp k}\begin{pmatrix} k_1 k_3 \\ k_2 k_3 \\ -k_\perp^2 \end{pmatrix}$

change of variables $A^a_k, \pi^a_k, A^3_k, \pi^3_k \longleftrightarrow A^{\parallel}_k, \pi^{\parallel}_k, A^{H,E}_k, \pi^{H,E}$

constraint $\partial_i \pi^i = 0$ Coulomb gauge $\partial_i A^i = 0$ $\Longleftrightarrow$ $A^{\parallel}_k, \pi^{\parallel}_k = 0$

oscillators $a^{H,E}_k = \sqrt{\frac{k}{2}}\left(A^{H,E}_k + \frac{i}{k}\pi^{H,E}_k\right)$, $a^H_{k_0, k_3 = 0} = 0$ more E than H modes

Poisson brackets $\lambda = (E, H)$ $\{a^\lambda_k, a^{*\lambda'}_{k'}\} = -i\,\delta^{\lambda\lambda'} \prod_{i=1}^{3} \delta_{n_i, n'_i}$

Hamiltonian $H = \sum'_{\lambda, n_0, n_3 \geq 0} \frac{k}{2}\left(a^\lambda_k a^{*\lambda}_k + a^{*\lambda}_k a^\lambda_k\right)$

Bromwich-Borgnis scalar fields

$$\phi^{H} = \sum_{n_0, n_3 > 0} \frac{1}{\sqrt{2k}\, k_\perp} \left(a_k^{H} \psi_k^{H} + c.c. \right) \quad , \quad \phi^{E} = -\sum_{n_0, n_3 \geq 0}{}' \frac{1}{\sqrt{2k}\, k\, k_\perp} \left(a_k^{E} \psi_k^{E} + c.c. \right)$$

Dirichlet

Neumann

Helmholtz equation : $(\Delta + k^2)\, \phi^{\lambda} = 0$

EM with Casimir boundary conditions

$\Longleftrightarrow$ 2 massless scalar on $x^3 \in [0, d]$ with Dirichlet & Neumann BC conditions

$\Longleftrightarrow$ 1 massless scalar on $x^3 \in [-d, d]$ with periodic BC conditions

$L_3 = 2d$

NB: non-local map in terms of modes

Digression: Hamiltonian BFV-BRST approach: Quartet Mechanism

$$S = \int dt \int d^3x[\dot{A}_\mu \pi^\mu + \dot{\eta}\mathcal{P} + \dot{\bar{C}}\rho - \mathcal{H}_0 - \{\Omega, K_\xi\}],$$

$$\{A_\mu(\vec{x}), \pi^\nu(\vec{y})\} = \delta^\nu_\mu \delta^{(3)}(\vec{x}, \vec{y}),$$
$$\{\eta(\vec{x}), \mathcal{P}(\vec{y})\} = -\delta^{(3)}(\vec{x}, \vec{y}) = \{\bar{C}(\vec{x}), \rho(\vec{y})\}.$$

$$\mathcal{H}_0 = \frac{1}{2}(\pi^i \pi_i + B^i B_i),$$

$$\Omega = -\int d^3x(i\rho\pi^0 + \eta\partial_i\pi^i),$$

$$K_\xi = -\int d^3x\left(i\bar{C}\partial_k A^k + \mathcal{P}A_0 - \xi\frac{i}{2}\bar{C}\pi^0\right),$$

($\xi = 1$, Feynman gauge)

