Geometrizing the Micro-Cosmos on a Supermanifold

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EPJC81 (2021) 572 [arXiv:2006.05831]

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PRD98 (2018) 016015

S. Karamitsos, AP, NPB927 (2018) 219

Outline:

- From Geometrizing the Cosmos to Micro-Cosmos
- Grand Covariance in Quantum Gravity
- The Fermion problem: Living on a Supermanifold?
- Grand Covariant Effective Action with Fermions
- Conclusions

• From Geometrizing the Cosmos to Micro-Cosmos

- **Geometrizing the Cosmos**: . . . , Pythagoras (5c BC)

- **Geocentric versus Heliocentric System**

  Geocentric: Anaximander (6c BC), . . . , Plato (4c BC), Aristotle (3c BC), Ptolemy (2c AD), . . . Tycho (16c AD)

  Heliocentric: Aristarchus (3c BC), Seleucus (2c BC), . . . , Copernicus (15c AD), Kepler (16c AD), Galileo (16c AD), . . .

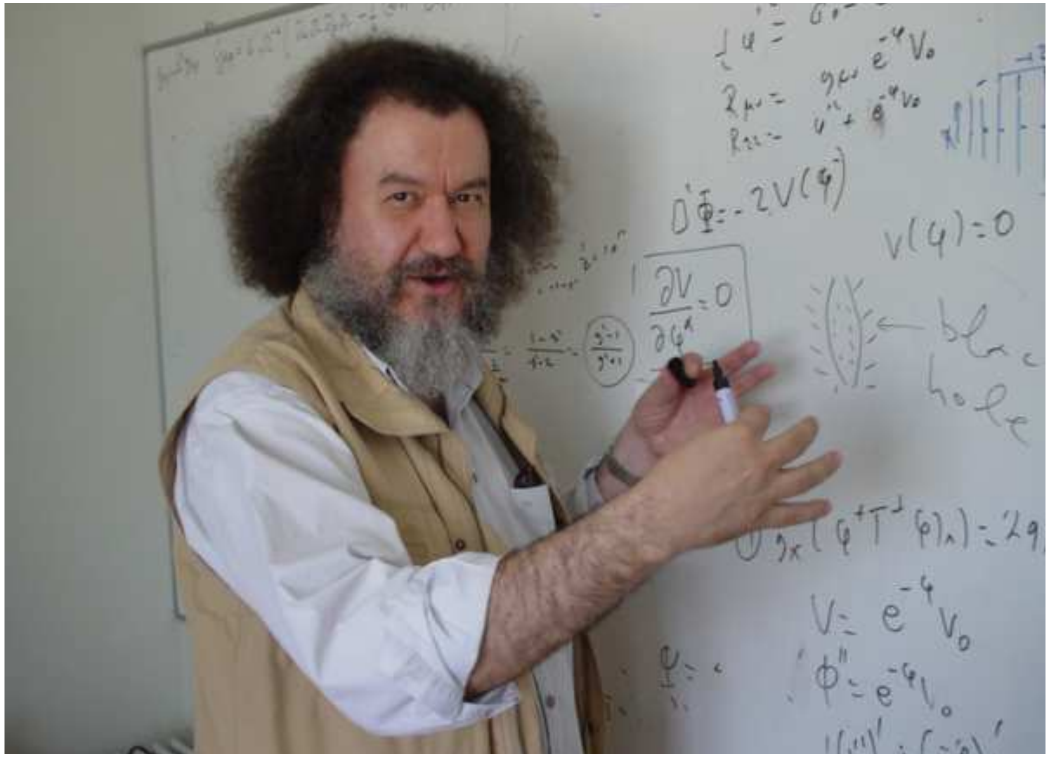
- **Absolute versus Relative/Local Inertial Frame in Gravitation**

  Absolute: Newton (17c AD), . . .

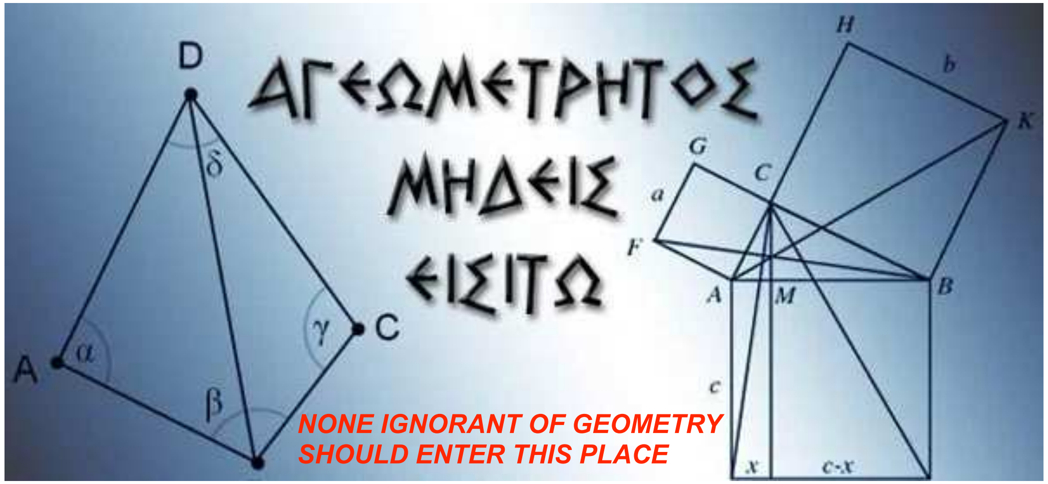
  Relative: Einstein (20c AD), . . .

- **Geometrizing the Micro-Cosmos** as a solution to frame problems in Quantum Field Theory and Quantum Gravity

– On the importance of Geometry



from Plato

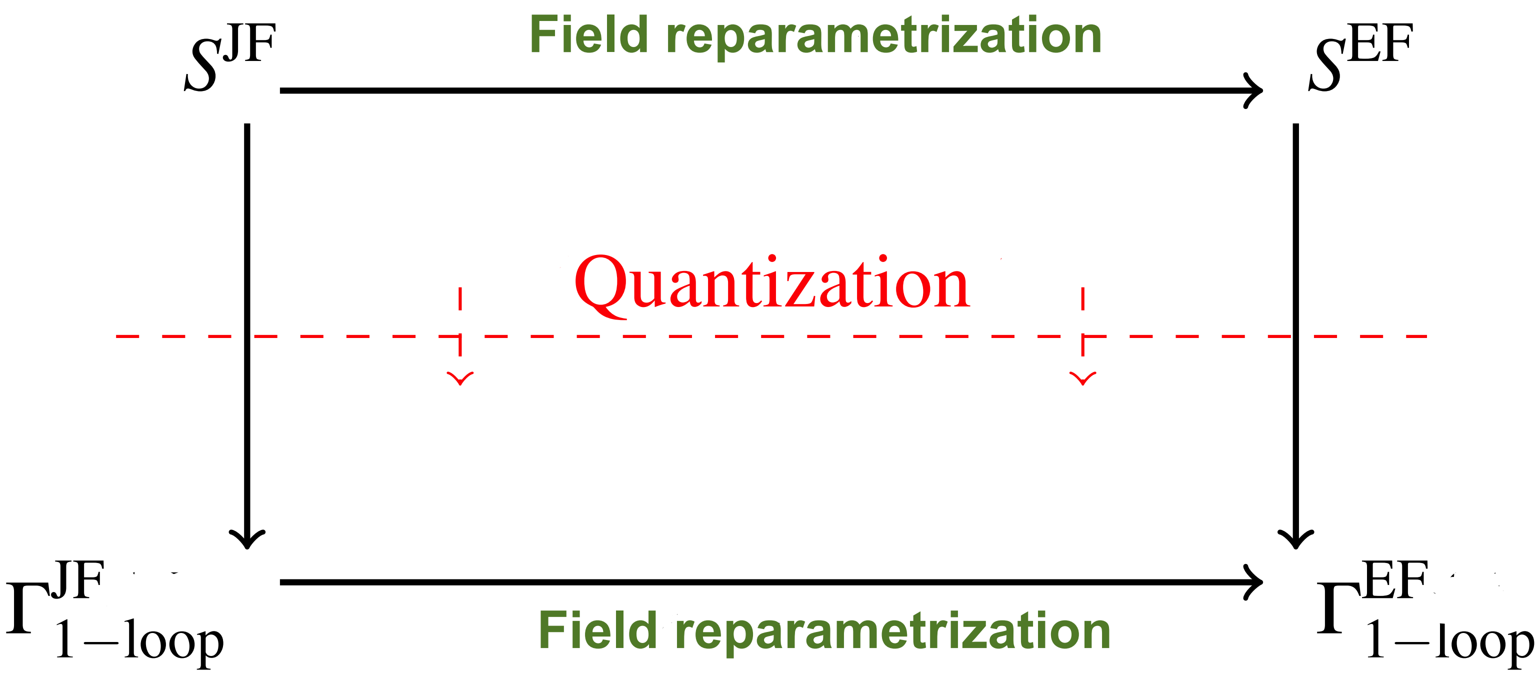


– Einstein versus Jordan Frame

Action in Einstein Frame: $S^{\rm EF}[g_{\mu\nu},\varphi] = \int_x \left[-\frac{1}{2}M_P^2 R + \frac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi)\right]$

Action in Jordan Frame: $S^{\rm JF}[\tilde{g}_{\mu\nu},\widetilde{\varphi}] = \int_x \left[-\frac{1}{2}f(\widetilde{\varphi})\widetilde{R} + \frac{1}{2}(\partial_\mu\widetilde{\varphi})^2 - \widetilde{V}(\widetilde{\varphi})\right]$

Frame equivalence $\Longrightarrow$ $S^{\rm JF}[\tilde{g}_{\mu\nu},\widetilde{\varphi}] = S^{\rm EF}[g_{\mu\nu},\varphi]$ [R. H. Dicke '62]



$\Gamma^{\rm JF}_{1-\rm loop}[\tilde{g}_{\mu\nu},\widetilde{\varphi}] \neq \Gamma^{\rm EF}_{1-\rm loop}[g_{\mu\nu},\varphi]$: **Effective action is frame dependent, except at extrema of the action.**

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• Grand Covariance in Quantum Gravity

[G. Vilkovisky '84, B. De Witt, '84;
K. Finn, S. Karamitsos, AP, '20]

$$S \;=\; \int d^4x\sqrt{-g}\;\left[-\frac{f(\boldsymbol{\varphi})}{2}R \;+\; \frac{1}{2}k_{AB}(\boldsymbol{\varphi})\,g^{\mu\nu}(\nabla_\mu\varphi^A)(\nabla_\nu\varphi^B) \;-\; V(\boldsymbol{\varphi})\right],$$

$S = S\big[g_{\mu\nu}, \boldsymbol{\varphi}; f(\boldsymbol{\varphi}), k(\boldsymbol{\varphi}), V(\boldsymbol{\varphi})\big]$: *classical action*

$f(\boldsymbol{\varphi}),\; k_{AB}(\boldsymbol{\varphi}),\, V(\boldsymbol{\varphi})$: *model functions*

Grand *or* Frame Covariance:

(i) *Spacetime diffeomorphisms*

$$x^\mu \;\to\; \tilde{x}^\mu = \tilde{x}^\mu(x^\nu), \quad \text{with} \;\; ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = \widetilde{g}_{\mu\nu}\,d\tilde{x}^\mu d\tilde{x}^\nu$$

(ii) *Field reparametrizations*

$$g_{\mu\nu} \;\to\; \widetilde{g}_{\mu\nu} = \widetilde{g}_{\mu\nu}(g_{\kappa\lambda}, \boldsymbol{\varphi}) \;=\; \Omega^2(\boldsymbol{\varphi})\,g_{\mu\nu}$$
$$\varphi^A \;\to\; \widetilde{\varphi}^A = \widetilde{\varphi}^A(g_{\mu\nu}, \boldsymbol{\varphi}) \;=\; \widetilde{\varphi}^A(\boldsymbol{\varphi})$$

• Grand Covariance in Quantum Gravity

[G. Vilkovisky '84, B. De Witt, '84; K. Finn, S. Karamitsos, AP, '20]

$$S \;=\; \int d^4x\sqrt{-g}\,\left[-\frac{f(\varphi)}{2}R \;+\; \frac{1}{2}k_{AB}(\varphi)\,g^{\mu\nu}(\nabla_\mu\varphi^A)(\nabla_\nu\varphi^B) \;-\; V(\varphi)\right],$$

$S = S\big[g_{\mu\nu}, \varphi; f(\varphi), k(\varphi), V(\varphi)\big]$: *classical action*

$f(\varphi),\; k_{AB}(\varphi),\; V(\varphi)$: *model functions*

Grand *or* Frame Covariance:

(i) *Spacetime diffeomorphisms*

$$x^\mu \;\to\; \tilde{x}^\mu = \tilde{x}^\mu(x^\nu), \quad \text{with} \quad \underbrace{\cancel{ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = \widetilde{g}_{\mu\nu}\,d\tilde{x}^\mu d\tilde{x}^\nu}}_{\text{Not invariant under (ii)}}$$