Dirichlet conditions for (A_0, π^0), $(\eta, \mathcal{P})$, $(\bar{C}, \rho)$

$$a_k = a_{k,0} + a_k^{\|}, \quad b_k = \frac{a_k^{\|} - a_{k,0}}{2}, \quad a_k^{\|} = \frac{a_k + 2b_k}{2}, \quad a_{k,0} = \frac{a_k - 2b_k}{2}$$

$$A_{k_a,k_3,0} = \frac{1}{\sqrt{2k}}(a_{k_a,k_3,0} + a^*_{-k_a,k_3,0}), \quad \pi^0_{k_a,k_3} = i\sqrt{\frac{k}{2}}(a_{k_a,k_3} - a^*_{-k_a,k_3}),$$

$$A^{\|}_{k_a,k_3} = -i\frac{1}{\sqrt{2k}}(a^{\|}_{k_a,k_3} - a^{*\|}_{-k_a,k_3}), \quad \pi^{\|}_{k_a,k_3} = -\sqrt{\frac{k}{2}}(a_{k_a,k_3} + a^*_{-k_a,k_3}),$$

$$\eta_{k_a,k_3} = -\frac{1}{\sqrt{2k^3}}(c_{k_a,k_3} + c^*_{-k_a,k_3}), \quad \mathcal{P}_{k_a,k_3} = i\sqrt{\frac{k^3}{2}}(\bar{c}_{k_a,k_3} + \bar{c}^*_{-k_a,k_3})$$

$$\bar{C}_{k_a,k_3} = -i\sqrt{\frac{k}{2}}(\bar{c}_{k_a,k_3} - \bar{c}^*_{-k_a,k_3}), \quad \rho_{k_a,k_3} = -\frac{1}{\sqrt{2k}}(c_{k_a,k_3} - c^*_{-k_a,k_3})$$

$$\Omega = -\int d^3x[i\rho\pi^0 + \eta\partial_i\pi^i] = \sum_{n_a,n_3>0}[-i\rho_k\pi^{0*}_k + k\eta_k\pi^{\|*}_k] = \sum_{n_a,n_3>0}[a^*_k c_k + c^*_k a_k]$$

$$K'_1 = -\int d^3x\,[i\bar{C}(\partial_i A^i - \frac{1}{2}\pi^0) + \mathcal{P}(A_0 + \frac{1}{2}\Delta^{-1}\partial_i\pi^i)]$$

$$= \sum_{n_a,n_3>0}[i\bar{C}_k(kA^{\|*}_k + \frac{1}{2}\pi^{0*}_k) - \mathcal{P}_k(A^*_{k,0} + \frac{1}{2k}\pi^{\|*}_k)] = \sum_{n_a,n_3>0} ik[\bar{c}^*_k b_k + b^*_k\bar{c}_k]$$

$$\int dx^0 \sum_{n_a,n_3>0}[\dot{A}^{\|}_k\pi^{*\|}_k + \dot{A}_{k,0}\pi^{*0}_k]$$

$$= \int dx^0\frac{1}{2i}\sum_{n_a,n_3>0}[-\dot{a}^*_{k,0}a_{k,0} + \dot{a}_{k,0}a^*_{k,0} + \dot{a}^{*\|}_k a^{\|}_k - \dot{a}^{\|}_k a^{*\|}_k]$$

$$= \int dx^0\frac{1}{2i}\sum_{n_a,n_3>0}[\dot{a}^*_k b_k - \dot{a}_k b^*_k + \dot{b}^*_k a_k - \dot{b}_k a^*_k]$$

$$\{a_{k,0}, a^*_{k',0}\} = i\delta_{k,k'}, \quad \{a^{\|}_{k,0}, a^{\|*}_{k',0}\} = -i\delta_{k,k'}$$

$$\{a_k, b^*_{k'}\} = -i\delta_{k,k'}, \quad \{c_k, \bar{c}^*_k\} = -i\delta_{k,k'}, \quad \{\bar{c}_k, c^*_{k'}\} = -i\delta_{k,k'}$$

$$-\int dx^0 \sum_{n_a,n_3>0}[\dot{\eta}_k\mathcal{P}^*_k + \dot{\bar{C}}_k\rho^*_k] = \int dx^0\sum_{n_a,n_3>0}\frac{1}{2i}[\dot{c}^*_k\bar{c}_k + \dot{c}_k\bar{c}^*_k + \dot{\bar{c}}_k c^*_k + \dot{\bar{c}}^*_k c_k]$$