(ii) *Field reparametrizations*

$$g_{\mu\nu} \;\to\; \widetilde{g}_{\mu\nu} = \widetilde{g}_{\mu\nu}(g_{\kappa\lambda}, \varphi) = \underline{\Omega^2(\varphi)\,g_{\mu\nu}}$$

$$\varphi^A \;\to\; \widetilde{\varphi}^A = \widetilde{\varphi}^A(g_{\mu\nu}, \varphi) = \widetilde{\varphi}^A(\varphi)$$

– **Introduce new model function $\ell(\varphi)$ to restore (i)**

$$ds^2 = \bar{g}_{\mu\nu}\, dx^\mu dx^\nu\,, \quad \text{with} \quad \bar{g}_{\mu\nu} \equiv \frac{g_{\mu\nu}}{\ell^2(\varphi)}$$

– **Transformation of model functions**

$$\begin{aligned}
\tilde{\ell}(\varphi) &= \Omega\, \ell(\varphi)\,, \\
\tilde{f}(\varphi) &= \Omega^{-2}\, f(\varphi)\,, \\
\tilde{k}_{\widetilde{A}\widetilde{B}}(\varphi) &= \Big[k_{AB} - 6f(\ln\Omega)_{,A}(\ln\Omega)_{,B} + 3f_{,A}(\ln\Omega)_{,B} + 3(\ln\Omega)_{,A}f_{,B}\Big]\partial^A\varphi_{\widetilde{A}}\, \partial^B\varphi_{\widetilde{B}}\,, \\
\widetilde{V}(\varphi) &= \Omega^{-4}\, V(\varphi)\,.
\end{aligned}$$

– **Frame invariance of the classical action S:**

$$\boxed{S[\tilde{g}_{\mu\nu}, \widetilde{\varphi}; \tilde{\ell}(\varphi), \tilde{f}(\varphi), \tilde{k}(\varphi), \widetilde{V}(\varphi)] \;=\; S[g_{\mu\nu}, \varphi; \ell(\varphi), f(\varphi), k(\varphi), V(\varphi)] \;\; *}$$

Models related by a frame transformation define an *equivalence class*

---

* $\ell(\varphi)$ can be *reparametrized* to $\ell = 1$ at the tree level, by choosing $\Omega = 1/\ell$.

– Coordinates of the Grand Configuration Space

$$\Phi^i \equiv \Phi^I(x_I) = \begin{pmatrix} g^{\mu\nu}(x) \\ \phi^A(x_A) \end{pmatrix}, \text{ with } i = \{I\,,\, x_I\}\,,\ I = \{\mu\nu\,,\, A\}\,,\ x_I = \{x\,,\, x_A\}\,.$$

– The Grand Configuration Space Metric

$$\mathcal{G}_{ij} \equiv \frac{\bar{g}_{\mu\nu}}{D} \frac{\bar{\delta}^2 S}{\bar{\delta}(\partial_\mu \Phi^i)\bar{\delta}(\partial_\nu \Phi^j)} = \ell^2 \begin{pmatrix} f P_{\mu\nu\rho\sigma} & -\frac{3}{4} f_{,B}\, g_{\mu\nu} \\ -\frac{3}{4} f_{,A}\, g_{\rho\sigma} & k_{AB} \end{pmatrix} \bar{\delta}^{(D)}(x_I - x_J),$$

where $\bar{\delta}^{(D)}(x_I - x_J) \equiv \delta^{(D)}(x_I - x_J)/\sqrt{-g}$ is *frame invariant*, and

$$P_{\mu\nu\rho\sigma} \equiv G_{(\mu\nu)(\rho\sigma)} = \frac{1}{2}\Big(g_{\mu\rho}\, g_{\sigma\nu} + g_{\mu\sigma}\, g_{\rho\nu} - \alpha\, g_{\mu\nu}\, g_{\rho\sigma}\Big)$$

is the *gravitational field-space metric*.

Condition on the inverse metric $G^{(\mu\nu)(\rho\sigma)}$:

$$G^{(\mu\nu)(\rho\sigma)} = g^{\alpha\mu} g^{\beta\nu} g^{\kappa\rho} g^{\lambda\sigma}\, G_{(\alpha\beta)\,(\kappa\lambda)} \implies \alpha = 0 \text{ or } 1.$$

– Quantum Effective Action

$$\exp\left(\frac{i}{\hbar}\Gamma[\boldsymbol{\varphi}]\right) \;=\; \int[\mathcal{D}\boldsymbol{\phi}]\;\exp\left\{\frac{i}{\hbar}\left[S[\boldsymbol{\phi}] + \frac{\delta\Gamma[\boldsymbol{\varphi}]}{\delta\varphi^a}(\varphi^a - \phi^a)\right]\right\}.$$

– Quantum Effective Action

$$\exp\left(\frac{i}{\hbar}\Gamma[\boldsymbol{\varphi}]\right) = \int \cancel{[\mathcal{D}\boldsymbol{\phi}]}\, \exp\left\{\frac{i}{\hbar}\left[S[\boldsymbol{\phi}] + \frac{\delta\Gamma[\boldsymbol{\varphi}]}{\delta\varphi^a}\cancel{(\varphi^a - \phi^a)}\right]\right\}.$$

Not invariant under frame transformations

– Quantum Effective Action

$$\exp\left(\frac{i}{\hbar}\Gamma[\boldsymbol{\varphi}]\right) = \int [\mathcal{D}\boldsymbol{\phi}] \exp\left\{\frac{i}{\hbar}\Big[S[\boldsymbol{\phi}] + \frac{\delta\Gamma[\boldsymbol{\varphi}]}{\delta\varphi^a}(\varphi^a - \phi^a)\Big]\right\}.$$

Not invariant under frame transformations.

– VDW Quantum Effective Action

$$\exp\left(\frac{i}{\hbar}\Gamma[\boldsymbol{\varphi}]\right) = \int [\overline{\mathcal{D}}\boldsymbol{\Phi}]\, \mathcal{M}[\boldsymbol{\Phi}] \exp\left\{\frac{i}{\hbar}\Big[S[\boldsymbol{\Phi}] + \frac{\bar{\delta}\Gamma[\boldsymbol{\varphi}]}{\bar{\delta}\varphi^i}\,\Sigma^i[\boldsymbol{\varphi},\boldsymbol{\Phi}]\Big]\right\},$$

with $\boldsymbol{\varphi} = (g^{\mu\nu},\ \boldsymbol{\phi})$,

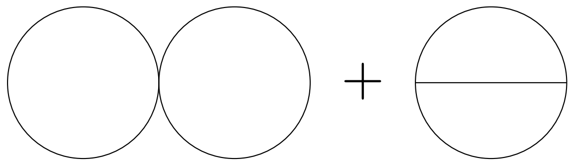
$$[\overline{\mathcal{D}}\boldsymbol{\Phi}] = \exp\Big[\sum_I \int d^Dx\,\sqrt{-\bar{g}(x)}\,\ln \mathcal{D}\Phi^I(x)\Big], \quad \mathcal{M}[\boldsymbol{\Phi}] = V_{\rm FP}\sqrt{\overline{\det}\,(\mathcal{G}_{ij})},$$

and $V_{\rm FP}$ is the *Faddeev–Popov determinant* [for SU(N), see Rebhan '87].