$$\{\Omega, K_1\} = \sum_{n_a, n_3>0} [\frac{1}{2}\pi_k^{\|}\pi_k^{*\|} - kA_k^{\|}\pi_k^{*0} + k\pi_k^{\|}A_{k,0}^{*} - \frac{1}{2}\pi_k^0\pi_k^{*0} - i\mathcal{P}_k\rho_k^* + ik^2\bar{C}_k\eta_k^*]$$

$$= \sum_{n_a, n_3>0} k[a_k^* b_k + b_k^* a_k + \bar{c}_k^* c_k + c_k^* \bar{c}_k]$$

$$\mathrm{Tr} e^{-\beta[\hat{\Omega}, \hat{K}_1]} = 0$$

completely decoupled sector that does not contribute to partition function

Partition function

3 standard ways to evaluate scalar partition functions

- Lagrangian PI, zeta function

$$\ln Z(\beta) = \frac{1}{2}\zeta'_{-\Delta^4}(0) + \frac{1}{2}\ln(2\pi\mu^2)\,\zeta_{-\Delta^4}(0)$$

$$\zeta_{\frac{-\Delta^4}{2\pi\mu^2}}(s) = \frac{\frac{\pi L_a}{a}}{(2\pi)^2}\int d^2 a \sum_{n_4, n_3}\left[\left(\frac{2\pi n_4}{\beta}\right)^2 + \left(\frac{2\pi n_3}{L_3}\right)^2 + k_a k^a\right]^{-s}(2\pi\mu^2)^s$$

$$\ln Z(\beta) = \frac{L_1 L_2}{L_3^2}\, t\, \frac{1}{2\pi^2}\underbrace{\sum_{l,m\in\mathbb{Z}^2/(0,0)}\frac{1}{[l^2 t^2 + m^2]^2}}_{\text{Epstein zeta function}} \qquad t = \frac{\beta}{L_3}$$

- canonical operator approach

$$\ln Z(\beta) = -\beta E_0 + \zeta(3)\frac{L_1 L_2}{\beta^2} + \frac{2 L_1 L_2}{L_3^{3/2}\beta^{1/2}} \sum_{n_3}{}' \sum_{\ell\in\mathbb{N}^*} \left(\frac{|n_3|}{\ell}\right)^{3/2} K_{3/2}\left(2\pi \ell |n_3| t\right)$$

$$E_0 = -\zeta(4)\frac{L_1 L_2}{L_3^3}$$

$$\zeta(4) = \frac{\pi^2}{90}$$

$$E_0 = \langle 0|\hat{H}|0\rangle_{\beta\to\infty}$$

zero point Casimir energy requires renormalization

$$\zeta(3) = \frac{\zeta(3)}{2\pi}$$

contribution of photons with $k_3 = 0$ $\Rightarrow$ propagation // plates

partition function of massless scalar in 2+1 dim

leading contribution to entropy at low temperature

$$S(\beta) = (1-\beta\,\partial_\beta)\ln Z(\beta) \approx 3\,\zeta(3)\frac{L_1 L_2}{\beta^2} + \dots$$ proportional to Area

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\left(1+O(z^{-1})\right)$$

exponentially suppressed at low temperature

Digression 1: Casimir force

Free energy $\quad F(\beta) = -\frac{1}{\beta} \ln Z(\beta) \approx E_0 - \frac{\zeta(3)\, L_1 L_2}{\beta^2} + \dots$

Attractive Casimir force on plates $\quad \text{Force} = -\frac{\partial F(\beta)}{\partial L_3} = -3\,\zeta(4)\,\frac{L_1 L_2}{(2d)^4} + \dots$

$$= -\frac{\pi^2}{240}\,\frac{L_1 L_2}{d^4} + \dots$$

no contribution from scalar in 2+1 d

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News

Popular physics myth is all at sea

Does the ghostly Casimir effect really cause ships to attract each other?

Philip Ball

"We have caught physicists in the very act of creating a myth."

Some stories are just too good to be true. And according to physicist Fabrizio Pinto, that's exactly the case for an analogy routinely used by physicists to illustrate the mysterious Casimir effect.

The Casimir effect is a strange force of attraction seen between two surfaces separated by empty space. And it has become almost obligatory for popular articles on the subject to mention that the same effect also operates between ships lying close together in a strong swell.

This notion stems from an article published in 1996 by Dutch scientist Sipko Boersma[1]. He pointed out that the French nautical writer P. C. Caussée stated in his 1836 book The Album of the Mariner that two ships should not be moored too close together because they are attracted one towards the other by a certain force of attraction. Boersma suggested that this early observation could be described by a phenomenon analogous to the Casimir effect.

• proper time – heat kernel – world line approach

$$\ln Z(\beta) = -\frac{1}{2} \mathrm{Tr} \ln \frac{\delta^2 S_L^E}{\delta\phi\,\delta\phi} = \frac{1}{2} \int_0^\infty \frac{dt_s}{t_s} \, \mathrm{Tr} \, e^{-t_s \hat{H}^P}$$

Hamiltonian of free particle $(m=\frac{1}{2})$ in 4 Euclidean dimensions

$$\hat{H}^P = (\hat{p}_4)^2 + (\hat{p}_3)^2 + p_a p^a$$

$$S^1_\beta \times S^1_{L_s} \times \mathbb{R}^2, \quad \beta = L_4$$

"2 small & 2 large directions"

heat kernel $\langle x' | e^{-t_s \hat{H}^P} | x \rangle = K_{-\Delta^4}(x', x; t_s)$

$$\frac{d}{dt_s} K_{-\Delta^4}(x', x; t_s) = \Delta^4 K_{-\Delta^4}(x', x, t_s)$$

$$K_{-\Delta^4}(x', x; 0) = \delta^4(x - x')$$

$$\mathbb{R}^2: \quad K_{-\Delta^2}(x', x; t_s) = \frac{1}{(4\pi t_s)} e^{-\frac{(x'^a - x^a)(x'_a - x_a)}{4 t_s}}$$

$$S^1_\beta: \quad K_{-\Delta^1_P}(x', x; t_s) = \frac{1}{\beta} \vartheta_3\left(\frac{x'-x}{\beta} \,\Big|\, i \frac{4\pi}{\beta^2} t_s\right)$$

combining $S^1_\beta \times S^1_{L_3} \times \mathbb{R}^2$ $\qquad \ln Z(t) = \frac{\frac{\pi}{a} L_a \, t}{L_3^2 \, 2^5 \pi^2} \sum'_{m,n} \int_0^\infty dt_s \, t_s^{-3} \, e^{-\frac{1}{4 t_s}[n^2 + m^2 t^2]}$

$$t = \frac{\beta}{L_3}$$

temperature inversion symmetry $\qquad \ln Z\left(\frac{1}{t}\right) = t^2 \ln Z(t)$

"swapping cycles"

Brown & Maclay 1969

$\Rightarrow$ high from low temperature behavior (& vice-versa)

high temperature expansion $\qquad \ln Z(\beta) \approx \underbrace{\frac{\pi}{a} L_a L_3}_{V} \, \zeta(4) \, \beta^{-3} + \frac{\frac{\pi}{a} L_a}{L_3^2} \zeta(3) + \ldots$

NB: only for the first term : EM = 2x massless scalar

Modular invariance

Massless scalar in 1+1 dim :

$$Z(\beta,\alpha) = \mathrm{Tr}\; e^{-\beta \hat{H} + i\alpha \hat{P}} \qquad H = \frac{1}{2}\int_0^L dx\,(\pi^2 + \phi'^2) \qquad P = -\int_0^L dx\, \pi \phi'$$

2d CFT result (Polchinski '86, Itzykson-Zuber '86)

$$(\beta,\alpha) \longleftrightarrow (\tau,\bar\tau)\,, \quad \tau = \frac{\alpha + i\beta}{L}$$ modular parameter

$$\ln Z(\tau,\bar\tau) = \frac{1}{\sqrt{2\,\mathrm{Im}(\tau)}}\;\frac{1}{\eta(\tau)\,\overline{\eta(\tau)}}$$

zero mode

SL(2,$\mathbb{Z}$) invariance $\tau' = \dfrac{a\tau + b}{c\tau + d}$

$a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$

$$\ln Z(\tau',\bar\tau') = \ln Z(\tau,\bar\tau)$$

(temperature inversion symmetry $\tau' = -\frac{1}{\tau}$

$\alpha = 0$, $\tau = it = i\frac{\beta}{L}$)

- Is there a suitable observable in EM that allows to boost temperature inversion symmetry to full modular invariance ?