– One- and Two-Loop VDW Effective Actions

$$\Gamma^{(1)}[\boldsymbol{\varphi}] \;=\; -\frac{i}{2}\ln \overline{\det}\big(\nabla^a\nabla_b S\big)\,,$$

$$\Gamma^{(2)}[\boldsymbol{\varphi}] \;=\;$$



$$= \frac{1}{8}\Delta^{ab}\Delta^{cd}\,\nabla_{(a}\nabla_b\nabla_c\nabla_{d)}S$$

$$-\frac{1}{12}\Delta^{ab}\Delta^{cd}\Delta^{ef}\big(\nabla_{(a}\nabla_c\nabla_{e)}S\big)\big(\nabla_{(b}\nabla_d\nabla_{f)}S\big)\,,$$

with $\Delta^{ab} = \big(\nabla_a\nabla_b S\big)^{-1}$.

– Grand Covariance of the VDW Effective Action

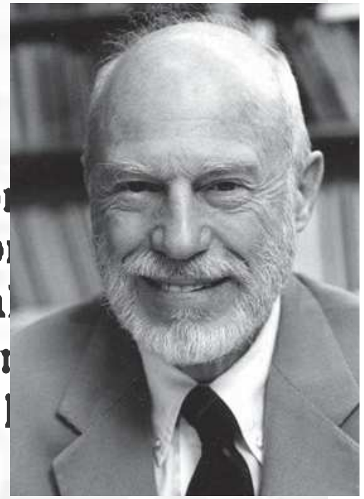
$$\boxed{\Gamma[\boldsymbol{\varphi};\ell(\phi),f(\phi),k_{AB}(\phi),V(\phi)\,] \;=\; \Gamma[\widetilde{\boldsymbol{\varphi}}(\boldsymbol{\varphi});\tilde{\ell}(\phi),\tilde{f}(\phi),\tilde{k}_{AB}(\phi),\widetilde{V}(\phi)\,]}$$

with $\boldsymbol{\varphi} = (g^{\mu\nu},\ \phi)$.

• The Fermion problem: Living on a Supermanifold?

The Effective Action

Bryce De Witt '84



14 DISCUSSION

This completes the basic outline of how Vilkovisky's idea fo invariant scalar effective action works, or can be made to wo discussing the advantages of such an effective action, I shall ta some of its possible defects. First of all, how unique is it? I do it is *in principle* as unique as Vilkovisky has claimed. Choices made for three quantities: the starting metric γ_{IJ}, the functional measure $\mu_K[I,K]$; all else follows from these. The last two have no effect on the final form of Γ. The measure μ_I, and hence μ, is determined by unitarity requirements. Expression (13.11) for μ appears to depend on a fourth arbitrary quantity, g^+, but in fact is independent of g^+. To show this just vary ${}_\alpha f_\beta$ and use (13.15).

That leaves γ_{IJ}. Vilkovisky (1984) has suggested that γ_{IJ} should be determined by the coefficient of the highest derivatives in the superclassial field equations. This cannot be correct in the fermion sector of supergravity theory, where the highest-order derivative is first order, because the coefficient of a first-order derivative cannot yield a tensor of even rank having

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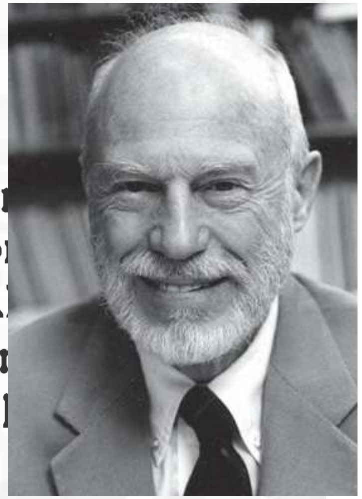

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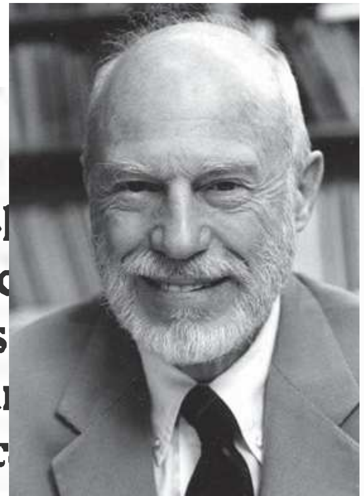
$\Longrightarrow$ **No known metric for theories with fermions:** $g_{XY}(\phi)\, \overline{\psi}^X i\gamma^\mu \partial_\mu \psi^Y$

The Effective Action

Bryce De Witt '84

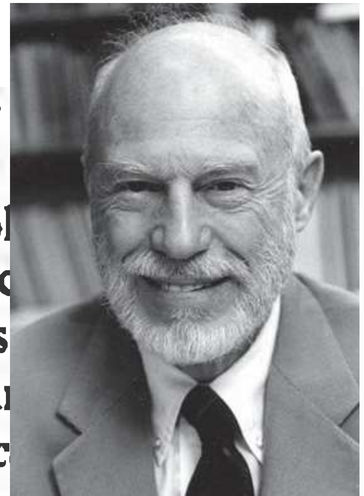


the right mass dimension. Such coefficients are much more
what they say about the singularity structures of the Green
and $\bar{\mathscr{G}}^{+}$, which are of relevance for unitarity. γ_{IJ} plays no rol
questions. Its only function is to provide (equation (11.18)) a c
Φ which respects the orbit decomposition and which enables
results to be obtained that are independent of how the va
chosen. Having said this, I must in fairness add that *in practic*
little freedom in the choice of γ_{IJ}. In all gauge theories certain choices stand
out as superior to all others for making the whole scheme work smoothly.
Although I cannot give a general algorithm for these metrics I believe that
Vilkovisky's effective action is *effectively* unique.

(**continued**)

The Effective Action

Bryce De Witt '84



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!

But, are we really living on a Supermanifold?