- Modular invariance in higher dimensions ? Cappelli - Coste '89, Cardy '91, Shaghoulian '16

- Simple answer for massless scalar on $S^1_\beta \times S^1_{Ld} \times \mathbb{R}^{d-1}$ (d spatial dim)

linear momentum in small x^d direction $P^S_d = -\int_V d^dx \; \pi \, \partial_d \phi$

$$Z(\beta, \alpha) = \mathrm{Tr}\, e^{-\beta \hat{H} + i\alpha \hat{P}_d} \qquad \tau = \frac{\alpha + i\beta}{L_d}$$

$$\ln Z(\tau', \bar{\tau}') = |c\tau + d|^{d-1} \ln Z(\tau, \bar{\tau})$$

Explicit expressions (Epstein zeta $\rightarrow$ real analytic Eisenstein series):

$$f_s(\tau,\bar\tau) = \sum_{(n_d,n_{d+1})\in\mathbb{Z}^2/(0,0)} \frac{\mathfrak{Im}(\tau)^s}{|n_{d+1}+n_d\tau|^{2s}}, \quad \mathfrak{Re}(s) > 1,$$

from functional approach

$$\ln \mathcal{Z}_d(\tau,\bar\tau) = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{\frac{d+1}{2}}} \frac{\prod_a L_a}{L_d^{d-1}} \frac{f_{\frac{d+1}{2}}(\tau,\bar\tau)}{\mathfrak{Im}(\tau)^{\frac{d-1}{2}}}$$

Fourier trsf from operator approach

$$\ln Z_d(\beta,\alpha) = -\beta E_0 + \frac{\Gamma(d-1)\zeta(d)}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})} \frac{\prod_a L_a}{\beta^{d-1}} + 2\frac{\prod_a L_a}{L_d^{\frac{d}{2}}\beta^{\frac{d-2}{2}}} \sum_{n_d}{}' \sum_{l\in\mathbb{N}*} (\frac{|n_d|}{l})^{\frac{d}{2}} K_{\frac{d}{2}}(2\pi l|n_d|\frac{\beta}{L_d}) e^{2\pi i l n_d \frac{\alpha}{L_d}}.$$

Mellin trsf from heat kernel

$$\ln Z_d(\beta,\alpha) = \frac{\prod_a L_a L_d \beta}{2^{d+2}\pi^{\frac{d+1}{2}}} \sum_{m,n}{}' \int_0^\infty dt\, t^{-\frac{d+3}{2}} e^{-\frac{1}{4t}L_d^2|m\tau+n|^2}$$

(after $t \rightarrow \frac{1}{t}$)

Comment: • existence of fully invariant $SL(2,\mathbb{Z})$ description in terms of $SL(2,\mathbb{Z})$ Eisenstein series and Riemann theta function.

• generalization to $\mathbb{R}^p \times \mathbb{T}^q \times S^1_\beta$ $\qquad p+q=d$

$\Rightarrow$ $SL(q+1,\mathbb{Z})$ Eisenstein series

skewed periodicities of $\mathbb{T}^q$ $\qquad x^a \sim x^a + \tilde{e}^a_\alpha$

constant vectors defining lattice

$y^\alpha = \tilde{e}_a{}^\alpha x^a \,, \quad y^\alpha \sim y^\alpha + \delta^\alpha_\beta$

unit periodicities

$$\tilde{g}_{\alpha\beta} = \hat{e}^a{}_\alpha \delta_{ab} \hat{e}^b{}_\beta$$ constant metric in y-space.