• Living on a Supermanifold

[K. Finn, S. Karamitsos, AP, EPJC81 (2021) 572]

– Fermions as Coordinates in the Field-Space Supermanifold

$$\mathbf{\Phi} \equiv \{\Phi^\alpha\} = \left(\phi^A,\quad \psi^1_a,\quad \overline{\psi}^1_{\dot{a}},\quad \psi^2_a,\quad \overline{\psi}^2_{\dot{a}},\quad \ldots\right)^{\mathsf{T}},$$

where ϕ^A are scalars and ψ^X_a are Dirac (or Weyl) fermions.

– Frame-covariant Lagrangian of a scalar theory with fermions

$$\mathcal{L} = \underbrace{\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi^\alpha\ {}_\alpha k_\beta(\mathbf{\Phi})\ \partial_\nu\Phi^\beta}_{:\ \text{scalars}} + \underbrace{\frac{i}{2}\zeta^\mu_\alpha(\mathbf{\Phi})\ \partial_\mu\Phi^\alpha}_{:\ \text{fermions}} - U(\mathbf{\Phi}).$$

Model functions:

${}_\alpha k_\beta(\mathbf{\Phi})$: *rank-2 tensor* of the would-be bosonic metric (with ${}_\alpha k_X = 0$)

$\zeta^\mu_\alpha(\mathbf{\Phi})$: mixed spacetime and field-space *vector*

$U(\mathbf{\Phi})$: a *scalar* describing the potential and Yukawa sector

– **Extracting the model functions ${}_\alpha k_\beta$ and ζ^μ_α**

$$ {}_\alpha k_\beta = \frac{g_{\mu\nu}}{4} \frac{\overrightarrow{\partial}}{\partial(\partial_\mu \Phi^\alpha)} \mathcal{L} \frac{\overleftarrow{\partial}}{\partial(\partial_\nu \Phi^\beta)} , \qquad \zeta^\mu_\alpha = \frac{2}{i} \left(\mathcal{L} - \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi^\alpha \, {}_\alpha k_\beta \, \partial_\nu \Phi^\beta \right) \frac{\overleftarrow{\partial}}{\partial(\partial_\mu \Phi^\alpha)} $$

– **Free theory as an example**

$$ \begin{aligned} \mathcal{L} \;&=\; \sum_{A \in N \text{scalars}} \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^A - \frac{1}{2} m_A^2 (\phi^A)^2 \\ &\quad + \sum_{X \in M \text{fermions}} \frac{i}{2} \left(\overline{\psi}^X \gamma^\mu \partial_\mu \psi^X - \partial_\mu \overline{\psi}^X \gamma^\mu \psi^X \right) - m_X \overline{\psi}^X \psi^X \, . \end{aligned} $$

Model functions:

$$ \begin{aligned} {}_\alpha k_\beta \;&=\; \begin{pmatrix} \delta_{AB} & \mathbf{0}_{N \times 8M} \\ \mathbf{0}_{8M \times N} & \mathbf{0}_{8M \times 8M} \end{pmatrix} , \\ \zeta^\mu_\alpha \;&=\; \left(\mathbf{0}_N , \; \overline{\psi}^1_{\dot{a}} \gamma^\mu_{\dot{a}a} , \; \gamma^\mu_{\dot{a}a} \psi^1_a , \; \overline{\psi}^2_{\dot{b}} \gamma^\mu_{\dot{b}b} , \; \gamma^\mu_{\dot{b}b} \psi^2_b , \; \ldots \right) \end{aligned} $$

– Deriving the Grand Metric

Define the rank-1 field-superspace tensor,

$$\zeta_\alpha(\mathbf{\Phi}) \;=\; \frac{\delta^\nu_\mu}{4}\,\frac{\delta\zeta^\mu_\alpha(\mathbf{\Phi})}{\delta\gamma^\nu} \;=\; \frac{1}{4}\,\frac{\delta\zeta^\nu_\alpha(\mathbf{\Phi})}{\delta\gamma^\nu}\,,$$

to derive the rank-2 anti-supersymmetric tensor (in analogy to $F_{\mu\nu}$ in QED)

$${}_\alpha\lambda_\beta(\mathbf{\Phi}) = \frac{\overrightarrow{\partial}}{\partial\Phi^\alpha}\zeta_\beta(\mathbf{\Phi}) \;-\; (-1)^{\alpha+\beta+\alpha\beta}\frac{\overrightarrow{\partial}}{\partial\Phi^\beta}\zeta_\alpha(\mathbf{\Phi})\,, \quad \text{with } \lambda^{\mathsf{sT}} = -\lambda.$$

Introduce the *non*-singular rank-2 tensor:

$${}_\alpha\Lambda_\beta \;\equiv\; {}_\alpha k_\beta \;+\; {}_\alpha\lambda_\beta \;\xrightarrow{\text{free theory}}\; {}_\alpha N_\beta \equiv \begin{pmatrix} 1_N & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1_4 & 0 & 0 & \cdots \\ 0 & 1_4 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1_4 & \cdots \\ 0 & 0 & 0 & 1_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

– Properties of the Grand Field-Space Metric ${}_\alpha G_\beta(\mathbf{\Phi})$

The Grand Metric ${}_\alpha G_\beta(\mathbf{\Phi})$ should:

1. Be *uniquely* determined from the *action*.
2. Transform as a proper *rank-2 field-space tensor*.
3. Be *supersymmetric* and *non-singular* to produce a non-zero line element.
4. Be *ultralocal*, i.e. it should not depend on $\partial_\mu \mathbf{\Phi}$.
5. Have the *local form* on each point of the field-space Supermanifold

$$
{}_a H_b \equiv \begin{pmatrix} 1_N & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1_4 & 0 & 0 & \cdots \\ 0 & -1_4 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1_4 & \cdots \\ 0 & 0 & 0 & -1_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .
$$

– The Grand Field-Space Metric

Determine first the *field-space vielbeins* ${}_{\alpha}e^{a}(\mathbf{\Phi})$ from

$$ {}_{\alpha}\Lambda_{\beta}(\mathbf{\Phi}) \;=\; {}_{\alpha}e^{a}(\mathbf{\Phi})\; {}_{a}N_{b}\; {}^{b}e^{\mathsf{sT}}_{\beta}(\mathbf{\Phi})\,, $$

and use these to obtain the Grand Field-Space Metric:

$$ \boxed{ {}_{\alpha}G_{\beta}(\mathbf{\Phi}) \;=\; {}_{\alpha}e^{a}(\mathbf{\Phi})\; {}_{a}H_{b}\; {}^{b}e^{\mathsf{sT}}_{\beta}(\mathbf{\Phi}) } $$