$$Z[\tilde{q}] = \int \mathcal{D}\phi \; e^{-S^E_L[\phi,\tilde{q}]}$$

$$S^E_L[\phi;\tilde{q}] = \frac{1}{2}\int d^p x \, d^{q+1} y \, \sqrt{\tilde{q}} \left[\partial^y_\alpha \phi \, \tilde{q}^{\alpha\beta} \partial^y_\beta \phi + \partial_I \phi \, \partial^I \phi \right]$$

- exact result in terms of $Sl(q+1,\mathbb{Z})$ invariant Eisenstein series
- in order to understand result in terms of operator quantization: use ADM parametrization!

For $d=3$, what is P_3^S in EM ?

use inverse map : $P_3^S = \sum_{k_\alpha \in \mathbb{Z}^2} \sum_{k_3>0} i k_3 \left(a^{*H}_k a^{\mp}_k - a^{*\mp}_k a^{H}_k \right)$

Spin angular momentum of light

$$\vec{J} = \int_V d^3x \; \vec{A}_\perp \times \vec{\Pi}_\perp$$

$$J^3 = \sum_{k_\alpha \in \mathbb{Z}^2} \sum_{k_3>0} \frac{i k_3}{k} \left(a^{*H}_k a^{\mp}_k - a^{*\mp}_k a^{H}_k \right)$$

$$P_3^S = \int_V d^3x \left[(-\Delta)^{1/2} \vec{A}_\perp \times \vec{\Pi}_\perp \right]^3$$

Gravitons in Casimir box

"perfectly conducting" boundary conditions on canonical variables h_{ij}, π^{ij}

$$h_{ab} = 0 = \pi^{ab} \qquad h_{33} = 0 = \pi^{33} \qquad (\partial_3 h_{a3}) = 0 = (\partial_3 \pi^{a3}) \qquad \text{at} \quad x^3 = 0, \; x^3 = d$$

→ determines mode expansion

Adapted polarization tensors in the change of basis

$$e_{TT+}{}^{ij} = \frac{1}{\sqrt{2}}\left(e_1{}^i e_1{}^j - e_2{}^i e_2{}^j\right), \qquad e_{TT\times}{}^{ij} = \frac{1}{\sqrt{2}}\left(e_1{}^i e_2{}^j + e_2{}^i e_1{}^j\right)$$

$$e_T{}^{ij} = \frac{1}{\sqrt{2}}\left(\delta^{ij} - e_\parallel{}^i e_\parallel{}^j\right), \qquad e_{LT\alpha}{}^{ij} = \frac{1}{\sqrt{2}}\left(e_\parallel{}^i e_\alpha{}^j + e_\alpha{}^i e_\parallel{}^j\right) \qquad \alpha = (H, E)$$

Hamiltonian form of Pauli-Fierz action = ADM action linearized around flat space

$$S = \int dx^0 \int d^3x \left(\partial_0 h_{ij} \pi^{ij} - n^i \mathcal{H}_i - n \mathcal{H}_\perp - \mathcal{H}_{PF} \right)$$

$$\mathcal{H}_i = -2 \partial^j \pi_{ij} \,, \quad \mathcal{H}_\perp = \Delta h - \partial^i \partial^j h_{ij} \,, \quad \mathcal{H}_{PF} = \pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2 + \frac{1}{4} \partial^l h^{ij} \partial_l h_{ij} - \frac{1}{2} \partial_i h^{ij} \partial^l h_{lj}$$
$$+ \frac{1}{2} \partial^i h \, \partial^j h_{ij} - \frac{1}{4} \partial^i h \, \partial_i h$$

solve the constraints in the action $\Rightarrow$ only $\underbrace{h^{TT\times}_{k}, \pi^{TT\times}_{k}}$ & $h^{TT+}_{k}, \pi^{TT+}_{k}$ remain

contain lower dim scalar field

at $k_3 = 0$, gravitons propagating // to the plates

same E, H modes but constructed out of h^{TTs}_{k}, π^{TTs}_{k}, $s = \times, +$

Same result for (extended) partition functions as EM or for periodic massless scalar on interval $L_3 = 2d$