– The Christoffell Symbols

$$ {}^{\alpha}\Gamma_{\beta\gamma} \;=\; \frac{1}{2}{}^{\alpha}G^{\delta}\Big[{}_{\delta}G_{\beta}\overleftarrow{\partial}_{\gamma} + (-1)^{\beta\gamma}\;{}_{\delta}G_{\gamma}\overleftarrow{\partial}_{\beta} - (-1)^{\beta}\overrightarrow{\partial}_{\delta}\;{}_{\beta}G_{\gamma}\Big] $$

– The Riemann Tensor

$$ \begin{aligned} {}^{\alpha}R_{\beta\gamma\delta} \;=\; & -{}^{\alpha}\Gamma_{\beta\gamma}\overleftarrow{\partial}_{\delta} + (-1)^{\gamma\delta}\;{}^{\alpha}\Gamma_{\beta\delta}\overleftarrow{\partial}_{\gamma} + (-1)^{\gamma(\beta+\epsilon)}\;{}^{\alpha}\Gamma_{\epsilon\gamma}\;{}^{\epsilon}\Gamma_{\beta\delta} \\ & - (-1)^{\delta(\epsilon+\beta+\gamma)}\;{}^{\alpha}\Gamma_{\epsilon\delta}\;{}^{\epsilon}\Gamma_{\beta\gamma} \end{aligned} $$

• Grand Covariant Effective Action with Fermions

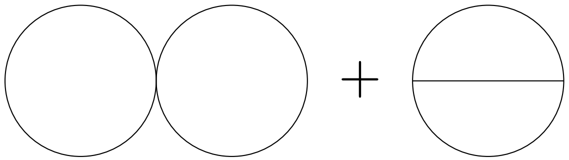
[K. Finn, S. Karamitsos, AP, EPJC81 (2021) 572]

$$\exp(i\Gamma[\mathbf{\Phi}]) = \int [D\mathbf{\Phi}_q]\,\sqrt{|\mathrm{sdet}G|}\,\exp\left(iS[\mathbf{\Phi}_q] - i\int d^4x\sqrt{-g}\,\Gamma[\mathbf{\Phi}]\frac{\overleftarrow{\partial}}{\partial\Phi^\alpha}\Sigma^\alpha[\mathbf{\Phi},\mathbf{\Phi}_q]\right)$$

– One- and Two-Loop Grand Covariant Effective Actions

$$\Gamma^{(1)}[\mathbf{\Phi}] = -\frac{i}{2}\ln \mathrm{sdet}\left(\overrightarrow{\nabla}^{\widehat{\alpha}}\,S\,\overleftarrow{\nabla}_{\widehat{\beta}}\right),$$ $\longleftarrow$ [e.g. see talk by A Dedes on SMEFT]

$$\Gamma^{(2)}[\mathbf{\Phi}] =$$



$$= \frac{1}{8}S\overleftarrow{\nabla}_{\{\widehat{\alpha}}\overleftarrow{\nabla}_{\widehat{\beta}}\overleftarrow{\nabla}_{\widehat{\gamma}}\overleftarrow{\nabla}_{\widehat{\delta}\}}\,{}^{\widehat{\delta}\widehat{\gamma}}\Delta\,{}^{\widehat{\beta}\widehat{\alpha}}\Delta$$
$$-(-1)^{\widehat{\gamma}\widehat{\beta}+\widehat{\epsilon}(\widehat{\delta}+\widehat{\beta})}\frac{1}{12}\left(S\overleftarrow{\nabla}_{\{\widehat{\epsilon}}\overleftarrow{\nabla}_{\widehat{\gamma}}\overleftarrow{\nabla}_{\widehat{\alpha}\}}\right){}^{\widehat{\alpha}}\Delta^{\widehat{\beta}}\,{}^{\widehat{\gamma}}\Delta^{\widehat{\delta}}\,{}^{\widehat{\epsilon}}\Delta^{\widehat{\zeta}}\left(\overrightarrow{\nabla}_{\{\widehat{\zeta}}\overrightarrow{\nabla}_{\widehat{\delta}}\overrightarrow{\nabla}_{\widehat{\beta}\}}S\right).$$

${}^{\widehat{\alpha}\widehat{\beta}}\Delta = (\overrightarrow{\nabla}_{\widehat{\alpha}}\overrightarrow{\nabla}_{\widehat{\beta}}\,S)^{-1}$, ${}^{\widehat{\alpha}}\Delta^{\widehat{\beta}} = (\overrightarrow{\nabla}_{\widehat{\alpha}}\,S\,\overleftarrow{\nabla}_{\widehat{\beta}})^{-1}$: rank-2 *frame-covariant props.*

– Single Fermion Model

$$\mathcal{L} \;=\; \frac{1}{2}k(\phi)\,\partial_\mu\phi\,\partial^\mu\phi \;+\; \frac{i}{2}g(\phi)\left(\overline{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\overline{\psi}\,\gamma^\mu\psi\right)$$
$$-\,\frac{1}{2}h(\phi)\,\overline{\psi}\gamma^\mu\psi\,\partial_\mu\phi \;-\; Y(\phi)\,\overline{\psi}\psi \;-\; V(\phi)$$

Supermanifold: $\Phi^\alpha = \big(\phi, \psi, \bar{\psi}\big)$, with grand field-space metric

$${}_\alpha G_\beta = \begin{pmatrix} k - \dfrac{g'^2 + h^2}{2g}\,\overline{\psi}\psi & -\frac{1}{2}(g' - ih)\,\overline{\psi} & \frac{1}{2}(g' + ih)\,\psi \\ \frac{1}{2}(g' - ih)\,\overline{\psi} & 0_4 & g1_4 \\ -\frac{1}{2}(g' + ih)\,\psi & -g1_4 & 0_4 \end{pmatrix}$$

But, ${}^\alpha R_{\beta\gamma\delta} = 0 \quad\Longrightarrow\quad$ field-space is flat

– Frame-reparametrization to a Cartesian Frame, $\widetilde{\Phi}^\alpha = (\widetilde{\phi},\ \widetilde{\psi},\ \widetilde{\overline{\psi}})^{\mathsf{T}}$:

$$\phi \ \to\ \widetilde{\phi} = \int_0^\phi \sqrt{k(\phi)}\, d\phi\,, \quad \psi\ \to\ \widetilde{\psi} = \sqrt{g(\phi)}\ \exp\bigg(\frac{i}{2}\int_0^\phi \frac{h(\phi)}{g(\phi)} d\phi\bigg)\ \psi$$

Lagrangian in the Cartesian Frame:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\widetilde{\phi}\,\partial^\mu\widetilde{\phi} + \frac{i}{2}\Big(\widetilde{\overline{\psi}}\gamma^\mu\partial_\mu\widetilde{\psi} - \partial_\mu\widetilde{\overline{\psi}}\,\gamma^\mu\widetilde{\psi}\Big) - \widetilde{Y}(\widetilde{\phi})\,\widetilde{\overline{\psi}}\widetilde{\psi} - \widetilde{V}(\widetilde{\phi})\,,$$

with $\widetilde{Y}(\widetilde{\phi}) = g(\phi)\,Y(\phi)$ and $\widetilde{V}(\widetilde{\phi}) = V(\phi)$.

– Grand Effective Action up to one-loop level

$$\Gamma[\Phi] = S[\Phi] - \frac{i}{2}\mathrm{Tr}\ln\bigg\{\Box + \widetilde{V}''(\widetilde{\phi}) + \widetilde{\overline{\psi}}\Big[2\,\widetilde{Y}'^{\,2}(\widetilde{\phi})\Big(\partial\!\!\!/ + \widetilde{Y}\Big)^{-1} - \widetilde{Y}''(\widetilde{\phi})\Big]\widetilde{\psi}\bigg\}$$
$$- i\mathrm{Tr}\ln\Big(\partial\!\!\!/ + \widetilde{Y}(\widetilde{\phi})\Big)$$

– Model with Multiple Fermions

The most general *frame-invariant* Lagrangian (up to quadratic kinetic terms)

$$\mathcal{L} \;=\; \frac{1}{2}\, g^{\mu\nu} k_{AB}(\mathbf{\Phi})\, \partial_\mu \phi^A\, \partial_\nu \phi^B \;+\; \frac{i}{2}\, g_{XY}(\mathbf{\Phi}) \left(\overline{\psi}^X \gamma^\mu \partial_\mu \psi^Y - \partial_\mu \overline{\psi}^X \gamma^\mu \psi^Y \right)$$
$$-\frac{1}{2}\, h_{AXY}(\mathbf{\Phi})\, \overline{\psi}^X \gamma^\mu \psi^Y \partial_\mu \phi^A \;+\; \frac{i}{2}\, j_{WXYZ}(\mathbf{\Phi})\, \overline{\psi}^W \gamma^\mu \psi^X \left(\overline{\psi}^Y \partial_\mu \psi^Z - \partial_\mu \overline{\psi}^Y \psi^Z \right)$$
$$- Y_{XY}(\mathbf{\Phi})\, \overline{\psi}^X \psi^Y - V(\boldsymbol{\phi})\,.$$

New kinetic model functions: $h_{AXY}(\mathbf{\Phi})$, $j_{WXYZ}(\mathbf{\Phi})$, with $j_{WXYY} = 0$.

Grand field-space metric (single scalar and $j_{WXYZ} = 0$):

$$ {}_\alpha G_\beta = \begin{pmatrix} k - \frac{1}{2}\overline{\boldsymbol{\psi}}(\boldsymbol{g}' - i\boldsymbol{h})\, \boldsymbol{g}^{-1}\, (\boldsymbol{g}' + i\boldsymbol{h})\boldsymbol{\psi} & -\frac{1}{2}\overline{\boldsymbol{\psi}}\, (\boldsymbol{g}' - i\boldsymbol{h}) & \frac{1}{2}\boldsymbol{\psi}^{\mathsf{T}}\, (\boldsymbol{g}' + i\boldsymbol{h}) \\ \frac{1}{2}\, (\boldsymbol{g}' - i\boldsymbol{h})\, \overline{\boldsymbol{\psi}}^{\mathsf{T}} & \mathbf{0} & \boldsymbol{g}\, 1_4 \\ -\frac{1}{2}\, (\boldsymbol{g}' + i\boldsymbol{h})\, \boldsymbol{\psi} & -\boldsymbol{g}\, 1_4 & \mathbf{0} \end{pmatrix}, $$

with $\boldsymbol{\psi} = \{\psi^X\}$, $\boldsymbol{g} = \{g_{XY}\} = \boldsymbol{g}^{\mathsf{T}}$ and $\boldsymbol{h} = \{h_{XY}\} = \boldsymbol{h}^{\mathsf{T}}$.

$${}^\alpha R_{\beta\gamma\delta} \neq 0 \quad \Longrightarrow \quad \text{field super-space has a non-zero curvature.}$$

• Conclusions

- Re-formulation of the Grand Covariant Effective Action for Scalar–Tensor Theories, with a complete set of model functions

- New model function $\ell = \ell(\Phi)$ determines the uniqueness of the VDW path-integral measure, with $ds^2 = g_{\mu\nu}/\ell^2\, dx^\mu dx^\nu$.

- Rigorous Algorithms for calculating the field-space metric from the Classical Action S for both bosons and fermions

- Extension of the VDW formalism on Supermanifolds to describe realistic theories that include fermions, such as the SM.

- Derivation of the Grand Covariant Effective Action for Theories with Fermions

Yes, we may well live on a Supermanifold

$$D\Sigma^2 \;=\; d\Phi^{\widehat{\alpha}}\;{}_{\widehat{\alpha}}G_{\widehat{\beta}}(\mathbf{\Phi})\; d\Phi^{\widehat{\beta}}$$

[K. Finn, S. Karamitsos, AP, EPJC81 (2021) 572]

Back-Up Slides

• Field-Space Riemann Tensor $\mathfrak{R}^{(\mu\nu)}{}_{(\alpha\beta)(\rho\sigma)(\gamma\delta)}$ for General Relativity

[K. Finn, S. Karamitsos, AP, PRD102 (2020) 045014]