Additional observable for modular invariance:

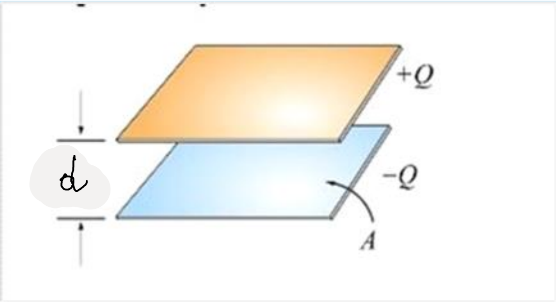
Spin angular momentum in linearized GR

$$J_i = \int d^3x \; \pi_{TT}^{mn} (\vec{e}_i \times h^{TT})^{mn}, \qquad (\vec{v} \times T)_{ij} = \tfrac{1}{2}(\epsilon_{ilm} v^l T^m{}_j + \epsilon_{jlm} v^l T^m{}_i)$$

generalized curl

$$P_3^S = \int d^3x \; (-\Delta)^{1/2} \; \pi_{TT}^{mn} (\vec{e}_i \times h^{TT})^{mn}$$

Digression 2: Gibbons-Hawking for charged capacitor



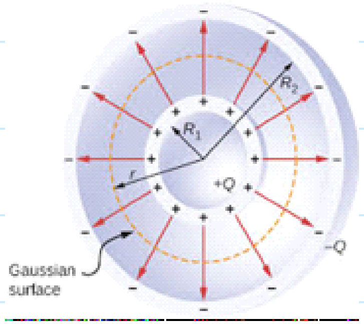
charged ∥ plates, improved action principle Regge-Teitelboim '74

$$S = \int dx^0 \left\{ \int d^3x \left[\dot{A}_i \pi^i - \tfrac{1}{2}\left(\pi^i \pi_i + B^i B_i\right) + A_0 \partial_i \pi^i \right] - \oint_{\partial V} d\sigma_i \, A_0 \, \pi^i \right.$$

electric charge observable $\quad Q = \oint_{\partial V} d\sigma_i \, \pi^i$

constant electric field $\quad \pi^i = -\delta^i_3 \dfrac{Q}{A}$ $\quad$ classical saddle point ($\Leftrightarrow$ BH solution)

$$\ln Z(\beta) \approx -S^E\Big|_{\text{saddle}} = \boxed{\frac{\beta \mu^2}{2d} A} \qquad \mu = -\left(A_0\Big|_{x^3=d} - A_0\Big|_{x^3=0}\right)$$



reproduces classical thermodynamics of planar capacitor

$$\ln Z(\beta) \approx C_P \frac{\beta \mu^2}{2}, \quad C_P = \frac{A}{d} \qquad \text{spherical case} \quad C_P \to C_S = \frac{4\pi R_1 R_2}{R_2 - R_1}$$

- microscopic derivation: remember the particle zero-mode (q,p) in the EM spectrum: when turning on the electric charge observable $Z(\beta,\mu) = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{Q})}$

$$Q = \int_{x^3=\mathrm{cte}} d^2x \, \Pi^3 = \sqrt{\frac{L_1 L_2}{d}}\, p$$

$$\ln Z(\beta,\mu)\big|_{\mathrm{particle}} = \boxed{\frac{\beta\mu^2}{2d} A}$$

reproduces Gibbons-Hawking result

no quantum corrections in Wigner-Kirkwood expansion because no potential

- still to be understood: more general soft sector due to general solution of boundary conditions $A_a = \partial_a \phi\big|_{\partial V}$

Conclusions & outlook

- At low temperature / small box, physical DoF in gauge & gravitational theories depend on the boundary conditions. Zero-modes & lower dimensional sectors are important, as in TFT.
- In linearized GR, gravitons propagating // to boundary give leading contribution to entropy at low temperature, scales like area.
- Investigate more complicated geometries, spin 0,1,2 fields in BH backgrounds, implications for BH physics in full GR ?

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