$$
\begin{aligned}
\mathfrak{R}^{(\mu\nu)}{}_{(\alpha\beta)(\rho\sigma)(\gamma\delta)} = & -\frac{1}{32}\delta^\mu_\rho\delta^\nu_\beta g_{\sigma\gamma}g_{\alpha\delta} - \frac{1}{32}\delta^\mu_\sigma\delta^\nu_\beta g_{\rho\gamma}g_{\alpha\delta} - \frac{1}{32}\delta^\mu_\beta\delta^\nu_\sigma g_{\rho\gamma}g_{\alpha\delta} - \frac{1}{32}\delta^\mu_\beta\delta^\nu_\rho g_{\sigma\gamma}g_{\alpha\delta} \\
& -\frac{1}{32}\delta^\mu_\rho\delta^\nu_\beta g_{\sigma\delta}g_{\alpha\gamma} - \frac{1}{32}\delta^\mu_\sigma\delta^\nu_\beta g_{\rho\delta}g_{\alpha\gamma} - \frac{1}{32}\delta^\mu_\beta\delta^\nu_\sigma g_{\rho\delta}g_{\alpha\gamma} - \frac{1}{32}\delta^\mu_\beta\delta^\nu_\rho g_{\sigma\delta}g_{\alpha\gamma} \\
& -\frac{1}{32}\delta^\mu_\alpha\delta^\nu_\rho g_{\sigma\gamma}g_{\beta\delta} - \frac{1}{32}\delta^\mu_\alpha\delta^\nu_\sigma g_{\rho\gamma}g_{\beta\delta} - \frac{1}{32}\delta^\mu_\rho\delta^\nu_\alpha g_{\sigma\gamma}g_{\beta\delta} - \frac{1}{32}\delta^\mu_\sigma\delta^\nu_\alpha g_{\rho\gamma}g_{\beta\delta} \\
& -\frac{1}{32}\delta^\mu_\alpha\delta^\nu_\rho g_{\sigma\delta}g_{\beta\gamma} - \frac{1}{32}\delta^\mu_\alpha\delta^\nu_\sigma g_{\rho\delta}g_{\beta\gamma} - \frac{1}{32}\delta^\mu_\rho\delta^\nu_\alpha g_{\sigma\delta}g_{\beta\gamma} - \frac{1}{32}\delta^\mu_\sigma\delta^\nu_\alpha g_{\rho\delta}g_{\beta\gamma} \\
& +\frac{1}{32}\delta^\mu_\gamma\delta^\nu_\beta g_{\rho\delta}g_{\sigma\alpha} + \frac{1}{32}\delta^\mu_\delta\delta^\nu_\beta g_{\rho\gamma}g_{\sigma\alpha} + \frac{1}{32}\delta^\mu_\beta\delta^\nu_\delta g_{\rho\gamma}g_{\sigma\alpha} + \frac{1}{32}\delta^\mu_\beta\delta^\nu_\gamma g_{\rho\delta}g_{\sigma\alpha} \\
& +\frac{1}{32}\delta^\mu_\gamma\delta^\nu_\beta g_{\rho\alpha}g_{\sigma\delta} + \frac{1}{32}\delta^\mu_\delta\delta^\nu_\beta g_{\rho\alpha}g_{\sigma\gamma} + \frac{1}{32}\delta^\mu_\beta\delta^\nu_\delta g_{\rho\alpha}g_{\sigma\gamma} + \frac{1}{32}\delta^\mu_\beta\delta^\nu_\gamma g_{\rho\alpha}g_{\sigma\delta} \\
& +\frac{1}{32}\delta^\mu_\alpha\delta^\nu_\gamma g_{\rho\delta}g_{\sigma\beta} + \frac{1}{32}\delta^\mu_\alpha\delta^\nu_\delta g_{\rho\gamma}g_{\sigma\beta} + \frac{1}{32}\delta^\mu_\gamma\delta^\nu_\alpha g_{\rho\delta}g_{\sigma\beta} + \frac{1}{32}\delta^\mu_\delta\delta^\nu_\alpha g_{\rho\gamma}g_{\sigma\beta} \\
& +\frac{1}{32}\delta^\mu_\alpha\delta^\nu_\gamma g_{\rho\beta}g_{\sigma\delta} + \frac{1}{32}\delta^\mu_\alpha\delta^\nu_\delta g_{\rho\beta}g_{\sigma\gamma} + \frac{1}{32}\delta^\mu_\gamma\delta^\nu_\alpha g_{\rho\beta}g_{\sigma\delta} + \frac{1}{32}\delta^\mu_\delta\delta^\nu_\alpha g_{\rho\beta}g_{\sigma\gamma} \\
& +\frac{1}{4D}g_{\rho\gamma}g^{\mu\nu}g_{\sigma\beta}g_{\alpha\delta} + \frac{1}{4D}g_{\rho\delta}g^{\mu\nu}g_{\sigma\beta}g_{\alpha\gamma} + \frac{1}{4D}g_{\rho\alpha}g^{\mu\nu}g_{\sigma\gamma}g_{\beta\delta} + \frac{1}{4D}g_{\rho\alpha}g^{\mu\nu}g_{\sigma\delta}g_{\beta\gamma} \\
& +\frac{1}{4D}g_{\rho\gamma}g^{\mu\nu}g_{\sigma\alpha}g_{\beta\delta} + \frac{1}{4D}g_{\rho\delta}g^{\mu\nu}g_{\sigma\alpha}g_{\beta\gamma} + \frac{1}{4D}g_{\rho\beta}g^{\mu\nu}g_{\sigma\delta}g_{\alpha\gamma} + \frac{1}{4D}g_{\rho\beta}g^{\mu\nu}g_{\sigma\gamma}g_{\alpha\delta} \\
& -\frac{1}{4D}g^{\mu\nu}g_{\rho\beta}g_{\sigma\gamma}g_{\alpha\delta} - \frac{1}{4D}g^{\mu\nu}g_{\rho\beta}g_{\sigma\delta}g_{\alpha\gamma} - \frac{1}{4D}g^{\mu\nu}g_{\rho\gamma}g_{\sigma\alpha}g_{\beta\delta} - \frac{1}{4D}g^{\mu\nu}g_{\rho\delta}g_{\sigma\alpha}g_{\beta\gamma} \\
& -\frac{1}{4D}g^{\mu\nu}g_{\rho\alpha}g_{\sigma\gamma}g_{\beta\delta} - \frac{1}{4D}g^{\mu\nu}g_{\rho\alpha}g_{\sigma\delta}g_{\beta\gamma} - \frac{1}{4D}g^{\mu\nu}g_{\rho\delta}g_{\sigma\beta}g_{\alpha\gamma} - \frac{1}{4D}g^{\mu\nu}g_{\rho\gamma}g_{\sigma\beta}g_{\alpha\delta}
\end{aligned}
$$

– Field-Space Ricci Tensor $\mathfrak{R}_{(\mu\nu)(\rho\sigma)}$

$$\mathfrak{R}_{(\mu\nu)(\rho\sigma)} \;=\; \frac{1}{4} g_{\mu\nu} g_{\rho\sigma} - \frac{D}{8} g_{\mu\rho} g_{\nu\sigma} - \frac{D}{8} g_{\mu\sigma} g_{\nu\rho}$$

– Field-Space Ricci Scalar $\mathfrak{R}$

$$\mathfrak{R} \;=\; \frac{D}{4} - \frac{D^2}{8} - \frac{D^3}{8} \;<\; 0\,,$$

for spacetime dimensions $D > 1$.

$\Longrightarrow$ Gravity has a curved field space, with *negative* scalar curvature.